Error analysis (§2 in Deuflhard/Hohmann)

input data x → algorithm / model f → output

input errors → algorithm errors → output errors

Sources of errors:

•) model errors (improve model)

•) data / input errors (improve measurement devices)

Comput. errors

∘ truncation / discretization errors (i.e. approx. by finite steps)

∘ rounding errors / errors in the algorithm due to finite precision

Conditioning of problems

How do perturbations / errors in input effect the output.
This concept is independent from the algorithm.

Examples for f: 1.) $f: d \in \mathbb{R} \to$ roots of $y^2 - d$
(input has two outputs)

2.) $x, b \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$

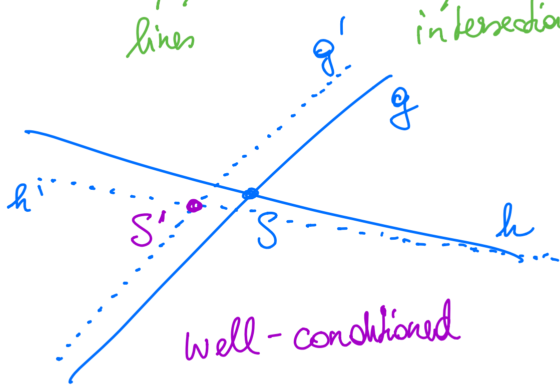
- $f: x \longmapsto Ax$ multiplication with a matrix
- $f: b \longmapsto A^{-1}b = x$ rhs to solution of $Ax = b$
- $f: A \longmapsto Ax$ perturbations w.r.to A.

3.) x, g functions $x, g: [0,1] \to \mathbb{R}$

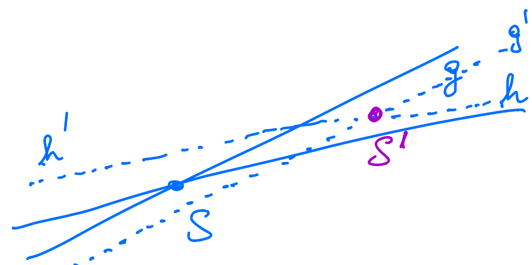
f: g to the solution of $-x'' = g$, $x(0) = x(1) = 0$

4.) $f: (g, h) \longrightarrow S$

lines intersection



well-conditioned



poorly conditioned

<u>Definition:</u> $f: U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $x \in U$, U open, $n, m \geq 1$

$\delta > 0$

$\|\tilde{x} - x\| \leq \delta$ absolute neighborhood around x

$\|\tilde{x} - x\| \leq \delta \|x\|$ relative neighborhood around x

Absolute condition number of f is the smallest $\kappa_{abs} \geq 0$ s.t.

$$\|f(\tilde{x}) - f(x)\| \leq \kappa_{abs} \|\tilde{x} - x\| \quad \text{for } \tilde{x} \to x$$

Relative condition number of f at x is the smallest $\kappa_{rel} \geq 0$ s.t.

$$\frac{\|f(\tilde{x})-f(x)\|}{\|f(x)\|} \leq \kappa_{rel} \frac{\|\tilde{x}-x\|}{\|x\|} \quad \text{for } \tilde{x} \to x$$

well-conditioned if $\kappa_{abs} / \kappa_{rel}$ small

poorly-conditioned if $\kappa_{abs} / \kappa_{rel}$ is large or infinity

Relation between condition numbers and derivatives:

f is differentiable : $f(x) = y$ original problem

$f(x+\delta x) = y + \delta y$ perturbed problem

$$\delta y = f(x+\delta x) - f(x) \overset{\text{Taylor}}{=} f'(x)\,\delta x + o(\|\delta x\|)$$

$$\overset{\text{use}}{\underset{\text{norms}}{\Longrightarrow}} \quad \frac{\|\delta y\|}{\|\delta x\|} \leq \|f'(x)\|$$

Jacobian, in $\mathbb{R}^{n\times m}$ if $f: \mathbb{R}^n \to \mathbb{R}^m$

$$\kappa_{abs} = \|f'(x)\|$$

$$\kappa_{rel} = \|f'(x)\| \frac{\|x\|}{\|f(x)\|}$$

Examples: $f: \mathbb{R}^2 \to \mathbb{R}$

$$\begin{pmatrix} a \\ b \end{pmatrix} \longmapsto a+b$$

$$f'\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} f_{,a} & f_{,b} \end{pmatrix} \in \mathbb{R}^{1\times 2}$$

$$= \begin{pmatrix} 1 & 1 \end{pmatrix}$$

We use the 1-norm in $\mathbb{R}^2$, i.e. $\left\|\begin{pmatrix} a \\ b \end{pmatrix}\right\| = |a| + |b|$

$$\left\| f'\begin{pmatrix} a \\ b \end{pmatrix} \right\| = \| (1, 1) \| \quad \text{matrix-norm induced by 1-norm}$$

$$= 1$$

$$\Longrightarrow \kappa_{abs} = 1$$

$$\kappa_{rel} = 1 \cdot \frac{|a|+|b|}{|a+b|}$$

Addition of numbers with same sign is well-conditioned

——— " ——— with different sign can be poorly conditioned if the numbers are close in absolute value.

As a consequence, round off can have bad effects, e.g.:

$a = 0.12345{*}\ldots$ $*$ means errors

$b = 0.12356{*}\ldots$

$a - b = 0.00011{*}\ldots$, so a, b were accurate for 5 digits, but $a-b$ is only accurate to 2 digits.

Warning: roundoff errors can have significant negative effect.

Example 2: $f : x \longmapsto Ax$

$f'(x) = A$, $\kappa_{abs} = \|A\|$

$$\kappa_{rel} = \|f'(x)\| \cdot \frac{\|x\|}{\|f(x)\|} = \|A\| \cdot \frac{\|x\|}{\|Ax\|}$$

Example 3: $f : b \longrightarrow A^{-1} b = x$ $Ax = b$

$$\kappa_{abs} = \|A^{-1}\|$$

$$\kappa_{rel} = \|A^{-1}\| \cdot \frac{\|b\|}{\|A^{-1}b\|} = \|A^{-1}\| \frac{\|Ax\|}{\|x\|} \leq$$

$$\leq \|A^{-1}\| \frac{\|A\|\,\|x\|}{\|x\|} = \|A\|\,\|A^{-1}\|$$

κ_A condition number of A

Example:

$$f: A \longrightarrow A^{-1}b = x$$

$\mathbb{R}^{n\times n}$, invertible

Derivative of f

$$F: A \longrightarrow A^{-1}$$

$$\mathbb{R}^{n\times n} \longrightarrow \mathbb{R}^{n\times n}$$

$C \in \mathbb{R}^{n\times n}$

$$F'(A)\,C = \underline{\underline{-A^{-1}CA^{-1}}}$$