1. **[Hermite interpolation]** Let \( x_0 = 0, x_1 = 1, x_2 = 2 \). Recall that the Hermite interpolation of a function \( f \) at the points \( x_0, x_1, x_2 \) has the form

\[
p(x) = \sum_{j=0}^{2} H_j(x)f(x_j) + \sum_{j=0}^{2} K_j(x)f'(x_j).
\]

(a) Show that the polynomial \( H_1(x) \) in this representation is given by

\[x^4 - 4x^3 + 4x^2.\]

(b) Verify that the polynomial \( K_1(x) \) in this representation is

\[x^5 - 5x^4 + 8x^3 - 4x^2.\]

(c) Sketch \( H_2(x) \) and \( K_2(x) \) in the same graph without computing their exact form explicitly.

2. **[Composite trapezoidal and Simpson sum, 4+2pt]** Write codes\(^1\) to approximate integrals of the form

\[
I(f) = \int_{a}^{b} f(t) \, dt
\]

using the trapezoidal and Simpson’s rule on the sub-intervals \([x_{i-1}, x_i], i = 1, \ldots, m\), where \( x_i = a + ih, i = 0, \ldots, m \) with \( h = (b - a)/m \).\(^2\)

(a) Hand in listings of your codes, and use them to approximate the integral

\[
\int_{0.1}^{1} \sqrt{x} \, dx.
\]

Compare the numerical errors \( E \) for both quadrature rules (the exact value of the integral is \( \frac{2}{3} - \frac{1}{15\sqrt{10}} \)). Try different \( m \) (e.g., \( m = 10, 20, 40, 80, \ldots \)) and plot the quadrature errors versus \( m \) in a double-logarithmic plot.

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\(^1\) Ideally, you write functions \( \text{trapez}(f,a,b,m) \) and \( \text{simpson}(f,a,b,m) \), where \( f \) is a function handle (see [http://www.mathworks.com/help/matlab/matlab_prog/creating-a-function-handle.html](http://www.mathworks.com/help/matlab/matlab_prog/creating-a-function-handle.html) if you are not familiar with that concept) or \( f \) is the vector \((f(x_0), \ldots, f(x_m))\).

\(^2\) For these composite rules, see Definitions 7.1 and 7.2 in the book.
(b) To numerically study how the errors $E$ decrease with $m$, we assume that the errors behaves like $Cm^{\kappa}$, with to-be-determined $C, \kappa \in \mathbb{R}$. Applying the logarithm to $E = Cm^{\kappa}$ results in
\[
\log(E) = D + \kappa \log(m),
\]
where $D = \log(C)$. Use the values for $m$ and $\log(E)$ you computed in (a) to find the best-fitting values for $D$ and $\kappa$ in (1) by solving a least squares problem. Compare your findings for $\kappa$ with the theoretical estimates for the composite trapezoidal and Simpson’s rules.\(^3\)

3. **[Best 2-norm approximation, 2pt]** The upper row in the below figure shows a function $f$ together with a polynomial approximation. For three plots, the optimal best 2-norm fit for three different weights $w(x)$ is used, and one is the result of an Lagrange interpolation. Match the approximations in the upper row with the information (weight functions or interpolation points) in the lower row.

4. **[Orthogonal polynomials, 2+2pt]** Remember that a function $f$ is called even if $f(-x) = f(x)$ and odd if $f(-x) = -f(x)$ for all $x$ in its domain. Let $w$ be an even weight function on the interval $(-a, a)$ and \{\varphi_0, \varphi_1, \ldots, \varphi_n\} be a system of orthogonal polynomials on $(-a, a)$ with respect to $w$, constructed from the monomial basis $1, x, x^2, \ldots$ using Gram-Schmidt-Orthogonalization.

(a) Show that, if $j$ is even, then $\varphi_j$ is an even function and if $j$ is odd, then $\varphi_j$ is an odd function.

\(^3\)Compare with (7.16) and (7.18) in the book. You can ignore the constants, just compare $\kappa$, the exponent of $m$, with the theoretical results.
(b) Let $f : [-a, a] \to \mathbb{R}$ and $p_n(x) = \gamma_0\varphi_0(x) + \ldots + \gamma_n\varphi_n(x)$ its best polynomial approximation of degree $n$ with respect to the weighted 2-norm. Show that if $f$ is an even function, then all the odd coefficients $\gamma_{2j-1}$ are zero and if $f$ is an odd function, then all the even coefficients $\gamma_{2j}$ are zero.

5. [Newton-Cotes vs. Gauss Quadrature, 2+2+2+1pt] We discussed two methods to integrate functions numerically, namely the Newton-Cotes formulas and Gauss quadrature.

(a) Recall that we calculated the first three orthogonal polynomials with respect to $w \equiv 1$ on $(0,1)$ in class to be $\{\varphi_0, \varphi_1, \varphi_2\} = \{1, x - 1/2, x^2 - x + 1/6\}$. Calculate $\varphi_3(x)$ using the ansatz $\varphi_3(x) = x^3 - a_2\varphi_2(x) - a_1\varphi_1(x) - a_0\varphi_0(x)$, with appropriately computed $a_2, a_1, a_0 \in \mathbb{R}$.

(b) Derive the Gaussian Quadrature formula for $n = 2$, i.e., calculate both the quadrature points $x_0, x_1, x_2$ (these are the roots of $\varphi_3$ and the corresponding weights $W_0, W_1, W_2$.

(c) Now we want to compare Gaussian quadrature derived in (b) with the Simpson’s Rule. Use both methods to numerically find

$$I_k = \int_0^1 x^k dx, \text{ for } k = 0, \ldots, 7.$$ 

Plot the errors arising in each method as a function of $k$. Note that to find the error, you will need to calculate the exact values for $I_k$ (by hand).

(d) Explain your findings using the results on the exact integration for polynomials up to certain degrees discussed in class.

6. [Orthogonal polynomials on $[0, \infty)$, 2+2+2pt extra credit]

(a) Find orthogonal polynomials $l_0, l_1, l_2, l_3$ for the unbounded interval $[0, \infty)$ with the weight function $\omega(x) = \exp(-x)$. Plot these polynomials (they are called Laguerre polynomials).

(b) As these are orthogonal polynomials, they correspond to a quadrature rule for weighted integrals on $[0, \infty)$. The resulting quadrature points and weight are given in Table 1. Verify that for $n = 2, n = 3$, the quadrature nodes $x_i$ are the roots of the polynomials $l_2(x), l_3(x)$ (up to round-off).

(c) Use the quadrature rules from Table 1 to approximate the integrals

$$\int_0^\infty \exp(-x) \exp(-x) dx \text{ and } \int_0^\infty \exp(-x^2) dx.$$ 

Note that, to take into account the weight $\omega(x) = \exp(-x)$, for the first integral $f(x) = \exp(-x)$ and for the second $f(x) = \exp(-x^2 + x)$. Report the errors for $n = 2, 3, 4$ using that the exact values for the integrals are $1/2$ and $\sqrt{\pi}/2$.

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4See equation (10.7) in the book.

5Feel free to look up the values for the indefinite integrals $\int_0^\infty \exp(-t) t^k dx \quad (k = 0, 1, 2, 3)$—I use Wolfram Alpha for looking up things like that: [http://www.wolframalpha.com/](http://www.wolframalpha.com/).
Table 1: Gauss quadrature points and weights for quadrature on $[0, \infty)$.

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