Mathematically, there are infinitely many real numbers, but computers can only work with a finite subset. Thus, there are gaps between numbers in a computer — this requires rounding. The gaps around the number 1 is machine epsilon. Its size depends on the type of number representation. It's about:

\[ \approx 10^{-6} \text{ for double precision, } \ll \text{double}^6 \]
\[ \approx 10^{-7} \text{ for single precision, } \ll \text{float}^4 \]

Computers store numbers in binary format, i.e., with a base of 2 instead of 10; A "bit" can have value 0, 1 such that

\[ 1001.101 \frac{1}{2} \text{ (base 2)} = 8 + 1 + \frac{1}{2} + \frac{1}{8} = 9.625 \text{ (base 10)} \]

Floating point representation:

\[
\text{base 10: } x = \pm S \times 10^E \quad 1 \leq S < 10
\]
\[
\text{base 2: } x = \pm S \times 2^E \quad 1 \leq S < 2
\]

So we only need to store the following:

Single

8-bit for \( E \)

23 bits for \( S \)

84 bits

32 bits
§ 2.7 Norms and Condition Numbers

We ask: What consequence has a small perturbation/error in $A \circ b$ on the solution $x$ of $Ax = b$?

Can small changes in $A \circ b$ potentially have a big influence on $x$? If yes, when?

**Matlab example:**

$$A = \begin{bmatrix} 4.5 & 2.1 \\ 1.6 & 1.1 \end{bmatrix}, \quad b = \begin{bmatrix} 19.249 \\ 6.943 \end{bmatrix}, \quad b_1 = \begin{bmatrix} 19.25 \\ 6.84 \end{bmatrix}$$

$$\Rightarrow x = \begin{bmatrix} 3.94 \\ 0.49 \end{bmatrix}, \quad x_1 = \begin{bmatrix} 2.9 \\ 2.0 \end{bmatrix}$$

Thus: The small change $\Delta b = \begin{bmatrix} 0.001 \\ -0.003 \end{bmatrix}$ has a big influence on $x$.

We consider the perturbed system

$$A(x + \Delta x) = b + \Delta b$$

where the perturbation $\Delta x$ is a consequence of the perturbation $\Delta b$.

We would like estimates like:

$$\|\Delta x\| \leq K \|\Delta b\|$$

for all $\Delta b$ with $K > 0$. 

```
\[ \frac{\|A x - b\|}{\|x\|} \leq K \frac{\|A b\|}{\|b\|} \quad \text{with } K > 0 \]

Natural questions:

- How can we find \( K, \hat{K} \) and how do they depend on \( A \)?
- What norms should we use?

Solving a 2x2 linear system is computing the intersection of two lines. This computation can be very sensitive to change.

\[ \text{Well-Conditioned Systems} \]

\[ \text{Ill-Conditioned Systems} \]

\[ k = \|A^T A\| \]

Condition number of \( A \)