1. [4pt] Let \( f(x) = e^x - x^2 - 2x - 1 \) and \( g(x) = 2 \ln(x + 1) \), where \( x \in (-1, \infty) \).

(a) Verify that the roots of \( f(x) \) are the same as the fixed points of \( g(x) \).

(b) Sketch \( y = g(x) \), \( y = x \) and indicate all fixed points. You don’t need to calculate them. (Hint for the sketch: Note that \( g'(0) > 1 \)).

(c) Use Brouwer’s fixed point theorem to argue the existence of a fixed point \( \xi \) in the interval \([a, b] = [e - 1, e^2 - 1] \).

(d) Use the contraction mapping theorem to show that \( \xi \) is the only fixed point in the interval \([e - 1, e^2 - 1] \).

2. [6pt] We attempt to find all solutions to \( f(x) = 0 \), where \( f(x) = e^x - 3x - 1 \).

(a) Sketch \( y = f(x) \) for \(-1 \leq x \leq 3 \). How many solutions \( \xi \) does \( f(x) = 0 \) have?

(b) Write code to implement the bisection method. Using the initial interval \([1, 3] \), write down the sequence of approximations \( x_1, x_2, x_3, x_4, x_5 \) produced from your code.
(c) What is the theoretical maximum value of $|x_5 - \xi|$? How large must we take $n$ to ensure that $|x_n - \xi| \leq 10^{-10}$?

(d) We now look at the fixed point problem $x = g(x)$ with $g(x) = \ln(3x + 1)$. Show that this is equivalent to finding the roots of $f$.

(e) Implement the fixed point iteration method for $x = g(x)$ given above. Using the initial point $x_0 = 1$, write down the iterates $x_1, x_2, x_3, x_4, x_5$.

(f) Plot the two sequences $(x_n)$ produced above as functions of $n$, with $n = 0, 1, \ldots, 100$. Is one method faster than the other?

3. [3pt] Let $\alpha \geq 0$ and consider the function

$$g(x) = x^3 - 2x^2 + 2x\alpha.$$  

(a) What are the fixed points of $g$ depending on $\alpha$ (calculate them analytically)? Make a plot with $\alpha$ as $x$-axis and the solution(s) as $y$-axis.

(b) Consider the fixed point iteration $x_{k+1} = g(x_k)$ for this $g$. What can you say about the stability of the fixed points in dependence of $\alpha$? You may assume that the initial guess is sufficiently close to the fixed point.

(c) Discuss the case $\alpha = 1$ either graphically, analytically or numerically.


(a) For each of the three functions (solid lines) depicted below,

(i) Write down the approximate values of the fixed points (as estimated by eye).

(ii) State for each fixed point, whether it is stable, unstable or neither of the two.

(b) You are given the first ten iterates of two sequences, $x_k$ and $y_k$, both of which converge to zero:
<table>
<thead>
<tr>
<th>$k$</th>
<th>$x_k$</th>
<th>$y_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.0000000000000</td>
<td>1.0000000000000</td>
</tr>
<tr>
<td>1</td>
<td>0.3000000000000</td>
<td>0.6648383611734</td>
</tr>
<tr>
<td>2</td>
<td>0.0900000000000</td>
<td>0.4404850619261</td>
</tr>
<tr>
<td>3</td>
<td>0.0270000000000</td>
<td>0.2895527955097</td>
</tr>
<tr>
<td>4</td>
<td>0.0081000000000</td>
<td>0.1869046766665</td>
</tr>
<tr>
<td>5</td>
<td>0.0024300000000</td>
<td>0.1155100169867</td>
</tr>
<tr>
<td>6</td>
<td>0.0007290000000</td>
<td>0.0638472856062</td>
</tr>
<tr>
<td>7</td>
<td>0.0002187000000</td>
<td>0.0254178900244</td>
</tr>
<tr>
<td>8</td>
<td>0.0000656100000</td>
<td>0.0032236709627</td>
</tr>
<tr>
<td>9</td>
<td>0.0000196830000</td>
<td>0.000008907744</td>
</tr>
<tr>
<td>10</td>
<td>0.0000059049000</td>
<td>0.0000000000001</td>
</tr>
</tbody>
</table>

(i) What do you think is the order of convergence of $x_k$? Explain your answer.
(ii) What do you think is the order of convergence of $y_k$? Explain your answer.

5. [3pt] Let $g$ be defined on $[5\pi/8, 11\pi/8]$.

$$g(x) = x + 0.8 \sin x.$$ 

determine the (smallest possible) Lipschitz constant $L$. What can you say about the asymptotic rate of convergence? How many iterations are required to increase the accuracy by one decimal place?

6. [3pt] We search for solutions in $[1, 2]$ to the equation

$$x^3 - 3x^2 + 3 = 0.$$ 

(a) Compute a solution using the secant method in $[1, 2]$, and write down $x_0, \ldots, x_5$.
(b) Find a solution using Newton’s method with starting value $x_0 = 1.5$, and write down $x_0, \ldots, x_5$.
(c) Find a solution using Newton’s method with starting value $x_0 = 2.1$. Sketch the equation graph and try to explain the behavior.

7. [3pt] Find the limit and order of convergence for the following sequences:

(a) $x_{k+1} = \alpha x_k$ for some $|\alpha| < 1$.
(b) $c_{k+1} = c_k - \tan c_k$.
(c) $b_k = 2^{-2^k}$.

8. [3pt] For $f : \mathbb{R} \to \mathbb{R}$ twice continuously differentiable, find the order of convergence of Steffensen’s method

$$x_{k+1} = x_k - \frac{[f(x_k)]^2}{f(x_k) + f(x_k) - f(x_k)}.$$ 

which is used to solve $f(x) = 0$. How does this iteration relate to Newton’s method?
9. [extra credit, up to 4pt] The logistic map \( g(x) = \alpha x(1 - x) \) with \( \alpha \in (0, 4) \) is a famous map modeling population dynamics.

(a) Show that for \( x_0 \in [0, 1] \) holds that \( x_{k+1} = g(x_k) \in [0, 1] \) for \( k = 1, 2, \ldots \) and that the only fixed points of \( g(\cdot) \) are \( \xi_1 = 0 \) and \( \xi_2 = 1 - 1/\alpha \).

(b) Show that \( \xi_1 \) is stable for \( \alpha \in (0, 1) \) and \( \xi_2 \) is stable for \( \alpha \in (1, 3) \).

(c) Definition: A period 2-cycle of a map \( g \) is a set of two distinct points \( \{x_0, x_1\} \), for which \( x_1 = g(x_0) \) and \( x_0 = g(x_1) \) holds. For \( \alpha \in [3, 1 + \sqrt{6}] \) calculate a period 2-cycle. Hint: Try to find fixed points of the map \( g^{(2)}(x) := g(g(x)) \).

(d) Implement a visualization of the bifurcation diagram for the logistic map by doing the following: Use at least 1000 equally-spaced values for \( \alpha \in [0, 4) \). Perform at least 1000 iterations per \( \alpha \)-value, always starting with \( x_0 = 0.5 \). Make a plot with \( \alpha \)-values plotted on the \( x \)-axis and the last roughly 100 values of your sequence on the \( y \)-axis.

(e) Plot the fixed points as well as the period 2-cycles from (a) and (c)—all are functions of \( \alpha \)—into the same figure as (d). What do you observe?

The resulting figure gives you a good idea of the attractive points of your map, i.e. values where the sequence \( (x_k)_k \) comes arbitrarily close, infinitely many times. To verify your figure, you can search the internet for Feigenbaum diagram.