Fall 2018: Numerical Analysis
Assignment 3 (due Oct. 23, 2019)

1. **[2+1+2pt+2pt (extra credit)]** Let us explore matrix norms and condition numbers.

   (a) For the following matrix given by
   $$A = \begin{bmatrix} 1 & -2 \\ 3 & -1 \end{bmatrix},$$
   calculate $\|A\|_1$, $\|A\|_2$, $\|A\|_\infty$ as well as the condition numbers for each norm by hand. Is A well or ill-conditioned?

   (b) Recall the formulas from Theorems 2.7 and 2.8 in the text book. If you assume that taking the absolute value and determining the maximum does not contribute to the overall computational cost, how many *flops* (floating point operations) are needed to calculate $\|A\|_1$ and $\|A\|_\infty$ for $A \in \mathbb{R}^{n\times n}$? By what factor will the calculation time increase when you double the size of matrix size?

   (c) Now implement a simple code that calculates $\|A\|_1$ and $\|A\|_\infty$ for a matrix of any size $n \geq 1$. Try to do this without using loops[1]! Using system sizes of $n_1 = 100$, $n_{k+1} = 2n_k, k = 1, \ldots, 7$, determine how long your code takes[2] to calculate $\|A\|_1$ and $\|A\|_\infty$ for a matrix $A \in \mathbb{R}^{n_i \times n_i}$ with random entries and report the results. Can you confirm the estimate from (b)?

   (d) **(extra credit)** MATLAB has the build-in function `norm` to calculate matrix norms.[3] Calculate for the system sizes in (c) $\|A\|_1$ and $\|A\|_\infty$ using both your implementation and MATLAB's `norm` function, determine for each n_i how long each code takes and plot the results in one graph. On average, by what factor is MATLAB's implementation faster than yours?

   Please also hand in your code.

2. **[4pt]** Let $A, B \in \mathbb{R}^{n\times n}$ and let the matrix norm $\|\cdot\|$ be induced by/subordinate of a vector norm $\|\cdot\|$.

   (a) Show that $\|AB\| \leq \|A\|\|B\|$.

   (b) For the identity matrix $I \in \mathbb{R}^{n\times n}$, show that $\|I\| = 1$.

   (c) For A invertible, show that $\kappa(A) \geq 1$, where $\kappa(A)$ is the condition number of that matrix A corresponding to the norm $\|\cdot\|$. Use the above two properties with $B := A^{-1}$ for your argument.

[1] The commands needed in MATLAB are `abs` and `sum`. Most commands can not only applied to numbers, but also to vectors, where they apply to each component.

[2] In MATLAB use the *stop watch* commands `tic` and `toc`.

[3] Use `help norm` to find out how to obtain the matrix norm that is induced by either the 1,2 or ∞-vector norm.

(d) Argue that the Frobenius matrix norm $\|A\|_F := \left(\sum_{i,j=1}^n a_{ij}^2\right)^{1/2}$ cannot be induced by a suitable vector norm.

3. **[3+3pt]** Estimates for vector and matrix norms.

   (a) Show that, for any $\boldsymbol{v} \in \mathbb{R}^n$, we have

   $$\|\boldsymbol{v}\|_\infty \leq \|\boldsymbol{v}\|_2 \quad \text{and} \quad \|\boldsymbol{v}\|_2^2 \leq \|\boldsymbol{v}\|_1 \|\boldsymbol{v}\|_\infty.$$

   In each case, give an example of a nonzero $\boldsymbol{v}$ for which equality is obtained.

   (b) Let us generalize the definition of matrix norms to non-square matrices. We define the $\|\cdot\|_p$ matrix norms ($p \in \{1, 2, \infty\}$) for an $m \times n$ matrix A by

   $$\|A\|_p = \sup_{\boldsymbol{v} \in \mathbb{R}^n \setminus \{\boldsymbol{0}\}} \frac{\|A\boldsymbol{v}\|_p}{\|\boldsymbol{v}\|_p}$$

   where the norm in the numerator is defined on $\mathbb{R}^m$ and the norm in the denominator is defined on $\mathbb{R}^n$.
   Using the problem above, show that

   $$\|A\|_\infty \leq \sqrt{n}\|A\|_2 \quad \text{and} \quad \|A\|_2 \leq \sqrt{m}\|A\|_\infty$$

   In each case, give an example of a nonzero matrix A for which equality is obtained.

4. **[3pt]** Let $A \in \mathbb{R}^{n \times n}$, let λ be an eigenvalue of A^TA and $\boldsymbol{x} \in \mathbb{R}^n \setminus \{\boldsymbol{0}\}$ be the corresponding eigenvector.

   (a) Show that $\|A\boldsymbol{x}\|_2^2 = \lambda \|\boldsymbol{x}\|_2^2$ and hence that $\lambda \geq 0$.

   (b) Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$ with associated subordinate matrix norm $\|\cdot\|$ on $\mathbb{R}^{n \times n}$. Show that $\|A\|_2 \leq \|A^TA\|^{1/2}$ (Hint: first show that $\lambda \leq \|A^TA\|$)

   (c) Using part (b) with matrix norm $\|\cdot\|_1$, show that $\kappa_2(A) \leq (\kappa_1(A)\kappa_\infty(A))^{1/2}$

5. **[3pt]** Let $A \in \mathbb{R}^{n \times n}$ be invertible. Let $\boldsymbol{b} \in \mathbb{R}^n \setminus \{\boldsymbol{0}\}$, and $A\boldsymbol{x} = \boldsymbol{b}$, $A\boldsymbol{x}' = \boldsymbol{b}'$ and denote the perturbations by $\Delta\boldsymbol{b} = \boldsymbol{b}' - \boldsymbol{b}$ and $\Delta\boldsymbol{x} = \boldsymbol{x}' - \boldsymbol{x}$. Show that the inequality obtained in Theorem 2.11 is *sharp*. That is, find vectors $\boldsymbol{b}, \Delta\boldsymbol{b}$ for which

   $$\frac{\|\Delta\boldsymbol{x}\|_2}{\|\boldsymbol{x}\|_2} = \kappa_2(A)\frac{\|\Delta\boldsymbol{b}\|_2}{\|\boldsymbol{b}\|_2}\ .$$

   (Hint: consider the eigenvectors of A^TA.)

6. **[3pt]** Using the `qr` function in MATLAB (or however else you like), find the QR factorization of

   $$A = \begin{bmatrix} 9 & -6 \\ 12 & -8 \\ 0 & 20 \end{bmatrix}.$$

Write down both formulations we discussed in class, i.e., $A = \hat{Q}\hat{R}$ with $\hat{Q} \in \mathbb{R}^{m\times n}$, $\hat{R} \in \mathbb{R}^{n\times n}$ as well as $A = QR$ with $Q \in \mathbb{R}^{m\times m}$, $\hat{R} \in \mathbb{R}^{m\times n}$. Use it to find the least squares solution to the system of linear equations

$$\begin{aligned} 9x - 6y &= 300 \\ 12x - 8y &= 600 \\ 20y &= 900 \ . \end{aligned}$$

Plot the three lines above and indicate the location of the least squares solution.

7. **[3pt]** We believe that a real number Y is approximately determined by X with the function

$$Y = a\exp(X) + bX^2 + cX + d \ .$$

We are given the following table of data connecting the values of X and Y:

X	0.0	0.5	1.0	1.5	2.0	2.5
Y	0.0	0.20	0.27	0.30	0.32	0.33

Using the above data points, write down five equations in the four unknowns a, b, c, d. The least squares solution to this system is the best fit function. Write down the normal equations for this system, solve them in MATLAB. Plot the data points (X, Y) as points[4] and the best fit function.

[4] Do not connect the points; in MATLAB you can do that using `plot(X,Y,'ro')`.