Theorem $A \in \mathbb{R}^{n\times n}$, there exists a permutation matrix $P \in \mathbb{R}^{n\times n}$, an L unit lower triangular, U upper triangular matrix such that

$$\boxed{PA = LU}$$

Proof: Induction over size n:

$n=2$ $\quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

last theorem

case 1: $a \neq 0 \longrightarrow A = LU$, $P = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Case 2: $a = 0$, $c \neq 0$: $\quad P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$PA = \begin{bmatrix} c & d \\ 0 & b \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} c & d \\ 0 & b \end{bmatrix}}_{U}$$

Case 3: $a = 0$, $c = 0$

$$\begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix}}_{U} \qquad P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Induction step: Choose a permutation matrix $P^{(1,r)}$ such that $P^{(1,r)}A$ has the element with the largest absolute value in the 1^{st} column at the 1^{st} spot.

$$P^{(1,r)}A = \left[\begin{array}{c|c} \alpha & w^T \\ \hline p & B \end{array}\right] = \left[\begin{array}{c|c} 1 & 0^T \\ \hline m & I \end{array}\right] \left[\begin{array}{c|c} \alpha & v^T \\ \hline 0 & C \end{array}\right]$$

I'll show that appropriate m, α, v, C can be found:

$v^T = w^T, \quad \alpha m = p, \quad C = B - m v^T$

case $\alpha = 0 \longrightarrow p = 0 \longrightarrow C = B, \quad m = 0$

$\alpha \neq 0 \quad m = \alpha^{-1} p$, with all entries of m have absolute value ≤ 1.

Induction hypothesis $\Longrightarrow \underbrace{P^*}_{\text{perm. matrix}} C = \underbrace{L^*}_{\text{unit lower tri'ang.}} U$ — upper triang.

$$P^{(1,r)} A = \begin{bmatrix} 1 & 0^T \\ 0 & (P^*)^T \end{bmatrix} \begin{bmatrix} 1 & 0^T \\ P^*_m & L^* \end{bmatrix} \begin{bmatrix} \alpha & v^T \\ 0 & U^* \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0^T \\ 0 & P^* \end{bmatrix} P^{(1,r)} \quad \text{permutation matrix}$$

$$PA = \begin{bmatrix} 1 & 0^T \\ P^*_m & L^* \end{bmatrix} \begin{bmatrix} \alpha & v^T \\ 0 & U^* \end{bmatrix}$$

$\square$.

§2.6 Computational cost

We count the number of elementary operation $(+, \cdot, /, -)$ as a measure of the cost of algorithms ("flops")

Consider LU without permutations (for simplicity):

$$l_{ij} = \frac{1}{u_{jj}} \left\{ a_{ij} - \sum_{k=1}^{j-1} l_{ik} u_{kj} \right\} \quad \begin{array}{l} i = 2, 3, \ldots, n \\ j = 1, \ldots, i-1 \end{array}$$

1 division
$j-1$ mult.
$j-1$ add/subst.

$$u_{ij} = a_{ij} - \sum_{k=1}^{i-1} l_{ik} u_{kj} \quad \begin{array}{l} i = 1, \ldots, n \\ j = i, \ldots, n \end{array}$$

$i-1$ mult.
$i-1$ adds/subst.

$$\sum_{i=2}^{n}\sum_{j=1}^{i-1}(2j-1) + \sum_{i=1}^{n}\sum_{j=i}^{n}(2i-2) = \frac{1}{6}n(n-1)(4n+1)$$

$$\sim \frac{2}{3}n^3 - \frac{1}{2}n^2$$

leading terms

"Complexity for LU-decomp."

<u>forward/backward subst:</u> $Ly = b$

$$y_i = b_i - \sum_{j=1}^{i-1} l_{ij} y_j \quad i = 2, 3, \dots n$$

$i-1$ mult.

$i-1$ subtracts/adds

$$\sum_{i=2}^{n} 2i-2 = 2\,\frac{n(n-1)}{2} \sim \underline{\underline{n^2}}$$

(analogue for backward subst.)

<u>Summary:</u> Solving linear system:

$\frac{2}{3}n^3 - \frac{1}{2}n^2$ (LU fact)

$2n^2$ (forw/backw. subst.)

<u>Inverse:</u> colums of A^{-1} are solutions to

$\underline{Ax_i = e_i}$

unit vectors