Infinite-Dimensional Semi-Smooth Newton and Augmented Lagrangian Methods for Friction and Contact Problems in Elasticity

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Abstract. In this thesis second-order methods for contact and friction problems in linear elasticity are developed and analyzed in infinite-dimensional Hilbert spaces. First, a scalar simplified friction problem written as non-differentiable minimization problem is considered. By means of the Fenchel duality theorem it is shown that its dual problem is an inequality-constrained maximization of a smooth functional. For the solution of a smoothed version of this problem a primal-dual active set strategy and a semi-smooth Newton are proposed, and the close relation between these techniques is analyzed. For the solution of the original problem these methods can be combined with a first-order augmented Lagrangian method. Local as well as global convergence results are given. In the second part of this thesis the Signorini contact problem without friction is discussed and a semi-smooth Newton method as well as an exact and inexact augmented Lagrangian method for the solution are analyzed. Finally, the Signorini problem with Tresca as well as Coulomb friction is considered. In two dimensions, results from the above problems can be extended to the problem with Tresca friction. In arbitrary dimension, the complementarity function involves additional nonlinearities which results in a different generalized Newton method for the problem with Tresca friction. The methods carry over to the Signorini problem with Coulomb friction by means of fixed point ideas. Comprehensive numerical tests discuss, among others, the dependence of the algorithm's performance on material and regularization parameters and on the mesh and yield a remarkable efficiency of the proposed methods for the solution of problems involving contact and friction.

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Contents

CHAPTER 1

Introduction

Problems involving contact and friction abound in many practical applications. These include biological processes, the design of machines and transportation systems, and metal forming. The importance of these phenomena motivates the development of methods allowing a reliable and fast simulation.

Contact and friction problems are inherently nonlinear, making modeling, analysis and numerical realization truly challenging. A closed form for the solution of these problems is generally not known. Thus, fast and reliable numerical solution techniques are extremely important. This and the fast development of computing power have led to an increased research interest in the field of numerical algorithms for contact problems with and without friction.

In contact problems, one is concerned with the deformation of an elastic body whose surface or boundary may hit a rigid foundation. These problems are often named after Signorini, who considered, as early as in 1933, the frictionless contact of an elastic body with a rigid foundation [101]. Since it is not known in advance which part of the body's surface will be in contact, the main difficulty is to determine the contact zone between elastic body and rigid foundation. In addition, at this contact boundary frictional forces are often too large to be neglected. Thus, besides the non-penetration condition, one also has to take into account a frictional behavior in the contact zone. This causes yet another nonlinearity in the problem formulation.

While in the engineering community finite-dimensional discretizations of contact and friction problems are usually studied, little attention has been paid to their infinite-dimensional counterparts, specifically to Newton-type methods. This thesis focuses on the formulation and analysis of second-order solution algorithms for the Signorini problem with and without friction in a function space framework. Such an infinite-dimensional analysis gives more insight into the problem, which is also of significant practical importance since the performance of a numerical algorithm is closely related to the infinite-dimensional problem structure. In particular, it is desirable that the numerical method can be considered as a discrete version of a well-defined and well-behaved algorithm for the continuous problem. A finite dimensional approach misses important features such as the regularity of Lagrange multipliers and its consequences, and smoothing and uniform definiteness properties of the involved operators. It is well-accepted that these properties significantly influence the behavior of numerical algorithms.

The approach taken in this thesis is to a large extent based on writing the problems under consideration as optimization problems. Then, we can derive the Fenchel dual problem (see [42]), which allows to transform a non-differentiable minimization problem into an inequality constrained minimization of a smooth functional. Whenever possible, in this work the problems are also seen from the optimizational point of view, i.e., aside from using just the first-order necessary conditions of the Signorini problem, which are usually the starting points of the analysis, we additionally use for our investigation alternately the primal and dual formulations of the problem. Another important aspect of this work is the use of certain nonlinear complementarity (NC) functions, which allow one to write complementarity conditions as nonsmooth operator equations in function spaces. The main algorithmic techniques used are semi-smooth Newton methods in function spaces (see [24, 58, 69, 81, 104]), the closely related primal-dual active set strategy (see, for example, [14, 15, 57, 58, 60, 70, 82]) and the first-order augmented Lagrangian method (see [16, 17, 67]).

The outline of this work is as follows. Frequently used results are provided in the next chapter. In Chapter 3, a simplified friction problem is discussed. On the basis of this scalar model problem dual formulations, a regularization technique and algorithms in Hilbert spaces and their convergence properties are discussed. In Chapter 4, the contact problem without friction is investigated and generalized Newton and exact as well as certain inexact augmented Lagrangian methods for the solution are analyzed. In Chapter 5, the results for this problem are generalized to the case that Coulomb or Tresca friction occur in the contact zone. Certain nonlinear complementarity functions for both, 2D and 3D frictional contact are discussed. All chapters include comprehensive numerical tests. By means of several numerical examples the dependence of the algorithm on material and regularization parameters and on the mesh is discussed.

CHAPTER 2

Preliminaries

In this chapter we introduce some basic definitions and summarize theoretical results that will frequently be used in this work. Besides results from convex analysis, a generalized differentiability concept is introduced and its main properties are summarized. Finally we give an overview of the notations used in this thesis.

1. Convex Functions

This section covers results from convex analysis and duality theory that will be of importance in the subsequent chapters. We first discuss orthogonal projections onto convex sets, before we turn to the definition of convex conjugate functions and the subdifferential of a convex function. The statement of the Fenchel duality theorem together with a brief discussion conclude this section.

1.1. Projections onto convex sets. Here we characterize projections onto convex sets and summarize some basic properties of these mappings in order to have them available for the subsequent sections. In the sequel let H be a real Hilbert space with inner product $\langle \cdot\,, \cdot \rangle$ and induced norm $\|\cdot\|$.

Definition 2.1 (Orthogonal Projection). Let $C \subset H$ be nonempty, closed and convex. Then there exists a unique mapping $P : H \to C$ satisfying

$$\|v - Pv\| = \inf_{u \in C} \|v - u\| \quad \text{for all } v \in H.$$

The mapping P is called *orthogonal projection of H onto C.*

A well known equivalent characterization of the orthogonal projection Pv in a Hilbert space is that

$$\big\langle v - Pv, u - Pv \big\rangle \leq 0 \quad \text{for all } u \in C. \tag{2.1}$$

We now derive some basic formulas for projections onto convex sets. Let $v, w \in H$ be given and P denote the orthogonal projection onto the nonempty and closed convex set C. Utilizing that $Pv, Pw \in C$, we obtain from (2.1) that

$$\big\langle v - Pv, Pw - Pv \big\rangle \leq 0,$$
$$\big\langle w - Pw, Pv - Pw \big\rangle \leq 0.$$

Summing up the above equations results in

$$\big\langle (v - Pv) - (w - Pw), Pv - Pw \big\rangle \geq 0. \tag{2.2}$$

Equation (2.2) yields that

$$\begin{aligned}\big\langle v-w, Pv-Pw\big\rangle &= \big\langle (v-Pv+Pv)-(w-Pw+Pw), Pv-Pw\big\rangle \\ &= \big\langle (v-Pv)-(w-Pw), Pv-Pw\big\rangle + \big\langle Pv-Pw, Pv-Pw\big\rangle \qquad (2.3)\\ &\geq \|Pv-Pw\| \geq 0.\end{aligned}$$

This observation shows that projections onto convex sets are monotone operators.

1.2. Convex conjugate functionals and the subdifferential. In this paragraph we define the conjugate of a convex function and we recall the definition of the subdifferential. We start with some basic definitions for convex functions, where we utilize a vector space V.

DEFINITION 2.2 (Proper, lower semicontinuity). Let $C \subset V$ be convex and $\varphi : C \to \mathbb{R} \cup \{-\infty, \infty\}$ be a convex function. Then

- φ is called *proper*, if it nowhere takes the value $-\infty$ and if it is not identically equal to $+\infty$.
- φ is called *lower semicontinuous* if

$$\liminf_{n\to\infty} \varphi(v_n) \geq \varphi(v)$$

for all sequences $(v_n)_{n\geq 0}$ with $\lim_{n\to\infty} v_n = v$.

Next we define the conjugate of a convex function, which is a mapping defined on the topological dual $V^\star$.

DEFINITION 2.3 (Convex conjugate). For a convex function $\varphi : V \to \mathbb{R} \cup \{-\infty, \infty\}$ the *convex conjugate function* $\varphi^\star : V^\star \to \mathbb{R} \cup \{-\infty, \infty\}$ is defined by

$$\varphi^\star(x^\star) = \sup_{x\in V} \big\{\big\langle x^\star, x\big\rangle - \varphi(x)\big\}$$

for $x^\star \in V^\star$.

As simple example we derive the convex conjugate functional corresponding to $\varphi : \mathbb{R} \to \mathbb{R}$ defined by $\varphi(x) := |x|$. From the definition we get for $x^\star \in \mathbb{R}$

$$\varphi^\star(x^\star) = \sup\{x^\star x - |x|\} = \begin{cases} 0 & \text{if } |x^\star| \leq 1, \\ \infty & \text{else.}\end{cases} \qquad (2.4)$$

Thus, the convex conjugate of the absolute value function is the indicator function of the interval $[-1, 1]$.

Convex conjugate functions are important for the investigation of optimization problems. Given an optimization problem, by means of convex conjugate functions a so-called dual optimization problem can be derived (see Section 1.3) that often allows to gain a deeper insight into the problem structure. For properties of conjugate functionals and a geometric interpretation we refer, e.g., to [87, p. 195–199] and to the discussions in [9, 31, 42]. The convex conjugate functional is closely related to the subdifferential of a convex function, which we turn to next. The subdifferential is a multivalued mapping from V into the dual $V^\star$.

DEFINITION 2.4 (Subdifferential). Let $\varphi : V \to \mathbb{R} \cup \{-\infty, \infty\}$ be proper and convex. The *subdifferential* $\partial\varphi$ *of* φ *at* $v \in V$ is defined by

$$\partial\varphi(v) := \{v^\star \in V^\star : \varphi(w) - \varphi(v) \geq \langle v^\star, w - v\rangle \text{ for all } w \in V\}. \tag{2.5}$$

The elements $v^\star \in \partial\varphi(v)$ are called *subgradients* of φ at v.

The set $\partial\varphi(v)$ is convex, closed and possibly empty, see [42, p. 21]. Provided φ is finite and continuous at $v \in V$, then $\partial\varphi(x)$ is nonempty [42, p. 22]. Moreover, if φ is Gâteaux differentiable at $v \in V$, then $\partial\varphi(v)$ only contains a single point that coincides with the Gâteaux derivative. Conversely, if φ is continuous, finite and has only one subgradient at v, then φ is Gâteaux differentiable at v, and the Gâteuax derivative coincides with the element in $\partial\varphi(v)$, see [42, p. 23].

The next basic property makes us anticipate the important role of the subdifferential in optimization problems:

$$\varphi(u) = \min_{v\in V} \varphi(v) \quad \text{if and only if} \quad 0 \in \partial\varphi(u).$$

We next formulate the dual of an optimization problem and recall the Fenchel duality theorem, that clarifies the relation between primal and dual problem.

1.3. Fenchel duality theory. In this section we state the Fenchel duality theorem in infinite-dimensional spaces. For a more complete discussion of duality theory we refer to [42].

Let V and Y be Banach spaces and denote their topological duals by $V^\star$ and $Y^\star$, respectively. Furthermore, let $\Lambda \in \mathcal{L}(V, Y)$, i.e., Λ is a bounded linear operator from V to Y and let $\mathcal{F} : V \to \mathbb{R} \cup \{\infty\}$, $\mathcal{G} : Y \to \mathbb{R} \cup \{\infty\}$ be convex. We consider the following optimization problem, henceforth called the *primal problem*:

$$\inf_{u\in V} \Big\{\mathcal{F}(u) + \mathcal{G}(\Lambda u)\Big\}. \tag{$\mathcal{P}$}$$

Corresponding to $(\mathcal{P})$ we define the so-called *dual problem*:

$$\sup_{p^\star\in Y^\star} \Big\{ - \mathcal{F}^\star(-\Lambda^\star p^\star) - \mathcal{G}^\star(p^\star)\Big\}, \tag{$\mathcal{P}^\star$}$$

where $\Lambda^\star \in \mathcal{L}(Y^\star, V^\star)$ is the linear adjoint operator of Λ and $\mathcal{F}^\star : V^\star \to \mathbb{R}\cup\{\infty\}$, $\mathcal{G}^\star : Y^\star \to \mathbb{R} \cup \{\infty\}$ denote the convex conjugates of $\mathcal{F}$ and $\mathcal{G}$, respectively. The Fenchel duality theorem now relates the problems $(\mathcal{P})$ and $(\mathcal{P}^\star)$, see, e.g., [42, p. 59] or [87, p. 201].

THEOREM 2.5. *Suppose that $\mathcal{F}$ and $\mathcal{G}$ are proper and lower semicontinuous and that there exists $v_0 \in V$ with $\mathcal{F}(v_0) < \infty$, $\mathcal{G}(\Lambda v_0) < \infty$ and $\mathcal{G}$ is continuous at Λv_0. Furthermore, assume that $\mathcal{F}(u) + \mathcal{G}(\Lambda u) \to \infty$ as $\|u\| \to \infty$ and that V is reflexive. Then the problems $(\mathcal{P})$ and $(\mathcal{P}^\star)$ admit (at least) one solution $\bar{u}$*

and $\bar{p}^\star$*, respectively, and further*

$$\begin{aligned}\mathcal{F}(\bar{u})+\mathcal{G}(\Lambda\bar{u}) &= \inf_{u\in V}\{\mathcal{F}(u)+\mathcal{G}(\Lambda u)\}\\ &= \sup_{p^\star\in Y^\star}\{-\mathcal{F}^\star(-\Lambda^\star p^\star)-\mathcal{G}^\star(p^\star)\} = -\mathcal{F}^\star(-\Lambda^\star\bar{p}^\star)-\mathcal{G}^\star(\bar{p}^\star).\end{aligned} \tag{2.6}$$

In general, it can be shown [42, p. 48] that

$$\inf_{u\in V}\{\mathcal{F}(u)+\mathcal{G}(\Lambda u)\} \geq \sup_{p^\star\in Y^\star}\{-\mathcal{F}^\star(-\Lambda^\star p^\star)-\mathcal{G}^\star(p^\star)\}$$

holds true, which explains why (2.6) is often referred to as "no duality gap occurs". The next theorem states conditions that relate solutions of the primal and the dual problem, for a proof see [42, p. 53].

THEOREM 2.6. *If* $(\mathcal{P})$ *and* $(\mathcal{P}^\star)$ *possess solutions, if further*

$$\inf_{u\in V}\{\mathcal{F}(u)+\mathcal{G}(\Lambda u)\} = \sup_{p^\star\in Y^\star}\{-\mathcal{F}^\star(-\Lambda^\star p^\star)-\mathcal{G}^\star(p^\star)\} \tag{2.7}$$

and this number is finite, then all solutions $\bar{u}$ *and* $\bar{p}^\star$ *of* $(\mathcal{P})$ *and* $(\mathcal{P}^\star)$ *satisfy the* extremality conditions

$$\begin{aligned}-\Lambda^\star\bar{p}^\star &\in \partial\mathcal{F}(\bar{u}),\\ \bar{p}^\star &\in \partial\mathcal{G}(\Lambda\bar{u}),\end{aligned} \tag{2.8}$$

where ∂ *denotes the subdifferential.*

Conversely, if $\bar{u}\in V$ *and* $\bar{p}^\star\in Y^\star$ *satisfy* (2.8)*, then* $\bar{u}$ *is a solution of* $(\mathcal{P})$*,* $\bar{p}^\star$ *is a solution of* $(\mathcal{P}^\star)$ *and* (2.7) *holds.*

2. Generalized Differentiability in Function Spaces

We now summarize results on a recently developed generalized differentiability concept in infinite-dimensional spaces. This will be essential for the methods and analysis developed in this thesis. In a first paragraph we comment on the development of these methods, before we summarize those facts about semi-smooth Newton methods that are essential for this work.

2.1. Semi-smooth Newton methods. The application of generalized Newton methods for semi-smooth problems in finite dimensions has a rather long history, see, e.g., [43, 44, 91, 92] and the references given there. Recently, in [24, 58, 81, 103, 104] concepts for generalized derivatives in *infinite* dimensions were introduced. Our work uses the notion of "slant differentiability in a neighborhood" as proposed in [58], which is a slight adaption of the terminology introduced in [24], where also the term "slant differentiability at a point" is introduced. A similar concept is proposed in [81], where the name Newton map is coined. Applications of such pointwise approaches to Newton's method, however, presupposes knowledge of the solution. The differentiability concept in [58] coincides with a specific application of the theory developed in [103, 104], we refer to the discussion on this relationship in [58]. As in the recent papers [69, 70] and also motivated from [81], we use instead of the notion "slant differentiability in a neighborhood" the name "Newton differentiability".

We remark, that the primal-dual active set strategy can be interpreted as a certain application of the semi-smooth Newton method to nonlinear complementarity functions, see [58]. The primal-dual active set strategy has been successfully applied to linear optimal control problems involving PDE's and ODE's with pointwise constraints (see [14,15,57,60,82]), and, more recently also to nonlinear control problems [35, 36, 70].

2.2. Definition and properties. In this section we define Newton differentiability according to [58] and summarize facts on semi-smooth Newton methods. Let X, Y and Z be Banach spaces and $F : D \subset X \to Z$ be a nonlinear mapping with open domain D.

DEFINITION 2.7. The mapping $F : D \subset X \to Z$ is called *Newton differentiable on the open subset* $U \subset D$ if there exists a mapping $G : U \to \mathcal{L}(X, Z)$ such that

$$\lim_{h \to 0} \frac{1}{\|h\|} \|F(x+h) - F(x) - G(x+h)h\| = 0 \tag{2.9}$$

for every $x \in U$. The mapping G in the above definition is referred to as *generalized derivative.*

Note that in the above definition G is not required to be unique to be a generalized derivative of F in U. We now give an example for a Newton differentiable function that will frequently be used in this work, namely we discuss Newton differentiability of the pointwise max- and min-operator in function space. For this purpose let X denote a function space of real-valued functions on some $\Omega \subset \mathbb{R}^n$, further $\max(0, y)$ and $\min(0, y)$ the pointwise max- and min-operations, respectively. As candidates for the generalized derivatives we introduce

$$G_{max}(y)(x) = \begin{cases} 1 & \text{if } y(x) \geq 0, \\ 0 & \text{if } y(x) < 0; \end{cases} \qquad G_{min}(y)(x) = \begin{cases} 1 & \text{if } y(x) \leq 0, \\ 0 & \text{if } y(x) > 0. \end{cases} \tag{2.10}$$

Then we have the following result, see [58].

THEOREM 2.8. *The mappings* $\max(0, \cdot) : L^q(\Omega) \to L^p(\Omega)$ *and* $\min(0, \cdot) : L^q(\Omega) \to L^p(\Omega)$ *with* $1 \leq p < q < \infty$ *are Newton differentiable on* $L^q(\Omega)$ *with generalized derivatives* G_{max} *and* G_{min}*, respectively.*

Note that Theorem 2.8 requires a norm gap (i.e., $p < q$) to hold true. In [58] it is shown that the functions in (2.10) cannot serve as generalized derivatives if $p \geq q$. We remark that one can choose an arbitrary real value for the generalized derivatives $G_{max}(y)$ and $G_{min}(y)$ in Definition (2.10) for points where $y(x) = 0$, and the above result still holds true, see [58].

We now focus on the generalized Newton method for the solution of the possibly nonsmooth equation $F(x) = 0$. Based on the above differentiability concept we are interested in the sequence $(x^k)_{k\geq 1}$ calculated from the Newton step

$$x^{k+1} = x^k - G(x^k)^{-1}F(x^k), \tag{2.11}$$

where G is a generalized derivative in the sense of Definition 2.7.

The next theorem aims at the convergence of this generalized Newton method (for a proof see [24, 58, 104]).

THEOREM 2.9. *Suppose that $\bar{x} \in D$ is a solution to $F(x) = 0$ and that F is Newton differentiable in an open neighborhood U of $\bar{x}$ and that $\{\|G(x)^{-1}\| : x \in U\}$ is bounded. Then the Newton-iteration* (2.11) *converges superlinearly to $\bar{x}$ provided that $\|x^0 - \bar{x}\|$ is sufficiently small.*

Next we turn to a first chain rule for Newton differentiable functions, for a proof we refer to [69].

THEOREM 2.10 (Chain rule 1). *Let $K : X \to Y$ be an affine mapping with $Ky = By + b$, $B \in \mathcal{L}(X, Y)$, $b \in Y$, and assume that $F : D \subset Y \to Z$ is Newton differentiable on the open subset $U \subset D$ with generalized derivative G. If $K^{-1}(U)$ is nonempty, then $H := F \circ K$ is Newton differentiable on $K^{-1}(U)$ with generalized derivative given by $G(By + b)B \in \mathcal{L}(X, Z)$.*

A more general version of Theorem 2.10 is proved in [70], where a nonlinear, but continuously Fréchet differentiable K is allowed. However, for this work the result of Theorem 2.10 suffices. We also need a chain rule for the case that a Fréchet differentiable mapping is decomposed with a Newton differentiable one.

THEOREM 2.11 (Chain rule 2). *Let $F : D \subset X \to Y$ be Newton differentiable in an open neighborhood $U \subset D$ with generalized derivative G such that $\{\|G(v)\|_{\mathcal{L}(X,Y)} : v \in U\}$ is bounded. Furthermore, let $K : Y \to Z$ be continuously Fréchet differentiable in $F(U)$ with derivative K'. Then $H := K \circ F$ is Newton differentiable with generalized derivative $K'(F)G \in \mathcal{L}(X, Z)$.*

PROOF. The Newton differentiability of F implies that for all $u \in U$

$$F(u + h) - F(u) = G(u + h)h + \|h\|_X a(h) \tag{2.12}$$

with $a(h) \in Y$ and $\|a(h)\|_Y \to 0$ as $\|h\|_X \to 0$. Due to the Fréchet differentiability of K we have for all $v \in F(U)$ that

$$K(v + k) - K(v) = K'(v)k + \|k\|_Y b(k) \tag{2.13}$$

with $b(k) \in Z$ and $\|b(k)\|_Z \to 0$ as $\|k\|_Y \to 0$. Setting $v := F(u)$ and $k := F(u + h) - F(u)$ in (2.13) results with (2.12) in

$$K(F(u + h)) - K(F(u)) = K'(F(u + h))G(u + h)h + c(h), \tag{2.14}$$

where

$$c(h) = (K'(F(u)) - K'(F(u + h)))G(u + h)h + \|h\|_X K'(F(u))a(h) + \|k\|_Y b(k).$$

Since K' is continuous and there exists $C > 0$ such that $\|G(v)\|_{\mathcal{L}(X,Y)} \le C$ for all $v \in U$, we obtain

$$\frac{\|c(h)\|_Y}{\|h\|_X} \le C\|K'(F(u)) - K'(F(u+h))\|_{\mathcal{L}(X,Z)} + \|K'(F(u))\|_{\mathcal{L}(X,Z)}\|a(h)\|_Y + \frac{\|F(u+h) - F(u)\|_Y}{\|h\|_X} b(k),$$

which tends to 0 as $h \to 0$. This proves the assertion of the theorem. □

3. Notation

In this section we summarize the notation and notational conventions used in this work.

3.1. Vector and function spaces. We usually work on a domain in $\mathbb{R}^n$ denoted by Ω with boundary Γ. This boundary may be separated into several parts denoted, for instance, by $\Gamma_d, \Gamma_f, \Sigma$.

Abstract vector spaces are denoted by capital letters such as V, Y and for function spaces we use the standard notation $H^1(\Omega), L^2(\Gamma), \ldots$. To distinguish products of function spaces we denote them by bold Latin letters, e.g.,

$$\mathbf{H}^1(\Omega) = H^1(\Omega) \times H^1(\Omega) \times \cdots \times H^1(\Omega),$$
$$\mathbf{L}^2(\Gamma) = L^2(\Gamma) \times L^2(\Gamma) \times \cdots \times L^2(\Omega).$$

The topological dual of a vector space V is denoted by $V^\star$.

For norms in a general vector space X we write $\|\cdot\|_X$. To shorten the notation we utilize abbreviations for norms in frequently appearing function spaces, namely: The norm in $\mathbb{R}^n$ is only denoted by $\|\cdot\|$, the absolute value function in $\mathbb{R}$ by $|\cdot|$. To shorten notation we frequently utilize the shorter notion $\|g\|_S$ instead of $\|g\|_{L^2(S)}$.

Duality pairings between elements in $V, V^\star$ are denoted by $\langle\cdot\,,\cdot\rangle_{V^\star,V}$, and unless otherwise specified, for the scalar product in a Hilbert space H we write $\langle\cdot\,,\cdot\rangle_H$. However, the scalar product in $L^2(S)$ is just denoted by $(\cdot\,,\cdot)_S$ and, if the set S is clear from the context we only write $(\cdot\,,\cdot)$.

The vector space of bounded linear mapping from a space X into Y is denoted by $\mathcal{L}(X,Y)$, and the corresponding norm by $\|\cdot\|_{\mathcal{L}(X,Y)}$ or shorter by $\|\cdot\|_{\mathcal{L}}$.

3.2. Variables and mappings. We usually denote scalar variables by Latin or Greek letters such as x, y and λ. For $n \ge 2$ vectors in $\mathbb{R}^n$ are denoted by bold letters, e.g., by $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{v}$. To distinguish matrices from vectors we denote them by underlined bold letters, such as $\underline{\boldsymbol{p}}, \underline{\boldsymbol{q}}, \underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\varepsilon}}$.

Mappings are denoted according to their image space. For instance, the trace mapping from $H^1(\Omega) \to L^2(\Gamma)$ is denoted by τ, while the trace mapping from $\mathbf{H}^1(\Omega) \to \mathbf{L}^2(\Gamma)$ is denoted by the bold $\boldsymbol{\tau}$. This vector valued trace mapping can be decomposed into a scalar valued map that only contains the component in direction normal to the boundary that is consequently denoted by τ_N. The vector valued tangential component trace mapping is denoted by $\boldsymbol{\tau}_T$.

For linear operators we frequently dismiss the brackets for the argument. For example, we write τy instead of $\tau(y)$, and the stress of a deformation $\boldsymbol{y}$ is denoted by $\underline{\boldsymbol{\sigma}}\boldsymbol{y}$ instead of $\underline{\boldsymbol{\sigma}}(\boldsymbol{y})$.

CHAPTER 3

A Simplified Friction Problem

This chapter is devoted to the development and analysis of iterative algorithms for the solution of mechanical problems involving friction. As a model problem we consider a simplified friction problem that can be stated as the minimization of the non-differentiable functional

$$(\mathcal{P}) \qquad \begin{cases} J(y) := \dfrac{1}{2}\|\nabla y\|_{\Omega}^2 + \dfrac{\mu}{2}\|y\|_{\Omega}^2 - (f,y)_{\Omega} + g\displaystyle\int_{\Gamma_f} |\tau y(x)|\,dx \\ \text{over the set } \; Y := \{y \in H^1(\Omega) : \tau y = 0 \; \text{ a.e. on } \; \Gamma_0\}, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$, $\Gamma_0 \subset \Gamma := \partial\Omega$ is a possibly empty open set, $\Gamma_f := \Gamma \setminus \Gamma_0$, $g > 0, \mu \geq 0$, $f \in L^2(\Omega)$ and τ denotes the trace operator. The precise problem formulation will be given in the first section.

While usually in engineering papers finite dimensional discretizations of $(\mathcal{P})$ and related problems are studied, little attention has been paid to their infinite-dimensional counter-parts, specifically to Newton-type methods. This contribution focuses on the formulation and analysis of second-order solution algorithms for $(\mathcal{P})$ in a function space framework. Such an infinite-dimensional analysis gives more insight into the problem, which is also of significant practical importance since the performance of a numerical algorithm is closely related to the infinite-dimensional problem structure. In particular, it is desirable that the numerical method can be considered as the discrete version of a well-defined and well-behaved algorithm for the continuous problem. A finite dimensional approach misses important features as for example the regularity of Lagrange multipliers and its consequences as well as smoothing and uniform definiteness properties of the involved operators. It is well accepted that these properties significantly influence the behavior of numerical algorithms.

In principal there are two approaches to overcome the difficulty associated with the non-differentiability in $(\mathcal{P})$. One is based on resolving the derivative of the absolute value function introducing a Lagrange multiplier, the other one is based on an appropriate smoothing of the non-differentiable term.
An over-relaxation method and the Uzawa algorithm are proposed in the classical monographs [46,47] for the solution of $(\mathcal{P})$ and convergence results for these first-order methods are given. The Uzawa method is also suggested for a variational inequality of the second kind in [6, 51], however, no numerical results are given there. The recent paper [26] discusses an inexact version of the Uzawa algorithm

for variational inequalities of the second kind and reports on numerical examples. In [62] iterative techniques for the solution of friction contact problems are presented and developed further in [54]. Those methods require to minimize a non-differentiable functional over a convex set in every iteration step, which also motivates our investigation of problem $(\mathcal{P})$.

In [28–30] a generalized differentiability concept (Pang's B-differential) is used that allows to apply a Newton-like method for discretizations of friction contact problems, whereas algorithm formulation and analysis is done in finite dimensional spaces and only few convergence rate results are given. The authors of those contributions report on good numerical results and, in [30], an almost mesh independent behavior of the algorithm is observed, which suggests that the finite dimensional method is induced by an infinite-dimensional one. A different approach towards numerical realization of discrete elliptic variational inequalities of the second kind was followed in [77, 78], where monotone multigrid methods are employed to derive an efficient solution method.
For a smoothed variational inequality of the second kind again in [47] the Uzawa method is proposed. More recent contributions apply classical Newton methods to the smoothed finite dimensional problems, see, e.g., [84].

While there is a large literature on finite dimensional constrained and non-differential optimization techniques (see, e.g., [43,44,91,92] for finite dimensional semi-smooth Newton methods), the systematic analysis of these methods in continuous function spaces started only rather recently [24, 58, 81, 104]. The methods proposed in this chapter are related to the primal-dual active set strategy for the solution of constrained optimal control problems [14, 15]. This algorithm is closely related to infinite-dimensional semi-smooth Newton methods as shown in [58]. This relation allows to establish fast local convergence [58] and mesh-independence results [56, 61]. While in the papers [14, 15, 58, 69] the above methodologies are applied to unilateral pointwise constrained optimization problems, the convergence analysis for bilaterally constrained problems (as is the dual of $(\mathcal{P})$) involves additional problems as will come out in this contribution (see also [57,60]). The first-order augmented Lagrangian method for nonsmooth convex optimization that can be seen as an implicit version of Uzawa's algorithm is investigated within a Hilbert space framework in [67].

This chapter is organized as follows: In Section 1 the exact formulation and basic results for $(\mathcal{P})$ are given, the dual problem and the extremality conditions are determined. Section 2 is devoted to a regularization procedure for the dual formulation, the corresponding primal problem and the convergence of the regularized problems. In Section 3 we state algorithms for the solution of the regularized and the original friction problem and investigate their close relation. Section 4 analyzes these algorithms and gives local as well as global convergence results. Section 5 summarizes our numerical testing for $(\mathcal{P})$ and Section 6 applies our findings to a dynamical version of $(\mathcal{P})$.

1. Simplified Friction Problem

In this section we precisely state the simplified friction problem and state basic existence, uniqueness and regularity results. Furthermore, we calculate the dual problem and the corresponding extremality conditions for $(\mathcal{P})$.

1.1. Problem statement. Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with Lipschitz continuous boundary Γ. We choose a possibly empty open measurable subset Γ_0 of Γ and set $\Gamma_f := \Gamma \setminus \Gamma_0$. Let there be given two constants $g > 0, \mu \geq 0$ and a real-valued $f \in L^2(\Omega)$. For $y \in H^1(\Omega)$ we define the functional

$$J(y) := \frac{1}{2}\|\nabla y\|_\Omega^2 + \frac{\mu}{2}\|y\|_\Omega^2 - (f, y)_\Omega + g \int_{\Gamma_f} |\tau y(x)|\, dx, \tag{3.1}$$

where $\|\cdot\|_\Omega$ and $(\cdot\,,\cdot)_\Omega$ denote the usual norm and scalar product in $L^2(\Omega)$ and $\tau : H^1(\Omega) \to H^{\frac{1}{2}}(\Gamma)$ denotes the trace operator. Using (3.1) and defining

$$Y = \{y \in H^1(\Omega) : \tau y = 0 \text{ a.e. on } \Gamma_0\} \tag{3.2}$$

the simplified friction $(\mathcal{P})$ problem is given by

$$\min_{y \in Y} J(y).$$

Note that problem $(\mathcal{P})$ is an unconstrained minimization problem of a non-differentiable functional. For $y, z \in H^1(\Omega)$ we introduce the notations

$$a(y, z) := (\nabla y, \nabla z)_\Omega + \mu(y, z)_\Omega \tag{3.3}$$

and

$$j(y) := g \int_{\Gamma_f} |\tau y(x)|\, dx. \tag{3.4}$$

Then we can equivalently formulate $(\mathcal{P})$ as an elliptic variational inequality of the second kind [46]:

$$\begin{cases} \text{Find } y \in Y \text{ such that} \\ a(y, z - y) + j(z) - j(y) \geq (f, z - y)_\Omega \text{ for all } z \in Y. \end{cases} \tag{3.5}$$

To emphasize our basic ideas we treat the rather simple model problem $(\mathcal{P})$. Many generalizations are possible. In particular, the bilinear form $a(\cdot\,,\cdot)$ defined in (3.3) can be replaced by any coercive scalar product (see, e.g., [102]). Another simple generalization of $(\mathcal{P})$ is to allow instead of a constant g in (3.4) a nonnegative function $g \in L^2(\Gamma_f)$, leading to $j(y) = \int_{\Gamma_f} g(x)|\tau y(x)|\, dx$.

1.2. Basic results. We first investigate existence and uniqueness of solutions to $(\mathcal{P})$ in the case that $a(\cdot\,,\cdot)$ is Y-coercive.

THEOREM 3.1. *Let $\Gamma_0 \neq \emptyset$ or $\mu > 0$. Then problem $(\mathcal{P})$ or equivalently* (3.5) *admits a unique solution $\bar{y} \in Y$.*

PROOF. In the case $\Gamma_0 \neq \emptyset, \mu \geq 0$ it follows from the Poincaré inequality that $a(\cdot\,,\cdot)$ defines a continuous and elliptic bilinear form on Y; if $\mu > 0$ and $\Gamma_0 = \emptyset$ this obviously also holds true. In [46, p. 49] it is proved that $j(\cdot)$ is lower semicontinuous on $H^1(\Omega)$, thus on Y. Now we can apply the general existence and uniqueness result for elliptic variational inequalities of the second kind, [46, p. 5], to obtain existence and uniqueness of a solution to $(\mathcal{P})$. □

In the case that $a(\cdot\,,\cdot)$ is not coercive we have

THEOREM 3.2. *If $\Gamma_0 = \emptyset$ and $\mu = 0$ we get existence and uniqueness of a solution to $(\mathcal{P})$ provided that*

$$\left| \int_\Omega f(x)\,dx \right| < g|\Gamma|. \tag{3.6}$$

PROOF. Existence of a solution under assumption (3.6) is proved in [39, p. 70], uniqueness in [39, p. 73]. □

In [39] it is also shown that one cannot expect existence nor uniqueness of a solution to $(\mathcal{P})$ when weakening condition (3.6). For the sake of completeness we finish this section with a regularity result, cf. [46].

THEOREM 3.3. *If Γ is sufficiently smooth and $\Gamma_0 = \emptyset$, then the solution $\bar{y} \in Y$ of $(\mathcal{P})$ is in $H^2(\Omega)$.*

1.3. The Fenchel dual. To get a deeper insight into problem $(\mathcal{P})$ we calculate the corresponding dual problem. To simplify notation we use the trace operator $\tau_f : Y \to L^2(\Gamma_f)$ defined by $\tau_f y = (\tau y)_{|\Gamma_f}$. We start with rewriting problem $(\mathcal{P})$ as

$$\inf_{y \in Y} \Big\{ \mathcal{F}(y) + \mathcal{G}(\Lambda y) \Big\} \tag{3.7}$$

with $\Lambda := \tau_f \in \mathcal{L}(Y, L^2(\Gamma_f))$,

$$\mathcal{F}(y) := \frac{1}{2} a(y,y) - (f,y)_\Omega \quad \text{and} \quad \mathcal{G}(\Lambda y) := g \int_{\Gamma_f} |\Lambda y(x)|\,dx.$$

The functionals $\mathcal{F} : Y \to \mathbb{R}$ and $\mathcal{G} : L^2(\Gamma_f) \to \mathbb{R}$ are convex and continuous. Following Section 1.3 or [42, p. 61] the Fenchel dual problem is

$$\sup_{\lambda \in L^2(\Gamma_f)} \Big\{ -\mathcal{F}^\star(-\Lambda^\star \lambda) - \mathcal{G}^\star(\lambda) \Big\}, \tag{3.8}$$

where $\mathcal{F}^\star : Y^\star \to \mathbb{R}$ and $\mathcal{G}^\star : L^2(\Gamma_f) \to \mathbb{R}$ denote the convex conjugate functionals to $\mathcal{F}$ and $\mathcal{G}$, see Definition 2.3, and $\Lambda^\star \in \mathcal{L}(L^2(\Gamma_f), Y^\star)$ the adjoint of the operator Λ. In (3.8) we already identified $L^2(\Gamma_f)$ with its dual.
Next we specify (3.8) for problem $(\mathcal{P})$. Let therefore $\lambda \in L^2(\Gamma_f)$ be given. Due to the definition of $\mathcal{F}^\star$ we have

$$\mathcal{F}^\star(-\Lambda^\star \lambda) = \sup_{w \in Y} \left\{ -(\lambda, \tau_f w)_{\Gamma_f} - \frac{1}{2} a(w,w) + (f,w)_\Omega \right\}.$$

It can easily be seen that the above supremum is attained by $w \in Y$ satisfying

$$a(w,v) - (f,v)_\Omega + (\lambda, \tau_f v)_{\Gamma_f} = 0 \quad \text{for all } v \in Y. \tag{3.9}$$

Thus, using (3.9) with $v := w$ leads to

$$\mathcal{F}^\star(-\Lambda^\star\lambda) = -(\lambda, \tau_f w)_{\Gamma_f} - \frac{1}{2}a(w,w) + (f,w)_\Omega = \frac{1}{2}a(w,w), \tag{3.10}$$

where $w = w(\lambda)$ satisfies (3.9).

Finally we calculate the conjugate convex functional of $\mathcal{G}$. For $\lambda \in L^2(\Gamma_f)$ we have

$$\mathcal{G}^\star(\lambda) = \sup_{\nu \in L^2(\Gamma_f)} \{(\lambda,\nu)_{\Gamma_f} - \mathcal{G}(\nu)\} = \begin{cases} 0 & \text{if } |\lambda| \le g \text{ a.e. on } \Gamma_f, \\ \infty & \text{otherwise,} \end{cases} \tag{3.11}$$

as shown in [42]. Plugging (3.10) and (3.11) into (3.8) we can now specify the dual problem:

$$(\mathcal{P}^\star) \qquad \begin{cases} \sup\limits_{|\lambda| \le g \text{ a.e. on } \Gamma_f} J^\star(\lambda) := -\dfrac{1}{2}a(w(\lambda), w(\lambda)), \\ \text{where } w(\lambda) \text{ satisfies} \\ a(w(\lambda), v) - (f,v)_\Omega + (\lambda, \tau_f v)_{\Gamma_f} = 0 \quad \text{for all } v \in Y. \end{cases}$$

It is easy to see that the functions $\mathcal{F}$ and $\mathcal{G}$ satisfy the conditions of Theorem 2.5, and thus it follows that

$$\inf_{y \in Y} J(y) = \sup_{|\lambda| \le g \text{ a.e. on } \Gamma_f} J^\star(\lambda),$$

that is, no duality gap occurs. As already shown, the primal problem $(\mathcal{P})$ admits a solution $\bar{y} \in Y$ if $a(\cdot\,,\cdot)$ is Y-coercive or condition (3.6) holds. Thus, under these assumptions existence of a solution $\bar{\lambda} \in L^2(\Gamma_f)$ for the dual problem $(\mathcal{P}^\star)$ follows from Fenchel duality theory, see Theorem 2.5 or [87, p. 201].

REMARK 3.4. By means of duality theory we have transformed $(\mathcal{P})$, the unconstrained minimization of a non-differentiable functional into $(\mathcal{P}^\star)$, the constrained maximization of a smooth functional. Constrained optimization problems like $(\mathcal{P}^\star)$ have been extensively studied in literature, see, e.g., [17, 87] and lots of theoretical results are available. This illustrates that considering the dual problem can result in new insight into the structure of optimization problems, that is of interest also for constructing efficient solution methods.

1.4. Extremality conditions. Following Theorem 2.6 the solutions $\bar{y}, \bar{\lambda}$ of primal and dual problem, respectively, satisfy the extremality conditions (2.8), namely

$$-\Lambda^\star\bar{\lambda} \in \partial\mathcal{F}(\bar{y}), \tag{3.12}$$

$$\bar{\lambda} \in \partial\mathcal{G}(\tau_f \bar{y}). \tag{3.13}$$

Due to the differentiability of $\mathcal{F}$ equation (3.12) can be written as

$$-\big\langle \Lambda^\star \bar\lambda, v \big\rangle_{Y^\star,Y} = \big\langle \mathcal{F}'(\bar y), v\big\rangle_{Y^\star,Y} \text{ for all } v \in Y,$$

or equivalently as

$$-(\bar\lambda, \tau_f v)_{\Gamma_f} = a(\bar y, v) - (f, v)_\Omega \text{ for all } v \in Y.$$

Hence condition (3.12) yields

$$a(\bar y, v) - (f, v)_\Omega + (\bar\lambda, \tau_f v)_{\Gamma_f} = 0 \text{ for all } v \in Y.$$

Next we turn our attention to (3.13). From the definition of the subdifferential it follows that

$$g \int_{\Gamma_f} (|\tau_f \bar y(x)| - |\nu(x)|)\, dx \le \int_{\Gamma_f} \bar\lambda(x)(\tau_f \bar y(x) - \nu(x))\, dx$$

for all $\nu \in L^2(\Gamma_f)$, which implies for $\nu = 0$ the relation

$$\int_{\Gamma_f} g|\tau_f \bar y(x)| - \bar\lambda(x)\tau_f \bar y(x)\, dx \le 0.$$

Since we have $|\bar\lambda| \le g$ a.e., it follows that

$$g|\tau_f \bar y| - \bar\lambda \tau_f \bar y = 0 \text{ a.e. on } \Gamma_f. \tag{3.14}$$

Introducing active and inactive sets for the dual problem $(\mathcal{P}^\star)$ by

$$\mathcal{A}_- := \{x \in \Gamma_f : \bar\lambda = -g \text{ a.e. on } \Gamma_f\}, \tag{3.15a}$$

$$\mathcal{I} := \{x \in \Gamma_f : |\bar\lambda| < g \text{ a.e. on } \Gamma_f\}, \tag{3.15b}$$

$$\mathcal{A}_+ := \{x \in \Gamma_f : \bar\lambda = g \text{ a.e. on } \Gamma_f\}, \tag{3.15c}$$

one can express the conditions $|\bar\lambda| \le g$ and (3.14) as

$$\begin{cases} \tau_f \bar y \le 0 & \text{a.e. on } \mathcal{A}_-, \\ \tau_f \bar y = 0 & \text{a.e. on } \mathcal{I}, \\ \tau_f \bar y \ge 0 & \text{a.e. on } \mathcal{A}_+. \end{cases} \tag{3.16}$$

The next lemma states that (3.15) and (3.16) can be expressed as one nonlinear equation.

LEMMA 3.5. *Conditions* (3.15), (3.16) *can equivalently be expressed as*

$$\tau_f \bar y = \max(0, \tau_f \bar y + \sigma(\bar\lambda - g)) + \min(0, \tau_f \bar y + \sigma(\bar\lambda + g)) \tag{3.17}$$

for every $\sigma > 0$.

PROOF. The equivalence follows from general results in convex analysis [67], but can also be verified by a direct computation. It can be seen easily that $(\bar\lambda, \tau_f \bar y)$ satisfying (3.16) also satisfies (3.17).
Conversely, let us assume that (3.17) holds and let us show (3.16) by contradiction. Assuming $\tau_f y > 0$ on $\mathcal{A}_-$ implies, using (3.17) that

$$\tau_f \bar y = \max(0, \tau_f \bar y - 2\sigma g) = \tau_f \bar y - 2\sigma g,$$

giving a contradiction. Hence, on $\mathcal{A}_-$ holds $\tau_f \bar{y} \le 0$. Similarly one gets that $\tau_f \bar{y} \ge 0$ on $\mathcal{A}_+$. Finally, assume that either $\tau_f \bar{y} > 0$ or $\tau_f \bar{y} < 0$ on $\mathcal{I}$. In the first case (3.17) implies

$$\tau_f \bar{y} = \max(0, \tau_f \bar{y} + \sigma(\bar{\lambda} - g)) = \tau_f \bar{y} + \sigma(\bar{\lambda} - g),$$

which is a contradiction since $\bar{\lambda} - g \neq 0$ on $\mathcal{I}$. Utilizing the same arguments the assumption $\tau_f \bar{y} < 0$ on $\mathcal{I}$ leads to a contradiction, which shows that (3.17) implies $\tau_f \bar{y} = 0$ on $\mathcal{I}$ and ends the proof. □

The function

$$\Phi_\sigma(\tau_f y, \lambda) := \tau_f y - \max(0, \tau_f y + \sigma(\lambda - g)) - \min(0, \tau_f y + \sigma(\lambda + g))$$

is called *nonlinear complementarity (NC) function* for (3.16) due to the fact that (3.16) is equivalent to $\Phi_\sigma(\tau_f \bar{y}, \bar{\lambda}) = 0$ for arbitrarily fixed $\sigma > 0$. Thus the extremality conditions (3.12), (3.13) yield

$$a(\bar{y}, v) - (f, v)_\Omega + (\bar{\lambda}, \tau_f v)_{\Gamma_f} = 0 \text{ for all } v \in Y, \tag{3.18a}$$

$$\Phi_\sigma(\tau_f \bar{y}, \bar{\lambda}) = 0 \text{ for all } \sigma > 0. \tag{3.18b}$$

Using Theorem 2.6 or [42, p. 59] we summarize our results in the next theorem.

THEOREM 3.6. *For $(y, \lambda) \in Y \times L^2(\Gamma_f)$ the following two conditions are equivalent to each other:*

(i) The variables y and λ solve the primal and dual problem $(\mathcal{P})$ and $(\mathcal{P}^\star)$, respectively, and

$$J(y) = -J^\star(\lambda).$$

(ii) The pair (y, λ) satisfies the conditions (3.18).

REMARK 3.7. If the solution variables $\bar{y}$ and $\bar{\lambda}$ of $(\mathcal{P})$ and $(\mathcal{P}^\star)$, respectively, are sufficiently smooth, then (3.18a) can be written as

$$\begin{cases} -\Delta\bar{y} + \mu\bar{y} & = & f & \text{a.e. in } \Omega, \\ \dfrac{\partial \bar{y}}{\partial n} + \bar{\lambda} & = & 0 & \text{a.e. on } \Gamma_f. \end{cases}$$

1.5. Comments on the numerical solution. In [46, 47] two methods for the numerical solution of $(\mathcal{P})$ are proposed. Firstly, an (over-)relaxation method for the discretized problem is described and tested, secondly a duality method (the Uzawa algorithm) is applied to $(\mathcal{P})$ and convergence results are given. The latter is a first-order update method for the dual variable [6]. For an inexact version of the Uzawa method applied to variational inequalities of the second kind we refer to [26].

In the sequel we focus on second-order (Newton-type) algorithms for the solution of $(\mathcal{P})$. In finite dimensions the application of generalized Newton methods for semi-smooth problems has a rather long history, see, e.g., [43, 44, 91, 92] and the references given there. Here we are interested in *infinite-dimensional* generalized Newton methods based on recent generalized differentiability concepts in function space [24, 58, 81, 104]. For this purpose it seems necessary to introduce a regularization procedure that allows to state and analyze our algorithms in infinite dimensions.

2. A regularization procedure

In this chapter we introduce a regularization procedure to overcome the difficulty associated with the non-differentiability of the functional J in (3.1). Therefore we consequently utilize results from duality theory and discuss relations between regularization, the primal and the dual problem. For the original simplified friction problem $(\mathcal{P})$ the iterates of the algorithms presented in the next section are not contained in spaces of square integrable functions, which presents a difficulty, both from the point of view of a numerical implementation and convergence analysis. Considering the regularized problem allows to formulate and analyze the algorithms in infinite-dimensional Hilbert spaces.
For the term $\int_{\Gamma_f} |\tau y(x)|\, dx$ that involves the absolute value function, many ways to construct sequences of differentiable approximations are possible, see, e.g., [64]. In [50] the non-differentiable term in (3.1) is replaced by

$$\int_{\Gamma_f} \sqrt{|\tau y(x)|^2 + \varepsilon^2}\, dx \tag{3.19}$$

with small $\varepsilon > 0$ and an a posteriori error estimate for the solution of the regularized simplified friction problem is established, but no numerical results are presented. Compared to the regularization (3.19) for the primal problem $(\mathcal{P})$ our approximation is motivated by considering the dual problem and by results in the context of semi-smooth Newton methods [58, 69, 104] and augmented Lagrangians [67, 68]. In the corresponding primal problem the regularization turns out to be a very natural one that is related to those used in [47, 51, 83].

This section is organized as follows. After presenting the regularization for the dual problem we calculate the corresponding primal problem and the optimality system, discuss the relation to [47,51,83] and [69] and investigate the convergence as the regularization parameter tends to infinity.

2.1. Regularization of the dual problem. From now on we assume that the bilinear form $a(\cdot\,,\cdot)$ is coercive in Y with coercitivity constant $\nu > 0$, i.e.,

$$a(y, y) \geq \nu \|y\|^2_{H^1(\Omega)} \text{ for all } y \in Y. \tag{3.20}$$

We can now introduce the regularized dual problem, where compared to $(\mathcal{P}^\star)$ we change the sign of the cost functional and thus the maximization becomes a minimization. For fixed $\hat{\lambda} \in L^2(\Gamma_f)$ and regularization parameter $\gamma > 0$ we consider

$$(\mathcal{P}_\gamma^\star)\qquad \begin{cases} \min\limits_{|\lambda|\le g \text{ a.e. on } \Gamma_f} J_\gamma^\star(\lambda) := \dfrac{1}{2}a(w(\lambda),w(\lambda)) + \dfrac{1}{2\gamma}\|\lambda-\hat\lambda\|^2_{\Gamma_f} - \dfrac{1}{2\gamma}\|\hat\lambda\|^2_{\Gamma_f},\\ \text{where } w(\lambda) \text{ satisfies}\\ a(w(\lambda),v) - (f,v)_\Omega + (\lambda,\tau_f v)_{\Gamma_f} = 0 \ \text{ for all } v\in Y. \end{cases}$$

Thus the regularized problem is obtained from $(\mathcal{P}^\star)$ by adding

$$\frac{1}{2\gamma}\|\lambda-\hat\lambda\|^2_{\Gamma_f} - \frac{1}{2\gamma}\|\hat\lambda\|^2_{\Gamma_f}, \tag{3.21}$$

to the objective functional. Choosing this regularization is motivated by augmented Lagrangians and results from optimal control problems. Note that the influence of the terms in (3.21) decreases as $\gamma\to\infty$. Standard arguments show that $(\mathcal{P}_\gamma^\star)$ admits a unique solution λ_γ for every $\gamma>0$. The second term in (3.21), which is a constant, can be neglected from the optimizational point of view; it has been introduced to get a simple connection with the corresponding primal problem (see Theorem 3.9).
In the sequel we shall use $e: Y\times L^2(\Gamma_f)\to Y^\star$ defined by

$$\langle e(y,\lambda), z\rangle_{Y^\star,Y} := a(y,z) - (f,z)_\Omega + (\lambda,\tau_f z)_{\Gamma_f}.$$

This notation allows us to write the variational equality in $(\mathcal{P}_\gamma^\star)$ as

$$e(y,\lambda) = 0 \ \text{ in } Y^\star.$$

We now derive the first-order optimality conditions for $(\mathcal{P}_\gamma^\star)$.

THEOREM 3.8. *Let $\lambda_\gamma\in L^2(\Gamma_f)$ be the unique solution of $(\mathcal{P}_\gamma^\star)$. Then there exists $y_\gamma\in Y$ and $\xi_\gamma\in L^2(\Gamma_f)$ such that*

$$e(y_\gamma,\lambda_\gamma) = 0 \text{ in } Y^\star, \tag{3.22a}$$

$$\tau_f y_\gamma + \gamma^{-1}(\hat\lambda-\lambda_\gamma) - \xi_\gamma = 0 \text{ in } L^2(\Gamma_f), \tag{3.22b}$$

and for every $\sigma>0$ holds

$$\xi_\gamma - \max(0,\xi_\gamma+\sigma(\lambda_\gamma-g)) - \min(0,\xi_\gamma+\sigma(\lambda_\gamma+g)) = 0 \text{ in } L^2(\Gamma_f). \tag{3.22c}$$

PROOF. We introduce the Lagrangian functional $\mathscr{L}: Y\times L^2(\Gamma_f)\times Y\to\mathbb{R}$ for $(\mathcal{P}_\gamma^\star)$ defined by

$$\mathscr{L}(y,\lambda,\beta) := J_\gamma^\star(y,\lambda) + \langle e(y,\lambda),\beta\rangle_{Y^\star,Y}.$$

First note that the derivative of $e(\cdot\,,\cdot)$ can easily be calculated and turns out to be surjective for all $(y,\lambda)\in H^1(\Omega)\times L^2(\Gamma_f)$. This ensures, following [88] the existence of $y_\gamma,\beta_\gamma\in Y$ such that

$$\mathscr{L}_y(y_\gamma,\lambda_\gamma,\beta_\gamma) = 0 \text{ and } e(y_\gamma,\lambda_\gamma) = 0.$$

More precisely, for all $\delta_y \in Y$ we have

$$\begin{aligned}\mathscr{L}_y(y_\gamma, \lambda_\gamma, \beta_\gamma)(\delta_y) &= a(y_\gamma, \delta_y) + a(\beta_\gamma, \delta_y)\\ &= a(\beta_\gamma, \delta_y) + (f, \delta_y)_\Omega - (\lambda_\gamma, \tau_f\, \delta_y)_{\Gamma_f} = 0,\end{aligned}$$

yielding that $\beta_\gamma = -y_\gamma$ due to the fact that the variational problem

$$\big\langle e(y, \lambda), z\big\rangle_{Y^\star, Y} = 0 \text{ for all } z \in Y$$

admits a unique solution. Optimality with respect to λ_γ implies

$$\mathscr{L}_\lambda(y_\gamma, \lambda_\gamma, \beta_\gamma)(\delta_\lambda - \lambda_\gamma) \geq 0$$

for all $\delta_\lambda \in L^2(\Gamma_f)$ with $|\delta_\lambda| \leq g$ a.e. on Γ_f. This yields

$$(\gamma^{-1}(\lambda_\gamma - \hat{\lambda}) + \tau_f\, \beta_\gamma, \delta_\lambda - \lambda_\gamma)_{\Gamma_f} = (-\gamma^{-1}(\hat{\lambda} - \lambda_\gamma) - \tau_f\, y_\gamma, \delta_\lambda - \lambda_\gamma)_{\Gamma_f} \geq 0$$

and after introducing the variable $\xi_\gamma := \gamma^{-1}(\hat{\lambda} - \lambda_\gamma) + \tau_f\, y_\gamma$ results in the complementarity condition

$$\begin{cases} \xi_\gamma \leq 0 & \text{a.e. on} \quad \mathcal{A}_{\gamma,-}, \\ \xi_\gamma = 0 & \text{a.e. on} \quad \mathcal{I}_\gamma, \\ \xi_\gamma \geq 0 & \text{a.e. on} \quad \mathcal{A}_{\gamma,+}, \end{cases} \tag{3.23}$$

where the sets $\mathcal{A}_{\gamma,+}, \mathcal{A}_{\gamma,-}, \mathcal{I}_\gamma \subset \Gamma_f$ are defined as

$$\mathcal{A}_{\gamma,-} := \{x \in \Gamma_f : \lambda_\gamma = -g \text{ a.e.}\}, \tag{3.24a}$$

$$\mathcal{I}_\gamma := \{x \in \Gamma_f : |\lambda_\gamma| < g \text{ a.e.}\}, \tag{3.24b}$$

$$\mathcal{A}_{\gamma,+} := \{x \in \Gamma_f : \lambda_\gamma = g \text{ a.e.}\}. \tag{3.24c}$$

Condition (3.23) is equivalent to (3.22c) as can be verified by a direct computation similarly as in the proof of Lemma 3.5. ☐

The necessary conditions (3.22a)-(3.22c) are also sufficient for λ_γ to be the solution of $(\mathcal{P}_\gamma^\star)$. This can be verified in several ways, we will use an argument form duality theory in Theorem 3.9 to argue sufficiency.

2.2. The corresponding regularized primal problem. Next we turn our attention to the primal formulation of problem $(\mathcal{P}_\gamma^\star)$. For $\alpha \in \mathbb{R}$ we define a function $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$h(x, \alpha) = \begin{cases} \dfrac{g}{\gamma}\, |\gamma x + \alpha| - \dfrac{g^2}{2\gamma} & \text{if} \quad |\gamma x + \alpha| > g, \\ \dfrac{1}{2\gamma}(\gamma x + \alpha)^2 & \text{if} \quad |\gamma x + \alpha| \leq g. \end{cases} \tag{3.25}$$

The function h is continuously differentiable and Figure 2.2 illustrates that h smoothes the absolute value function. We can now define the problem $(\mathcal{P}_\gamma)$ that

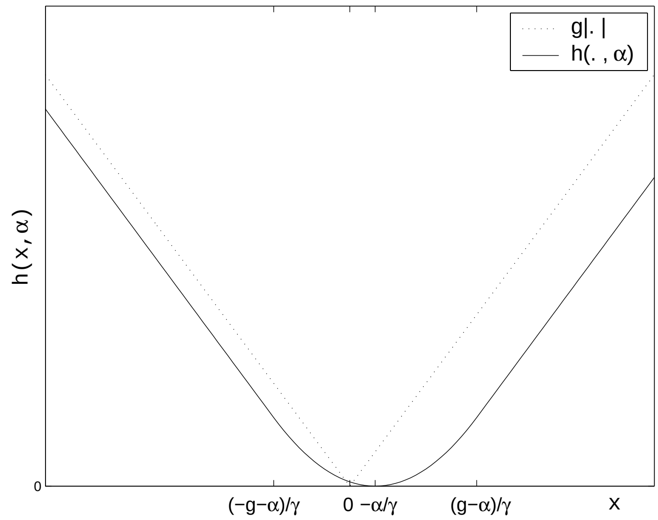


FIGURE 1. *Approximation h of the absolute value function as defined in* (3.25).

will turn out to be the primal problem corresponding to $(\mathcal{P}_\gamma^\star)$:

$$(\mathcal{P}_\gamma) \qquad \min_{y\in Y} J_\gamma(y) := \frac{1}{2}a(y,y) - (f,y)_\Omega + \int_{\Gamma_f} h(\tau_f\, y(x), \hat\lambda(x))\, dx.$$

Note that the functional J_γ is uniformly convex and continuously differentiable. The next theorem clarifies the connection between $(\mathcal{P}_\gamma)$ and $(\mathcal{P}_\gamma^\star)$.

THEOREM 3.9. *Problem $(\mathcal{P}_\gamma^\star)$ is the dual problem of $(\mathcal{P}_\gamma)$ and we have*

$$J_\gamma^\star(\lambda_\gamma) = -J_\gamma(y_\gamma), \tag{3.26}$$

where λ_γ and y_γ denote the solutions of $(\mathcal{P}_\gamma^\star)$ and $(\mathcal{P}_\gamma)$, respectively. Furthermore, introducing the variable $\xi_\gamma := \tau_f\, y_\gamma + \gamma^{-1}(\hat\lambda - \lambda_\gamma) \in L^2(\Gamma_f)$, the extremality conditions relating $(\mathcal{P}_\gamma)$ and $(\mathcal{P}_\gamma^\star)$ yield (3.22a)-(3.22c) *and these conditions are sufficient for λ_γ and y_γ to be solutions of $(\mathcal{P}_\gamma^\star)$ and $(\mathcal{P}_\gamma)$, respectively.*

PROOF. For calculating the dual problem to $(\mathcal{P}_\gamma)$ we use the notation introduced in Section 1.3, where the dual problem for the (original) simplified friction problem is derived. We only have to replace the functional $\mathcal{G}$ in (3.7) by

$$\tilde{\mathcal{G}}(\tau_f\, y) := \int_{\Gamma_f} h(\tau_f\, y(x), \hat\lambda(x))\, dx$$

with the definition of h as given in (3.25). For calculating the convex conjugate functional $\tilde{\mathcal{G}}^\star$ to $\tilde{\mathcal{G}}$ let $\lambda \in L^2(\Gamma_f)$. Then,

$$\begin{aligned}
\tilde{\mathcal{G}}^\star(\lambda) &= \sup_{\nu \in L^2(\Gamma_f)} \Big\{ (\lambda,\nu)_{\Gamma_f} - \int_{\Gamma_f} h(\nu(x))\,dx \Big\} \\
&= \sup_{\nu \in L^2(\Gamma_f)} \Big\{ \int_{|\gamma\nu+\hat{\lambda}|>g} \frac{\lambda}{\gamma}(\gamma\nu+\hat{\lambda}) - \frac{g}{\gamma}\big|\gamma\nu+\hat{\lambda}\big| + \frac{g^2}{2\gamma} - \frac{\lambda\hat{\lambda}}{\gamma}\,dx \qquad (3.27)\\
&\quad + \int_{|\gamma\nu+\hat{\lambda}|\le g} \lambda\nu - \frac{1}{2\gamma}(\gamma\nu+\hat{\lambda})^2\,dx \Big\}. \qquad (3.28)
\end{aligned}$$

From (3.27) one gets that $\tilde{\mathcal{G}}^\star(\lambda) = \infty$ unless $|\lambda| \le g$ a.e. on Γ_f. Suppose now that $|\lambda| \le g$ a.e. on Γ_f. We will show that when evaluating the above supremum one only has to take into account the term in (3.28). For given $\lambda \in L^2(\Gamma_f)$, $|\lambda| \le g$ a.e., we define a mapping $H : L^2(\Gamma_f) \to \mathbb{R}$ by

$$\begin{aligned}
H(\nu) := \int_{|\gamma\nu+\hat{\lambda}|>g} \frac{\lambda}{\gamma}(\gamma\nu+\hat{\lambda}) - \frac{g}{\gamma}\big|\gamma\nu+\hat{\lambda}\big| + \frac{g^2}{2\gamma} - \frac{\lambda\hat{\lambda}}{\gamma}\,dx \\
+ \int_{|\gamma\nu+\hat{\lambda}|\le g} \lambda\nu - \frac{1}{2\gamma}(\gamma\nu+\hat{\lambda})^2\,dx.
\end{aligned}$$

For $\nu_0 \in L^2(\Gamma_f)$ we introduce $\tilde{\nu}_0 \in L^2(\Gamma_f)$ by

$$\tilde{\nu}_0(x) := \begin{cases} \nu_0(x) & \text{if } \ |\gamma\nu_0(x)+\hat{\lambda}(x)| \le g, \\ \gamma^{-1}(g-\hat{\lambda}(x)) & \text{if } \ \gamma\nu_0(x)+\hat{\lambda}(x) > g, \\ \gamma^{-1}(-g-\hat{\lambda}(x)) & \text{if } \ \gamma\nu_0(x)+\hat{\lambda}(x) < -g. \end{cases}$$

Then it follows that

$$\begin{aligned}
&H(\nu_0) - H(\tilde{\nu}_0) \\
&\quad = \frac{1}{\gamma}\int_{\gamma\nu_0+\hat{\lambda}>g} \gamma\lambda\nu_0 - \gamma g\nu_0 - g\hat{\lambda} + g^2 - \lambda(g-\hat{\lambda})\,dx \\
&\quad + \frac{1}{\gamma}\int_{\gamma\nu_0+\hat{\lambda}<-g} \gamma\lambda\nu_0 + \gamma g\nu_0 + g\hat{\lambda} + g^2 - \lambda(-g-\hat{\lambda})\,dx.
\end{aligned}$$

Since $|\lambda| \le g$ we get, using $\nu_0 > \gamma^{-1}(g - \hat{\lambda})$ under the first integral and $\nu_0 < \gamma^{-1}(-g - \hat{\lambda})$ under the second that

$$\begin{aligned} H(\nu_0) - H(\tilde{\nu}_0) &\le \frac{1}{\gamma} \int_{\gamma\nu_0 + \hat{\lambda} > g} (\lambda - g)(g - \hat{\lambda}) - g\hat{\lambda} + g^2 - \lambda(g - \hat{\lambda})\, dx \\ &\quad + \frac{1}{\gamma} \int_{\gamma\nu_0 + \hat{\lambda} < -g} (\lambda + g)(-g - \hat{\lambda}) + g\hat{\lambda} + g^2 - \lambda(-g - \hat{\lambda})\, dx \\ &= 0. \end{aligned}$$

The above consideration shows that

$$\sup_{\nu \in L^2(\Gamma_f)} H(\nu) = \sup_{\substack{\nu \in L^2(\Gamma_f) \\ |\gamma\nu + \hat{\lambda}| \le g \text{ a.e. on } \Gamma_f}} H(\nu)$$

yielding that in calculating the supremum in (3.27), (3.28) only the term (3.28) has to be considered. Hence, the maximizer is easily calculated as $\nu_0 = \gamma^{-1}(\lambda - \hat{\lambda})$, which shows that

$$\tilde{\mathcal{G}}(\lambda) = \begin{cases} \dfrac{1}{2\gamma} \|\lambda - \hat{\lambda}\|^2_{\Gamma_f} - \dfrac{1}{2\gamma} \|\hat{\lambda}\|^2_{\Gamma_f} & \text{if } |\lambda| \le g \text{ a.e. on } \Gamma_f, \\ \infty & \text{else.} \end{cases} \tag{3.29}$$

Plugging (3.29) and (3.10) into (3.8) results in problem $(\mathcal{P}^\star_\gamma)$, which verifies that $(\mathcal{P}^\star_\gamma)$ is the dual of $(\mathcal{P}_\gamma)$. In addition, the conditions of Theorem 2.5 are satisfied and thus (3.26) holds true. Evaluating the extremality conditions can be done in a similar way as for the simplified friction problem (see page 15) and results in system (3.22a)-(3.22c). This shows, using Theorem 2.6 that the conditions (3.22a)-(3.22c) are also sufficient for λ_γ and y_γ to be a solution to $(\mathcal{P}^\star_\gamma)$ and $(\mathcal{P}_\gamma)$, respectively. □

2.3. Convergence as $\gamma \to \infty$. The aim of this chapter is to establish a convergence result with respect to the regularization parameter γ. For related results we refer, e.g., to [47, 69].

THEOREM 3.10. *For any $\hat{\lambda} \in L^2(\Gamma_f)$ the solutions y_γ of the regularized problems $(\mathcal{P}_\gamma)$ converge to the solution $\bar{y}$ of the original problem $(\mathcal{P})$ strongly in Y as $\gamma \to \infty$. Furthermore, the solutions λ_γ of the dual problems $(\mathcal{P}^\star_\gamma)$ converge to the solution $\bar{\lambda}$ of $(\mathcal{P}^\star)$ weakly in $L^2(\Gamma_f)$.*

PROOF. Recall the complementarity conditions and the definition of the active and inactive sets (3.16), (3.15) for the original and (3.23), (3.24) for the regularized problem. Furthermore, recall that $(y_\gamma, \lambda_\gamma)$ as well as $(\bar{y}, \bar{\lambda})$ satisfy the variational equality

$$\big\langle e(y, \lambda), z \big\rangle_{Y^\star, Y} = 0 \text{ for all } z \in Y \tag{3.30}$$

with the definition of $e(\cdot\,,\cdot)$ given in Section 2.1. Note that for all $\gamma > 0$ we have $|\lambda_\gamma| \leq g$ a.e. on Γ_f. We now choose an arbitrary sequence γ_n such that $\gamma_n \to \infty$ for $n \to \infty$. From the weak compactness of the unit sphere in a Hilbert space we can infer the existence of $\lambda^* \in L^2(\Gamma_f)$ and a subsequence $\lambda_{\gamma_{n_k}}$ in $L^2(\Gamma_f)$ such that

$$\lambda_{\gamma_{n_k}} \rightharpoonup \lambda^* \text{ weakly in } L^2(\Gamma_f).$$

Since closed convex sets in Hilbert spaces are weakly closed, [42, p. 6], we have $|\lambda^*| \leq g$ a.e. in $L^2(\Gamma_f)$. The weak convergence of $(\lambda_{\gamma_{n_k}})$ in $L^2(\Gamma_f)$ implies $y_{\gamma_{n_k}} \rightharpoonup y^*$ weakly in Y for some $y^* \in Y$ and that the pair (y^*, λ^*) also satisfies (3.30). We henceforth drop the subscript n_k with γ_{n_k}. From (3.30) it follows that

$$\begin{aligned} a(y_\gamma - \bar{y}, y_\gamma - \bar{y}) &= -(\lambda_\gamma - \bar{\lambda}, \tau_f(y_\gamma - \bar{y}))_{\Gamma_f} \\ &= (\tau_f\bar{y}, \lambda_\gamma - \bar{\lambda})_{\Gamma_f} + (\tau_f y_\gamma, \bar{\lambda} - \lambda_\gamma)_{\Gamma_f}. \end{aligned} \tag{3.31}$$

We are now going to estimate the above two terms separately. Let us first turn our attention to the term $(\tau_f\bar{y}, \lambda_\gamma - \bar{\lambda})_{\Gamma_f}$. We have that

$$\tau_f\bar{y}(\lambda_\gamma - \bar{\lambda}) = \tau_f\bar{y}(\lambda_\gamma - g) \leq 0 \text{ a.e. on } \mathcal{A}_+,$$

since $\tau_f\bar{y} \geq 0$ and $\lambda_\gamma \leq g$. Similarly we find that $\tau_f\bar{y}(\lambda_\gamma - \bar{\lambda}) \leq 0$ on $\mathcal{A}_-$ utilizing $\tau_f\bar{y} \leq 0$ and $\lambda_\gamma \geq -g$. Finally, on $\mathcal{I}$ we have $\tau_f\bar{y} = 0$ which yields, since $\Gamma_f = \mathcal{A}_- \cup \mathcal{A}_+ \cup \mathcal{I}$ that

$$(\tau_f\bar{y}, \lambda_\gamma - \bar{\lambda})_{\Gamma_f} \leq 0. \tag{3.32}$$

Next we consider $\tau_f y_\gamma(\bar{\lambda} - \lambda_\gamma)$ on the sets $\mathcal{A}_{\gamma,-}, \mathcal{A}_{\gamma,+}$ and $\mathcal{I}_\gamma$, which also form a disjoint splitting of Γ_f. On $\mathcal{A}_{\gamma,-}$ the variable λ_γ is equal to $-g$ and $\gamma\tau_f y_\gamma + \hat{\lambda} \leq -g$ holds. This implies

$$\tau_f y_\gamma(\bar{\lambda} - \lambda_\gamma) = \tau_f y_\gamma(\bar{\lambda} + g) \leq \gamma^{-1}(-g - \hat{\lambda})(\bar{\lambda} + g) \text{ a.e. on } \mathcal{A}_{\gamma,-}. \tag{3.33}$$

By a similar calculation one finds

$$\tau_f y_\gamma(\bar{\lambda} - \lambda_\gamma) \leq \gamma^{-1}(g - \hat{\lambda})(\bar{\lambda} - g) \text{ a.e. on } \mathcal{A}_{\gamma,+}. \tag{3.34}$$

On $\mathcal{I}_\gamma$ we have $\lambda_\gamma = \gamma\tau_f y_\gamma + \hat{\lambda}$ and thus $|\gamma\tau_f y_\gamma + \hat{\lambda}| < g$, which shows that

$$\frac{-g - \hat{\lambda}}{\gamma} < \tau_f y_\gamma < \frac{g - \hat{\lambda}}{\gamma}.$$

Thus almost everywhere on Γ_f we have

$$\tau_f y_\gamma(\bar{\lambda} - \lambda_\gamma) = \tau_f y_\gamma(\bar{\lambda} - \gamma\tau_f y_\gamma - \hat{\lambda}) \tag{3.35}$$

$$= -\gamma|\tau_f y_\gamma|^2 + \tau_f y_\gamma(\bar{\lambda} - \hat{\lambda})$$

$$\leq -\gamma|\tau_f y_\gamma|^2 + |\tau_f y_\gamma||\bar{\lambda} - \hat{\lambda}| \tag{3.36}$$

$$\leq -\gamma|\tau_f y_\gamma|^2 + \gamma^{-1}(g + |\hat{\lambda}|)|\bar{\lambda} - \hat{\lambda}|. \tag{3.37}$$

Hence, using (3.33),(3.34) and (3.35) one gets

$$(\tau_f y_\gamma, \bar\lambda - \lambda_\gamma)_{\Gamma_f} \le \gamma^{-1}(g + |\hat\lambda|, |\bar\lambda| + |\hat\lambda| + g)_{\Gamma_f}. \tag{3.38}$$

Using the coercivity of $a(\cdot\,,\cdot)$, (3.31),(3.32) and (3.38) we can now estimate

$$\begin{aligned} 0 &\le \limsup_{\gamma\to\infty} \nu \|y_\gamma - \bar y\|^2_{H^1(\Omega)} && (3.39)\\ &\le \limsup_{\gamma\to\infty} a(y_\gamma - \bar y, y_\gamma - \bar y) \\ &\le \lim_{\gamma\to\infty} (\tau_f y_\gamma, \bar\lambda - \lambda_\gamma)_{\Gamma_f} && (3.40)\\ &\le \lim_{\gamma\to\infty} \gamma^{-1}(g + |\hat\lambda|, |\bar\lambda| + |\hat\lambda| + g)_{\Gamma_f} = 0. && (3.41)\end{aligned}$$

It follows that $y_\gamma \to \bar y$ strongly in Y and hence $y^* = \bar y$. Passing to the limit in (3.30) for $(y, \lambda) = (y_\gamma, \lambda_\gamma)$ and using that weak limits are unique implies that $\lambda^* = \bar\lambda$. Thus we have proved that every sequence γ_n with $\gamma_n \to \infty$ for $n \to \infty$ contains a subsequence γ_{n_k} such that $\lambda_{\gamma_{n_k}} \rightharpoonup \bar\lambda$ in $L^2(\Gamma_f)$ and $y_{\gamma_{n_k}} \to \bar y$ in Y. Since $(\bar y, \bar\lambda)$ is the unique solution to (3.22a)-(3.22c) the whole family $\{(y_\gamma, \lambda_\gamma)\}$ converges in the sense given in the statement of the theorem. □

As a corollary to the proof of Theorem 3.10 one can obtain a convergence rate of y_γ to $\bar y$.

COROLLARY 3.11. *Let y_γ and $\bar y$ be solutions of $(\mathcal{P}_\gamma)$ and $(\mathcal{P})$, respectively and denote by $\nu > 0$ the coercivity constant of $a(\cdot\,,\cdot)$ on Y. Then there exists a constant $C > 0$ independent of γ such that*

$$\nu \|y_\gamma - \bar y\|^2_{H^1(\Omega)} \le a(y_\gamma - \bar y, y_\gamma - \bar y) \le \frac{C}{\gamma}.$$

PROOF. The left inequality follows form the coercivity of $a(\cdot\,,\cdot)$ on Y, the right inequality from (3.31) and (3.38). □

2.4. Relation to other regularizations. First consider the case that $\hat\lambda \equiv 0$. Then the smoothing of the absolute value function in $(\mathcal{P}_\gamma)$ results in

$$h(x) = \begin{cases} g|x| - \dfrac{1}{2\gamma} g^2 & \text{if } |x| \ge \dfrac{g}{\gamma} \\ \dfrac{\gamma}{2} x^2 & \text{if } |x| < \dfrac{g}{\gamma}. \end{cases}$$

Problem $(\mathcal{P}_\gamma)$ with the above smoothing of the absolute value function has also been studied in [47, p. 249-287], where it is seen as problem of determining a thermal control. The above smoothing has further been used in [51, 83] for the numerical solution of related problems. Note that $\hat\lambda \equiv 0$ and $f \equiv 0$ implies $J_\gamma(y_\gamma) = 0$, which appears natural if one considers J_γ as an energy functional.

Setting $\hat\lambda := \bar\lambda$ (the solution of $(\mathcal{P}^\star)$) in the regularized problems $(\mathcal{P}_\gamma^\star)$ and $(\mathcal{P}_\gamma)$, the solution variables λ_γ and y_γ of $(\mathcal{P}_\gamma^\star)$ and $(\mathcal{P}^\star)$ coincide with $\bar\lambda$ and $\bar y$, the solutions to $(\mathcal{P}_\gamma^\star)$ and $(\mathcal{P}_\gamma)$, respectively. The above choice for $\hat\lambda$ is surely not very realistic, but it motivates an update strategy in λ for the iterative solution of the simplified friction problem, which leads to the augmented Lagrangian method, [67, 68]. We return to this method in Section 3.4.

Let us now argue the relation of the above regularization to the one in [69], where $\hat\lambda$ is now arbitrary. For this purpose we choose $\sigma := \gamma^{-1}$ in the complementarity condition (3.22c) and eliminate the variable ξ_γ using (3.22b). This gives

$$\tau_f\, y_\gamma + \frac{1}{\gamma}(\hat\lambda - \lambda_\gamma) - \max(0, \tau_f\, y_\gamma + \frac{1}{\gamma}(\hat\lambda - g)) - \min(0, \tau_f\, y_\gamma + \frac{1}{\gamma}(\hat\lambda + g)) = 0. \tag{3.42}$$

The specific choice of $\sigma = \gamma^{-1}$ results in eliminating the variable λ_γ in the max- and min- function, which is of interest regarding semi-smooth Newton methods, [58, 104], as will become clear in the next section. In [69] a formulation related to (3.42) was successfully used to construct an effective algorithm for unilaterally constrained variational problems of the first kind. However, in the case of $(\mathcal{P}_\gamma^\star)$, which is a bilaterally constrained optimization problem, (3.42) may mislead us to an algorithm, which is less efficient. Obviously, from the theoretical point of view, the two formulations are equivalent, but the splitting of (3.42) into (3.22b) and (3.22c) contains the additional parameter σ and thus motivates a slightly different algorithm, as will be discussed in the next section.

3. Primal-Dual, Semi-Smooth Newton and Augmented Lagrangian Methods

In this section we present iterative second-order algorithms to solve the optimality system (3.22a)-(3.22c) and discuss some of their basic properties. To simplify notation we drop the subscript 'γ' for the iterates $(y_\gamma^k, \lambda_\gamma^k, \xi_\gamma^k)$ of the algorithms. The solution variables of the regularized problem are still denoted by $(y_\gamma, \lambda_\gamma, \xi_\gamma)$.

3.1. Primal-dual active set algorithm.

The primal-dual active set strategy (PDAS) is related to the algorithms in [14, 15, 57, 60, 70] in the context of constrained optimal control problems and to the one in [69] for the solution of obstacle problems. It is an iterative algorithm which uses the current primal variable λ^k for $(\mathcal{P}_\gamma^\star)$ and the current dual variable ξ^k to predict new active sets $\mathcal{A}_-^{k+1}$, $\mathcal{A}_+^{k+1}$ for the constrained optimization problem $(\mathcal{P}_\gamma^\star)$, whereas this prediction is motivated from expressing the complementarity condition in the form (3.22c). On these active sets the variable λ^{k+1} is fixed. In each iteration step the method requires to solve an equality constrained problem, which in our case turns out to be linear. Note that, compared to inequality constrained optimization, equality constrained problems are significantly easier to handle, both theoretically and numerically. The whole algorithm is specified next.

Algorithm 1: (PDAS)

(1) Choose $y^0 \in \{y \in Y : \frac{\partial y}{\partial n}_{|\Gamma_f} \in L^2(\Gamma_f)\}$, $\sigma > 0$ and set $\lambda^0 := -\frac{\partial y^0}{\partial n}_{|\Gamma_f}$, $\xi^0 := \tau_f y^0 + \gamma^{-1}(\hat{\lambda} - \lambda^0)$, $k := 0$.

(2) Determine

$$\begin{aligned}
\mathcal{A}_-^{k+1} &= \{x \in \Gamma_f : \xi^k + \sigma(\lambda^k + g) < 0\},\\
\mathcal{A}_+^{k+1} &= \{x \in \Gamma_f : \xi^k + \sigma(\lambda^k - g) > 0\},\\
\mathcal{I}^{k+1} &= \Gamma_f \setminus (\mathcal{A}_-^{k+1} \cup \mathcal{A}_+^{k+1}).
\end{aligned}$$

(3) If $k \geq 1$, $\mathcal{A}_-^{k+1} = \mathcal{A}_-^k$ and $\mathcal{A}_+^{k+1} = \mathcal{A}_+^k$ stop, else

(4) Solve

$$\begin{aligned}
&a(y^{k+1}, z) - (f, z)_\Omega + (\lambda^{k+1}, \tau_f z)_{\Gamma_f} = 0 \text{ for all } z \in Y,\\
&\tau_f y^{k+1} + \gamma^{-1}(\hat{\lambda} - \lambda^{k+1}) = 0 \text{ on } \mathcal{I}^{k+1},\\
&\lambda^{k+1} = -g \text{ on } \mathcal{A}_-^{k+1}, \text{ and } \lambda^{k+1} = g \text{ on } \mathcal{A}_+^{k+1}.
\end{aligned}$$

(5) Set

$$\lambda^{k+1} = \hat{\lambda} + \gamma\tau_f y^{k+1} \text{ on } \mathcal{I}^{k+1},$$

$$\xi^{k+1} = \begin{cases} \tau_f y^{k+1} + \gamma^{-1}(\hat{\lambda} + g) & \text{on } \mathcal{A}_-^{k+1},\\ \tau_f y^{k+1} + \gamma^{-1}(\hat{\lambda} - g) & \text{on } \mathcal{A}_+^{k+1},\\ 0 & \text{on } \mathcal{I}^{k+1}, \end{cases}$$

$k := k + 1$ and go to Step 2.

Note that the system in Step 4 of the algorithm constitutes the first-order optimality system for the equality constrained optimization problem

$$\min_{\substack{\lambda \in L^2(\Gamma_f)\\ \lambda = -g \text{ on } \mathcal{A}_-^{k+1}\\ \lambda = g \text{ on } \mathcal{A}_+^{k+1}}} J_\gamma^\star(\lambda) \quad \text{s.t. } e(w_0, \lambda) = 0 \text{ in } Y^\star \tag{3.43}$$

and hence existence of a solution follows. Note that ξ^{k+1} is the Lagrange multiplier for the equality constraints in (3.43). The justification of the stopping criterion in Step 3 of (PDAS) is given by the following lemma.

LEMMA 3.12. *If Algorithm* (PDAS) *stops, the last iterate is the solution to system* (3.22a)-(3.22c).

PROOF. First note that, by construction, the iterates fulfill (3.22a), (3.22b). It remains to show that (3.22c) is satisfied, too. Let $(y^{k-1}, \lambda^{k-1}, \xi^{k-1})$ denote the iterate upon termination. On $\mathcal{I}^k = \mathcal{I}^{k-1}$ we have $|\xi^{k-1} + \sigma\lambda^{k-1}| < \sigma g$ and $\xi^{k-1} = 0$, thus $|\lambda^{k-1}| < g$. On $\mathcal{A}_-^{k-1}$ the variable λ^{k-1} is set to $-g$ and one can infer from $\mathcal{A}_-^k = \mathcal{A}_-^{k+1}$ that $\xi^{k-1} + \sigma(\lambda^{k-1} + g) = \xi^{k-1} < 0$. A similar argument yields $\xi^{k-1} > 0$ on $\mathcal{A}_+^{k-1}$. This shows that $(y^{k-1}, \lambda^{k-1}, \xi^{k-1})$ satisfies (3.23) or equivalently (3.22c), which ends the proof. $\square$

REMARK 3.13. Let us now discuss the influence of the parameter σ on the iteration sequence for $k \geq 1$. On $\mathcal{I}^k$ we have that $\xi^k = 0$ and thus σ has no influence when determining the new active and inactive sets. On $\mathcal{A}_-^k$ we have $\lambda^k = -g$ and distinguish two cases: The set where $\xi^k < 0$ belongs to $\mathcal{A}_-^{k+1}$ for the next iteration independently from σ. In case $\xi^k > 0$ we have

$$\xi^k + \sigma(\lambda^k - g) = \xi^k - 2\sigma g.$$

The set where $\xi^k - 2\sigma g \leq 0$ moves to $\mathcal{I}^{k+1}$, if $\xi^k - 2\sigma g > 0$ to $\mathcal{A}_+^{k+1}$ for the next iteration. Hence, only in this case σ influences the sequence of iterates. Smaller values for σ make it more likely that points belong to $\mathcal{A}_-^k \cap \mathcal{A}_+^{k+1}$. A similar observation as for $\mathcal{A}_-^k$ holds true for $\mathcal{A}_+^k$, which shows that with $\sigma > 0$ one can control the probability that points are shifted from one active set to the other within one iteration. We also remark that if for some $\sigma := \sigma_1 > 0$ one has $\mathcal{A}_-^k \cap \mathcal{A}_+^{k+1} = \mathcal{A}_+^k \cap \mathcal{A}_-^{k+1} = \emptyset$, then for every $\sigma \geq \sigma_1$ also $\mathcal{A}_-^k \cap \mathcal{A}_+^{k+1} = \mathcal{A}_+^k \cap \mathcal{A}_-^{k+1} = \emptyset$. Furthermore, in this case the sets $\mathcal{A}_-^{k+1}, \mathcal{A}_+^{k+1}$ and $\mathcal{I}^{k+1}$ are the same for all $\sigma \geq \sigma_1$.

The above observation will be of interest in connection with our local convergence analysis of (PDAS). We now turn our attention to the reduced primal-dual active set algorithm.

3.2. Reduced primal-dual active set algorithm. We now utilize (3.42) to write the optimality system (3.22a)-(3.22c) for the regularized problem as one nonlinear equation. For this purpose we denote by $\tilde{y}$ the solution to the problem

$$a(y, v) - (f, v)_\Omega = 0 \text{ for all } v \in Y.$$

Furthermore, we introduce $B^{-1} \in \mathcal{L}(H^{-\frac{1}{2}}, Y)$, the solution mapping for the variational equality

$$a(y, v) - \left\langle \lambda, \tau_f v \right\rangle_{H^{-\frac{1}{2}}, H^{\frac{1}{2}}} = 0 \text{ for all } v \in Y$$

for given $\lambda \in H^{-\frac{1}{2}}(\Gamma_f)$. We can now define the Neumann-to-Dirichlet operator

$$C := \tau_f B^{-1}_{|L^2(\Gamma_f)} \in \mathcal{L}(L^2(\Gamma_f), L^2(\Gamma_f)) \tag{3.44}$$

and summarize some of its properties in the next lemma.

LEMMA 3.14. *The Neumann-to-Dirichlet operator C defined in* (3.44) *is self-adjoint, positive definite, injective and compact.*

PROOF. Let $\lambda_1, \lambda_2 \in L^2(\Gamma_f)$, then

$$(C\lambda_1, \lambda_2)_{\Gamma_f} = (\tau_f y_1, \lambda_2)_{\Gamma_f}$$

with $y_1 \in Y$ satisfying

$$a(y_1, z) - (\lambda_1, \tau_f z)_{\Gamma_f} = 0 \text{ for all } z \in Y. \tag{3.45}$$

In addition, we introduce $y_2 \in Y$ that solves (3.45) with λ_1 replaced by λ_2. Thus

$$\begin{aligned}(C\lambda_1, \lambda_2)_{\Gamma_f} &= (\tau_f\, y_1, \lambda_2)_{\Gamma_f} = a(y_2, y_1),\\ (\lambda_1, C\lambda_2)_{\Gamma_f} &= (\lambda_1, \tau_f\, y_2)_{\Gamma_f} = a(y_1, y_2),\end{aligned}$$

which shows, utilizing the symmetry of $a(\cdot\,,\cdot)$ that C is self-adjoint. Taking $0 \neq \lambda_1 = \lambda_2$ results in

$$(C\lambda_1, \lambda_1)_{\Gamma_f} = a(y_1, y_1) > 0$$

yielding the positivity of C, which implies that C is injective. To verify the compactness of C we observe that $Rg(C) \subset H^{\frac{1}{2}}(\Gamma_f)$ due to the trace theorem, [86]. Thus, utilizing that the embedding

$$H^{\frac{1}{2}}(\Gamma_f) \hookrightarrow\hookrightarrow L^2(\Gamma_f)$$

is compact [1], compactness of C follows. $\square$

With the help of the operators B^{-1} and C one can write the solution y to

$$a(y, z) - (f, z)_\Omega + (\lambda, \tau_f\, z) = 0 \text{ for all } z \in Y$$

for given $\lambda \in L^2(\Gamma_f)$ as $y = -B^{-1}\lambda + \tilde{y}$ and $\tau_f\, y$ as $-C\lambda + \tau_f\, \tilde{y}$. This allows to eliminate the variable $\tau_f\, y$ in (3.42). For this purpose we utilize the mapping $\bar{F} : L^2(\Gamma_f) \to L^2(\Gamma_f)$ defined by

$$\begin{aligned}\bar{F}(\lambda) &= C\lambda - \tau_f\, \tilde{y} - \gamma^{-1}(\hat{\lambda} - \lambda)\\ &\quad + \max(0, -C\lambda + \tau_f\, \tilde{y} + \gamma^{-1}(\hat{\lambda} - g))\\ &\quad + \min(0, -C\lambda + \tau_f\, \tilde{y} + \gamma^{-1}(\hat{\lambda} + g)).\end{aligned} \tag{3.46}$$

Note that $\bar{F}(\lambda) = 0$ characterizes λ as the solution of $(\mathcal{P}_\gamma^\star)$. In the sequel we utilize for $\mathcal{S} \subset \Gamma_f$ the extension-by-zero operator $E_\mathcal{S} : L^2(\mathcal{S}) \to L^2(\Gamma_f)$ defined by

$$E_\mathcal{S}(g)(x) := \begin{cases} g(x) & \text{if } x \in \mathcal{S}\\ 0 & \text{else.}\end{cases} \tag{3.47}$$

Its adjoint operator $E_\mathcal{S}^\star : L^2(\Gamma_f) \to L^2(\mathcal{S})$ is the restriction operator onto $\mathcal{S}$. To simplify the notation we utilize $g_\mathcal{S} := E_\mathcal{S}^\star g$ for $g \in L^2(\Gamma_f)$. The primal-dual active set method for the reduced problem is specified next.

Algorithm 2: (RPDAS)

(1) Choose $y^0 \in \{y \in Y : \frac{\partial y}{\partial n}_{|\Gamma_f} \in L^2(\Gamma_f)\}$, $\sigma > 0$ and set $\lambda^0 := -\frac{\partial y^0}{\partial n}_{|\Gamma_f}$ and $k := 0$.

(2) Determine

$$\begin{aligned}
\mathcal{A}_-^{k+1} &= \{x \in \Gamma_f : -C\lambda^k + \tau_f \tilde{y} + \gamma^{-1}(\hat{\lambda} + g) < 0\}, \\
\mathcal{A}_+^{k+1} &= \{x \in \Gamma_f : -C\lambda^k + \tau_f \tilde{y} + \gamma^{-1}(\hat{\lambda} - g) > 0\}, \\
\mathcal{I}^{k+1} &= \Gamma_f \setminus (\mathcal{A}_-^{k+1} \cup \mathcal{A}_+^{k+1}).
\end{aligned}$$

(3) Set

$$\lambda^{k+1} = -g \text{ on } \mathcal{A}_-^{k+1},\ \lambda^{k+1} = g \text{ on } \mathcal{A}_+^{k+1}$$

and solve for λ^{k+1} on $\mathcal{I}^{k+1}$

$$\frac{1}{\gamma}\lambda_{\mathcal{I}^{k+1}}^{k+1} + C(E_{\mathcal{I}^{k+1}}\lambda_{\mathcal{I}^{k+1}}^{k+1} - E_{\mathcal{A}_-^{k+1}} g_{\mathcal{A}_-^{k+1}} + E_{\mathcal{A}_+^{k+1}} g_{\mathcal{A}_+^{k+1}}) = (\tau_f \tilde{y} + \frac{1}{\gamma}\hat{\lambda})_{\mathcal{I}^{k+1}}$$

(4) Update $k := k+1$ and go to Step 2.

The equation in Step 3 of the algorithm arises from fixing λ^{k+1} on $\mathcal{A}_-^{k+1}$, $\mathcal{A}_+^{k+1}$ and calculating λ utilizing $\bar{F}(\lambda) = 0$. This equation admits a solution due to the positivity of the operator C, which implies positivity of the operator $\frac{1}{\gamma} + E_{\mathcal{I}^{k+1}}^\star C E_{\mathcal{I}^{k+1}}$. Furthermore, note that - neglecting possible errors in a numerical realization of the algorithm - (RPDAS) coincides with (PDAS) when $\sigma = \gamma^{-1}$. Hence the simple stopping criterion from Step 3 of (PDAS) can also be used for (RPDAS). Then the assertion of Lemma 3.12 holds for (RPDAS) as well.

3.3. Relation between (RPDAS), (PDAS) and a semi-smooth Newton method. Writing the optimality system as a single nonlinear equation as done in (3.46) suggests to apply a generalized Newton method to solve $\bar{F}(\lambda) = 0$. Recall from Section 2 that for the application of the semi-smooth Newton method to an equation involving the max- and min-operator the relevant variables inside the max- and min- function have to appear under a smoothing operator. We can now show that the function $\bar{F}$ fulfills this condition, and that the resulting semi-smooth Newton algorithm is related to (RPDAS). First, observe that $Rg(C) \subset H^{\frac{1}{2}}(\Gamma_f)$ and that

$$H^{\frac{1}{2}}(\Gamma_f) \hookrightarrow L^q(\Gamma_f) \text{ for } \begin{cases} q = \dfrac{2(n-1)}{n-2} & \text{if } n \geq 3, \\ q < \infty & \text{if } n = 2, \end{cases} \tag{3.48}$$

where $n \geq 2$ denotes the dimension of Ω. Note that $q > 2$ for all $n \geq 2$. From the Theorems 2.8 and 2.10 it follows that $\bar{F}$ is Newton differentiable on $L^2(\Gamma_f)$. A generalized derivative of $\bar{F}$ is given by

$$G_{\bar{F}}(\lambda)(\delta) = C\delta + \frac{1}{\gamma}\delta - E_{\mathcal{A}_-}(C\delta)_{\mathcal{A}_-} - E_{\mathcal{A}_+}(C\delta)_{\mathcal{A}_+} = (E_{\mathcal{I}}E_{\mathcal{I}}^\star C + \frac{1}{\gamma})\delta, \tag{3.49}$$

where the extension-by-zero operator E is defined as in (3.47), and $\mathcal{A}_-$, $\mathcal{A}_+$, $\mathcal{I}$ are given by

$$\mathcal{A}_- = \{x \in \Gamma_f : -C\lambda + \tau_f\, \tilde{y} + \gamma^{-1}(\hat{\lambda} - g) \geq 0\},$$
$$\mathcal{A}_+ = \{x \in \Gamma_f : -C\lambda + \tau_f\, \tilde{y} + \gamma^{-1}(\hat{\lambda} + g) \leq 0\},$$
$$\mathcal{I} = \Gamma_f \setminus (\mathcal{A}_- \cup \mathcal{A}_+).$$

For given $\lambda^k \in L^2(\Gamma_f)$ the resulting Newton step

$$x^{k+1} = x^k - G_{\bar{F}}(x^k)^{-1}\bar{F}(x^k) \tag{3.50}$$

for $\bar{F}$ is

$$\begin{aligned}(\frac{1}{\gamma} + E_{\mathcal{I}^{k+1}}E^\star_{\mathcal{I}^{k+1}}C)\delta^{k+1} = -C\lambda^k + \tau_f\, \tilde{y} + \frac{1}{\gamma}(\hat{\lambda} - \lambda^k)\\ - E_{\mathcal{A}_+^{k+1}}(-C\lambda^k + \tau_f\, \tilde{y} + \frac{1}{\gamma}(\hat{\lambda} - g))_{\mathcal{A}_+^{k+1}}\\ - E_{\mathcal{A}_-^{k+1}}(-C\lambda^k + \tau_f\, \tilde{y} + \frac{1}{\gamma}(\hat{\lambda} + g))_{\mathcal{A}_-^{k+1}},\end{aligned}$$

where $\delta^{k+1} = \lambda^{k+1} - \lambda^k$ and $\mathcal{A}_-^{k+1}, \mathcal{A}_+^{k+1}, \mathcal{I}^{k+1}$ are defined as in Step 2 of (RPDAS). On $\mathcal{A}_-^{k+1}$ the above Newton step gives

$$\frac{1}{\gamma}\delta_\lambda = -C\lambda^k + \tau_f\, \tilde{y} + \frac{1}{\gamma}(\hat{\lambda} - \lambda^k) + C\lambda^k - \tau_f\, \tilde{y} - \frac{1}{\gamma}(\hat{\lambda} + g)),$$

which shows that

$$\frac{1}{\gamma}(\lambda^{k+1} - \lambda^k) = -\frac{1}{\gamma}(\lambda^k + g)$$

yielding $\lambda^{k+1} = -g$ on $\mathcal{A}_-^{k+1}$. Similarly one deduces $\lambda^{k+1} = g$ on $\mathcal{A}_+^{k+1}$. Finally, on $\mathcal{I}^{k+1}$ we find

$$(C + \frac{1}{\gamma})\delta = -C\lambda^k + \tau_f\, \tilde{y} + \frac{1}{\gamma}(\hat{\lambda} - \lambda^k)$$

resulting in

$$(C + \frac{1}{\gamma})\lambda^{k+1} = \tau_f\, \tilde{y} + \frac{1}{\gamma}\hat{\lambda}. \tag{3.51}$$

Hence, one semi-smooth Newton iteration step for the solution of $\bar{F}(\lambda) = 0$ is given as follows: For a given iterate $\lambda^k \in L^2(\Gamma_f)$ calculate the active sets according to Step 2 of (RPDAS). Then set

$$\lambda^{k+1} = -g \text{ on } \mathcal{A}_-^{k+1},\ \lambda^{k+1} = g \text{ on } \mathcal{A}_+^{k+1},$$

and solve for λ^{k+1} on $\mathcal{I}^{k+1}$

$$\frac{1}{\gamma}\lambda^{k+1}_{\mathcal{I}^{k+1}} + C(E_{\mathcal{I}^{k+1}}\lambda^{k+1}_{\mathcal{I}^{k+1}} - E_{\mathcal{A}_-^{k+1}}g_{\mathcal{A}_-^{k+1}} + E_{\mathcal{A}_+^{k+1}}g_{\mathcal{A}_+^{k+1}}) = (\tau_f\, \tilde{y} + \frac{1}{\gamma}\hat{\lambda})_{\mathcal{I}^{k+1}}.$$

This iteration step is equal to Steps 2 and 3 of (RPDAS). An analogous result for unilaterally constrained optimization problems was established in [58]. Note that the norm gap required for Newton differentiability of the max- and min-function results from directly exploiting the smoothing property of the operator C. This has become possible since we chose $\sigma := \gamma^{-1}$ in (3.22c) which allowed us to eliminate the explicit appearance of λ in the max- and min-function. Taking advantage of this fact, (RPDAS) does not require a smoothing step as the semi-smooth Newton method in [104].

We now investigate whether (PDAS) can also be interpreted as generalized Newton method. We introduce $F : Y \times L^2(\Gamma_f) \times L^2(\Gamma_f) \to Y^\star \times L^2(\Gamma_f) \times L^2(\Gamma_f)$ by

$$F(y,\lambda,\xi) := \begin{pmatrix} e(y,\lambda) \\ \tau_f\, y + \gamma^{-1}(\hat\lambda - \lambda) - \xi \\ \xi - \max(0, \xi + \sigma(\lambda - g)) - \min(0, \xi + \sigma(\lambda + g)) \end{pmatrix}, \tag{3.52}$$

and observe that $F(y,\lambda,\xi) = 0$ characterizes y and λ as solutions to $(\mathcal{P}_\gamma)$ and $(\mathcal{P}_\gamma^\star)$, respectively. Applying the Newton iteration (3.50) with the generalized derivative of the max- and min-function as given in (2.10) to the mapping F results in Algorithm (PDAS). This can be seen similarly as for (RPDAS). However, note that for (PDAS) this procedure is purely formal, since we do not have the norm gap required for infinite-dimensional Newton differentiability of the max- and min-function. Hence, in principle we cannot expect properties coming from the interpretation as Newton method for (PDAS). Nevertheless, in Section 4.2 it is shown that for certain problems (PDAS) converges locally superlinearly without the necessity of a smoothing step as used in [104] to get local superlinear convergence of semi-smooth Newton methods.

3.4. Augmented Lagrangian methods for the solution of $(\mathcal{P}^\star)$. Augmented Lagrangian methods combine ordinary Lagrangian methods and penalty methods without suffering of the disadvantages of these methods. For instance, the augmented Lagrangian method converges without requiring that the penalty parameter tends to infinity. The first-order augmented Lagrangian method that we state here can be considered as an implicit version of the Uzawa method, see [67]. For a detailed discussion of these methods we also refer to [16].

To argue the close relation of the regularization for $(\mathcal{P}^\star)$ to augmented Lagrangians recall that (3.22b),(3.22c) can equivalently be expressed as (3.42) and, after multiplication with γ as

$$\lambda_\gamma = \gamma\tau_f\, y_\gamma + \hat\lambda - \max(0, \gamma\tau_f\, y_\gamma + \hat\lambda - g) - \min(0, \gamma\tau_f\, y_\gamma + \hat\lambda + g). \tag{3.53}$$

The augmented Lagrangian method is an iterative algorithm for the calculation of λ in $(\mathcal{P}^\star)$. Given an iterate λ^l for $(\mathcal{P}^\star)$, the next iterate λ^{l+1} can be determined setting $\hat\lambda := \lambda^l$ in the right side of (3.53). The whole method is specified next.

Algorithm 3: (ALM)

(1) Choose $\gamma > 0$, $\lambda^0 \in L^2(\Gamma_f)$ and set $l := 0$.
(2) Solve for $(y^{l+1}, \lambda^{l+1}, \xi^{l+1}) \in Y \times L^2(\Gamma_f) \times L^2(\Gamma_f)$ system (3.22a)-(3.22c) with $\hat{\lambda} := \lambda^l$.
(3) Update $l := l + 1$ and go to Step 2.

The auxiliary problem in Step 2 of (ALM) has exactly the form of our regularized problem and can thus efficiently be solved using (PDAS) or (RPDAS). The question arises concerning the precision to which the system in Step 2 of (ALM) should be solved. Several strategies are possible, such as solving the system exactly for all l or performing only one iteration step of the semi-smooth Newton method in each iteration. We tested several strategies and report on them in Section 5. Note that in (ALM) the regularization parameter γ plays the role of a penalty parameter, which is not necessarily taken to infinity, nevertheless (ALM) detects the solution of $(\mathcal{P}^\star)$, as will be shown in the next section.

4. Convergence Analysis

4.1. Local convergence analysis of (RPDAS). In this section we give a local convergence result for (RPDAS) for the solution of the regularized friction problem. For this purpose we utilize the interpretation of (RPDAS) as a semi-smooth Newton method.

Theorem 3.15. *If $\|\lambda^0 - \lambda_\gamma\|_{\Gamma_f}$ is sufficiently small, then for all $\hat{\lambda} \in L^2(\Gamma_f)$ and $\gamma > 0$ the iterates λ^k of* (RPDAS) *converge to (λ_γ) superlinearly in $L^2(\Gamma_f)$. Furthermore, the corresponding primal iterates y^k converge superlinearly in Y to y_γ.*

Proof. We only have to show superlinear convergence of λ^k to λ_γ in $L^2(\Gamma_f)$. Then, the superlinear convergence of y^k to y_γ in $Y \subset H^1(\Omega)$ follows since $B^{-1} \in \mathcal{L}(L^2(\Gamma_f), Y)$ is continuous.

We already argued Newton differentiability of the mapping $\bar{F} : L^2(\Gamma_f) \to L^2(\Gamma_f)$. To apply Theorem 2.9 it remains to verify that the generalized derivative $G_{\bar{F}} \in \mathcal{L}(L^2(\Gamma_f), L^2(\Gamma_f))$ of $\bar{F}$ have uniformly bounded inverses. Recall for $\mathcal{S} \subset \Gamma_f$ the definition of the extension-by-zero operator $E_\mathcal{S}$ and its adjoint $E_\mathcal{S}^\star$ as given in (3.47). Let $(h_{\mathcal{A}_-}, h_{\mathcal{A}_+}, h_\mathcal{I}) \in L^2(\mathcal{A}_-) \times L^2(\mathcal{A}_+) \times L^2(\mathcal{I})$ and consider the equation

$$G_{\bar{F}}(\lambda)(\delta) = G_{\bar{F}}(\lambda)(\delta_{\mathcal{A}_-}, \delta_{\mathcal{A}_+}, \delta_\mathcal{I}) = (h_{\mathcal{A}_-}, h_{\mathcal{A}_+}, h_\mathcal{I}). \tag{3.54}$$

Recalling the explicit form (3.49) of $G_{\bar{F}}$ we get from (3.54) that $\delta_{\mathcal{A}_-} = \gamma h_{\mathcal{A}_-}$ and $\delta_{\mathcal{A}_+} = \gamma h_{\mathcal{A}_+}$ must hold. Furthermore,

$$\Big(\frac{1}{\gamma} + E_\mathcal{I}^\star C E_\mathcal{I}\Big)\delta_\mathcal{I} = h_\mathcal{I} - \gamma E_\mathcal{I}^\star C E_{\mathcal{A}_-} h_{\mathcal{A}_-} - \gamma E_\mathcal{I}^\star C E_{\mathcal{A}_+} h_{\mathcal{A}_+}. \tag{3.55}$$

Due to the positivity of C we can define a new scalar product $\langle\langle \cdot\,, \cdot \rangle\rangle$ on $L^2(\mathcal{I})$ by

$$\langle\langle x, y \rangle\rangle := \Big\langle \big(\frac{1}{\gamma} + E_\mathcal{I}^\star C E_\mathcal{I}\big)x, y \Big\rangle_\mathcal{I} \text{ for } x, y \in L^2(\mathcal{I}).$$

Utilizing the positivity of C we have

$$\langle\langle x, x\rangle\rangle \geq \frac{1}{\gamma}\langle x, x\rangle_{\mathcal{I}} \text{ for all } x \in L^2(\mathcal{I}),$$

i.e., the product $\langle\langle \cdot\,, \cdot\rangle\rangle$ is coercive with constant γ^{-1} independently from $\mathcal{I}$. Applying the Lax-Milgram lemma one finds that not only (3.54) admits a unique solution $\delta_{\mathcal{I}}$, but also that

$$\|\delta_{\mathcal{I}}\|_{\mathcal{I}} \leq \gamma\|h_{\mathcal{I}}\|_{\mathcal{I}} + \gamma^2\|C\|_{\mathcal{L}(L^2(\Gamma_f))}\left\{\|h_{\mathcal{A}_-}\|_{\mathcal{A}_-} + \|h_{\mathcal{A}_+}\|_{\mathcal{A}_+}\right\}.$$

This proves the uniform boundedness of $G_{\bar{F}}^{-1}(\lambda)$ for all $\lambda \in L^2(\Gamma_f)$ and ends the proof. □

4.2. Local convergence analysis of (PDAS). As observed at the end of Section 3.3 algorithm (PDAS) cannot directly be interpreted as locally superlinear convergent semi-smooth Newton method if no smoothing steps are used. However, utilizing Remark 3.13 local superlinear convergence holds for (PDAS) as well, provided the dimension n of Ω is 2.

COROLLARY 3.16. *Assume that $n = 2$ and $\Gamma_0 \subset \Gamma$ is sufficient regular. If $\|\lambda^0 - \lambda_\gamma\|_{\Gamma_f}$ is sufficiently small, the iterates (y^k, λ^k) of* (PDAS) *with $\sigma \geq \gamma^{-1}$ converge superlinearly in $Y \times L^2(\Gamma_f)$.*

PROOF. The idea of this proof is, to show that in a neighborhood of the solution λ_γ the iterates λ^k of (PDAS) coincide with λ^k from (RPDAS), which allows to apply Theorem 3.15 also for (PDAS).
Step 1. We first only consider (RPDAS) and denote by $\delta > 0$ the convergence radius of this semi-smooth Newton method. We introduce a δ_0 with $0 < \delta_0 \leq \delta$ that will be further specified below and choose $\lambda^0 \in L^2(\Gamma_f)$ such that $\|\lambda^0 - \lambda_\gamma\|_{\Gamma_f} \leq \delta_0$. Since $\delta_0 \leq \delta$ the method converges and $\|\lambda^k - \lambda^{k+1}\|_{\Gamma_f} \leq 2\delta_0$ for $k \geq 1$. Note that the difference of the corresponding variables $y^k - y^{k+1}$ solves

$$a(y^k - y^{k+1}, v) + (\lambda^k - \lambda^{k+1}, \tau_f v)_{\Gamma_f} = 0 \text{ for all } v \in Y.$$

Then it follows from regularity results for mixed elliptic problems [97, 100] that

$$\|y^k - y^{k+1}\|_{C^0(\bar{\Omega})} \leq C\|\lambda^k - \lambda^{k+1}\|_{\Gamma_f} \text{ for some } C > 0.$$

For the corresponding traces we have

$$\|\tau_f (y^k - y^{k+1})\|_{C^0(\Gamma_f)} \leq C\|\lambda^k - \lambda^{k+1}\|_{\Gamma_f} \leq 2C\delta_0. \tag{3.56}$$

We now show that, for δ_0 sufficiently small $\mathcal{A}_-^k \cap \mathcal{A}_+^{k+1} = \mathcal{A}_+^k \cap \mathcal{A}_-^{k+1} = \emptyset$. We prove this claim by contradiction, i.e., we assume that $J = \mathcal{A}_+^k \cap \mathcal{A}_-^{k+1} \neq \emptyset$. Then, almost everywhere on J we have

$$\tau y^{k-1} + \gamma^{-1}(\hat{\lambda} - g) > 0 \;\text{ and }\; \tau y^k + \gamma^{-1}(\hat{\lambda} + g) < 0,$$

which implies

$$\tau(y^{k+1} - y^k) > \frac{2g}{\gamma}.$$

Thus, utilizing (3.56)

$$\frac{2g}{\gamma} < \|\tau(y^{k+1} - y^k)\|_{C^0(\Gamma_f)} \leq 2C\delta_0. \tag{3.57}$$

If we choose $\delta_0 \leq \frac{g}{C\gamma}$, relation (3.57) cannot hold true and therefore $J = \emptyset$; An analogous observation holds true for $\mathcal{A}_-^k \cap \mathcal{A}_+^{k+1}$, which shows that

$$\mathcal{A}_-^k \cap \mathcal{A}_+^{k+1} = \mathcal{A}_+^k \cap \mathcal{A}_-^{k+1} = \emptyset \text{ if } \delta_0 \leq \frac{g}{C\gamma}. \tag{3.58}$$

Step 2. Recall that the iterates of (PDAS) with $\sigma = \gamma^{-1}$ coincide with those of (RPDAS). Thus, if $\|\lambda^0 - \lambda_\gamma\|_{\Gamma_f} \leq \delta_0$, then $\mathcal{A}_-^k \cap \mathcal{A}_+^{k+1} = \mathcal{A}_+^k \cap \mathcal{A}_-^{k+1} = \emptyset$ for (PDAS) with $\sigma = \gamma^{-1}$. It follows from Remark 3.13 that for the active sets calculated from (PDAS) using $\sigma \geq \gamma^{-1}$ also $\mathcal{A}_-^k \cap \mathcal{A}_+^{k+1} = \mathcal{A}_+^k \cap \mathcal{A}_-^{k+1} = \emptyset$ holds. This shows that (RPDAS) and (PDAS) determine the same iterates for the variable λ_γ provided that $\|\lambda^0 - \lambda_\gamma\|_{\Gamma_f} < \delta_0$. Hence, superlinear L^2-convergence for λ^k determined from (PDAS) holds. For the variables y^k superlinear convergence in Y follows from the continuity of solution mapping $B^{-1} \in (L^2(\Gamma_f), Y)$. □

4.3. Global conditional convergence of (RPDAS). Our global convergence result is based on an appropriately defined functional which decays when evaluated along the iterates of the algorithm. A related strategy to prove global convergence (i.e., convergence from arbitrary initialization) is used in [70] in the context of optimal control problems. For unilateral constrained problems global convergence results can be gained using monotonicity properties of the operators involved, see, e.g., [69]. For bilateral constraints (as in $(\mathcal{P}_\gamma^\star)$) such properties cannot be utilized. In the sequel we use the notation from (PDAS) with $\sigma := \gamma^{-1}$ for (RPDAS). For $(\lambda, \xi) \in L^2(\Gamma_f) \times L^2(\Gamma_f)$ we define the functional

$$M(\lambda, \xi) := \frac{1}{\gamma^2} \int_{\Gamma_f} |(\lambda - g)^+|^2 + |(\lambda + g)^-|^2 \, dx + \int_{\mathcal{A}_+^*} |(\xi)^-|^2 \, dx + \int_{\mathcal{A}_-^*} |(\xi)^+|^2 \, dx, \tag{3.59}$$

where $\mathcal{A}_+^* = \{x \in \Gamma_f : \lambda(x) \geq g\}$ and $\mathcal{A}_-^* = \{x \in \Gamma_f : \lambda(x) \leq -g\}$. By $(\cdot)^+$ and $(\cdot)^-$ we denote the positive and negative part, i.e.,

$$(\cdot)^+ := \max(0, \cdot) \text{ and } (\cdot)^- := -\min(0, \cdot).$$

As a preparatory step for the following estimates we prove a lemma on compact operators in Hilbert spaces.

Lemma 3.17. *Let X be a real Hilbert space with inner product $(\cdot\,,\cdot)$ and $C \in \mathcal{L}(X)$ injective, self-adjoint, positive and compact. Then*

$$(y, y) \leq \|C\|_{\mathcal{L}(X)} (C^{-1}y, y)$$

for all $y \in Rg(C)$.

PROOF. Self-adjointness, positivity and compactness of C imply that C can be written in the form

$$C = \sum_{n=1}^{\infty} \lambda_n P_n$$

with projection operators P_n and $0 < \lambda_n \leq \|C\|_{\mathcal{L}(X)}$ for $n = 1, 2, \ldots$, see, e.g., [33]. We have $y = Cx$ with $x \in X$ and thus

$$\begin{aligned}(y, y) = (Cx, Cx) &= \sum_{n=1}^{\infty} \lambda_n^2 \|P_n(x)\|^2_{\mathcal{L}(X)} \leq \|C\|_{\mathcal{L}(X)} \sum_{n=1}^{\infty} \lambda_n \|P_n(x)\|^2_{\mathcal{L}(X)} \\ &= \|C\|_{\mathcal{L}(X)} \sum_{n=1}^{\infty} \lambda_n (x, P_n(x)) = \|C\|_{\mathcal{L}(X)} (x, y) = \|C\|_{\mathcal{L}(X)} (C^{-1} y, y),\end{aligned}$$

where $(P_n(x), P_n(x)) = (x, P_n(x))$ was used, which holds since P_n is a projection operator on a subspace and thus $(P_n(x) - x, P_n(x)) = 0$. $\square$

Following Lemma 3.14 the Neumann-to-Dirichlet mapping C, as given in (3.44) fulfills the conditions of Lemma 3.17. Utilizing the operator C the Steps 4 and 5 of (PDAS) imply

$$C^{-1} \tau_f y^{k+1} = -\lambda^{k+1} = -\begin{cases} g & \text{on } \mathcal{A}_+^{k+1}, \\ \gamma \tau_f y^{k+1} + \hat{\lambda} & \text{on } \mathcal{I}^{k+1}, \\ -g & \text{on } \mathcal{A}_-^{k+1}, \end{cases}$$

$$\tau_f y^{k+1} + \gamma^{-1}(\hat{\lambda} - \lambda^{k+1}) - \xi^{k+1} = 0. \tag{3.60}$$

With the above notation we get

$$C^{-1}(\tau_f (y^k - y^{k+1})) = \lambda^{k+1} - \lambda^k = \begin{cases} R^k_{\mathcal{A}^+} & \text{on } \mathcal{A}_+^{k+1}, \\ \gamma(\tau_f (y^{k+1} - y^k)) + R^k_{\mathcal{I}} & \text{on } \mathcal{I}^{k+1}, \\ R^k_{\mathcal{A}_-} & \text{on } \mathcal{A}_-^{k+1}, \end{cases} \tag{3.61}$$

where

$$R^k_{\mathcal{A}_+} = \begin{cases} 0 & \text{on } \mathcal{A}_+^{k+1} \cap \mathcal{A}_+^k, \\ g - \lambda^k < 0 & \text{on } \mathcal{A}_+^{k+1} \cap \mathcal{I}^k, \\ 2g < \gamma \xi^k & \text{on } \mathcal{A}_+^{k+1} \cap \mathcal{A}_-^k; \end{cases}$$

$$R^k_{\mathcal{I}} = \begin{cases} \gamma \tau_f y^k + \hat{\lambda} - g = \gamma \xi^k \leq 0 & \text{on } \mathcal{I}^{k+1} \cap \mathcal{A}_+^k, \\ 0 & \text{on } \mathcal{I}^{k+1} \cap \mathcal{I}^k, \\ \gamma \xi^k \geq 0 & \text{on } \mathcal{I}^{k+1} \cap \mathcal{A}_-^k; \end{cases}$$

$$R^k_{\mathcal{A}_+} = \begin{cases} -2g > \gamma\xi^k & \text{on } \mathcal{A}^{k+1}_- \cap \mathcal{A}^k_+, \\ -g - \lambda^k > 0 & \text{on } \mathcal{A}^{k+1}_- \cap \mathcal{I}^k, \\ 0 & \text{on } \mathcal{A}^{k+1}_- \cap \mathcal{A}^k_-. \end{cases}$$

Let us denote by R^k the function defined on Γ_f, whose restrictions to $\mathcal{A}^{k+1}_-$, $\mathcal{I}^{k+1}$ and $\mathcal{A}^{k+1}_+$ coincide with $R^k_{\mathcal{A}_-}$, $R^k_{\mathcal{I}}$ and $R_{\mathcal{A}^k_-}$, respectively. Note that, from the definition of R^k we have

$$\|R^k\|^2_{\Gamma_f} \leq \gamma^2 \|\xi^k\|^2_{\mathcal{B}^k} + \|g - \lambda^k\|^2_{\mathcal{A}^{k+1}_+ \cap \mathcal{I}^k} + \|g + \lambda^k\|^2_{\mathcal{A}^{k+1}_- \cap \mathcal{I}^k} \tag{3.62}$$

$$\leq \gamma^2 M(\lambda^k, \xi^k), \tag{3.63}$$

where $\mathcal{B}^k := \mathcal{I}^{k+1} \cap (\mathcal{A}^k_- \cup \mathcal{A}^k_+) \cup (\mathcal{A}^{k+1}_- \cap \mathcal{A}^k_+) \cup (\mathcal{A}^{k+1}_+ \cap \mathcal{A}^k_-)$. To shorten the notation we introduce $\delta^k_y := \tau_f\,(y^{k+1} - y^k)$. Multiplying (3.61) with $-\delta^k_y$ results in

$$C^{-1}(\delta^k_y)(\delta^k_y) = -R^k \delta^k_y - \chi_{\mathcal{I}^{k+1}} \gamma (\delta^k_y)^2 \ \text{ a.e. on } \Gamma_f,$$

where $\chi_{\mathcal{I}^{k+1}}$ denotes the characteristic function for $\mathcal{I}^{k+1}$. Thus,

$$(C^{-1}\delta^k_y, \delta^k_y)_{\Gamma_f} = \int_{\Gamma_f} -R^k \delta^k_y \, dx - \gamma \int_{\mathcal{I}^{k+1}} (\delta^k_y)^2 \, dx$$

$$\leq \|R^k\|_{\Gamma_f} \|\delta^k_y\|_{\Gamma_f}, \tag{3.64}$$

where we used the Cauchy-Schwarz inequality and $\int_{\mathcal{I}^{k+1}} (\delta^k_y)^2 \, dx \geq 0$. Utilizing Lemma 3.17 for the Neumann-to-Dirichlet mapping C yields

$$\|\delta^k_y\|^2_{\Gamma_f} \leq \|C\|_{\mathcal{L}(L^2(\Gamma_f))} (C^{-1}\delta^k_y, \delta^k_y)_{\Gamma_f}. \tag{3.65}$$

Combining (3.64) and (3.65) implies that

$$\|\delta^k_y\|_{\Gamma_f} \leq \|C\|_{\mathcal{L}(L^2(\Gamma_f))} \|R^k\|_{\Gamma_f}. \tag{3.66}$$

We can now prove the following convergence theorem for (RPDAS), or equivalently for (PDAS) with $\sigma = \gamma^{-1}$.

THEOREM 3.18. *If $\gamma < \|C\|^{-1}_{\mathcal{L}(L^2(\Gamma_f))}$, then*

$$M(\lambda^{k+1}, \xi^{k+1}) < M(\lambda^k, \xi^k)$$

for $k = 0, 1, 2, \ldots$ with $(\lambda^k, \xi^k) \neq (\lambda_\gamma, \xi_\gamma)$, where (λ^k, ξ^k) denote the iterates of (PDAS) *with $\sigma = \gamma^{-1}$. Moreover, (y^k, λ^k, ξ^k) converges to $(y_\gamma, \lambda_\gamma, \xi_\gamma)$ strongly in $Y \times L^2(\Gamma_f) \times L^2(\Gamma_f)$.*

PROOF. Recall that form the definition of (PDAS) with $\sigma = \gamma^{-1}$ one gets

$$\begin{aligned} \lambda^{k+1} &= \gamma \tau_f\, y^{k+1} + \hat{\lambda} && \text{on } \mathcal{I}^{k+1}, \\ \xi^{k+1} &= \tau_f\, y^{k+1} + \gamma^{-1}(\hat{\lambda} + g) && \text{on } \mathcal{A}^{k+1}_-, \\ \xi^{k+1} &= \tau_f\, y^{k+1} + \gamma^{-1}(\hat{\lambda} - g) && \text{on } \mathcal{A}^{k+1}_+. \end{aligned}$$

We therefore have

$$\begin{aligned}\xi^{k+1} &= \delta_y^k + \tau_f y^k + \gamma^{-1}(\hat{\lambda} - g)\\ &= \delta_y^k + \begin{cases} \xi^k - \gamma^{-1}(g - \lambda^k) > 0 & \text{on } \mathcal{A}_+^{k+1} \cap \mathcal{A}_-^k,\\ \gamma^{-1}(\lambda^k - g) > 0 & \text{on } \mathcal{A}_+^{k+1} \cap \mathcal{I}^k,\\ \xi^k > 0 & \text{on } \mathcal{A}_+^{k+1} \cap \mathcal{A}_+^k.\end{cases}\end{aligned} \tag{3.67}$$

Thus

$$|(\xi^{k+1})^-| \le |\delta_y^k| \text{ a.e. on } \mathcal{A}_+^{k+1}.$$

Note that

$$\mathcal{A}_+^{*,k+1} := \{x \in \Gamma_f : \lambda^{k+1}(x) \ge g\} = \mathcal{A}_+^{k+1} \cup \{x \in \mathcal{I}^{k+1} : \lambda^{k+1}(x) \ge g\},$$

which implies, using $\xi^{k+1} = 0$ on $\mathcal{I}^{k+1}$,

$$|(\xi^{k+1})^-| \le |\delta_y^k| \text{ a.e. on } \mathcal{A}_+^{*,k+1}. \tag{3.68}$$

Analogously, it follows that

$$|(\xi^{k+1})^+| \le |\delta_y^k| \text{ a.e. on } \mathcal{A}_-^{*,k+1}, \tag{3.69}$$

where $\mathcal{A}_-^{*,k+1} := \{x \in \Gamma_f : \lambda^{k+1}(x) \le -g\}$. Moreover, on $\mathcal{I}^{k+1}$,

$$\begin{aligned}\lambda^{k+1} - g &= \gamma\delta_y^k + \gamma\tau_f y^k + \hat{\lambda} - g\\ &= \gamma\delta_y^k + \begin{cases} \gamma(\xi^k + \gamma^{-1}(\lambda^k - g)) \le 0 & \text{on } \mathcal{I}^{k+1} \cap \mathcal{A}_-^k,\\ \lambda^k - g \le 0 & \text{on } \mathcal{I}^{k+1} \cap \mathcal{I}^k,\\ \gamma\xi^k \le 0 & \text{on } \mathcal{I}^{k+1} \cap \mathcal{A}_+^k.\end{cases}\end{aligned}$$

The above estimate shows that

$$|(\lambda^{k+1} - g)^+| \le \gamma|\delta_y^k| \text{ a.e. on } \mathcal{I}^{k+1}, \tag{3.70}$$

and analogously one can show

$$|(\lambda^{k+1} + g)^-| \le \gamma|\delta_y^k| \text{ a.e. on } \mathcal{I}^{k+1}. \tag{3.71}$$

Since on active sets λ^{k+1} is set either to g or $-g$, we get that

$$(\lambda^{k+1} - g)^+ = (\lambda^{k+1} + g)^- = 0 \text{ on } \mathcal{A}_-^{k+1} \cup \mathcal{A}_+^{k+1}. \tag{3.72}$$

Furthermore, at most one of the expressions at a.a. $x \in \Gamma_f$

$$|(\lambda^{k+1} - g)^+|,\ |(\lambda^{k+1} + g)^-|,\ |(\xi^{k+1})^-|,\ |(\xi^{k+1})^+|$$

can be strictly positive, which shows, combining (3.68)-(3.72) that

$$M(\lambda^{k+1}, \xi^{k+1}) \le \|\delta_y^k\|_{\Gamma_f}^2. \tag{3.73}$$

Combining (3.62) and (3.66) with (3.73) shows that

$$M(\lambda^{k+1}, \xi^{k+1}) \le \gamma^2 \|C\|_{\mathcal{L}(L^2(\Gamma_f))}^2 M(\lambda^k, \xi^k). \tag{3.74}$$

Our assumption on γ implies that

$$\|C\|^2_{\mathcal{L}(L^2(\Gamma_f))}\gamma^2 < 1, \tag{3.75}$$

which shows that

$$M(\lambda^{k+1}, \xi^{k+1}) < M(\lambda^k, \xi^k), \tag{3.76}$$

unless $(\lambda^k, \xi^k) = (\lambda_\gamma, \xi_\gamma)$. Combining (3.62), (3.66), (3.73) and (3.74) it follows that

$$\begin{aligned}\|\delta_y^k\|^2_{\Gamma_f} &\le \|C\|^2_{\mathcal{L}(L^2(\Gamma_f))}\|R^k\|^2_{\Gamma_f} \le \gamma^2\|C\|^2_{\mathcal{L}(L^2(\Gamma_f))}M(\lambda^k, \xi^k) \\ &\le \gamma^2\|C\|^2_{\mathcal{L}(L^2(\Gamma_f))}\|\delta_y^{k-1}\|^2_{\Gamma_f} \le \left(\gamma\|C\|_{\mathcal{L}(L^2(\Gamma_f))}\right)^{2(k+1)} M(\lambda^0, \xi^0),\end{aligned} \tag{3.77}$$

which shows, utilizing (3.75) that

$$\lim_{k\to\infty} M(\lambda^k, \xi^k) = \lim_{k\to\infty} \|R^k\|_{\Gamma_f} = 0.$$

Moreover, summing up equations (3.77) over k and utilizing (3.75) shows

$$\sum_{k=1}^{\infty} \|\delta_y^k\|^2_{\Gamma_f} < \infty,$$

which implies that $\tau_f y^k$ is a Cauchy sequence and thus there exists $z \in L^2(\Gamma_f)$ such that

$$\lim_{k\to\infty} \tau_f y^k = z \tag{3.78}$$

in $L^2(\Gamma_f)$. Using (3.61) results in

$$\begin{aligned}\|\lambda^{k+1} - \lambda^k\|_{\Gamma_f} &\le \gamma\|\delta_y^k\|_{\Gamma_f} + \|R^k\|_{\Gamma_f} \\ &\le \left(\gamma\|C\|_{\mathcal{L}(L^2(\Gamma_f))}\right)^k (\gamma^2\|C\|_{\mathcal{L}(L^2(\Gamma_f))} + \gamma)M(\lambda^0, \xi^0)^{\frac{1}{2}},\end{aligned}$$

and therefore, with the same argument as above, there exists $\check{\lambda} \in L^2(\Gamma_f)$ such that

$$\lim_{k\to\infty} \lambda^k = \check{\lambda} \text{ in } L^2(\Gamma_f),$$

and from (3.60), (3.78)

$$\lim_{k\to\infty} \xi^k = \check{\xi} \in L^2(\Gamma_f).$$

Since $0 = \lim_{k\to\infty} M(\lambda^k, \xi^k) = M(\check{\lambda}, \check{\xi})$, the pair $(\check{\lambda}, \check{\xi})$ satisfies condition (3.22c). From (3.22a) follows the existence of $\check{y} \in Y$ such that

$$\lim_{k\to\infty} y^k = \check{y} \text{ in } Y.$$

Note that, since (y^k, λ^k, ξ^k) satisfies (3.22a), (3.22b), this is also the case for $(\check{y}, \check{\lambda}, \check{\xi})$. Hence, due to the uniqueness of a solution to (3.22a)-(3.22c) (see Theorem 3.9), $(\tilde{y}, \tilde{\lambda}, \tilde{\xi}) = (y_\gamma, \lambda_\gamma, \xi_\gamma)$, which ends the proof. $\square$

4.4. Global convergence of (ALM). The next theorem states global convergence of (ALM) for all $\gamma > 0$ and shows that large γ increases the speed of convergence. In the statement of the next theorem we denote the coercivity constant of $a(\cdot\,,\cdot)$ on Y by $\nu > 0$.

THEOREM 3.19. *The iterates λ^l of* (ALM) *and the corresponding variables y^l satisfy*

$$\nu \|y^{l+1} - \bar{y}\|^2_{H^1(\Omega)} + \frac{1}{2\gamma}\|\lambda^{l+1} - \bar{\lambda}\|^2_{\Gamma_f} \le \frac{1}{2\gamma}\|\lambda^l - \bar{\lambda}\|^2_{\Gamma_f} \tag{3.79}$$

and thus

$$\nu \sum_{k=1}^{\infty} \|y^k - \bar{y}\|^2_{H^1(\Omega)} \le \frac{1}{2\gamma}\|\lambda^0 - \bar{\lambda}\|^2_{\Gamma_f}, \tag{3.80}$$

which implies that $y^l \to \bar{y}$ strongly in Y and $\lambda^l \rightharpoonup \bar{\lambda}$ weakly in $L^2(\Gamma_f)$.

PROOF. From the fact that $\bar{y}$ and $\bar{\lambda}$ are the solutions to $(\mathcal{P})$ and $(\mathcal{P}^\star)$, respectively, it follows that

$$a(\bar{y}, y^{l+1} - \bar{y}) - (f, y^{l+1} - \bar{y})_\Omega + (\bar{\lambda}, \tau_f\,(y^{l+1} - \bar{y}))_{\Gamma_f} = 0, \tag{3.81}$$

and since y^{l+1} and λ^{l+1} solve $(\mathcal{P}_\gamma)$ and $(\mathcal{P}_\gamma^\star)$ with $\hat{\lambda} := \lambda^l$, we infer

$$a(y^{l+1}, y^{l+1} - \bar{y}) - (f, y^{l+1} - \bar{y})_\Omega + (\lambda^{l+1}, \tau_f\,(y^{l+1} - \bar{y}))_{\Gamma_f} = 0. \tag{3.82}$$

Subtracting (3.81) from (3.82) results in

$$a(y^{l+1} - \bar{y}, y^{l+1} - \bar{y}) + (\lambda^{l+1} - \bar{\lambda}, \tau_f\,(y^{l+1} - \bar{y}))_{\Gamma_f} = 0. \tag{3.83}$$

Note that one can write (3.53) and (3.17) as

$$\lambda^{l+1} = P(\gamma\tau_f\, y^{l+1} + \lambda^l) \;\text{ and }\; \bar{\lambda} = P(\gamma\tau_f\,\bar{y} + \bar{\lambda}), \tag{3.84}$$

where $P : L^2(\Gamma_f) \to L^2(\Gamma_f)$ denotes the pointwise projection onto the convex set $\{v \in L^2(\Gamma_f) : |v| \le g \text{ a.e. on } \Gamma_f\}$. Thus we get

$$\begin{aligned}
(\lambda^{l+1} - \bar{\lambda}, \tau_f\,(y^{l+1} - \bar{y}))_{\Gamma_f} &= \gamma^{-1}(\lambda^{l+1} - \bar{\lambda}, (\gamma\tau_f\, y^{l+1} + \lambda^l) - (\gamma\tau_f\,\bar{y} + \bar{\lambda}))_{\Gamma_f} \\
&\quad - \gamma^{-1}(\lambda^{l+1} - \bar{\lambda}, \lambda^l - \bar{\lambda})_{\Gamma_f} \\
&\ge \gamma^{-1}\|\lambda^{l+1} - \bar{\lambda}\|^2_{\Gamma_f} - \gamma^{-1}(\lambda^{l+1} - \bar{\lambda}, \lambda^l - \bar{\lambda})_{\Gamma_f},
\end{aligned}$$

where we used that

$$(\lambda^{l+1} - \bar{\lambda}, (\gamma\tau_f\, y^{l+1} + \lambda^l - \lambda^{l+1}) - (\gamma\tau_f\,\bar{y} + \bar{\lambda} - \bar{\lambda}))_{\Gamma_f} \ge 0,$$

which holds using (3.84) and since P is a projection onto a convex set (compare with (2.2)). Using (3.83) and the coercivity of $a(\cdot\,,\cdot)$ on Y we get

$$\begin{aligned}\nu\|y^{l+1}-\bar{y}\|^2_{H^1(\Omega)} &\le a(y^{l+1}-\bar{y},y^{l+1}-\bar{y}) = -(\lambda^{l+1}-\bar{\lambda},\tau_f\,(y^{l+1}-\bar{y}))_{\Gamma_f}\\ &\le -\frac{1}{\gamma}\|\lambda^{l+1}-\bar{\lambda}\|^2_{\Gamma_f}+\frac{1}{\gamma}(\lambda^{l+1}-\bar{\lambda},\lambda^l-\bar{\lambda})_{\Gamma_f}\\ &\le -\frac{1}{2\gamma}\|\lambda^{l+1}-\bar{\lambda}\|^2_{\Gamma_f}+\frac{1}{2\gamma}\|\lambda^l-\bar{\lambda}\|^2_{\Gamma_f},\end{aligned}$$

which proves (3.79). Summing up (3.79) with respect to l we obtain (3.80). Thus $y^l \to \bar{y}$ in Y and $\lambda^l \rightharpoonup \bar{\lambda}$ from (3.22a). □

5. Numerical Results

In this section we present our test examples for the algorithms proposed in Section 3 for the solution of the regularized and the original simplified friction problem. For simplicity we use for all examples the unit square as domain, i.e., $\Omega = (0,1)\times(0,1)$. The set of admissible deformations is

$$Y = \{y \in H^1(\Omega) : \tau y = 0 \text{ on } \Gamma_0\},$$

where $\Gamma_0 = \partial\Omega \setminus \Gamma_f$ with $\Gamma_f \subset \partial\Omega$ specified separately for each example. For our calculations we utilize a finite difference discretization with the usual five-point stencil approximation to the Laplace operator. The discretization of the normal derivative is based either on one-sided or symmetric differences. We denote by N the number of gridpoints in one of the space dimensions, that is we work on an $N\times N$-grid. The implementation is done in MATLAB 6.1. To investigate convergence properties we frequently report on

$$d^l_\lambda := \|\bar{\lambda}-\lambda^l\|_{\Gamma_f}, \tag{3.85}$$

where $\bar{\lambda} := \lambda_{10^{16}}$ is the solution of $(\mathcal{P}^\star_\gamma)$ with $\gamma = 10^{16}$ and λ^l denotes the actual iterate. We compare our results with those obtained using the Uzawa algorithm, which can be interpreted as an explicit form of the augmented Lagrangian method [67]. While (ALM) converges for every $\gamma > 0$ the Uzawa method converges only for $\gamma \in [\alpha_1,\alpha_2]$ with $0 < \alpha_1 < \alpha_2$, where α_1 and α_2 are in general not known [47]. We initialize both the Uzawa and the first-order augmented Lagrangian method with $\lambda^0 := 0$. We report on the number of iterations for the Uzawa method (one iteration requires one linear solve) and stop the iteration if $d^l_\lambda < 10^{-4}$. Unless otherwise specified we use $\sigma = 1$ for (PDAS), $\hat{\lambda} = 0$ and, as initialization for (PDAS), (SS) and (ALM) the solution to (3.22a)-(3.22c) with $\xi^0 = 0$, which corresponds to the solution of $(\mathcal{P}^\star_\gamma)$ neglecting the constraints on λ.

5.1. Example 1. This example is taken from [47, p. 301], the data is as follows: The part Γ_f of the boundary, where friction occurs is $([0,1]\times\{0\})\cup$

TABLE 1. *Example 1: Number of iterations for different values of γ.*

γ	10^2	10^4	10^6	10^8
$\#iter$	2	3	3	3

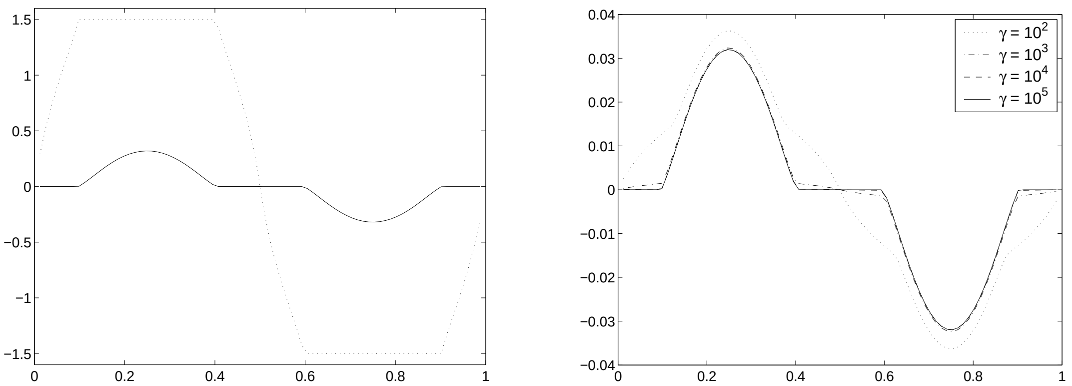


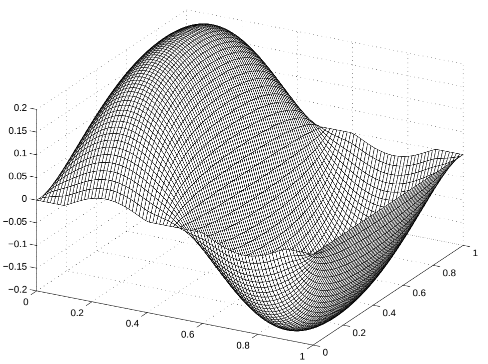
FIGURE 2. *Example 1: Solution variables λ_γ (dotted), τy_γ (multiplied by 10, solid) for $\gamma = 10^8$ (left) and λ_γ for $\gamma = 10^2, 10^3, 10^4, 10^5$ (right).*

$([0,1] \times \{1\})$. Furthermore, $\mu = 0$, $g = 1.5$ and

$$f(x) = \begin{cases} 10 & \text{for } x \in (0, 1/2) \times (0,1), \\ -10 & \text{for } x \in [1/2, 1) \times (0,1). \end{cases}$$

We choose $N = 80$ discretization points per space dimension. First, we state the results obtained for the Uzawa algorithm. This method requires 32 iterations for $\gamma = 10$, 17 iterations for $\gamma = 20$ and does not converge for $\gamma = 30$. In our tests for (PDAS) and (RPDAS) we vary the value of the regularization parameter γ and investigate the convergence as $\gamma \to \infty$. Table 1 reports on the number $\#iter$ of iterations (RPDAS) needs for various values of γ. It can be seen that the algorithm requires only very few iterations to find the solution of the problem. For this example increasing the regularization parameter does not increase the number of iterations required to detect the solution, compare with [69], where a different behavior for obstacle problems is observed. We remark that for (RPDAS) no points are shifted from the lower active set to the upper or conversely within one iteration and thus (PDAS) determines the same iterates as (RPDAS) for all $\sigma \geq \gamma^{-1}$, see Remark 3.13. For $\gamma = 10^8$ the variables λ_γ and $\tau_f y_\gamma$ on $[0,1] \times \{0\}$ are shown in Figure 2 (left), the result y_γ is plotted in Figure 3 (left).

We now investigate the convergence as $\gamma \to \infty$. In Figure 2 (right) we plot τy_γ for $\gamma = 10^2, 10^3, 10^4, 10^5$ on $[0,1] \times \{0\}$, for $\gamma > 10^5$ the changes are beyond the graphical resolution. It can be seen that, as γ increases, $\tau_f y_\gamma$ approaches 0, e.g., on $[0.4, 0.6]$, such that the complementarity condition of the original friction problem is almost satisfied for large γ.



γ	$\|\|\|y_\gamma - \bar{y}\|\|\|$
1	4.46e-1
10	1.79e-1
10^2	2.30e-2
10^3	2.49e-3
10^4	2.54e-4
10^5	2.54e-5
10^6	2.54e-6

Figure 3. *Example 1: Solution y_γ for $\gamma = 10^8$ (left) and convergence w.r. to γ (right).*

Finally, the table in Figure 3 reports on the value of

$$|||y_\gamma - \bar{y}||| := (a(y_\gamma - \bar{y}, y_\gamma - \bar{y}))^{\frac{1}{2}}$$

for various γ, where we take $\bar{y} := y_{10^{16}}$ as approximation to the solution of the simplified friction problem and use a 160×160-grid to keep the discretization error small. Note that $|||\cdot|||$ is a norm equivalent to the usual one in $H^1(\Omega)$. The result suggests the convergence rate γ^{-1}, while from Corollary 3.11 we only get the rate $\gamma^{-1/2}$.

5.2. Example 2. For this example, taken from [47, p. 281] the data is as follows:

$$\Gamma_f = [1/4, 3/4] \times \{0\}, \;\; \mu = 0 \text{ and } \; f(x) = 10 \cdot \chi_C$$

with $C = (\frac{3}{8}, \frac{5}{8})^2$. The results y_γ for $g = 1$ and $\gamma = 1, 10, 100$ are shown in Figure 4, compare with [47, p. 283], where the same problem is solved using an iterative overrelaxation method. To detect the solution (RPDAS) as well as (PDAS) requires only one iteration step, since no active points occur for the dual solution variable λ_γ. This shows another positive feature of the primal-dual active set method: If the constraint is nowhere active at the solution, then the algorithms (using the unconstrained solution as initialization) terminate after the first iteration step at the exact solution of the discretized problem. For other initializations (e.g., setting λ^0 equal to one of the constraints) the algorithms usually terminate after two iterations. For $g = 0.25$ the upper constraint on λ is active in some interval for all tested values of γ and all grids, nevertheless (RPDAS) and (PDAS) never required more than three iterations to terminate at the solution.

5.3. Example 3. This example investigates the behavior of (PDAS), (RPDAS) and (ALM) for a more complicated structure of the active and inactive sets. Uzawa's method faces serious troubles with this example: In all test examples with $\gamma > 0.2$ the method did not converge. For $\gamma = 0.2$ we stopped our test

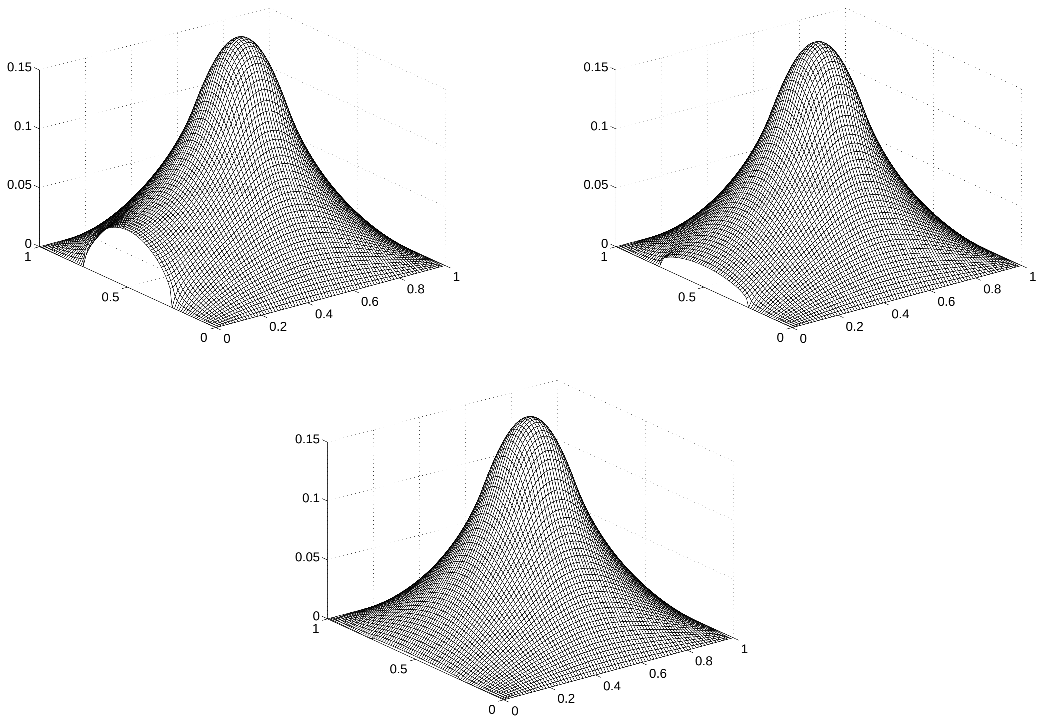


FIGURE 4. *Example 2: Solution y_γ for $g = 1$ and $\gamma = 1$ (left, top), $\gamma = 10$ (right, top) and $\gamma = 100$ (bottom).*

TABLE 2. *Example 3: Number of iterations of* (RPDAS) *and* (PDAS) *for different values of γ and $N = 160$.*

γ	3	5	10	50	100	150	160	10^3	10^{10}
$\#iter_{(\mathrm{RPD})}$	3	4	4	5	6	7	*div*	*div*	*div*
$\#iter_{(\mathrm{PD})}$	3	4	4	5	6	5	5	6	6

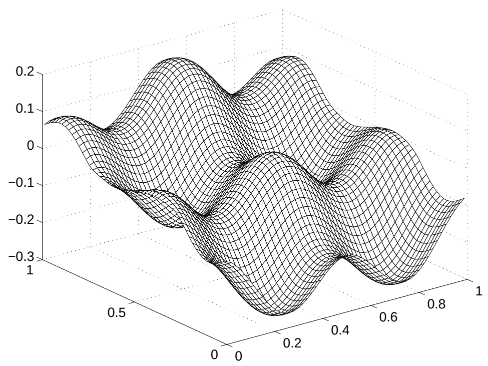
after 400 iterations at d_λ^{400} =1.03e-2. We choose $\Gamma_f = \partial\Omega$, which implies that $Y = H^1(\Omega)$. Furthermore, $\mu = 0.5$, $g = 0.4$ and the external force is

$$f(x) = 10(\sin 4\pi x + \cos 4\pi x).$$

Figure 5 (left) shows the solution y_γ for $\gamma = 50$. In the table in Figure 5 we report on

$$q_\lambda^k := \frac{\|\lambda^k - \lambda_\gamma\|_{\Gamma_f}}{\|\lambda^{k-1} - \lambda_\gamma\|_{\Gamma_f}},\ k = 1, 2, \dots \tag{3.86}$$

for $\gamma = 50$ to investigate if one can observe local superlinear convergence of the iterates determined with (RPDAS). We observe a monotone decrease of q_λ^k, which corresponds to superlinear convergence of λ^k in $L^2(\Gamma_f)$.



k	q_λ^k
1	0.58
2	0.45
2	0.25
4	0.18
5	0.00

FIGURE 5. *Example 3: Solution y_γ for $\gamma = 50$ (left) and values for d_λ^k (right).*

Table 2 shows the number of iterations $\#iter_{(\mathrm{RPD})}$ and $\#iter_{(\mathrm{PD})}$ required by (RPDAS) and (PDAS), respectively, to find the solution for different values of γ, where 160×160 inner gridpoints are used. We observe a slight increase in the number of iterations as γ increases. For $\gamma \geq 160$ (RPDAS) does not detect the solution, whereas (PDAS) does. Using (RPDAS) for these examples we can observe the following behavior: Points in Γ_f move from $\mathcal{A}_-^{k-1}$ to $\mathcal{A}_+^k$, and then from $\mathcal{A}_+^k$ back to $\mathcal{A}_-^{k+1}$ for some $k \geq 2$, and due to this scattering the algorithm does not find the solution. Algorithm (PDAS) does not experience such problems and finds the solution after a few iterations for all γ that we tested.

To avoid possible difficulties due to local convergence of the semi-smooth Newton method (RPDAS) we test the following globalization strategies: First we use a continuation procedure with respect to γ, motivated from the local convergence result for (RPDAS): We solve for $\gamma = 150$ and use the solution as initialization for the algorithm with larger γ. This procedure turns out to be successful only for a moderate increase in γ. Increasing γ moderately, typically only one or two more iterations are needed to find the solution for larger γ. However, this method appears inconvenient and costly. Next we test backtracking with J_γ as merit function to globalize (RPDAS). This strategy works successfully, but in particular for larger γ several backtracking steps are necessary in each iteration. The resulting stepsize is very small and thus overall up to 50 iterations are needed to find the solution. This behavior becomes more distinct for large γ.

We also apply algorithm (ALM) (see Section 3.4) for the solution of this example. Recall that in (ALM) γ is fixed, but the variable λ is updated. Recall further that (ALM) is a solution method for $(\mathcal{P}^\star)$, the dual of the original simplified friction problem. In each iteration of (ALM) one has to solve an auxiliary problem, which is of the form of our regularized problem with the specific choice $\hat\lambda := \lambda^l$. In a first attempt we solve this auxiliary problem exactly using (PDAS), where this method is initialized with the solution of the auxiliary problem in the previous iteration step of (ALM). Due to the local superlinear convergence of this semi-smooth Newton method the auxiliary problem is solved in very few

TABLE 3. *Example 3: Tests for* (ALM) *with exact solve of the auxiliary problem,* $N = 160$, $\gamma = 10^2, 10^4$.

	l	1	2	3	4	5	6	7
$\gamma = 10^2$	$\#iter_{PD}$	6	3	2	1	1	1	1
	d^l_λ	2.55e-2	8.54e-3	3.62e-3	1.62e-3	7.45e-4	3.58e-4	1.77e-4
$\gamma = 10^4$	$\#iter_{PD}$	6	1	1	1	1		
	d^l_λ	4.00e-4	3.06e-6	2.67e-8	2.62e-10	2.84e-12		

iterations, as can be seen in Table 3, where for $\gamma = 10^2$ and $\gamma = 10^4$ we report on the number of iterations $\#iter_{PD}$ required by (PDAS) in every step l of (ALM). Moreover, we report on

$$d^l_\lambda := \|\lambda^l - \bar{\lambda}\|_{\Gamma_f}, \tag{3.87}$$

where λ^l denotes the iterates of (ALM) and $\bar{\lambda} := \lambda_{10^{16}}$ is the approximate solution of $(\mathcal{P}^\star)$. As expected we observe faster convergence of λ^l in the case $\gamma = 10^4$.

In a second approach we test (ALM) with only one iteration step of (PDAS) in every (ALM)-iteration. The results for $\gamma = 10^2, 10^4$ are shown in Table 4. Again we report on the value of d^l_λ. We observe that for $\gamma = 10^2$ the first iterates present a better approximation to $\bar{\lambda}$ than for $\gamma = 10^4$, whereas then the case $\gamma = 10^4$ shows a faster convergence behavior. This leads us to the idea of increasing the parameter γ in every step of (ALM). We start with $\gamma = 10$ and multiply γ by 10 in every step of (ALM). The results for this test run are shown in the last line of Table 4.

We now compare the above results for (ALM) with those of the primal-dual active set algorithms for large γ (e.g., $\gamma = 10^{10}$). In the case of exact solving the auxiliary problems in (ALM) the overall number of system solves is significantly higher than if (PDAS) with large γ is used. The second strategy, where we only apply one semi-smooth iteration in the inner loop turns out to be more efficient, and the results in the case where we also increase γ are remarkably good. In this case the number of overall system solves is the same as for the primal-dual active set strategies with large γ. However, these semi-smooth Newton methods determine the solution of the regularized problem $(\mathcal{P}^\star_\gamma)$, which is - for large γ - close to the solution of $(\mathcal{P}^\star)$, while (ALM) is a solution method for the original simplified friction problem $(\mathcal{P}^\star)$. We finally remark the advantage of (RPDAS) and (PDAS) with large γ compared to (ALM) that one has a simple stopping criterion available, which guarantees that the exact solution of the discretized (regularized) problem is found.

5.4. Example 4. For the last example the Uzawa algorithm again only converges for small γ which results in an extremely slow convergence. The data are as follows: $\Gamma_f = \partial\Omega$ and thus $Y = H^1(\Omega)$, furthermore $\mu = 0.5$, $g = 0.3$ and the

Table 4. *Example 3: Values for d_λ^l (as defined in* (3.87)*) applying* (ALM) *with one iteration step of* (PDAS), $N = 160$, $\gamma = 10^2, 10^4, 10^l$.

l	1	2	3	4	5	6	7	8
$\gamma = 10^2$	2.96e-1	1.61e-1	5.84e-2	1.65e-2	3.95e-3	1.88e-3	9.78e-4	5.19e-4
$\gamma = 10^4$	3.81e-1	2.15e-1	1.46e-1	3.79e-2	6.14e-3	1.51e-5	1.47e-7	1.59e-9
$\gamma = 10^l$	1.41e-1	1.84e-1	9.67e-2	3.44e-2	5.65e-3	1.33e-7	1.45e-12	

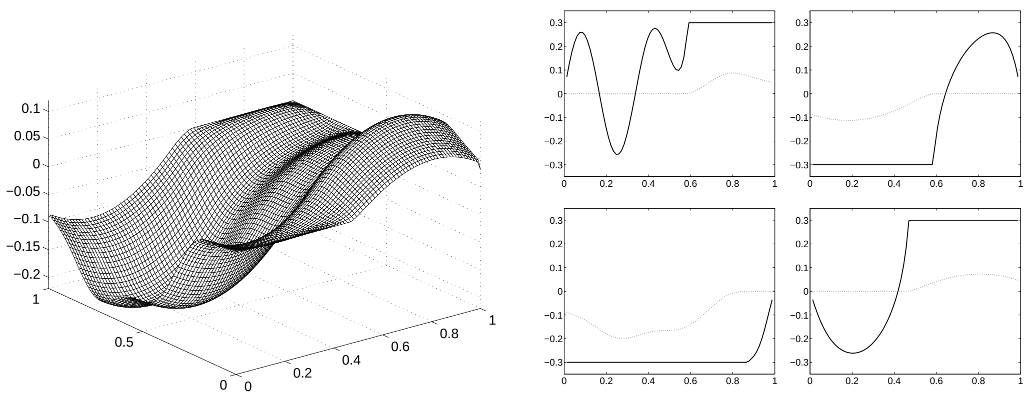


Figure 6. *Example 4: Solution y_γ (left) and variables λ_γ (solid), τy_γ (dotted) on $(0,1)\times\{0\}$, $(0,1)\times\{1\}$, $\{0\}\times(0,1)$, $\{1\}\times(0,1)$ for $\gamma = 10^3$ (right).*

external force is

$$f(x) = |3x - 1| + 2\text{sgn}(2y - 1) + 2\text{sgn}(x - 0.75) + 5\sin(6\pi x).$$

In Figure 6 (left) we show the solution y_γ for $\gamma = 10^3$. The corresponding variables λ_γ and τy_γ on the different parts of Γ_f are shown in Figure 6 (right).

In a series of test runs (Table 5) we investigate the number of iterations for various grids (N corresponds to an $N \times N$-grid) and different values for γ. We observe that the number of iterations required to detect the solution is rather small for all mesh-sizes and choices for γ. For the calculations we use (RPDAS), except for those indicated by *. For these examples with a rather large values of γ (RPDAS) starts to chatter due effects described in the previous example. Thus we utilize (PDAS) to solve these problems, which is always successful. Utilizing (PDAS) also for the examples with smaller γ yields the same number of iterations as (RPDAS). Furthermore, note from Table 6 that for small γ ($\gamma = 5, 10$) a mesh-independent behavior can be observed, whereas for larger γ this is not the case. A possible explanation for this behavior is the decreasing convergence domain for the infinite-dimensional semi-smooth Newton method (RPDAS) as γ decreases.

We also tested whether one can observe superlinear convergence of the iterates calculated with (RPDAS). The results in Table 6 yield that q_λ^k (as defined in

TABLE 5. *Example 4: Number of iterations for different values of* γ *on* $N \times N$*-meshes.*

γ		5	10	30	50	10^2	10^3	10^5	10^{10}
	20	3	3	3	3	3	3	3	3*
	40	3	3	4	5	4	4	4	4*
N	80	3	4	4	5	5	6	6	6*
	160	3	3	5	5	6	8	7*	7*
	320	3	4	5	6	7	7	8*	8*

TABLE 6. *Example 4: Values for* q_λ^k *for* $\gamma = 10$ *and* $\gamma = 100$*,* $N = 80$*.*

k	1	2	3	4	5
q_λ^k for $\gamma = 10$	0.39	0.12	0.08	0.00	-
q_λ^k for $\gamma = 100$	0.64	0.61	0.13	0.07	0.00

TABLE 7. *Example 4: Values for* d_λ^l *(as defined in* (3.87)*) applying* (ALM)*,* $N = 160$*,* $\gamma = 10^2, 10^4, 10^l$*.*

l	1	2	3	4	5	6	7	8
$\gamma = 10^2$	1.62e-1	1.07e-1	2.22e-2	6.42e-3	1.37e-3	4.47e-4	1.84e-4	8.87e-5
$\gamma = 10^4$	2.14e-1	1.38e-1	1.10e-1	1.47e-2	2.88e-3	4.13e-4	8.88e-7	8.01e-9
$\gamma = 10^l$	8.54e-2	8.78e-2	3.89e-2	1.04e-2	2.07e-3	4.17e-4	8.85e-10	

(3.86)) decreases, which corresponds to superlinear convergence of the iterates λ^k in $L^2(\Gamma_f)$.

We also tested (ALM) for the solution of this example, where we used (PDAS) to solve the auxiliary problem. In the case that this problem was solved exactly the overall number of system solves was 16 for $\gamma = 50$ and between 12 and 20 for other tested values of γ. The results for the case where only one iteration step of (PDAS) is performed in every (ALM)-iteration are summarized in Table 7, where we again report on d_λ^l as defined in (3.87). Increasing γ as in Example 3 turns out to result in a very efficient method. In this case the number of system solves is similar as when the primal-dual active set methods are applied to the regularized problem with large γ, but (ALM) has the advantage that the iterates converge to the solution of the original (i.e., not regularized) simplified friction problem.

5.5. Summary of the numerical results. In our numerical testing we observe a remarkable efficiency of algorithms (RPDAS) and (PDAS) for the solution of the regularized simplified friction problem ($\mathcal{P}_\gamma^\star$) and of (ALM) for the solution of ($\mathcal{P}^\star$). For moderate values of γ the iterates of (RPDAS) and (PDAS) coincide and these algorithms converge superlinearly. For large values of γ (RPDAS) may start to chatter, while (PDAS) always detects the solution. The two tested globalization strategies for (RPDAS) turn out to be successful but inconvenient. The number of iterations of the semi-smooth Newton methods increases only slightly for finer grids and larger regularization parameters. The efficiency of (RPDAS) and (PDAS) is interesting also with respect to augmented Lagrangian methods since these algorithms present a powerful tool to solve or approximately solve the auxiliary problem in (ALM). Our tests show that, solving the auxiliary problem in (ALM) only approximately using (RPDAS) or (PDAS), the overall number of system solves for (ALM) is rather the same as for the semi-smooth Newton methods with large γ. However, (ALM) has the advantage that it detects the solution of the dual of the original simplified friction problem $(\mathcal{P}^\star)$ without requiring that $\gamma \to \infty$. Finally, we remark the advantage of (PDAS) with large γ that one has a simple stopping criterion at hand that guarantees that the exact solution of $(\mathcal{P}_\gamma^\star)$ is detected.

6. A Dynamical Simplified Friction Problem

In this section we apply our findings for $(\mathcal{P})$ to a dynamical version of the simplified friction problem. First we state the problem and give basic result. Then we apply a discretization with respect to the time variable and end up with a problem similar to $(\mathcal{P})$ that has to be solved in every time step. A numerical example shows that the semi-smooth Newton algorithms are an efficient tool also for the solution of time-dependent friction problems.

6.1. Problem formulation. Let Z be a Hilbert space with inner product $(\cdot\,,\cdot)_Z$ and corresponding norm $\|\cdot\|_Z$. For $T > 0$ we introduce the spaces

$$\begin{aligned} L^2(0,T;Z) &:= \left\{\varphi : [0,T] \to Z : \int_0^T \|\varphi(t)\|_Z^2\,dt < \infty\right\}, \\ L^\infty(0,T;Z) &:= \{\varphi : [0,T] \to Z \text{ s.t. } \exists C > 0\text{: } \|\varphi(t)\|_Z < C \text{ a.e. in } [0,T]\} \end{aligned}$$

with corresponding norms

$$\begin{aligned} \|\varphi\|_{L^2(0,T;Z)} &:= \left(\int_0^T \|\varphi(t)\|_Z^2\,dt\right)^{\frac{1}{2}}, \\ \|\varphi\|_{L^\infty(0,T;Z)} &:= \inf\{C > 0 : \|\varphi(t)\|_Z \le C \text{ a.e. in } [0,T]\}. \end{aligned}$$

It can be shown that the above spaces are Banach spaces and that $L^2(0,T;Z)$ is a Hilbert space, [47]. In the sequel we denote by the prime $'$ the derivative with respect to the time variable t and use the notations for Y, $a(\cdot\,,\cdot)$ and $j(\cdot)$ as defined in (3.2), (3.3) and (3.4). We can now formulate $(\mathcal{P}_{dyn})$, the dynamical simplified friction problem as follows.

$$(\mathcal{P}_{dyn}) \quad \begin{cases} \text{Find } y \in L^2(0,T;Y) \text{ s.t.} \\ y' \in L^2(0,T;Y), y'' \in L^2(0,T;L^2(\Omega)), y(0) = y_0,\ y'(0) = y_1 \text{ and} \\ \big(y''(t) - f(t), z - y'(t)\big) + a(y, z - y'(t)) + j(z) - j(y'(t)) \geq 0 \\ \text{for a.a. } t \in [0,T] \text{ and all } z \in Y, \end{cases}$$

where $f \in L^2(0,T;H^1(\Omega))$ and $y_0, y_1 \in Y$ are given. Then the following existence and uniqueness result for problem $(\mathcal{P}_{dyn})$ holds.

THEOREM 3.20. *Assume that $f', f'' \in L^2(0,T;L^2(\Omega))$, $y_0 \in H^2(\Omega)$ and $y_1 \in H^1(\Omega)$. Then there exists a unique solution y of $(\mathcal{P}_{dyn})$ with $y' \in L^\infty(0,T;Y)$ and $y'' \in L^\infty(0,T;H^1(\Omega))$.*

PROOF. A verification for the above theorem follows from Theorem 5.7 in [39, p. 157]. Notice that the rather lengthy proof for existence of a solution consists of three main steps: Approximating $j(\cdot)$ by a smooth functional j_ε, establishing a priori estimates for the solution u_ε independent of ε and then passing to the limit as $\varepsilon \to \infty$. □

Next we summarize the first-order necessary conditions for a solution $\bar{y}$ of $(\mathcal{P}_{dyn})$: There exists $\bar{\lambda} : [0,T] \to L^2(\Gamma_f)$ such that for a.a. $t \in [0,T]$ the following equations are satisfied:

$$a(\bar{y}(t), z) + (\bar{y}''(t) - f(t), z)_{\Gamma_f} + (\bar{\lambda}(t), \tau_f z)_{\Gamma_f} = 0 \text{ for all } z \in Y \tag{3.88a}$$

$$\tau_f \bar{y}'(t) = \max(0, \tau_f \bar{y}'(t) + \sigma(\bar{\lambda} - g)) + \min(0, \tau_f \bar{y}'(t) + \sigma(\bar{\lambda} + g)) \tag{3.88b}$$

in $L^2(\Gamma_f)$ for every $\sigma > 0$.

Note that the optimality condition for $(\mathcal{P}_{dyn})$ concerns an inequality of evolution of second order in t. Systems such as (3.88a),(3.88b) with $\bar{y}''$ replaced by $\bar{y}'$ in (3.88a) correspond to problems of, e.g., temperature control, see [39, p. 46].

6.2. Semi-discretization in time. In this section we discretize $(\mathcal{P}_{dyn})$ with respect to the time variable. We divide the interval $[0,T]$ into M equidistant subintervals, i.e., into $[t_{i-1}, t_i]$ for $i = 1, \ldots, M$ with $t_i = Hi$, $H = TM^{-1}$. By y^i we denote the approximation to the solution $\bar{y}$ of $(\mathcal{P}_{dyn})$ on time level i and we introduce the abbreviations

$$d^i = \frac{y^{i+1} - y^{i-1}}{2H} \quad \text{and} \quad \delta^i = \frac{y^{i+1} - y^i}{H}.$$

Note that $(\delta^i + \delta^{i-1})/2 = d^i$ and that

$$\frac{\delta^i - \delta^{i-1}}{H} = \frac{y^{i+1} - 2y^i + y^{i-1}}{H^2} = \frac{2(d^i - \delta^{i-1})}{H}. \tag{3.89}$$

We denote by

$$y^{i+\Theta} := (1 - \Theta)y^i + \Theta y^{i+1} = 2\Theta H d^i - \Theta H \delta^{i-1} + y^i,$$

and analogously

$$f^{i+\Theta} := (1-\Theta)f^i + \Theta f^{i+1},$$

where $f^i = f(t_i,\cdot) \in L^2(\Omega)$. For $\Theta \in [0,1]$ we discretize the variational inequality in $(\mathcal{P}_{dyn})$ as follows:

$$\left(\frac{y^{i+1}-2y^i+y^{i+1}}{H^2} - f^{i+\Theta}, z-d^i\right)_\Omega + a(y^{i+\Theta}, z-d^i) + j(z) - j(d^i) \geq 0 \tag{3.90}$$

for all $z \in Y$. This is, using (3.89) equivalent to

$$\begin{aligned}&\left(\frac{2(d^i-\delta^{i-1})}{H} - f^{i+\Theta}, z-d^i\right)_\Omega + a(2\Theta H d^i - \Theta H\delta^{i-1} + y^i, z-d^i)\\ &+ j(z) - j(d^i) \geq 0 \text{ for all } z \in Y,\end{aligned}$$

and further to

$$\begin{aligned}&\frac{2}{H}(d^i-\delta^{i-1}, z-d^i)_\Omega - (f^{i+\Theta}, z-d^i)_\Omega + 2\Theta H a(d^i, z-d^i)\\ &- a(\Theta H\delta^{i-1} - y^i, z-d^i) + j(z) - j(d^i) \geq 0 \quad \text{for all } z \in Y.\end{aligned}$$

The solution of the above variational inequality is also characterized as solution to the optimization problem

$$(\mathcal{P}^i) \qquad \begin{cases} \max\limits_{d^i\in Y} & \frac{1}{2}(d^i,d^i)_\Omega + \frac{\Theta H^2}{2}a(d^i,d^i) - \frac{1}{2}(2\delta^{i-1} + Hf^{i+\Theta}, d^i)_\Omega \\ & -\frac{H}{2}a(\Theta H\delta^{i-1} - y^i, d^i) + \frac{H}{2}j(d^i). \end{cases}$$

This enables us one by one to compute the approximate solution of $(\mathcal{P}_{dyn})$ at the time levels t_i as follows:

(1) Set $y^0 := y_0$, $y^1 := y_0 + Hy_1$.
(2) For $i = 1,\dots,M-1$ solve $(\mathcal{P}^i)$ and set $y^{i+1} := y^{i-1} + 2Hd^i$.

In [47, p.484] the above scheme is investigated for the case $g = 0$ (i.e., $j(\cdot) = 0$), where unconditional stability for $\Theta = 1$ and conditional stability for $\Theta < 1$ is shown. For $\Theta > 0$ problem $(\mathcal{P}^i)$ has a similar form as the static problem $(\mathcal{P})$ and can thus be analyzed and solved similarly as $(\mathcal{P})$.

6.3. Numerical results. For a numerical realization we regularize problem $(\mathcal{P}^i)$ as done for the static case and apply (PDAS) for its solution. As initialization for the solution of $(\mathcal{P}^i)$ we choose $d^i = 0$ on $\partial\Omega$ and calculate the other variables from the linear equation (3.88a). Utilizing this initialization the same linear system with changing right hand side has to be solved in each time step. To exploit this fact we a priori calculate the Choleski-factorization of the system matrix (which is an M-matrix, see [49]) before we start our calculations for the time evolution. Thus the initialization step for (PDAS) only requires one forward and one backward substitution in every time step.

We report on an example with the following data: $\Omega = [0,1]^2$, $\Gamma_f = \partial\Omega$, $f \equiv 0$, $g = 1$, $\mu = 0.5$, $T = 2$ and

$$y_0(x,y) = 4y(y-1)\sqrt{\sin(\pi x)}, \;\; y_1(x,y) = -y_0(x,y).$$

TABLE 8. *Average number of* (PDAS)*-iterations for various values of* Θ *and* H, $N = 50$.

Θ	0.5	0.5	0.5	0.5	1	1	1	1
H	0.01	0.02	0.05	0.1	0.01	0.02	0.05	0.1
aver. #iter	1.005	1.08	1.325	1.50	1.005	1.12	1.43	1.50

We choose the regularization parameter $\gamma = 10^6$ and $N = 50$ (i.e., the state variables are discretized on a 50×50 grid). For the time discretization we apply a semi-implicit scheme (i.e., $\Theta = 0.5$) and the time increment $H = 0.01$. Figure 7 shows the solution $y(t_i, \cdot)$ for $t_i = 0.1, 0.4, 0.7, 1.3$. The semi-smooth Newton method usually only requires one iteration step to detect the solution for problem $(\mathcal{P}^i)$. With the above parameters only one of the 200 solved problems $(\mathcal{P}^i)$ requires two iterations that is an average number of iterations of 1.005. For tested values of Θ smaller than 0.5 leaving the other parameters unchanged the scheme turns out to be unstable, while for all tested $\Theta \geq 0.5$ we observe a stable behavior, also for larger timesteps such as $H = 0.1$. Table 8 reports on the average number of (PDAS)-iterations (*aver. #iter*) for different Θ and H.

A possible explanation for the remarkable efficiency of the semi-smooth Newton method (PDAS) for the solution of $(\mathcal{P}^i)$ may be that - compared to the static problem $(\mathcal{P})$ - the differential operator in $(\mathcal{P}^i)$ involves an additional identity operator which results — after discretization — in an M-matrix for which all eigenvalues are larger than 1. Furthermore, for small timesteps this matrix is close to the identity matrix. We remark that for unilateral constrained problems M-matrix properties can be utilized to gain monotonicity and thus convergence results for the iterates, see [58, 69].

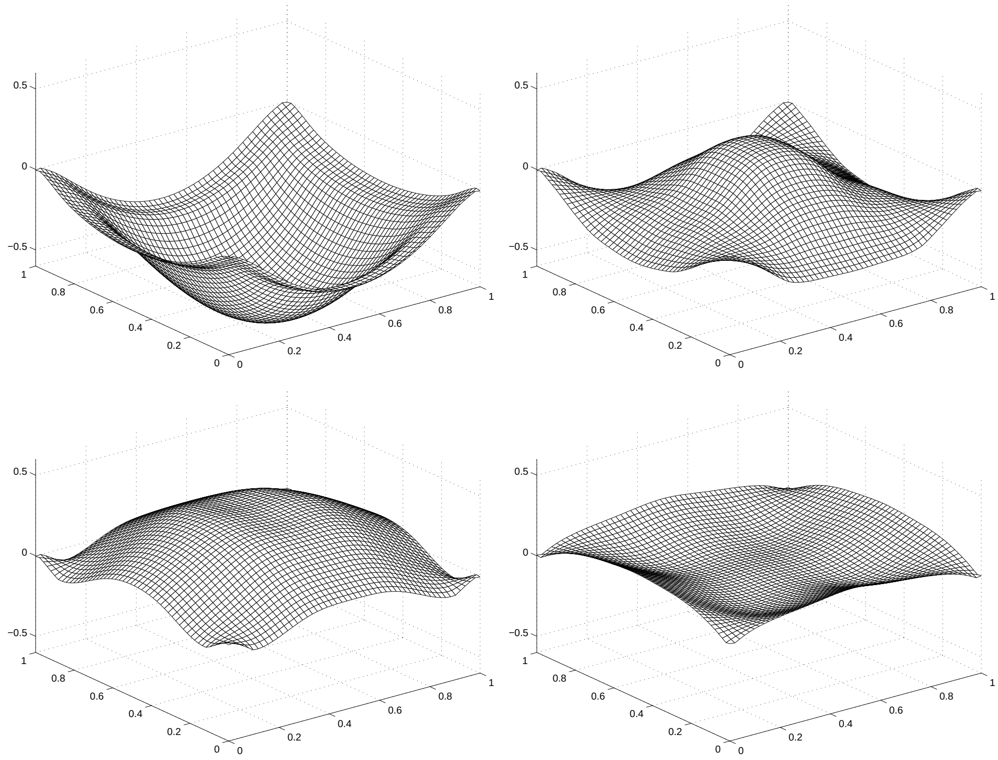


FIGURE 7. *Solution for* $t_i = 0.1, 0.4, 0.7, 1.3$ *(from upper left to lower right) for* $(\mathcal{P}_{dyn})$, $N = 50$, $\gamma = 10^6$.

CHAPTER 4

Contact Problems in Linear Elasticity

Contact problems in elasticity appear in important processes in nature and in many technical applications. Firstly, they are used in mechanical problems if no or negligible friction in the contact zone occurs. Secondly, they are crucial ingredients for the investigation and simulation of more realistic frictional contact problems. And, thirdly, they are also of theoretical mathematical interest due to their relation to variational inequalities and constrained optimization problems.

In contact problems (also known as Signorini problems), one is concerned with the deformation of an elastic body whose surface or boundary possibly hits a rigid foundation. It is not known in advance which part of the body's surface will be in contact with the foundation. Such problems are known as unilateral contact problems in contrast to so-called bilateral contact problems, where the contact region is a priori known. The main difficulty in Signorini problems is to identify the contact zone. Then — provided the material law is linear — the problem reduces to a linear one. The problems discussed in this chapter do not involve friction in the contact zone, i.e., the deformation in tangential direction is unrestricted.

Classical references for contact problems are [62, 73], while a general treatment of variational inequalities can be found in [39, 46, 47, 75]. For a rather new contribution on computational contact problems we refer to [107].

Let us now discuss various approaches towards the pointwise inequality constraints in Signorini problems. Pure regularization techniques [23, 46, 47] require careful handling of regularization parameters in order to find a reasonable compromise between efficiency and accuracy. Duality techniques are based on the introduction of a Lagrange multiplier (cf., [17,46,47]), and result in a system that includes a complementarity condition. Active set strategies (see, e.g., [63, 107]) iteratively provide approximations of the contact set. A domain decomposition method combined with an inexact projection method for the numerical treatment of discrete contact problems is suggested and analyzed in [98, 99]. Monotone multigrid methods (see, e.g., [63,76,78–80]) also represent efficient methods for the numerical solution of discrete Signorini problems. Recently, a finite dimensional primal-dual active set strategy combined with a multigrid approach has been applied to multibody contact problems in [65, 66]. The methods used in these papers are related to [59], and also to the techniques developed in this chapter.

For convergence rates of the finite element approximations of Signorini problems, we refer to [10, 12] and the references given therein.

The approach taken in this chapter is motivated by the variational formulation of the Signorini problem in a Hilbert space framework. We present a regularization procedure that is mainly of penalty type, which allows us to apply a superlinearly convergent generalized Newton method in infinite dimensions. The resulting algorithm turns out to have the form of an active set strategy and is observed to converge in numerical practice regardless of the initialization. Combining the regularized problem with a first-order augmented Lagrangian method results in the convergence of the solutions to the solution of the original contact problem. We also discuss a certain inexact variant of this method and give convergence results. The methods presented in this chapter are motivated by semi-smooth Newton methods in function spaces [58, 104] and their application to optimal control and obstacle problems [58, 69].

Let us give a brief outline of this chapter. In Section 1, the contact problem is stated in a functional analytic framework, equivalent statements and basic properties are discussed. A regularization procedure is presented in Section 2, where the convergence as the regularization parameter tends to infinity is investigated as well. In Section 3 and 4 we present the semi-smooth Newton algorithm, the exact and inexact first-order augmented Lagrangian method, and give convergence results for these methods. In the concluding Section, our numerical testing of the algorithms is summarized.

1. The Contact Problem

In this section the contact problem in linear elasticity is formulated in an appropriate functional analytic framework and existence and uniqueness results are summarized. A dual formulation of the problem is derived and a Lagrange multiplier that resolves the contact condition is introduced.

1.1. Problem statement. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be an open bounded domain with $\mathcal{C}^{1,1}$-boundary $\Gamma := \partial\Omega$ (see Figure 1). We define $\mathbf{H}^1(\Omega) := \prod_{j=1}^n H^1(\Omega)$ and use the analogous notation for the product spaces $\mathbf{L}^2(\Omega)$, $\mathbf{H}^{\frac{1}{2}}(\Gamma), \ldots$ and their duals. The set of admissible deformations is introduced as

$$\mathbf{Y} := \{\boldsymbol{v} \in \mathbf{H}^1(\Omega) : \boldsymbol{\tau v} = 0 \text{ a.e. on } \Gamma_d\},$$

where $\Gamma_d \subset \Gamma$ is open, nonempty and $\boldsymbol{\tau} : \mathbf{H}^1(\Omega) \to \mathbf{H}^{\frac{1}{2}}(\Gamma)$ denotes the (componentwise) trace operator. Furthermore, we denote by $\Sigma := \operatorname{int}(\Gamma \setminus \Gamma_d)$ the interior of $\Gamma \setminus \Gamma_d$, by $\Gamma_c \subset \Sigma$ the nonempty open region of possible contact and by $\Gamma_n := \operatorname{int}(\Sigma \setminus \Gamma_c)$ the (possible empty) set with given Neumann conditions. We assume that $\overline{\Gamma}_c \subset \Sigma$, that $\partial\Gamma_c, \partial\Sigma \subset \Gamma$ are smooth and define a closed subspace of $\mathbf{H}^{\frac{1}{2}}(\Sigma)$ by

$$\mathbf{H}^{\frac{1}{2}}_{00}(\Sigma) := \{\boldsymbol{\xi} \in \mathbf{L}^2(\Sigma) : \text{ there exists } \boldsymbol{v} \in \mathbf{H}^1(\Omega) : \boldsymbol{\tau v}_{|\Gamma_d} = 0, \boldsymbol{\tau v}_{|\Sigma} = \boldsymbol{\xi}\}.$$

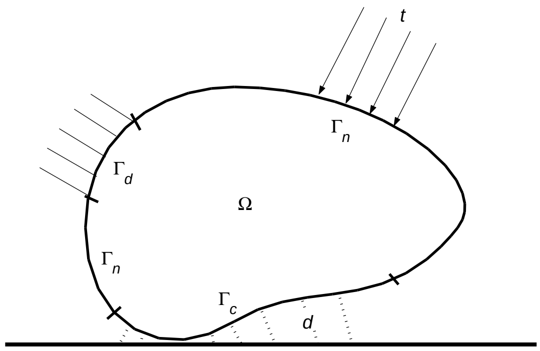


FIGURE 1. *Elastic body with rigid obstacle.*

For equivalent ways to define this space we refer to [34,39]. Next we decompose the elements in $\mathbf{H}_{00}^{1/2}(\Sigma)$ into normal and tangential components. For this reason the $\mathcal{C}^{1,1}$-regularity of the boundary is required, which guarantees that the components of the unit outward vector $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_n)$ on Σ are Lipschitz-continuous. We define the surjective, continuous and linear normal trace mapping

$$\tau_N : \mathbf{Y} \to H_{00}^{\frac{1}{2}}(\Sigma)$$

by $\tau_N \boldsymbol{v} := (\boldsymbol{\tau v})^\top \boldsymbol{\nu}$. The corresponding tangential trace mapping

$$\boldsymbol{\tau}_T : \mathbf{Y} \to \mathbf{H}_{T00}^{\frac{1}{2}}(\Sigma) := \{\boldsymbol{v} \in \mathbf{H}_{00}^{\frac{1}{2}}(\Sigma) : \tau_N \boldsymbol{v} = 0\}$$

defined by $\boldsymbol{\tau}_T \boldsymbol{v} := \boldsymbol{v} - (\tau_N \boldsymbol{v})\boldsymbol{\nu}$ is also linear, continuous and surjective, see [73, p. 88]. Since by assumption $\overline{\Gamma}_c \subset \Sigma$ it follows that

$$\mathbf{H}^{\frac{1}{2}}(\Gamma_c) = \{\boldsymbol{\xi}_{|\Gamma_c} : \boldsymbol{\xi} \in \mathbf{H}_{00}^{\frac{1}{2}}(\Sigma)\}.$$

The restrictions of τ_N and $\boldsymbol{\tau}_T$ to Γ_c

$$\begin{aligned} \tau_{N_c} &:= \tau_{N|\Gamma_c} : \mathbf{Y} \to H^{\frac{1}{2}}(\Gamma_c), \\ \boldsymbol{\tau}_{T_c} &:= \boldsymbol{\tau}_{T|\Gamma_c} : \mathbf{Y} \to \mathbf{H}_T^{\frac{1}{2}}(\Gamma_c) = \{\boldsymbol{v} \in \mathbf{H}^{\frac{1}{2}}(\Gamma_c) : \tau_{N_c} \boldsymbol{v} = 0\} \end{aligned}$$

are also linear, continuous and surjective. In the sequel we use the notations τ_{N_c} and $\boldsymbol{\tau}_{T_c}$ only if we want to highlight that we are dealing with restricted functions. If no confusion can occur we use τ_N and $\boldsymbol{\tau}_T$ for τ_{N_c} and $\boldsymbol{\tau}_{T_c}$, respectively.

Next we define the strain and stress tensors for linear elasticity. For $\boldsymbol{y} \in \mathbf{Y}$ the components of the strain tensor $\underline{\boldsymbol{\varepsilon}}(\boldsymbol{y}) \in (L^2(\Omega))^{n\times n}$ are given by

$$\varepsilon_{ij}(\boldsymbol{y}) := \frac{1}{2}\left(\frac{\partial y_i}{\partial x_j} + \frac{\partial y_j}{\partial x_i}\right) \quad \text{for } 1 \le i, j \le n.$$

The stress tensor $\underline{\boldsymbol{\sigma}}(\boldsymbol{y}) \in (L^2(\Omega))^{n\times n}$ is calculated from $\underline{\boldsymbol{\varepsilon}}(\boldsymbol{y})$ according to

$$\underline{\boldsymbol{\sigma}}(\boldsymbol{y}) := \mathbb{C}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{y}) = (\lambda \operatorname{tr}(\underline{\boldsymbol{\varepsilon}}(\boldsymbol{y}))\mathrm{I} + 2\mu\underline{\boldsymbol{\varepsilon}}(\boldsymbol{y})),$$

where I denotes the $n \times n$-identity matrix and $\operatorname{tr}(\cdot)$ the trace of a matrix. The Lamé constants λ, μ are given by

$$\mu = \frac{E}{2(1+\nu)}, \quad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)},$$

with $E > 0$ denoting Young's modulus and $\nu \in (0, \frac{1}{2})$ the Poisson ratio. Above, the fourth order isotropic material tensor $\mathbb{C}$ that describes the mapping from $\underline{\boldsymbol{\varepsilon}}(\boldsymbol{y})$ to $\underline{\boldsymbol{\sigma}}(\boldsymbol{y})$ has been introduced. In linear elasticity its inverse $\mathbb{C}^{-1}$ exists and can be calculated explicitly, namely

$$\underline{\boldsymbol{\varepsilon}}(\boldsymbol{y}) = \mathbb{C}^{-1}\underline{\boldsymbol{\sigma}}(\boldsymbol{y}) = \frac{1}{2\mu}\underline{\boldsymbol{\sigma}}(\boldsymbol{y}) - \frac{\lambda}{n\lambda + 2\mu}\operatorname{tr}(\underline{\boldsymbol{\sigma}}(\boldsymbol{y}))\mathrm{I}, \tag{4.1}$$

where n denotes the dimension of Ω. We can now define the symmetric bilinear form for the Navier-Lamé system. Let $\boldsymbol{y}, \boldsymbol{z} \in \mathbf{Y}$, then

$$a(\boldsymbol{y}, \boldsymbol{z}) := \int_\Omega \underline{\boldsymbol{\varepsilon}}(\boldsymbol{y}) : \underline{\boldsymbol{\sigma}}(\boldsymbol{z})\, dx = \int_\Omega \underline{\boldsymbol{\sigma}}(\boldsymbol{y}) : \underline{\boldsymbol{\varepsilon}}(\boldsymbol{z})\, dx,$$

where ':' means the sum of the componentwise product of two matrices, that is the above expression denotes the usual scalar product $(\underline{\boldsymbol{\varepsilon}}(\boldsymbol{y}), \underline{\boldsymbol{\sigma}}(\boldsymbol{y}))$ between $\underline{\boldsymbol{\varepsilon}}(\boldsymbol{y})$ and $\underline{\boldsymbol{\sigma}}(\boldsymbol{y})$ in $(L^2(\Omega))^{n\times n}$. For given $\boldsymbol{f} \in \mathbf{L}^2(\Omega)$ and $\boldsymbol{t} \in \mathbf{L}^2(\Gamma_n)$ we define the continuous linear form $L : \mathbf{Y} \to \mathbb{R}$ by

$$L(\boldsymbol{y}) = \int_\Omega \boldsymbol{f}\boldsymbol{y}\, dx + \int_{\Gamma_n} \boldsymbol{t}\,\boldsymbol{\tau}\boldsymbol{y}\, dx$$

for $\boldsymbol{y} \in \mathbf{Y}$. Furthermore, we denote by

$$\mathbf{K} := \{\boldsymbol{v} \in \mathbf{Y} : \tau_N \boldsymbol{v} \le 0 \text{ a.e. on } \Gamma_c\}$$

the cone of functions in $\mathbf{Y}$ with non-positive normal trace component on Γ_c. To model a possible gap between the elastic body and the rigid foundation we introduce $\boldsymbol{d} \in \mathbf{Y}$ with $\tau_N \boldsymbol{d} \ge 0$ a.e. on Γ_c and denote by $d := \tau_{N_c}\boldsymbol{d} \in H^{\frac{1}{2}}(\Gamma_c)$. Then the Signorini problem can be written as minimization of the potential energy over the set of admissible deformations, i.e., as

$$\min_{\boldsymbol{y} \in \boldsymbol{d} + \mathbf{K}} J(\boldsymbol{y}) := \frac{1}{2}a(\boldsymbol{y}, \boldsymbol{y}) - L(\boldsymbol{y}), \tag{$\mathcal{P}$}$$

or equivalently as elliptic variational inequality of the first kind [46]:

$$\begin{cases} \text{Find } \boldsymbol{y} \in \boldsymbol{d} + \mathbf{K} \text{ such that} \\ a(\boldsymbol{y}, \boldsymbol{z} - \boldsymbol{y}) \ge L(\boldsymbol{z} - \boldsymbol{y}) \text{ for all } \boldsymbol{z} \in \boldsymbol{d} + \mathbf{K}. \end{cases} \tag{4.2}$$

1.2. Basic results. We now state existence and uniqueness results for the contact problem in linear elasticity.

THEOREM 4.1. *Problem* ($\mathcal{P}$) *or equivalently* (4.2) *admits a unique solution* $\boldsymbol{y} \in \boldsymbol{d} + \mathbf{K}$.

PROOF. (Sketch) The proof follows from the so-called Korn inequality, which states that there exists a constant $c > 0$ such that

$$\int_\Omega \sum_{i,j=1}^{n} \varepsilon_{ij}(\boldsymbol{y})\varepsilon_{ij}(\boldsymbol{y})\,dx + \|\boldsymbol{y}\|_\Omega^2 \geq c\|\boldsymbol{y}\|_{\mathbf{H}^1(\Omega)}^2 \tag{4.3}$$

for all $\boldsymbol{y} \in \mathbf{H}^1(\Omega)$. For the involved proof of (4.3) we refer to [39, p. 110] or [73, p. 103]. Since the elements in $\mathbf{Y}$ are fixed on the nonempty set Γ_d, Poincare's inequality implies together with (4.3) that $a(\cdot\,,\cdot)$ is coercive on $\mathbf{Y}$. This yields that

$$J(\boldsymbol{y}) \to \infty \text{ as } \|\boldsymbol{y}\|_{\mathbf{Y}} \to \infty,$$

and implies, since $\mathbf{K}$ is weakly closed, the existence of a solution to $(\mathcal{P})$. Uniqueness of the solution follows from the uniform convexity of $J(\cdot)$. $\square$

Finally we give the strong formulation of the contact problem $(\mathcal{P})$. The Navier-Lamé equation is given by

$$-\operatorname{Div}\underline{\boldsymbol{\sigma}}(\boldsymbol{y}) = -\mu\boldsymbol{\Delta}\boldsymbol{y} - (\lambda+\mu)(\nabla \operatorname{div}\boldsymbol{y}) = \boldsymbol{f} \text{ in } \Omega,$$

where Div denotes the rowwise div-operator and $\boldsymbol{\Delta}$ the vector-Laplacian. We have the Dirichlet boundary conditions

$$\boldsymbol{\tau}\boldsymbol{y} = 0 \text{ on } \Gamma_d$$

and Neumann-type boundary conditions

$$\boldsymbol{\sigma}_{\boldsymbol{\nu}}\boldsymbol{y} = \boldsymbol{t} \text{ on } \Gamma_n,$$

where $\boldsymbol{\sigma}_{\boldsymbol{\nu}}\boldsymbol{y} = (\underline{\boldsymbol{\sigma}}\boldsymbol{y})^\top\boldsymbol{\nu}$. To complete the strong formulation we give the contact condition on Γ_c:

$$\boldsymbol{\sigma}_T\boldsymbol{y} = 0,\ \ \tau_N\boldsymbol{y} - d \leq 0,\ \ \sigma_N\boldsymbol{y} \leq 0,\ \ (\tau_N\boldsymbol{y} - d)\sigma_N\boldsymbol{y} = 0 \ \text{ on } \ \Gamma_c,$$

where $\sigma_N\boldsymbol{y} := (\boldsymbol{\sigma}_{\boldsymbol{\nu}}\boldsymbol{y})^\top\boldsymbol{\nu}$ and $\boldsymbol{\sigma}_T\boldsymbol{y} = \boldsymbol{\sigma}_{\boldsymbol{\nu}}\boldsymbol{y} - (\sigma_N\boldsymbol{y})\boldsymbol{\nu}$. The above condition on Γ_c expresses that the deformation on the boundary is stress-free in tangential direction (i.e., $\boldsymbol{\sigma}_T\boldsymbol{y} = 0$), that the nonpenetration condition $\tau_N\boldsymbol{y} - d \leq 0$ on Γ_c is satisfied and that the normal stress $\sigma_N\boldsymbol{y}$ is nonpositive in the zone of contact (i.e., where $\tau_N\boldsymbol{y} - d = 0$), and 0 outside this region.

1.3. Optimality system. In the next theorem we characterize the solution of $(\mathcal{P})$ introducing a Lagrange multiplier that will turn out to be the negative normal stress on the boundary. In the sequel we abbreviate the notation of the duality pairing for elements

$$\xi \in H^{\frac{1}{2}}(\Gamma_c), \varrho \in H^{-\frac{1}{2}}(\Gamma_c) \ \text{ by } \ \left\langle \varrho, \xi \right\rangle_{\Gamma_c} \ \text{ instead of } \ \left\langle \varrho, \xi \right\rangle_{H^{-\frac{1}{2}}(\Gamma_c), H^{\frac{1}{2}}(\Gamma_c)}.$$

THEOREM 4.2. *The solution* $\bar{\boldsymbol{y}} \in \boldsymbol{d} + \mathbf{K}$ *of* $(\mathcal{P})$ *is characterized by the existence of* $\bar{\lambda} \in H^{-\frac{1}{2}}(\Gamma_c)$ *such that*

$$a(\bar{\boldsymbol{y}}, \boldsymbol{z}) - L(\boldsymbol{z}) + \langle \bar{\lambda}, \tau_{N_c}\boldsymbol{z} \rangle_{\Gamma_c} = 0 \quad \text{for all } \boldsymbol{z} \in \mathbf{Y}, \tag{4.4a}$$

$$\langle \bar{\lambda}, \tau_{N_c}\boldsymbol{z} \rangle_{\Gamma_c} \leq 0 \quad \text{for all } \boldsymbol{z} \in \mathbf{K}, \tag{4.4b}$$

$$\langle \bar{\lambda}, \tau_{N_c}\bar{\boldsymbol{y}} - d \rangle_{\Gamma_c} = 0. \tag{4.4c}$$

PROOF. Since $\tau_{N_c} : \mathbf{Y} \to H^{\frac{1}{2}}(\Gamma_c)$ is surjective, it follows for instance from [88] that there exists a Lagrange multiplier $\bar{\lambda} \in H^{-\frac{1}{2}}(\Gamma_c)$ such that (4.4a)-(4.4c) hold. □

Note that, provided $\bar{\lambda} \in L^1(\Gamma_c)$, the conditions $\bar{\boldsymbol{y}} \in \boldsymbol{d} + \mathbf{K}$, (4.4b) and (4.4c) can equivalently be written as

$$\bar{\lambda} = \max(0, \bar{\lambda} + \sigma(\tau_N \bar{\boldsymbol{y}} - d)) \quad \text{for each } \sigma > 0. \tag{4.5}$$

This can be verified by a direct computation, and it also follows from results in convex analysis. Writing the complementarity conditions (4.4b), (4.4c) as (4.5) motivates the application of a semi-smooth Newton method for the solution of the Signorini problem, see Section 3.

1.4. The dual problem. Next we calculate the Fenchel dual for $(\mathcal{P})$. This is done following the theory in Section 1.3, see also [42]. A related discussion for contact problems can be found in [73, p. 201].

We start the calculation of the dual problem with some definitions. Let $\mathcal{F} : \mathbf{Y} \to \mathbb{R}$ be defined by

$$\mathcal{F}(\boldsymbol{y}) := \begin{cases} -L(\boldsymbol{y}) & \text{if } \boldsymbol{y} \in \boldsymbol{d} + \mathbf{K}, \\ \infty & \text{else,} \end{cases}$$

and define the set of symmetric matrices of L^2-functions by

$$\mathbf{V} = \{\underline{\boldsymbol{p}} \in (L^2(\Omega))^{n\times n} : p_{ij} = p_{ji} \text{ for all } 1 \leq i, j \leq n\},$$

furthermore $\Lambda \in \mathcal{L}(\mathbf{Y}, \mathbf{V})$ is defined by $\Lambda \boldsymbol{y} := \underline{\boldsymbol{\varepsilon}}(\boldsymbol{y})$. Finally, $\mathcal{G} : \boldsymbol{V} \to \mathbb{R}$ is given by

$$\mathcal{G}(\underline{\boldsymbol{p}}) := \frac{1}{2}\int_\Omega \underline{\boldsymbol{p}} : \mathbb{C}\underline{\boldsymbol{p}}\, dx.$$

It is easy to check that $\mathcal{F}$ and $\mathcal{G}$ are proper, convex and lower semicontinuous functions. Utilizing the above definitions the contact problem $(\mathcal{P})$ can be written in the form

$$\min_{\boldsymbol{y}\in\mathbf{Y}} \Big\{ \mathcal{F}(\boldsymbol{y}) + \mathcal{G}(\Lambda \boldsymbol{y}) \Big\}. \tag{4.6}$$

We now turn to the dual problem corresponding to $(\mathcal{P})$ which obeys the abstract form

$$\sup_{\underline{\boldsymbol{p}} \in \mathbf{V}^\star} \Big\{ -\mathcal{F}^\star(-\Lambda^\star \underline{\boldsymbol{p}}) - \mathcal{G}^\star(\underline{\boldsymbol{p}}) \Big\}, \tag{4.7}$$

where $\mathcal{F}^\star$ and $\mathcal{G}^\star$ denote the convex conjugate functions for $\mathcal{F}$ and $\mathcal{G}$, respectively, and $\Lambda^\star \in \mathcal{L}(\mathbf{V}^\star, \mathbf{Y}^\star)$ is the linear adjoint of Λ. Note that $\mathcal{F}$ and $\mathcal{G}$ satisfy the conditions of Theorem 2.5, and thus the solutions $\bar{\boldsymbol{y}}$ and $\underline{\bar{\boldsymbol{p}}}$ of (4.6) and (4.7), respectively, satisfy the extremality conditions

$$\begin{aligned} -\Lambda^\star \underline{\bar{\boldsymbol{p}}} &\in \partial \mathcal{F}(\bar{\boldsymbol{y}}), \\ \underline{\bar{\boldsymbol{p}}} &\in \partial \mathcal{G}(\Lambda \bar{\boldsymbol{y}}), \end{aligned} \tag{4.8}$$

with ∂ denoting the subdifferential. In the following discussion we identify $\mathbf{V}$ with its dual $\mathbf{V}^\star$. To evaluate the dual problem for $(\mathcal{P})$ we start with the calculation of $\mathcal{F}^\star(-\Lambda^\star \underline{\boldsymbol{p}})$. Due to the definition of the convex conjugate functional we have

$$\begin{aligned} \mathcal{F}^\star(-\Lambda^\star \underline{\boldsymbol{p}}) &= \sup_{\boldsymbol{y} \in \mathbf{Y}} \Big\{ \big\langle -\Lambda^\star \underline{\boldsymbol{p}}, \boldsymbol{y} \big\rangle_{\mathbf{Y}^\star,\mathbf{Y}} - \mathcal{F}(\boldsymbol{y}) \Big\} \\ &= \sup_{\substack{\boldsymbol{y} \in \mathbf{Y} \\ \tau_N \boldsymbol{y} - d \le 0 \text{ a.e. on } \Gamma_c}} \Big\{ \big\langle -\Lambda^\star \underline{\boldsymbol{p}}, \boldsymbol{y} \big\rangle_{\mathbf{Y}^\star,\mathbf{Y}} + (\boldsymbol{f}, \boldsymbol{y})_\Omega + (\boldsymbol{t}, \boldsymbol{\tau} \boldsymbol{y})_{\Gamma_n} \Big\} \\ &\ge \sup_{\boldsymbol{y} \in \mathbf{H}_0^1(\Omega)} \Big\{ \big\langle \operatorname{Div} \underline{\boldsymbol{p}}, \boldsymbol{y} \big\rangle_{\mathbf{Y}^\star,\mathbf{Y}} + (\boldsymbol{f}, \boldsymbol{y})_\Omega \Big\}. \end{aligned}$$

The latter supremum (and thus $\mathcal{F}^\star(-\Lambda^\star \underline{\boldsymbol{p}})$) equals $+\infty$ unless

$$-\operatorname{Div} \underline{\boldsymbol{p}} = \boldsymbol{f} \tag{4.9}$$

and thus $\operatorname{Div} \underline{\boldsymbol{p}} \in \mathbf{L}^2(\Omega)$. We now state a trace theorem for elements in $\mathbf{V}_{Div} := \{\underline{\boldsymbol{p}} \in \mathbf{V} : \operatorname{Div} \underline{\boldsymbol{p}} \in \mathbf{L}^2(\Omega)\}$, cp. [73, p. 93].

LEMMA 4.3. *There exists a uniquely determined linear continuous mapping* $\boldsymbol{\pi} : \mathbf{V}_{Div} \to \mathbf{H}^{-\frac{1}{2}}(\Gamma)$ *such that*

$$\boldsymbol{\pi}(\underline{\boldsymbol{p}}) = \underline{\boldsymbol{p}} \cdot \boldsymbol{\nu} \text{ if } \underline{\boldsymbol{p}} \text{ is continuously differentiable on } \overline{\Omega}$$

and such that the following Green's formula holds (the subindex $(\cdot)_i$ denotes the i-th row of $\underline{\boldsymbol{p}}$)

$$(\operatorname{div} \boldsymbol{p}_i, z)_\Omega + (\boldsymbol{p}_i, \nabla z)_\Omega - \big\langle (\boldsymbol{\pi} \underline{\boldsymbol{p}})_i, \tau z \big\rangle_{H^{-\frac{1}{2}}(\Gamma), H^{\frac{1}{2}}(\Gamma)} = 0 \text{ for all } z \in H^1(\Omega). \tag{4.10}$$

Furthermore, one can split the trace operator $\boldsymbol{\pi}$ in its normal and tangential component that is, there exist linear operators

$$\pi_N : \mathbf{V}_{Div} \to H^{-\frac{1}{2}}(\Gamma), \qquad \boldsymbol{\pi}_T : \mathbf{V}_{Div} \to \mathbf{H}_T^{-\frac{1}{2}}(\Gamma) := \{\boldsymbol{y} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma) : \pi_N \boldsymbol{y} = 0\},$$

such that

$$\pi_N \underline{\boldsymbol{q}} = (\underline{\boldsymbol{q}} \boldsymbol{\nu})^\top \boldsymbol{\nu} \quad \text{and} \quad \boldsymbol{\pi}_T \underline{\boldsymbol{q}} = \underline{\boldsymbol{q}} \boldsymbol{\nu} - (\pi_N \underline{\boldsymbol{q}})\, \nu$$

for all $\underline{\boldsymbol{q}}$ that are continuously differentiable on $\overline{\Omega}$.

Let us now suppose that (4.9) holds. Then it follows, utilizing (4.10) that

$$\mathcal{F}^\star(-\Lambda^\star \underline{\boldsymbol{p}}) = \sup_{\substack{\boldsymbol{y} \in \mathbf{Y} \\ \tau_N \boldsymbol{y} - d \le 0 \text{ a.e. on } \Gamma_c}} \Big\{ \big\langle -\boldsymbol{\pi} \underline{\boldsymbol{p}}, \boldsymbol{\tau} \boldsymbol{y} \big\rangle_{\mathbf{H}_{00}^{-\frac{1}{2}}(\Sigma), \mathbf{H}_{00}^{\frac{1}{2}}(\Sigma)} + (\boldsymbol{t}, \boldsymbol{\tau} \boldsymbol{y})_{\Gamma_n} \Big\}$$

which is $+\infty$ unless

$$\boldsymbol{\pi}\underline{\boldsymbol{p}} = \boldsymbol{t} \text{ in } \mathbf{L}^2(\Gamma_n). \tag{4.11}$$

Finally, the splitting

$$\big\langle -\boldsymbol{\pi}\underline{\boldsymbol{p}}, \boldsymbol{\tau y}\big\rangle_{\Gamma_c} = \big\langle -\pi_N\underline{\boldsymbol{p}}, \tau_N\boldsymbol{y}\big\rangle_{\Gamma_c} + \big\langle -\boldsymbol{\pi}_T\underline{\boldsymbol{p}}, \boldsymbol{\tau}_T\boldsymbol{y}\big\rangle_{\Gamma_c}$$

into normal and tangential part shows that

$$\mathcal{F}^\star(-\Lambda^\star\underline{\boldsymbol{p}}) = \begin{cases} -\big\langle \pi_N\underline{\boldsymbol{p}}, d\big\rangle_{\Gamma_c} & \text{if (4.9), (4.11), } \boldsymbol{\pi}_T\underline{\boldsymbol{p}} = 0 \text{ and} \\ & \pi_N\underline{\boldsymbol{p}} \le 0 \text{ in } H^{-\frac{1}{2}}(\Gamma_c), \\ \infty & \text{else.} \end{cases}$$

Next we calculate $\mathcal{G}^\star(\underline{\boldsymbol{p}})$:

$$\begin{aligned} \mathcal{G}^\star(\underline{\boldsymbol{p}}) &= \sup_{\underline{\boldsymbol{q}} \in \mathbf{V}} \big\{(\underline{\boldsymbol{p}}, \underline{\boldsymbol{q}}) - \mathcal{G}(\underline{\boldsymbol{q}})\big\} = \sup_{\underline{\boldsymbol{q}} \in \mathbf{V}} \int_\Omega \Big\{\underline{\boldsymbol{p}} : \underline{\boldsymbol{q}} - \frac{1}{2}\underline{\boldsymbol{q}} : \mathbb{C}\underline{\boldsymbol{q}}\Big\}\,dx \\ &= \frac{1}{2}\int_\Omega \mathbb{C}^{-1}\underline{\boldsymbol{p}} : \underline{\boldsymbol{p}}\,dx. \end{aligned}$$

We can now summarize our results for the dual problem in the next lemma.

LEMMA 4.4. *The dual problem corresponding to $(\mathcal{P})$ is given by*

$$(\mathcal{P}^\star) \qquad \sup_{\substack{s.t.\ (4.9),\ (4.11) \\ \boldsymbol{\pi}_T\underline{\boldsymbol{p}}=0 \text{ and } \pi_N\underline{\boldsymbol{p}}\le 0 \text{ a.e. on } \Gamma_c}} -\frac{1}{2}\int_\Omega \mathbb{C}^{-1}\underline{\boldsymbol{p}} : \underline{\boldsymbol{p}}\,dx + \big\langle \pi_N\underline{\boldsymbol{p}}, d\big\rangle_{\Gamma_c}.$$

The relation between primal and dual variables is given by the extremality conditions (4.8). Evaluating the condition $\underline{\boldsymbol{p}} \in \partial\mathcal{G}(\Lambda\boldsymbol{y})$ yields that

$$\mathcal{G}(\Lambda\boldsymbol{y}) - \mathcal{G}(\Lambda\boldsymbol{z}) \le \int_\Omega \underline{\boldsymbol{p}} : \underline{\boldsymbol{\varepsilon}}(\boldsymbol{y} - \boldsymbol{z})\,dx$$

for all $\boldsymbol{z} \in \mathbf{Y}$, which implies that

$$\frac{1}{2}\int_\Omega (\mathbb{C}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{y}) - 2\underline{\boldsymbol{p}}) : \underline{\boldsymbol{\varepsilon}}(y)\,dx \le \frac{1}{2}\int_\Omega (\mathbb{C}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{z}) - 2\underline{\boldsymbol{p}}) : \underline{\boldsymbol{\varepsilon}}(z)\,dx.$$

From the first-order necessary optimality conditions to the above minimization problem one gets that

$$\underline{\boldsymbol{p}} = \mathbb{C}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{y}) = \underline{\boldsymbol{\sigma}}(\boldsymbol{y}). \tag{4.12}$$

The extremality condition $-\Lambda^\star\underline{\boldsymbol{p}} \in \partial\mathcal{F}(\boldsymbol{y})$ leads to the variational inequality

$$(\underline{\boldsymbol{p}}, \underline{\boldsymbol{\varepsilon}}(\boldsymbol{z} - \boldsymbol{y})) - L(\boldsymbol{z} - \boldsymbol{y}) \ge 0 \text{ for all } \boldsymbol{z} \in \boldsymbol{d} + \mathbf{K}, \tag{4.13}$$

and after introducing a Lagrange multiplier $\lambda \in H^{-\frac{1}{2}}(\Gamma_c)$ to

$$(\underline{\boldsymbol{p}}, \underline{\boldsymbol{\varepsilon}}(\boldsymbol{z})) - L(\boldsymbol{z}) + \big\langle \lambda, \tau_N\boldsymbol{z}\big\rangle_{\Gamma_c} = 0 \text{ for all } \boldsymbol{z} \in \boldsymbol{d} + \mathbf{K}, \tag{4.14}$$

and to (4.4b) and (4.4c). Finally, (4.12) together with (4.14) results in (4.4a). Thus, we have shown that the extremality conditions lead to the optimality system (4.4) as well. We conclude this section with giving a physical interpretation for the Lagrange multiplier λ. This variable, introduced as a multiplier, will turn out to be the negative normal stress along the boundary. To prove this assertion we deduce from (4.14), (4.4c) and Lemma 4.3 that

$$\begin{aligned} 0 &= \int_\Omega \underline{\boldsymbol{p}} : \underline{\boldsymbol{\varepsilon}}(\boldsymbol{y})\, dx - L(\boldsymbol{z}) + \langle \lambda, \tau_N \boldsymbol{z} \rangle_{\Gamma_c} \\ &= \langle \boldsymbol{\pi} \underline{\boldsymbol{p}}, \boldsymbol{\tau} \boldsymbol{z} \rangle_\Gamma + \langle \boldsymbol{t}, \boldsymbol{\tau} \boldsymbol{z} \rangle_{\Gamma_n} + \langle \lambda, \tau_N \boldsymbol{z} \rangle_{\Gamma_c} \\ &= \langle \pi_N \underline{\boldsymbol{p}} + \lambda, \tau_N \boldsymbol{z} \rangle_{\Gamma_c} + \langle \boldsymbol{\pi}_T \underline{\boldsymbol{p}}, \boldsymbol{\tau}_T \boldsymbol{z} \rangle_{\Gamma_c} \end{aligned}$$

for all $\boldsymbol{z} \in \mathbf{Y}$. This implies that

$$\lambda = -\pi_N \underline{\boldsymbol{p}}, \tag{4.15}$$

i.e., λ is the negative normal boundary stress. Using this observation we can replace $\pi_N \underline{\boldsymbol{p}}$ in $(\mathcal{P}^\star)$ by $-\lambda$ (with the same λ as in (4.4)) and further "sup" by "min" to obtain an alternative formulation for the dual problem $(\mathcal{P}^\star)$, namely

$$\begin{cases} -\min\limits_{\lambda \geq 0 \text{ in } H^{-\frac{1}{2}}(\Gamma_c)} J^\star(\lambda) := \dfrac{1}{2} a\big(\boldsymbol{y}(\lambda), \boldsymbol{y}(\lambda)\big) + \langle \lambda, d \rangle_{\Gamma_c}, \\ \text{where } \boldsymbol{y}(\lambda) \text{ satisfies} \\ a(\boldsymbol{y}(\lambda), \boldsymbol{z}) - L(\boldsymbol{z}) + \langle \lambda, \tau_N \boldsymbol{z} \rangle_{\Gamma_c} = 0 \text{ for all } \boldsymbol{z} \in \mathbf{Y}. \end{cases} \tag{4.16}$$

Observe that in the above formulation the variable $\boldsymbol{y}(\lambda)$ only enters the formulation as auxiliary variable that depends on λ.

2. Regularization

The regularization procedure that will be presented in this section allows us to apply a semi-smooth Newton method for the solution of the contact problem in an infinite-dimensional framework. Furthermore, for certain examples it will turn out to have a positive influence onto the numerical performance of our algorithms.

2.1. Regularization and the corresponding dual and primal problem. To introduce the regularized problem we replace the complementarity condition (4.5) for given $\hat{\lambda} \in L^2(\Gamma_c)$ and $\gamma > 0$ by

$$\lambda = \max(0, \hat{\lambda} + \gamma(\tau_N \boldsymbol{y} - d)). \tag{4.17}$$

This replacement is motivated from semi-smooth Newton methods for the solution of nonlinear complementarity functions such as (4.5), see [58, 69]. The introduction of $\hat{\lambda}$ is motivated by first-order augmented Lagrangians, cf. [67] and the discussion in the next sections.

Considering in (4.17) the variable $\boldsymbol{y}$ as a function of λ defined by means of (4.4a), we observe that $\tau_N \boldsymbol{y}$ is smoother than λ. This property is necessary for semi-smoothness of the max-function (see Theorem 2.8). In the original problem (4.5) we cannot expect any smoothing of the expression inside the max-function due to the explicit appearance of λ. We now turn to the primal and dual problem corresponding to (4.4a) and (4.17). The regularized primal problem is the unconstrained minimization problem

$$(\mathcal{P}_\gamma) \qquad \min_{\boldsymbol{y} \in \mathbf{Y}} J_\gamma(\boldsymbol{y}) := \frac{1}{2} a(\boldsymbol{y}, \boldsymbol{y}) - L(\boldsymbol{y}) + \frac{1}{2\gamma} \| \max(0, \hat{\lambda} + \gamma(\tau_N \boldsymbol{y} - d)) \|_{\Gamma_c}^2,$$

and the corresponding dual problem (in the min-notation (4.16)) is given by

$$(\mathcal{P}_\gamma^\star) \quad \begin{cases} -\displaystyle\min_{\lambda \geq 0 \text{ in } L^2(\Gamma_c)} \frac{1}{2} a\big(\boldsymbol{y}(\lambda), \boldsymbol{y}(\lambda)\big) + \big\langle \lambda, d \big\rangle_{\Gamma_c} + \frac{1}{2\gamma} \|\lambda - \hat{\lambda}\|_{\Gamma_c}^2 - \frac{1}{2\gamma} \|\hat{\lambda}\|_{\Gamma_c}^2, \\ \text{where } \boldsymbol{y}(\lambda) \text{ satisfies} \\ a(\boldsymbol{y}(\lambda), \boldsymbol{z}) - L(\boldsymbol{z}) + \big\langle \lambda, \tau_N \boldsymbol{z} \big\rangle_{\Gamma_c} = 0 \text{ for all } \boldsymbol{z} \in \mathbf{Y}. \end{cases}$$

Note that the last term in $(\mathcal{P}_\gamma^\star)$ only involves the constant $\hat{\lambda}$ and can thus be neglected for the minimization. However, it is necessary to ascertain the usual relation (2.6) between the values of primal and dual functional. By a calculation similar to the one for the original contact problem $(\mathcal{P})$ it can be verified that $(\mathcal{P}_\gamma^\star)$ represents the dual problem of $(\mathcal{P}_\gamma)$. Note that in $(\mathcal{P}_\gamma)$ the parameter γ plays the role of a penalty parameter that penalizes the violation of the (primal) constraint. Hence, $(\mathcal{P}_\gamma)$ is an unconstrained problem while the original problem $(\mathcal{P})$ involves a pointwise inequality constraint. Thus, a solution to $(\mathcal{P}_\gamma)$ is not necessary feasible for $(\mathcal{P})$. On the other hand, the dual problems $(\mathcal{P}^\star)$, $(\mathcal{P}_\gamma^\star)$ are both constrained optimization problems and a solution λ_γ to $(\mathcal{P}_\gamma^\star)$ is feasible for $(\mathcal{P}^\star)$ and vice versa provided that $\bar{\lambda} \in L^2(\Gamma_c)$. For later reference we summarize the above results in the following lemma.

LEMMA 4.5. *Problem $(\mathcal{P}_\gamma^\star)$ is the dual of $(\mathcal{P}_\gamma)$ and the solutions $\boldsymbol{y}_\gamma$ of $(\mathcal{P}_\gamma)$ and λ_γ of $(\mathcal{P}_\gamma^\star)$ are characterized by the extremality conditions*

$$a(\boldsymbol{y}_\gamma, \boldsymbol{z}) - L(\boldsymbol{z}) + (\lambda_\gamma, \tau_N \boldsymbol{z})_{\Gamma_c} = 0 \text{ for all } \boldsymbol{z} \in \mathbf{Y}, \tag{4.18}$$

$$\lambda_\gamma - \max(0, \hat{\lambda} + \gamma(\tau_N \boldsymbol{y}_\gamma - d)) = 0 \text{ on } \Gamma_c. \tag{4.19}$$

2.2. Convergence as $\gamma \to \infty$. In this section we investigate the convergence of the solution variables $(\boldsymbol{y}_\gamma, \lambda_\gamma)$ of the regularized problems $(\mathcal{P}_\gamma)$ and $(\mathcal{P}_\gamma^\star)$ towards $(\bar{\boldsymbol{y}}, \bar{\lambda})$, the solution of $(\mathcal{P})$ and $(\mathcal{P}^\star)$, respectively.

THEOREM 4.6. *With the above notation we have that $\boldsymbol{y}_\gamma \to \bar{\boldsymbol{y}}$ strongly in $\mathbf{Y}$ and $\lambda_\gamma \to \bar{\lambda}$ weakly in $H^{-\frac{1}{2}}(\Gamma_c)$.*

PROOF. Recall that both, the variables $(\boldsymbol{y}_\gamma, \lambda_\gamma)$ and $(\bar{\boldsymbol{y}}, \bar{\lambda})$ satisfy equation (4.4a). Furthermore, (4.4b) and (4.4c) hold for $(\bar{\boldsymbol{y}}, \bar{\lambda})$, while $(\boldsymbol{y}_\gamma, \lambda_\gamma)$ satisfies

$$\lambda_\gamma = \max(0, \hat{\lambda} + \gamma(\tau_N \boldsymbol{y}_\gamma - d)). \tag{4.20}$$

Setting $\boldsymbol{z} := \boldsymbol{y}_\gamma - \bar{\boldsymbol{y}}$ in (4.18) results in

$$a(\boldsymbol{y}_\gamma, \boldsymbol{y}_\gamma - \bar{\boldsymbol{y}}) - L(\boldsymbol{y}_\gamma - \bar{\boldsymbol{y}}) + (\lambda_\gamma, \tau_N(\boldsymbol{y}_\gamma - \bar{\boldsymbol{y}}))_{\Gamma_c} = 0. \tag{4.21}$$

Next we estimate

$$\begin{aligned}(\lambda_\gamma, \tau_N(\boldsymbol{y}_\gamma - \bar{\boldsymbol{y}}))_{\Gamma_c} &= (\lambda_\gamma, \tau_N \boldsymbol{y}_\gamma - d)_{\Gamma_c} - (\lambda_\gamma, \tau_N \bar{\boldsymbol{y}} - d)_{\Gamma_c} \\ &\geq \gamma^{-1}\big(\lambda_\gamma, \hat{\lambda} + \gamma(\tau_N \boldsymbol{y}_\gamma - d)\big)_{\Gamma_c} - \gamma^{-1}(\lambda_\gamma, \hat{\lambda})_{\Gamma_c},\end{aligned}$$

where $(\lambda_\gamma, \tau_N \bar{\boldsymbol{y}} - d) \leq 0$ a.e. on Γ_c was used. Thus,

$$\begin{aligned}\big(\lambda_\gamma, \tau_N(\boldsymbol{y}_\gamma - \bar{\boldsymbol{y}})\big)_{\Gamma_c} &\geq \gamma^{-1}\big(\lambda_\gamma, \max(0, \hat{\lambda} + \gamma(\tau_N \boldsymbol{y}_\gamma - d))\big)_{\Gamma_c} - \gamma^{-1}(\lambda_\gamma, \hat{\lambda})_{\Gamma_c} \\ &= \gamma^{-1}\|\lambda_\gamma\|^2_{\Gamma_c} - \gamma^{-1}(\lambda_\gamma, \hat{\lambda})_{\Gamma_c} \qquad (4.22) \\ &= \frac{1}{2\gamma}\|\lambda_\gamma - \hat{\lambda}\|^2_{\Gamma_c} + \frac{1}{2\gamma}\|\lambda_\gamma\|^2_{\Gamma_c} - \frac{1}{2\gamma}\|\hat{\lambda}\|^2_{\Gamma_c} \geq -\frac{1}{2\gamma}\|\hat{\lambda}\|^2_{\Gamma_c}. \qquad (4.23)\end{aligned}$$

Equation (4.21) together with (4.22) imply that

$$a(\boldsymbol{y}_\gamma, \boldsymbol{y}_\gamma) + \frac{1}{\gamma}\|\lambda_\gamma\|^2_{\Gamma_c} \leq a(\boldsymbol{y}_\gamma, \bar{\boldsymbol{y}}) + \frac{1}{\gamma}(\lambda_\gamma, \hat{\lambda})_{\Gamma_c} + L(\boldsymbol{y}_\gamma - \bar{\boldsymbol{y}}). \tag{4.24}$$

Using the coercivity (with constant $c > 0$) and the continuity (with constant $C > 0$) of $a(\cdot\,, \cdot)$ in (4.24) results in

$$c\|\boldsymbol{y}_\gamma\|^2_{\mathbf{Y}} + \frac{1}{\gamma}\|\lambda_\gamma\|^2_{\Gamma_c} \leq C\|\boldsymbol{y}_\gamma\|_{\mathbf{Y}}\|\bar{\boldsymbol{y}}\|_{\mathbf{Y}} + \|L\|_{\mathcal{L}(\mathbf{Y},\mathbb{R})}\|\boldsymbol{y}_\gamma - \bar{\boldsymbol{y}}\|_{\mathbf{Y}} + \frac{1}{\gamma}\|\lambda_\gamma\|_{\Gamma_c}\|\hat{\lambda}\|_{\Gamma_c},$$

which shows that

$$c\|\boldsymbol{y}_\gamma\|_{\mathbf{Y}} + \frac{1}{\gamma}\|\lambda_\gamma\|_{\Gamma_c}$$

is uniformly bounded with respect to $\gamma \geq 1$. Hence $\boldsymbol{y}_\gamma$ is bounded in $\mathbf{Y}$ and λ_γ in $H^{-\frac{1}{2}}(\Gamma_c)$ from (4.18). Consequently, there exist $(\tilde{\boldsymbol{y}}, \tilde{\lambda}) \in \mathbf{Y} \times H^{-\frac{1}{2}}(\Gamma_c)$ and a sequence γ_k with $\lim_{k\to\infty} \gamma_k = \infty$ such that

$$\boldsymbol{y}_{\gamma_k} \rightharpoonup \tilde{\boldsymbol{y}} \text{ weakly in } \mathbf{Y} \quad \text{and} \quad \lambda_{\gamma_k} \rightharpoonup \tilde{\lambda} \text{ weakly in } H^{-\frac{1}{2}}(\Gamma_c). \tag{4.25}$$

In the sequel we dismiss the subscript 'k' with γ_k. Note that, due to the definition of λ_γ,

$$\frac{1}{\gamma}\|\lambda_\gamma\|^2_{\Gamma_c} = \gamma\|\max(0, \tfrac{1}{\gamma}\hat{\lambda} + \tau_N \boldsymbol{y}_\gamma - d)\|^2_{\Gamma_c}. \tag{4.26}$$

Since $H^{\frac{1}{2}}(\Gamma_c)$ embeds compactly into $L^2(\Gamma_c)$, $\tau_N \boldsymbol{y}_\gamma$ converges to $\tau_N \tilde{\boldsymbol{y}}$ almost everywhere on Γ_c. Thus (4.26) implies that $\tau_N \tilde{\boldsymbol{y}} - d \leq 0$ a.e. on Γ_c.

Subtracting equation (4.4a) for $(\boldsymbol{y}_\gamma, \lambda_\gamma)$ from the same equation for $(\bar{\boldsymbol{y}}, \bar{\lambda})$ and setting $\boldsymbol{z} := \boldsymbol{y}_\gamma - \bar{\boldsymbol{y}}$ yields

$$a(\boldsymbol{y}_\gamma - \bar{\boldsymbol{y}}, \boldsymbol{y}_\gamma - \bar{\boldsymbol{y}}) = -\left\langle \lambda_\gamma - \bar{\lambda}, \tau_N(\boldsymbol{y}_\gamma - \bar{\boldsymbol{y}})\right\rangle_{\Gamma_c}, \tag{4.27}$$

where, as before $\left\langle \cdot\,, \cdot\right\rangle_{\Gamma_c}$ denotes the duality pairing between elements in $H^{-\frac{1}{2}}(\Gamma_c)$ and $H^{\frac{1}{2}}(\Gamma_c)$. Using (4.23), the coercivity of $a(\cdot\,,\cdot)$ and (4.27) shows that

$$\begin{aligned}
0 \le \limsup_{\gamma\to\infty} c\|\boldsymbol{y}_\gamma - \bar{\boldsymbol{y}}\|_{\mathbf{Y}}^2 &\le \lim_{\gamma\to\infty} \left\langle \bar{\lambda}, \tau_N(\boldsymbol{y}_\gamma - \bar{\boldsymbol{y}})\right\rangle_{\Gamma_c} \\
&= \lim_{\gamma\to\infty} \left\langle \bar{\lambda}, \tau_N\tilde{\boldsymbol{y}} - d\right\rangle_{\Gamma_c} - \left\langle \bar{\lambda}, \tau_N\bar{\boldsymbol{y}} - d\right\rangle_{\Gamma_c} \\
&= \lim_{\gamma\to\infty} \left\langle \bar{\lambda}, \tau_N\tilde{\boldsymbol{y}} - d\right\rangle_{\Gamma_c} \le 0,
\end{aligned}$$

where $\tau_N\tilde{\boldsymbol{y}} - d \le 0$ a.e. on Γ_c is used. From the above estimate follows that $\boldsymbol{y}_\gamma \to \bar{\boldsymbol{y}}$ strongly in $\mathbf{Y}$ and thus $\tilde{\boldsymbol{y}} = \bar{\boldsymbol{y}}$. Passing to the limit in

$$a(\boldsymbol{y}_\gamma, \boldsymbol{z}) - L(\boldsymbol{z}) + (\lambda_\gamma, \tau_N\boldsymbol{z})_{\Gamma_c} = 0 \text{ for all } \boldsymbol{z} \in \mathbf{Y}$$

yields

$$a(\bar{\boldsymbol{y}}, \boldsymbol{z}) - L(\boldsymbol{z}) + \left\langle \tilde{\lambda}, \tau_N\boldsymbol{z}\right\rangle_{\Gamma_c} = 0 \text{ for all } \boldsymbol{z} \in \mathbf{Y}. \tag{4.28}$$

Comparing (4.28) with (4.4a) shows that $\tilde{\lambda} = \bar{\lambda}$. Thus, every sequence γ_n with $\gamma_n \to \infty$ for $n \to \infty$ contains a subsequence γ_{n_k} such that

$$\boldsymbol{y}_{\gamma_{n_k}} \to \bar{\boldsymbol{y}} \text{ in } \mathbf{Y} \text{ and } \lambda_{\gamma_{n_k}} \rightharpoonup \bar{\lambda} \text{ in } H^{-\frac{1}{2}}(\Gamma_c).$$

This implies, due to the uniqueness of the solution variables $\bar{\boldsymbol{y}}, \bar{\lambda}$ that the whole family $\{(\boldsymbol{y}_\gamma, \lambda_\gamma)\}$ converges as stated in the theorem. □

3. The semi-smooth Newton method

In this section we apply a semi-smooth Newton method for the solution of $(\mathcal{P}_\gamma)$ and discuss properties of the algorithm. In the sequel we denote the solution of $(\mathcal{P}_\gamma)$ and $(\mathcal{P}_\gamma^\star)$ by $\boldsymbol{y}_\gamma$ and λ_γ, respectively, and for simplicity we dismiss the subscript 'γ' for the iterates, i.e., we use $\boldsymbol{y}^k, \lambda^k$ instead of $\boldsymbol{y}_\gamma^k, \lambda_\gamma^k$.

3.1. Presentation of the algorithm. We now focus on the solution of the regularized contact problem, that is we search for $(\boldsymbol{y}_\gamma, \lambda_\gamma) \in \mathbf{Y} \times L^2(\Gamma_c)$ solving (4.18) and (4.19). The algorithm given in this section results from the application of the generalized Newton method (see Section 2, page 6) to the mapping $F : L^2(\Gamma_c) \to L^2(\Gamma_c)$ given by

$$F(\lambda) := \lambda - \max(0, \hat{\lambda} + \gamma(\tau_N\boldsymbol{y}(\lambda) - d)), \tag{4.29}$$

where $\boldsymbol{y}(\lambda) \in \mathbf{Y}$ is the unique solution $\boldsymbol{y}$ of (4.18) for given $\lambda \in L^2(\Gamma_c)$. Note that $\tau_N\boldsymbol{y} \in H^{\frac{1}{2}}(\Gamma_c)$, which embeds continuously into $L^q(\Gamma_c)$ for every $q < \infty$ in the case $n = 2$ and for $q = 2(n-1)/(n-2)$ if $n \ge 3$. Thus, $\tau_N\boldsymbol{y} \in L^q(\Gamma_c)$ for some $q > 2$ and we obtain the norm gap required for Newton differentiability of the max-function (see Theorem 2.8). Applying the semi-smooth Newton method with the derivative of the max-function as given in (2.10) to the equation $F(\lambda) = 0$ results in the following algorithm, where $\chi_{\mathcal{S}}$ denotes the characteristic function for a set $\mathcal{S} \subset \Gamma_c$.

Algorithm: (C-SS)

(1) Choose $(\lambda^0, \boldsymbol{y}^0) \in L^2(\Gamma_c) \times \mathbf{Y}$ satisfying (4.18) and set $k := 0$.

(2) Determine

$$\mathcal{A}^{k+1} = \{x \in \Gamma_c : \hat{\lambda} + \gamma(\tau_N \boldsymbol{y}^k - d) > 0\},$$
$$\mathcal{I}^{k+1} = \Gamma_c \setminus \mathcal{A}^{k+1}.$$

(3) If $k \geq 1$ and $\mathcal{A}^{k+1} = \mathcal{A}^k$ stop, else

(4) Solve

$$a(\boldsymbol{y}^{k+1}, \boldsymbol{z}) - L(\boldsymbol{z}) + \big(\hat{\lambda} + \gamma(\tau_N \boldsymbol{y}^{k+1} - d), \chi_{\mathcal{A}^{k+1}} \tau_N \boldsymbol{z}\big)_{\Gamma_c} = 0$$

$$\text{for all } \boldsymbol{z} \in \mathbf{Y}, \text{ set } \lambda^{k+1} = \begin{cases} \hat{\lambda} + \gamma(\tau_N \boldsymbol{y}^{k+1} - d) & \text{on } \mathcal{A}^{k+1}, \\ 0 & \text{on } \mathcal{I}^{k+1}, \end{cases} \tag{4.30}$$

and $k := k + 1$ and go to Step 2.

Note that the solution to (4.30) is unique, since (4.30) represents the necessary and sufficient optimality condition for the auxiliary problem

$$\min_{\boldsymbol{y} \in \mathbf{Y}} \frac{1}{2} a(\boldsymbol{y}, \boldsymbol{y}) - L(\boldsymbol{y}) + \frac{1}{2\gamma} \|\hat{\lambda} + \gamma(\tau_N \boldsymbol{y} - d)\|^2_{\mathcal{A}^{k+1}},$$

which is uniquely solvable. Properties of the algorithm are analyzed next.

LEMMA 4.7. *If Algorithm* (C-SS) *stops, the last iterates* $\boldsymbol{y}^k, \lambda^k$ *are the solutions to* $(\mathcal{P}_\gamma)$ *and* $(\mathcal{P}_\gamma^\star)$*, respectively.*

PROOF. First note that all iterates $(\boldsymbol{y}^k, \lambda^k)$ satisfy (4.18). It remains to show that if the conditions of the Lemma are satisfied, then (4.19) holds true as well. From the uniqueness of the solution to (4.30) $\mathcal{A}^{k+1} = \mathcal{A}^k$ implies $\boldsymbol{y}^{k+1} = \boldsymbol{y}^k$ and $\lambda^{k+1} = \lambda^k$. This yields that $\lambda^k > 0$ on $\mathcal{A}^{k+1} = \mathcal{A}^k$ and $\lambda^k = 0$ on $\mathcal{I}^{k+1} = \mathcal{I}^k$ and thus

$$\lambda^k = \max(0, \lambda^k) = \max(0, \hat{\lambda} + \gamma(\tau_N \boldsymbol{y}^k - d)),$$

which shows that $(\boldsymbol{y}^k, \lambda^k)$ also satisfies the condition (4.19). □

Note that $(\mathcal{P})$ is a unilaterally box-constrained optimization problem, in contrast to the simplified friction problem (see Chapter 3), which is (in its dual form) bilaterally constrained. It is not difficult to verify that for unilaterally constrained problems the determination of the active and inactive sets as the original primal-dual active set strategy does (see [14, 15]) is independent of the factor σ multiplied with $(\tau_N \boldsymbol{y} - d)$, except possibly for the first iteration step.

Thus, this value can be set to γ^{-1}, which results in the above algorithm, cp. [58]. For this reason a separated discussion of the primal-dual active set strategy and the semi-smooth Newton method as for the simplified friction problem is redundant.

3.2. Convergence analysis. In the next theorem local superlinear convergence of the iterates of (C-SS) is proved.

THEOREM 4.8. *Suppose that* $\|\lambda_0 - \lambda_\gamma\|_{\Gamma_c}$ *is sufficiently small. Then, for all* $\hat{\lambda} \in L^2(\Gamma_c)$ *and* $\gamma > 0$ *the iterates* $(\boldsymbol{y}^k, \lambda^k)$ *of* (C-SS) *converge to* $(\boldsymbol{y}_\gamma, \lambda_\gamma)$ *superlinearly in* $\mathbf{Y} \times L^2(\Gamma_c)$.

PROOF. The proof is similar to the one for Theorem 2.2 in [69], see also the proof of Theorem 3.15. □

4. Exact and Inexact Augmented Lagrangian Methods

Augmented Lagrangian methods apply for the solution of contact problems provided the solution variable $\bar{\lambda} \in L^2(\Gamma_c)$. In this chapter we give conditions that guarantee this regularity of the Lagrange multiplier and describe an exact as well as an inexact augmented Lagrangian method for the solution of the Signorini problem $(\mathcal{P})$. We use the fact that the auxiliary problem in every iteration step of the first-order augmented Lagrangian method coincides with the regularized problem discussed in the previous sections. Note that in the augmented Lagrangian approach the penalties can be chosen small, nevertheless the iterates converge to the solution of the original problem $(\mathcal{P})$. This can be advantageous in numerical practice, since large penalty parameters γ in $(\mathcal{P}_\gamma)$ may lead to an ill-conditioning of the problem.

4.1. Regularity of $\bar{\lambda}$. For the application of augmented Lagrangian methods the multiplier $\bar{\lambda}$ (or equivalently the solution of $(\mathcal{P}^\star)$) must belong to $L^2(\Gamma_c)$. We now comment on conditions that guarantee such a regularity. From [74, Thm. 2.2] it follows that

$$\bar{\boldsymbol{y}} \in \mathbf{H}^2(\Omega_\delta) \text{ for each } \delta > 0,$$

where

$$\Omega_\delta = \{x \in \Omega : \ \operatorname{dist}(x, \partial\Sigma \cup \partial\Gamma_c) > \delta\}$$

provided that $\boldsymbol{f} \in \mathbf{L}^2(\Omega)$, $\boldsymbol{g} \in \mathbf{H}^{\frac{1}{2}}(\Gamma_n)$ and Σ, Γ_c are sufficiently regular. Thus, $\sigma_N \bar{\boldsymbol{y}} \in L^2_{loc}(\Gamma_c)$, i.e., $\sigma_N \bar{\boldsymbol{y}}$ is square integrable on compact subsets of Γ_c. Let us now make the assumption that the active set at the solution

$$\mathcal{A}(\bar{\boldsymbol{y}}) = \{x \in \Gamma_c : \tau_N \bar{\boldsymbol{y}} - d = 0 \text{ a.e.}\}$$

is strictly contained in Γ_c, that is

$$\overline{\mathcal{A}(\bar{\boldsymbol{y}})} \subset \Gamma_c. \tag{A}$$

Then it follows from the complementarity conditions that $\sigma_N \bar{\boldsymbol{y}} = 0$ a.e. on $\Gamma_c \setminus \mathcal{A}(\bar{\boldsymbol{y}})$. This implies that $\sigma_N \bar{\boldsymbol{y}} \in L^2(\Gamma_c)$ and thus $\bar{\lambda} \in L^2(\Gamma_c)$. In general (i.e., without assumption (A)) $\sigma_N \bar{\boldsymbol{y}} \in L^1(\Gamma_c)$ for sufficiently smooth data [74, p. 633], but $\sigma_N \bar{\boldsymbol{y}} \notin L^2(\Gamma_c)$ even for arbitrarily smooth data. Note that assumption (A) involves the unknown solution $\bar{\boldsymbol{y}}$ and, hence, cannot be verified a priori rigorously. However, for many examples it is clear form the geometry of the problem that (A) holds and the augmented Lagrangian method can be applied. For the rest of this section we assume that (A) holds.

4.2. The first-order augmented Lagrangian method. This method is, such as the Uzawa algorithm, an update strategy for the multiplier in $(\mathcal{P})$. It can be considered as an implicit version of the Uzawa algorithm, cp. [67]. Its main advantage compared to the latter strategy is its unconditional convergence for all penalty (or regularization) parameters $\gamma > 0$. The Uzawa method only converges for sufficiently small (and possibly very small) $\gamma > 0$, which may lead to extremely slow convergence. However, the drawback of the augmented Lagrangian method is that in every iteration step it requires to solve a nonlinear problem compared to the linear problem in every iteration of the Uzawa algorithm. Since this nonlinear problem is exactly of the form $(\mathcal{P}_\gamma)$, we can apply strategies presented in the previous section for its solution. The whole method is specified next.

Algorithm: (C-ALM)

(1) Choose $\lambda^0 \in L^2(\Gamma_c)$ and set $l := 0$.

(2) Choose $\gamma^{l+1} > 0$ and solve $(\mathcal{P}_\gamma)$ with $\hat{\lambda} := \lambda^l$, i.e., determine $(\boldsymbol{y}^{l+1}, \lambda^{l+1})$ such that

$$a(\boldsymbol{y}^{l+1}, \boldsymbol{z}) - L(\boldsymbol{z}) + (\lambda^{l+1}, \tau_N \boldsymbol{z})_{\Gamma_c} = 0 \text{ for all } \boldsymbol{z} \in \mathbf{Y},$$
$$\lambda^{l+1} = \max(0, \lambda^l + \gamma^{l+1}(\tau_N \boldsymbol{y}^{l+1} - d)).$$

(3) Update $l := l + 1$ and go to Step 2.

The following convergence result for (C-ALM) holds true, where $\bar{\boldsymbol{y}}, \bar{\lambda}$ denote the solution of $(\mathcal{P})$ and $(\mathcal{P}^\star)$, respectively.

Theorem 4.9. *For every choice of parameters* $0 < \gamma^0 \le \gamma^1 \le \gamma^2 \le \cdots$ *the iterates* λ^l *of* (C-ALM) *converge weakly to* $\bar{\lambda}$ *in* $L^2(\Gamma_c)$. *Furthermore, the corresponding iterates* $\boldsymbol{y}^l$ *converge strongly to* $\bar{\boldsymbol{y}}$ *in* $\mathbf{Y}$.

Proof. The proof follows from the convergence proof for the inexact version of Algorithm (C-ALM) presented in the next section. $\square$

Let us commend on the role of the parameters γ^l in (C-ALM). Due to possible ill-conditioning of $(\mathcal{P}_\gamma^\star)$ for large penalty parameters γ one may start with a moderate value for γ in Step 2 of (C-ALM) and increase this value during the iteration. However, note that λ^l converges to $\bar{\lambda}$ without the requirement that γ tends to infinity, which is not the case for pure penalty methods. The idea to solve the problem in Step 2 of (C-ALM) only approximately leads to inexact augmented Lagrangian methods that will be discussed next.

4.3. An inexact first-order augmented Lagrangian method. In this section we formulate an inexact augmented Lagrangian method in infinite dimensions, where inexactness is allowed in the linear system. The result can be utilized regarding to a preconditioned version of the augmented Lagrangian method and is also of interest in connection with the stability of numerical implementations of the first-order augmented Lagrangian method discussed in the previous section. The formulation of the inexact method is motivated from inexact Uzawa algorithms for saddle point problems, [19,25,27] and variational inequalities, [26]. Let $\tilde{a}(\cdot\,,\cdot)$ be a scalar product and $\omega \geq 1$ such that

$$a(\boldsymbol{z},\boldsymbol{z}) \leq \tilde{a}(\boldsymbol{z},\boldsymbol{z}) \leq \omega a(\boldsymbol{z},\boldsymbol{z}) \text{ for all } \boldsymbol{z} \in \mathbf{Y}. \tag{4.31}$$

To discuss the above condition, let us denote the matrices belonging to $\tilde{a}(\cdot\,,\cdot)$ and $a(\cdot\,,\cdot)$ in a discrete setting by $\tilde{A}$ and A, respectively. Then $\tilde{A}$ can be chosen as a preconditioner of A, that is

$$\tilde{A}x = f$$

should be easier to solve than

$$Ax = f.$$

The preconditioner $\tilde{A}$ can be constructed, e.g., from incomplete LU or Choleski decomposition, multigrid or domain decomposition methods [18,96,106]. In this context property (4.31) can be understood as a condition on how well $\tilde{A}$ shall approximate A. This condition is a standard condition for preconditioning of linear systems (see, e.g., [18]).

We now state the inexact augmented Lagrangian method.

Algorithm: (C-IALM)

(1) Choose $\lambda^0 \in L^2(\Gamma_c)$ and set $l := 0$.
(2) Choose $\gamma^{l+1} > 0$ and determine $(\boldsymbol{y}^{l+1}, \lambda^{l+1})$ such that

$$\tilde{a}(\boldsymbol{y}^{l+1},\boldsymbol{z}) - \tilde{a}(\boldsymbol{y}^{l},\boldsymbol{z}) + \left\{a(\boldsymbol{y}^{l},\boldsymbol{z}) - L(\boldsymbol{z}) + (\lambda^{l+1}, \tau_N\boldsymbol{z})_{\Gamma_c}\right\} = 0 \text{ for all } \boldsymbol{z} \in \mathbf{Y},$$

$$\lambda^{l+1} = \max(0, \lambda^l + \gamma^{l+1}(\tau_N \boldsymbol{y}^{l+1} - d)).$$

(3) Update $l := l + 1$ and go to Step 2.

In the next theorem we prove global convergence of (C-IALM) provided the bilinear form $\tilde{a}(\cdot\,,\cdot)$ is a sufficiently good approximation for $a(\cdot\,,\cdot)$. Observe that, in case the scalar product $\tilde{a}(\cdot\,,\cdot)$ equals $a(\cdot\,,\cdot)$, the above algorithm coincides with (C-ALM). Hence, the next theorem contains Theorem 4.9 as a special case.

THEOREM 4.10. *Suppose that* (4.31) *holds for some* $\omega < 5$. *Then, for every choice of parameters* $0 < \gamma^0 \le \gamma^1 \le \gamma^2 \le \cdots$ *the iterates* λ^l *of* (C-IALM) *converge weakly to* $\bar\lambda$ *in* $L^2(\Gamma_c)$, *furthermore the corresponding iterates* $\boldsymbol{y}^l$ *converge strongly to* $\bar{\boldsymbol{y}}$ *in* $\mathbf{Y}$.

PROOF. The proof of this theorem is inspired by the convergence proof for the inexact Uzawa algorithm in [26]. Let us denote by $\delta_y^l := \boldsymbol{y}^l - \bar{\boldsymbol{y}} \in \mathbf{Y}$ and $\delta_\lambda^l := \lambda^l - \bar\lambda \in L^2(\Gamma_c)$, where $\bar{\boldsymbol{y}}, \bar\lambda$ denote the solution variables of $(\mathcal{P})$ and $(\mathcal{P}^\star)$, respectively. From the fact that $(\bar{\boldsymbol{y}}, \bar\lambda)$ satisfies (4.4a) we have for $l \ge 1$

$$a(\bar{\boldsymbol{y}}, \delta_y^{l+1}) - L(\delta_y^{l+1}) + (\bar\lambda, \tau_N \delta_y^{l+1})_{\Gamma_c} = 0, \tag{4.32}$$

and Step 2 of (C-IALM) implies

$$\tilde a(\boldsymbol{y}^{l+1} - \boldsymbol{y}^l, \delta_y^{l+1}) + a(\boldsymbol{y}^l, \delta_y^{l+1}) - L(\delta_y^{l+1}) + (\lambda^{l+1}, \tau_N \delta_y^{l+1})_{\Gamma_c} = 0. \tag{4.33}$$

Subtracting (4.32) from (4.33) results in

$$0 = \tilde a(\boldsymbol{y}^{l+1} - \boldsymbol{y}^l, \delta_y^{l+1}) + a(\boldsymbol{y}^l - \bar{\boldsymbol{y}}, \delta_y^{l+1}) + (\lambda^{l+1} - \bar\lambda, \tau_N \delta_y^{l+1})_{\Gamma_c} \tag{4.34}$$

$$= \tilde a(\delta_y^{l+1} - \delta_y^l, \delta_y^{l+1}) + a(\delta_y^l, \delta_y^{l+1}) + (\delta_\lambda^{l+1}, \tau_N \delta_y^{l+1})_{\Gamma_c}. \tag{4.35}$$

Note that

$$\lambda^{l+1} = P\big(\lambda^l + \gamma^{l+1}(\tau_N \boldsymbol{y}^{l+1} - d)\big) \quad \text{and} \quad \bar\lambda = P\big(\bar\lambda + \gamma^{l+1}(\tau_N \bar{\boldsymbol{y}} - d)\big), \tag{4.36}$$

where $P : L^2(\Gamma_c) \to L^2(\Gamma_c)$ denotes the pointwise projection onto the convex set $K = \{\xi \in L^2(\Gamma_c) : \xi \ge 0 \text{ a.e.}\}$. Thus, following (2.2) we obtain

$$\Big(\lambda^{l+1} - \bar\lambda, \big(\lambda^l + \gamma^{l+1}(\tau_N \boldsymbol{y}^{l+1} - d)\big) - \lambda^{l+1}\big) - \big(\bar\lambda + \gamma^{l+1}(\tau_N \bar{\boldsymbol{y}} - d) - \bar\lambda\big)\Big)_{\Gamma_c} \ge 0.$$

This implies that

$$\begin{aligned}
&(\delta_\lambda^{l+1}, \tau_N \delta_y^{l+1})_{\Gamma_c} \\
&= (\gamma^{l+1})^{-1}\Big(\lambda^{l+1} - \bar\lambda, \big(\lambda^l + \gamma^{l+1}(\tau_N \boldsymbol{y}^{l+1} - d)\big) - \big(\bar\lambda + \gamma^{l+1}(\tau_N \bar{\boldsymbol{y}} - d)\big)\Big)_{\Gamma_c} \\
&\quad - (\gamma^{l+1})^{-1}(\lambda^{l+1} - \bar\lambda, \lambda^l - \bar\lambda)_{\Gamma_c} \\
&\ge (\gamma^{l+1})^{-1}\|\lambda^{l+1} - \bar\lambda\|_{\Gamma_c}^2 - (\gamma^{l+1})^{-1}(\lambda^{l+1} - \bar\lambda, \lambda^l - \bar\lambda)_{\Gamma_c}
\end{aligned}$$

$$\ge \frac{1}{2\gamma^{l+1}}\|\delta_\lambda^{l+1}\|_{\Gamma_c}^2 - \frac{1}{2\gamma^{l+1}}\|\delta_\lambda^l\|_{\Gamma_c}^2. \tag{4.37}$$

Let us now turn to the estimation of $\tilde{a}(\delta_y^{l+1} - \delta_y^l, \delta_y^{l+1}) + a(\delta_y^l, \delta_y^{l+1})$:

$$\begin{aligned}
&\tilde{a}(\delta_y^{l+1} - \delta_y^l, \delta_y^{l+1}) + a(\delta_y^l, \delta_y^{l+1}) \\
&=\tilde{a}(\frac{1}{2}\delta_y^l - \delta_y^{l+1}, \frac{1}{2}\delta_y^l - \delta_y^{l+1}) - \frac{1}{4}\tilde{a}(\delta_y^l, \delta_y^l) + a(\delta_y^l, \delta_y^{l+1}) \\
&\geq a(\frac{1}{2}\delta_y^l - \delta_y^{l+1}, \frac{1}{2}\delta_y^l - \delta_y^{l+1}) - \frac{\omega}{4}a(\delta_y^l, \delta_y^l) + a(\delta_y^l, \delta_y^{l+1}) \\
&=\frac{1}{4}a(\delta_y^l, \delta_y^l) + a(\delta_y^{l+1}, \delta_y^{l+1}) - \frac{\omega}{4}a(\delta_y^l, \delta_y^l) \\
&=\frac{1}{4}(1-\omega)a(\delta_y^l, \delta_y^l) + a(\delta_y^{l+1}, \delta_y^{l+1}).
\end{aligned}$$

Utilizing the above estimate, (4.35) and (4.37) yield that

$$\frac{1}{2\gamma^{l+1}}\|\delta_\lambda^{l+1}\|_{\Gamma_c}^2 \leq \frac{1}{2\gamma^{l+1}}\|\delta_\lambda^l\|_{\Gamma_c}^2 + \frac{1}{4}(\omega - 1)a(\delta_y^l, \delta_y^l) - a(\delta_y^{l+1}, \delta_y^{l+1}). \tag{4.38}$$

Introducing the auxiliary variable

$$\kappa^l := \frac{1}{2\gamma^l}\|\delta_\lambda^l\|_{\Gamma_c}^2 + a(\delta_y^l, \delta_y^l),$$

the estimate (4.38) implies

$$\begin{aligned}
\kappa^{l+1} &\leq \frac{1}{2\gamma^{l+1}}\|\delta_\lambda^l\|_{\Gamma_c}^2 + \frac{1}{4}(\omega - 1)a(\delta_y^l, \delta_y^l) \\
&\leq \kappa^l - a(\delta_y^l, \delta_y^l) + \frac{1}{4}(\omega - 1)a(\delta_y^l, \delta_y^l) \\
&= \kappa^l + \frac{1}{4}(\omega - 5)a(\delta_y^l, \delta_y^l),
\end{aligned}$$

where $\gamma^l \leq \gamma^{l+1}$ was used. Since $1 \leq \omega < 5$, the sequence κ^l is monotonically decreasing, it is obviously bounded from below and thus convergent. Hence,

$$\lim_{l\to\infty} a(\delta_y^l, \delta_y^l) = 0, \tag{4.39}$$

resulting in $\boldsymbol{y}^l \to \bar{\boldsymbol{y}}$ strongly in $\mathbf{Y}$ and from Step 2 of (C-IALM) $\lambda^l \to \bar{\lambda}$ weakly in $L^2(\Gamma_c)$. □

The above proof shows that in the augmented Lagrangian method the bilinear form $a(\cdot\,,\cdot)$ can be replaced by the approximation $\tilde{a}(\cdot\,,\cdot)$, provided the residuum is added to the right hand side of the linear equation. This may be advantageous, if the matrix A (belonging to $a(\cdot\,,\cdot)$) is badly conditioned. If one applies (C-SS) for the solution of the system in Step 2 of (C-IALM) one has to solve a linear system involving A several times. Replacing A by the preconditioner $\tilde{A}$ may lead to an auxiliary problem that is easier to solve.

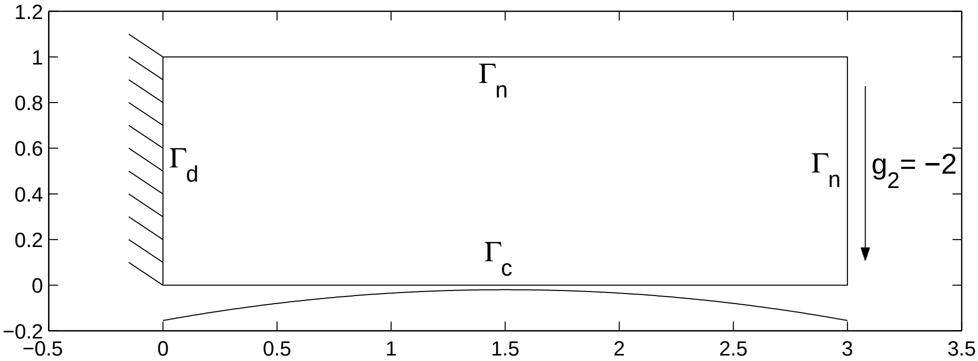


Figure 2. *Geometry for Example 1.*

5. Numerical Results

In this section we present results of our numerical testing for contact problems in plane elasticity utilizing the algorithms described above. In a first subsection we state the examples and discuss the implementation, before we report on the results of our tests of the semi-smooth Newton as well as the first-order augmented Lagrangian methods.

5.1. Presentation of examples and implementation.

Example 1. The geometry for this example is shown in Figure 2, where for reasons of graphical presentation the gap function d was multiplied by a factor of 20. The geometry is given as follows: $\Omega = [0,3] \times [0,1]$, $\Gamma_d = \{0\} \times [0,1]$ and $\Gamma_c = [0,3] \times \{0\}$. Furthermore, $\boldsymbol{f} = 0$ and

$$\boldsymbol{g} = \begin{cases} \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \text{on } [0,3] \times \{1\}, \\ \begin{pmatrix} 0 \\ -2 \end{pmatrix} & \text{on } \{3\} \times [0,1]. \end{cases}$$

We choose $E = 5000$ and report on tests for $\nu = 0.4, 0.49, 0.499, 0.4999$. The distance to the obstacle is given by $d(x_1) = 0.003(x_1 - 1.5)^2 + 0.001$. We discuss results on meshes of 30×10 up to 240×80 elements. Considering the geometry of the problem, it is clear that condition (A) discussed in Section 4.1 is fulfilled, i.e., the application of the (exact and inexact) augmented Lagrangian method is justified even from the infinite-dimensional perspective. Note that elasticity problems exhibit the phenomenon of "locking" with respect to the parameter ν, that is, for conforming finite elements the accuracy of the finite element solution deteriorates as $\nu \to 0.5$ ($\nu = 0.5$ characterizes incompressible materials), see [7,8]. Nevertheless for every fixed $\nu < 0.5$ the finite element schemes are converging, but the convergence may be very slow as the degrees of freedom increase.

5.1.1. *Example 2.* The geometry and the initial mesh for this example are shown in Figure 3: The curved part of the elastic body is given by a Bézier curve

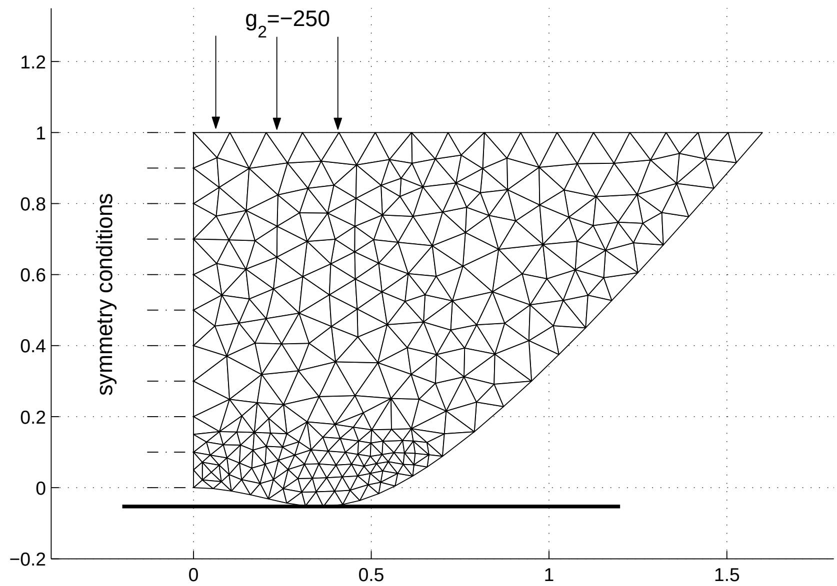


FIGURE 3. *Geometry and undeformed mesh for Example 2.*

with control points

$$\begin{pmatrix} 0.0 \\ 0.0 \end{pmatrix}, \begin{pmatrix} 0.4 \\ 0.0 \end{pmatrix}, \begin{pmatrix} 0.4 \\ -0.4 \end{pmatrix}, \begin{pmatrix} 1.6 \\ 1.0 \end{pmatrix}.$$

This part of the boundary is also the region of possible contact Γ_c, and the obstacle is given by the straight line $x_2 = -0.053$. We use a Young's modulus of $E = 5000$ and the Poisson's ration $\nu = 0.4$. Furthermore, $\boldsymbol{f} = 0$ and an external force of $g_2 = -250$ is applied on $[0,5] \times \{1\}$, whereas $\boldsymbol{g} = 0$ on the rest of the Neumann boundary $\Gamma_n := [0, 1.6] \times \{1\}$. On the left boundary we assume symmetry conditions, i.e., $\tau_N \boldsymbol{y} = 0$ and $\boldsymbol{\sigma}_T \boldsymbol{y} = 0$. Note that in this example $\Gamma_0 = \emptyset$, nevertheless a unique solution for the contact problems exists. This can be seen intuitively considering the geometry, but can also be argued rigorously, see [41, 71, 73]. Again, we can expect condition (A) to hold true, thus the application of the augmented Lagrangian method is justified from the infinite-dimensional point of view. For the construction of the initial mesh (shown in Figure 3) the mesh generator of FEMLAB [32] was used. This mesh is refined using an averaging a posteriori error estimator that has been shown to be reliable for elasticity problems (without contact) [20–22]. The undeformed meshes after 2,4,6 and 8 refinement steps are plotted in Figure 4.

Software and setting of the parameters. For the discretization of the elasticity problems we use P_1 and Q_1 finite elements. We modified the MATLAB-code for elasticity problems published in [2] in such a way that it applies for contact problems as well: For the application of the semi-smooth Newton method a MATLAB routine as an outer loop was written that requires in every iteration

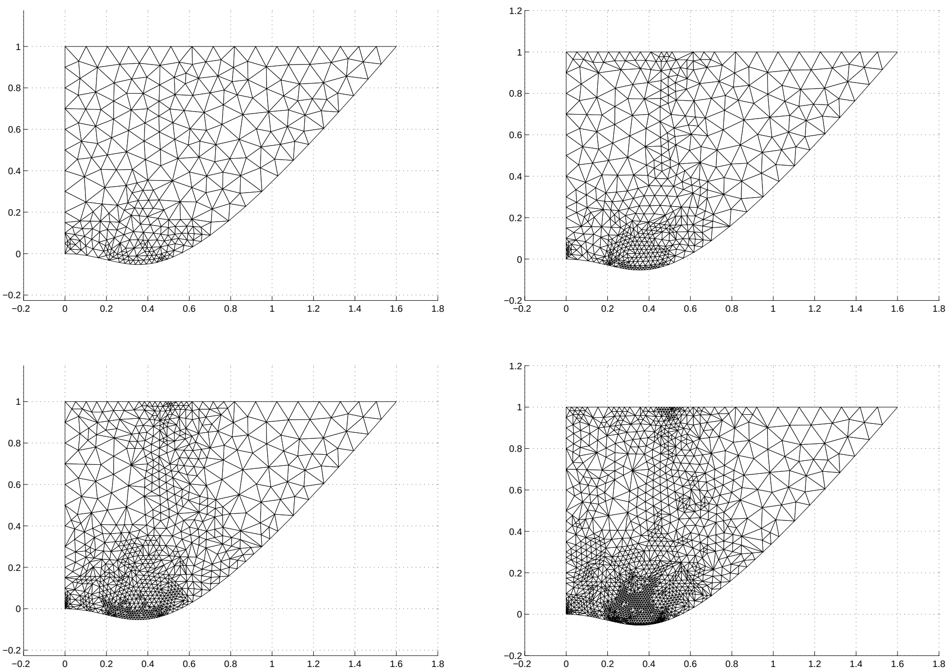


FIGURE 4. *Example 2: Undeformed mesh after 2, 4 (upper row) and 6, 8 (lower row) refinement steps,* $\gamma = 10^{10}$, $\nu = 0.4$.

the solution of a problem with given Dirichlet and Neumann conditions. The code in [2] implements Dirichlet conditions as constraints, which makes the adaptation for contact problems easier. For the refinement of the mesh all elements with an error of at least 50% of the maximum error are marked. Then a red-refinement routine implemented in MATLAB by Stefan A. Funken (written for the summer school "Effiziente Algorithmen und adaptive FEM" in Benediktbeuern, Germany, 2001) is used. This refinement strategy leads to regular FE-meshes, i.e., it does not allow hanging nodes and only adds triangles whose angles are bounded from below. Unless otherwise specified all linear systems are solved exactly using MATLAB's backslash that makes use of the properties of sparse, symmetric matrices.

The semi-smooth Newton method is always initialized with the solution of the unconstrained problem (i.e., the solution of (4.18) with $\lambda = 0$) and, unless otherwise specified we use $\hat{\lambda} = 0$ for (C-SS). The augmented Lagrangian method is always initialized with $\lambda^0 = 0$.

5.2. Results for Example 1. First we apply the semi-smooth Newton method to the regularized problem $(\mathcal{P}_\gamma)$. The algorithm always converges after a few iterations and a monotone behavior can be observed in the sense that the size of the active set decreases in every iteration. The deformed mesh for $\gamma = 10^{10}$ and $\nu = 0.49$ is shown in Figure 5, where the displacement $\boldsymbol{y}_\gamma$ as well

TABLE 1. *Example 1: Number of iterations for different values of ν and γ.*

γ		10^2	10^4	10^6	10^{10}
ν	0.4	4	6	7	7
	0.49	3	7	8	8
	0.499	4	8	12	12
	0.4999	4	9	25	29

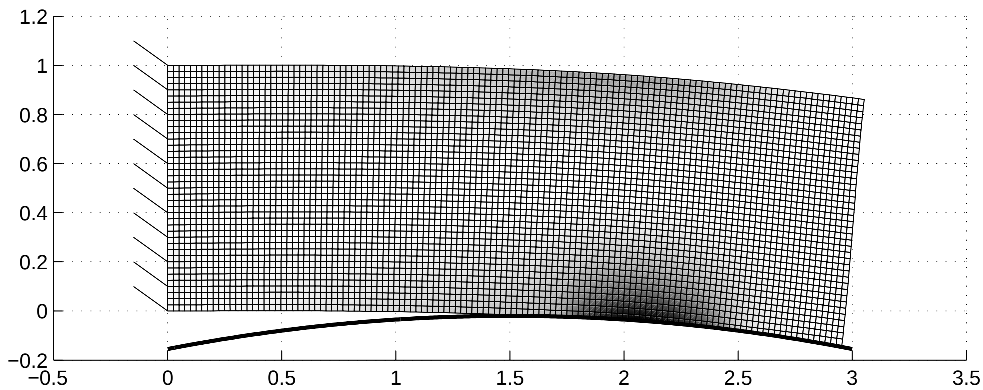


FIGURE 5. *Example 1: Deformed mesh, gray tones visualize the elastic shear energy density, $\nu = 0.49$, $\gamma = 10^{10}$.*

as the gap-function d is magnified by the factor 20. The gray tones show the elastic shear energy density, see [2]. In Table 1 we give the number of iterations for $\nu = 0.4, 0.49, 0.499, 0.4999$ and various values of γ on a mesh of 120×40 elements. Observe that the number of iterations increases as $\nu \to 0.5$.

We now investigate the superlinear convergence of (C-SS). In Figure 6 we plot

$$q^k := \frac{\|\lambda_\gamma - \lambda^k\|_{\Gamma_c}}{\|\lambda_\gamma - \lambda^{k-1}\|_{\Gamma_c}} \tag{4.40}$$

versus the distance of the iterates to the solution

$$d^k = \|\lambda_\gamma - \lambda^k\|_{\Gamma_c}, \quad \text{for } k = 2, 3, \ldots \tag{4.41}$$

for $\nu = 0.499$ and various regularization parameters γ. Firstly, we observe a decrease of q^k close to the solution λ_γ indicating local superlinear convergence of (C-SS). Secondly, the region where q^k decreases is larger for smaller regularization parameter γ. Hence, the regularization parameter γ influences the convergence behavior significantly. Problems with smaller γ require fewer iterations, and a decrease of q^k can be observed in the iteration process. Moderate values for γ seem to increase the region of superlinear convergence of (C-SS). For other tested values of the Poisson ration ν we observe a similar behavior. However, if ν is not

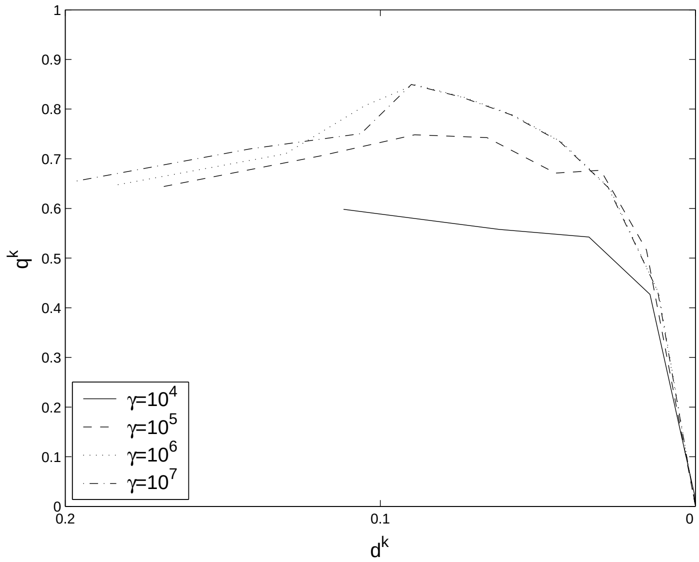


Figure 6. *Example 1: Quotient q^k versus d^k as defined in* (4.40) *and* (4.41) *for various γ on a grid of* 120×40 *elements.*

close to the critical value 0.5 the number of iterations of (C-SS) is more stable with respect to large values of γ and fine grids.

Table 1 and Figure 6 suggest the application of a continuation procedure with respect to γ: Thereby one solves the problem for rather small value of γ, and uses the solution as initialization for the next larger γ. As can be seen from Table 2 this strategy reduces the overall number of iterations significantly, and furthermore, it stabilized the algorithm as $\nu \to 0.5$, i.e., a nearly ν-independent number of iterations is necessary for the solution. Furthermore, the continuation procedure makes the algorithm almost mesh-independent, as can be seen from Table 3, where the number of iterations for various grids is shown for $\nu = 0.4999$ utilizing the prolongation procedure. Table 2 as well as Table 3 show that the overall number of iterations decreases significantly if the prolongation procedure is used.

We next discuss the inexact augmented Lagrangian method for the solution of $(\mathcal{P})$. We therefore solve the linear system in (C-SS) iteratively using the MATLAB function `symmlq` that solves systems with symmetric matrices, with tolerance `tol`$=10^{-3}$ and an incomplete Choleski factor as preconditioner. The (C-SS) iteration is stopped either if the active sets are the same for two consecutive iterations or after 3 (inexact) iterations. Table 4 shows the number of (C-SS)-iterations and the number of active points $\#\mathcal{A}^l$ in every (C-IALM)-iteration for fixed $\gamma = 10^3$ and $\nu = 0.499$. The same test run for increasing γ is shown in Table 5, and it can be seen that increasing γ in every iteration of the augmented Lagrangian method improves the behavior of the method.

TABLE 2. *Example 1: Number of iterations for different values of* ν *(first column), using prolongation w.r.t. to* γ *(2nd to 5th column), the resulting overall number of iteration* ($\sum$)*, and the result without prolongation (last column) on a grid of* 120×40 *elements.*

		γ					γ
		10^2	$\to 10^3$	$\to 10^4$	$\to 10^{10}$	$\sum$	10^{10}
ν	0.4	4	+3	+2	+1	10	7
	0.49	3	+3	+2	+1	9	8
	0.499	4	+3	+3	+1	11	12
	0.4999	4	+3	+3	+1	11	29

TABLE 3. *Example 1: Number of iterations on different grids (given in the first column) using prolongation w.r.t. to* γ *(2nd to 5th column), the resulting overall number of iterations* ($\sum$) *and the result without prolongation strategy (last column),* $\nu = 0.4999$.

	γ					γ
	10^2	$\to 10^3$	$\to 10^4$	$\to 10^{10}$	$\sum$	10^{10}
60×20	3	+3	+2	+1	9	21
120×40	4	+3	+3	+1	11	29
240×80	4	+3	+3	+2	12	31

TABLE 4. *Example 1: Number of semi-smooth Newton iterations* ((C-SS)-iter) *in the* l*-th iteration of the inexact augmented Lagrangian method with* $\gamma = 10^3$ *and number of active points* $\#\mathcal{A}^l$ *on* 120×40*-grid,* $\nu = 0.499$.

l	1	2	3	4	5	6
(C-SS)-iter	3	3	2	2	2	1
γ	10^3	10^3	10^3	10^3	10^3	10^3
$\#\mathcal{A}^l$	45	34	32	31	30	30

TABLE 5. *Same as Table 4 but* γ *is increased in every iteration of* (C-IALM).

l	1	2	3	4
(C-SS)-iter	3	3	1	1
γ	10^3	10^4	10^5	10^6
$\#\mathcal{A}^l$	45	31	30	30

5.3. Results for Example 2. Table 6 shows the number of (C-SS)-iterations $\#iter$ for $\gamma = 10^{10}$ and the number of active points $\#\mathcal{A}$ for increasingly fine grids. The original grid (see Figure 3) corresponds to grid no. 1 and ten grids

TABLE 6. *Example 2: Number of iterations of* (C-SS) *on various grids, number of active points* $\#\mathcal{A}^l$ *at the solution and number of unknowns* $\#var$, $\nu = 0.4$.

grid no.	1	2	3	4	5	6	7	8	9	10	11
$\#iter$	3	3	4	4	4	5	5	5	6	6	6
$\#\mathcal{A}$	4	6	8	8	16	16	17	18	30	30	32
$\#var$	530	548	634	766	1074	1282	1784	2482	3052	3752	4738

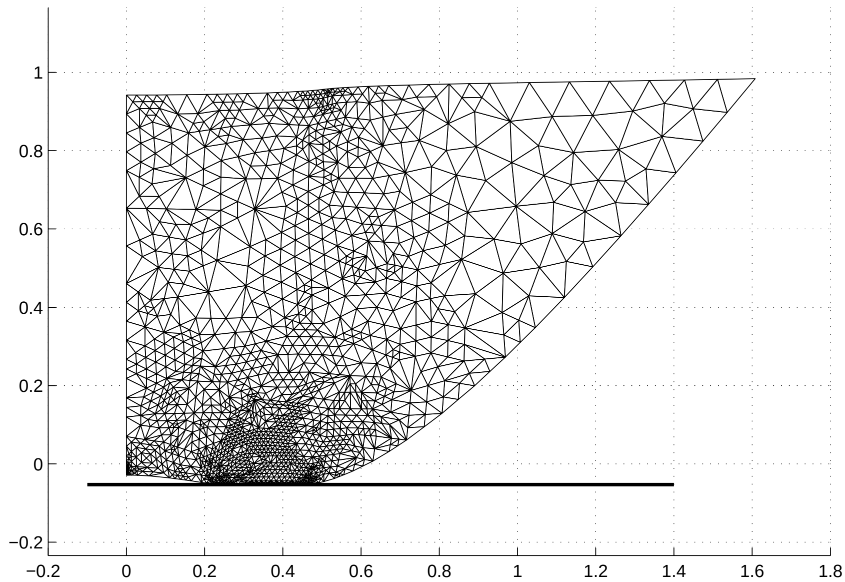


FIGURE 7. *Example 2. Deformed mesh,* $\nu = 0.4$, $\gamma = 10^{10}$.

are constructed by iterative refinement (grid no. $2, \dots, 11$). Table 6 also shows the number $\#var$ of unknowns of each FE-mesh. Note that, for this example (where $\nu = 0.4$) the number of iterations increases only moderately on finer grids. For all examples we observe a monotone behavior of the active sets in the algorithm, i.e., the active set of an iterate is always a subset of the active set from the previous iteration. The deformed mesh (with grid no. 9) for $\gamma = 10^{10}$ and the rigid foundation are shown in Figure 7.

Next we report on local superlinear convergence properties of the iterates of (C-SS) (see Theorem 4.8). For this reason we introduce

$$q_y^k := \frac{a(\boldsymbol{y}^{k+1} - \boldsymbol{y}_\gamma, \boldsymbol{y}^{k+1} - \boldsymbol{y}_\gamma)^{\frac{1}{2}}}{a(\boldsymbol{y}^k - \boldsymbol{y}_\gamma, \boldsymbol{y}^k - \boldsymbol{y}_\gamma)^{\frac{1}{2}}} \quad \text{for } k = 1, 2, \dots,$$

TABLE 7. *Example 2: Values for q_y^k for $\gamma = 10^5, 10^{10}$ on grid no. 9.*

γ	q_y^1	q_y^2	q_y^3	q_y^4	q_y^5
10^5	0.394	0.419	0.361	0.283	0
10^{10}	0.398	0.423	0.284	0.205	0

TABLE 8. *Example 2: Convergence parameters of (C-ALM) on grid no. 9, $\gamma = 10^3$: Number of active points $\#\mathcal{A}^l$ and number of* (C-SS) *iterations.*

l	1	2	3	4	5
$\#\mathcal{A}^l$	37	33	32	30	30
(C-SS)-iter	3	2	2	1	1

where $\boldsymbol{y}_\gamma$ denotes the solution for the problem corresponding to the regularization parameter γ, and $\boldsymbol{y}^k$ the iterates. Since the norm induced by the scalar product $a(\cdot\,,\cdot)$ is equivalent to the usual norm on $\mathbf{Y}$, superlinear convergence of the iterates holds if the sequence q_y^k decreases for $k \geq k_0 \geq 1$. Table 7 shows the values of q_y^k for $\gamma = 10^5, 10^{10}$. We observe a decrease of q_y^k beginning from the second iteration.

Next we turn to tests for the augmented Lagrangian method. For this reason we utilize the grid obtained after 8 refinement steps. Table 8 summarizes interesting data from the application of the first-order augmented Lagrangian method, where $\gamma = 10^3$ is fixed and in every (C-ALM)-iteration the semi-smooth Newton method is initialized with the solution variables of the previous iteration.

CHAPTER 5

Frictional Contact Problems in Linear Elasticity

Problems involving friction *and* contact are among the most difficult in mechanics but certainly are of crucial importance in many different areas such as machine dynamics, metal forming and implants in biomechanics. The main difficulty of these problems lies in the contact and friction conditions, which are inherently nonlinear thus making both theoretical analysis as well as efficient numerical realization truly challenging.

In pure contact problems, also known as Signorini problems, the contact between an elastic body and a rigid foundation is incorporated in the model, but friction in the contact zone is neglected. The main difficulty of contact problems is that the deformation has to satisfy the nonpenetration condition, i.e. one has to figure out the actual contact zone between elastic body and rigid foundation that is a priori unknown.

However, at the contact boundary between the rigid foundation and the elastic body, frictional forces are often too large to be neglected. Thus, besides the nonpenetration condition, one also has to take into account the frictional behavior in the contact zone.

The predominant friction laws used in literature are the Tresca and Coulomb law. Both add, besides the contact condition, further nonlinearity to the problem. While the contact problem with Tresca friction leads to a classical variational inequality, the Coulomb friction problem results in a quasivariational inequality. This makes proving theoretical results for Coulomb's friction difficult, e.g., the proof for existence of a solution to the quasistatic contact problem with Coulomb friction is lengthy and only holds true under certain conditions. The question whether or not a solution for the contact problem with Coulomb's friction law exists has been raised in [39]. First answers are given in [53, 72, 90], where the existence of a solution is shown, provided the friction coefficient is sufficiently small. The estimates for the smallness of this coefficient are weakened in [40] utilizing a different technique of proof. For an overview on existence results for contact problems with Coulomb friction see the review papers [4, 5, 94]. Instead of the Coulomb law, frequently the Tresca friction law is used, since this law is simpler to analyze (see, e.g., [4, 39, 52, 53, 72, 90]). Moreover, a commonly chosen approach towards the solution of Coulomb frictional contact problems is to define the solution as a fixed point of a sequence of solution to the Tresca problem.

Concerning the numerical realization of Coulomb frictional contact problems, two main approaches can be found in the literature. First, one may treat the discretized system directly as in [29,30,84]. A drawback of this approach is that it can only be used in finite dimensions and that convergence results are difficult to obtain. Nevertheless, the authors of the above mentioned papers report on good numerical results. The second and more commonly used approach is to utilize a sequence of Tresca friction problems and a fixed point idea (see, e.g., [38,54,55,80,85]). Thus, the crucial requirement to obtain an efficient numerical algorithm for Coulomb frictional contact problems lies in a fast and reliable algorithm for the solution of the contact problem with Tresca friction. Besides the fact that frequently the Tresca friction model is directly used in applications this highly motivates the development of fast solvers for contact problems with Tresca friction. To summarize papers concerned with this question we start with the contributions [85, 93]. They utilize a successive overrelaxation method and the Uzawa algorithm to solve the Tresca friction problem. The authors of the more recent articles [38, 54] use a dual formulation of the problem and quadratic programming methods with proportioning and projections (see [37]) for the solution of discrete 2D Tresca frictional contact problems. This approach is generalized to 3D in [55]. A different idea is followed in [80], where monotone multigrid methods (see [76–78]) are applied to construct an efficient and globally convergent solver for discrete Tresca frictional contact problems. However, the implementation of this method is rather complicated and no convergence rate results are available. A good survey on the development of numerical methods for contact problems with Coulomb friction is [94].

This chapter is devoted to the development and analysis of algorithms for frictional contact problems in *infinite-dimensional* function spaces. A recent generalized differentiability concept in a Hilbert space framework is applied to derive second order methods for elasticity problems subject to unilateral contact with friction. The approach taken in this chapter is to a large extent based on the Fenchel duality theorem (see [42]) that allows to transform a non-differentiable minimization problem into an inequality constrained minimization of a smooth functional. This approach is applied to the contact problem with Tresca friction, which can be formulated as constraint non-differentiable minimization. Aside from using just the first-order necessary conditions of this problem, which are usually the starting points of the analysis, we additionally use for our investigation alternately the primal and dual formulations of the problem. Another important aspect of this work is the use of certain nonlinear complementarity (NC) functions that allow one to write complementarity conditions as nonsmooth operator equations in function spaces. An application of the semi-smooth Newton methods as developed in [58, 69, 104] to the (smoothed) set of necessary optimality conditions leads to new algorithms for the solution of both 2D and 3D contact problems with Tresca friction. In the 2D–case a specific application of this method turns out to be related to the primal-dual active set strategy that is recently successfully used, e.g., for optimal control problems [14, 15, 58]. In numerical practice the resulting algorithms yield a remarkable efficiency for contact problems with Tresca friction that also carries over to contact problems with Coulomb friction.

Let us briefly outline the structure of this chapter. In the first section the contact problems with Coulomb and Tresca friction are presented. The Fenchel dual for the Tresca friction problem and the corresponding extremality conditions are derived. Several formulations of these conditions utilizing NC-functions are discussed. In Section 2, we investigate a regularization procedure for the dual problem, derive the corresponding primal problem and the extremality conditions, and prove convergence of the solutions as the regularization parameters tend to infinity. In the subsequent section generalized Newton methods in Hilbert spaces for the solution of the regularized contact problem with Tresca friction are presented. The regularization allows us to prove local superlinear convergence of the iterates in infinite dimensions. Besides generalized Newton methods for the regularized problem, we also discuss first-order augmented Lagrangian methods for the original problem. In Section 4, we focus on a regularized contact problem with Coulomb friction, prove that this problem always has a solution and propose algorithms for its numerical realization. The last two sections summarize our numerical testing of the algorithms for Tresca as well as Coulomb friction contact problems.

1. Contact Problems with Coulomb and Tresca Friction

In this section we state the problem of determining the deformation of a linear-elastic body subject to contact and friction conditions. We start with giving a strong formulation of the contact problem with Coulomb friction and discuss difficulties inherent in the problem. Then, we restrict ourselves to the problem with given friction, also known as Tresca friction problem. We show existence and uniqueness of the solution, derive the Fenchel-dual problem and relate primal and dual variables by means of the extremality conditions. Moreover, we investigate the influence of the regularity of the given friction on the regularity of the dual solution variables. The investigation of the problem with given friction leads to a mathematically precise weak formulation of the problem with Coulomb friction. A brief summary of existence results for the solution of the Coulomb contact problem concludes this section.

1.1. The contact problem with Coulomb friction. The main assumptions for this section are as for Chapter 4, but for the reader's convenience we briefly repeat them here.

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be an open bounded domain with $\mathcal{C}^{1,1}$-boundary $\Gamma := \partial\Omega$. Let this boundary be divided into three disjoint parts, namely the Dirichlet part Γ_d, furthermore the part Γ_n, where Neumann data are given and the part Γ_c, where contact and friction with a rigid foundation may occur. We assume that

$\overline{\Gamma}_c \cap \overline{\Gamma}_d = \emptyset$. The vector space of admissible deformations is

$$\mathbf{Y} := \{\boldsymbol{v} \in \mathbf{H}^1(\Omega) : \boldsymbol{\tau v} = 0 \text{ a.e. on } \Gamma_d\},$$

and we assume that $\boldsymbol{f} \in \mathbf{L}^2(\Omega)$, $\boldsymbol{t} \in \mathbf{L}^2(\Gamma_n)$, $\boldsymbol{d} \in \mathbf{Y}$ with $d := \tau_N \boldsymbol{d} \geq 0$ on Γ_c. To realize the contact conditions we define the cone of functions in $\mathbf{Y}$ with non-positive trace on Γ_c

$$\mathbf{K} := \{\boldsymbol{v} \in \mathbf{Y} : \tau_N \boldsymbol{v} \leq 0 \text{ a.e. in } \Gamma_c\}.$$

Furthermore, we define the strain and stress tensors $\underline{\boldsymbol{\varepsilon}}, \underline{\boldsymbol{\sigma}}$ for linear elasticity as done in Chapter 4 and denote the corresponding symmetric bilinear form by $a(\cdot\,,\cdot)$. Let $\mathfrak{F} : \Gamma_c \to \mathbb{R}$, $\mathfrak{F} \geq 0$ be the coefficient of friction, for the discussion of its regularity we refer to Section 1.2. We denote by $\boldsymbol{y}$ the unknown deformation and by $\boldsymbol{\sigma}_T \boldsymbol{y}$ and $\sigma_N \boldsymbol{y}$ the corresponding boundary stress in tangential and normal direction, respectively. Similarly, we split the deformation along the boundary into its tangential part $\boldsymbol{\tau}_T \boldsymbol{y}$ and normal part $\tau_N \boldsymbol{y}$. Then, the Coulomb friction law is given by

$$\|\boldsymbol{\sigma}_T \boldsymbol{y}\| < \mathfrak{F}|\sigma_N \boldsymbol{y}| \qquad \text{on } \{x \in \Gamma_c : \boldsymbol{\tau}_T \boldsymbol{y} = 0\}, \tag{5.1}$$

$$\boldsymbol{\sigma}_T \boldsymbol{y} = -\mathfrak{F}|\sigma_N \boldsymbol{y}| \frac{\boldsymbol{\tau}_T \boldsymbol{y}}{\|\boldsymbol{\tau}_T \boldsymbol{y}\|} \quad \text{on } \{x \in \Gamma_c : \boldsymbol{\tau}_T \boldsymbol{y} \neq 0\}, \tag{5.2}$$

where above and in the sequel $\|\cdot\|$ stands for the Euclidean norm in $\mathbb{R}^n$ and $|\cdot|$ for the absolute value function. Coulomb's friction law is a local friction law, since the frictional behavior in a point only depends on the tangential and normal stress developed at this point. In case of (5.1) a point of the elastic body is called *sticky*, and it is called *sliding* if (5.2) holds.

We can now state the strong formulation of the contact problem with Coulomb friction in linear elasticity. In the following we denote the unit outward vector along the boundary Γ_c by $\boldsymbol{\nu}$.

$$-\mu\boldsymbol{\Delta y} - (\lambda + \mu)(\nabla \operatorname{div} \boldsymbol{y}) = \boldsymbol{f} \qquad \text{in } \Omega, \tag{5.3a}$$

$$\boldsymbol{\tau y} = 0 \qquad \text{on } \Gamma_d, \tag{5.3b}$$

$$\underline{\boldsymbol{\sigma}}(\boldsymbol{y})\boldsymbol{\nu} = \boldsymbol{t} \qquad \text{on } \Gamma_n, \tag{5.3c}$$

$$\tau_N \boldsymbol{y} - d \leq 0, \;\; \sigma_N \boldsymbol{y} \leq 0, \;\; (\tau_N \boldsymbol{y} - d)\sigma_N \boldsymbol{y} = 0 \quad \text{on } \Gamma_c, \tag{5.3d}$$

$$\|\boldsymbol{\sigma}_T \boldsymbol{y}\| < \mathfrak{F}|\sigma_N \boldsymbol{y}| \qquad \text{on } \{x \in \Gamma_c : \boldsymbol{\tau}_T \boldsymbol{y} = 0\}, \tag{5.3e}$$

$$\boldsymbol{\sigma}_T \boldsymbol{y} = -\mathfrak{F}|\sigma_N \boldsymbol{y}| \frac{\boldsymbol{\tau}_T \boldsymbol{y}}{\|\boldsymbol{\tau}_T \boldsymbol{y}\|} \qquad \text{on } \{x \in \Gamma_c : \boldsymbol{\tau}_T \boldsymbol{y} \neq 0\}. \tag{5.3f}$$

Above, $\boldsymbol{\Delta}$ denotes the componentwise Laplace operator. Next we turn to a variational formulation of the above problem. We introduce the nonlinear functional $\omega : \mathbf{Y} \times \mathbf{Y} \to \mathbb{R}$ by

$$\omega(\boldsymbol{y}, \boldsymbol{z}) := \int_{\Gamma_c} \mathfrak{F}|\sigma_N \boldsymbol{y}| \|\boldsymbol{\tau}_T \boldsymbol{z}\| \, dx. \tag{5.4}$$

By means of the bilinear form $a(\cdot\,,\cdot)$ and the functional ω we can rewrite (5.3) as variational inequality:

$$\text{(5.5)} \qquad \begin{cases} \text{Find } \boldsymbol{y} \in \boldsymbol{d} + \mathbf{K} \text{ such that} \\ a(\boldsymbol{y}, \boldsymbol{z} - \boldsymbol{y}) + \omega(\boldsymbol{y}, \boldsymbol{z}) - \omega(\boldsymbol{y}, \boldsymbol{y}) \geq L(\boldsymbol{z} - \boldsymbol{y}) \text{ for all } \boldsymbol{z} \in \boldsymbol{d} + \mathbf{K}. \end{cases}$$

In [62, 73] it is shown, that "sufficiently smooth" solutions of (5.5) and of (5.3) coincide. However, there are major mathematical difficulties inherent in the above problems. On the one hand, the functional ω is not well-defined for $\boldsymbol{y} \in \boldsymbol{d} + \mathbf{K} \subset \mathbf{H}^1(\Omega)$, since $\sigma_N \boldsymbol{y}$ is in general only an element in $\in H^{-\frac{1}{2}}(\Gamma_c)$ and thus the expression $|\sigma_N \boldsymbol{y}|$ has no meaning unless $\sigma_N \boldsymbol{y}$ is a pointwise almost everywhere defined function. Furthermore, (5.3) cannot be associated to an optimization problem for which standard a priori estimates would guarantee existence or uniqueness of a solution.

In the next sections we shall discuss various versions of the contact problem with Coulomb and Tresca friction that allow an exact statement in a functional analytic framework. In Section 1.4 we give an exact and well-defined weak formulation of (5.3) that circumvents the above mentioned problems. We now turn to the problem with Tresca friction, i.e., in the friction conditions (5.1), (5.2) the term $|\sigma_N \boldsymbol{y}|$ is replaced by a given function that does not depend on $\boldsymbol{y}$.

1.2. The contact problem with Tresca friction. Due to the above mentioned problems with formulating and analyzing the contact problem with Coulomb friction, often the contact problem with so-called given friction, also known as Tresca friction is considered (see, e.g., [52, 62]). To incorporate friction in contact problems with Tresca's law is widely accepted in practice and it is known to give good results, at least if one has a reasonable friction bound available. In this friction law the bound between slip and stick is given a priori and does not depend on $\sigma_N \boldsymbol{y}$ such as in (5.3). Hence, the problem can be stated as optimization problem, which allows the application of arguments from convex analysis to argue existence of a unique solution. The Tresca problem can also be utilized to obtain a mathematical precise weak formulation for the contact problem with Coulomb friction, see Section 1.4. This formulation relies on a fixed point argument that can also be exploited numerically to calculate solutions for the model with Coulomb friction.

We shall now comment on the regularity of the friction coefficient $\mathfrak{F}$, where we follow [4] and assume that

$$\mathfrak{F} \in L^\infty(\Gamma_c),$$

and, in addition that $\mathfrak{F}$ belongs to the space of factors on $H^{\frac{1}{2}}(\Gamma_c)$. Then, the mapping

$$H^{\frac{1}{2}}(\Gamma_c) \ni \lambda \mapsto \mathfrak{F}\lambda \in H^{\frac{1}{2}}(\Gamma_c)$$

is welldefined and bounded. By duality it follows that, if $\mathfrak{F}$ is a factor on $H^{\frac{1}{2}}(\Gamma_c)$, it is a factor on the dual $H^{-\frac{1}{2}}(\Gamma_c)$ as well. In [48, p. 21] it is shown that, if $\mathfrak{F}$ is uniformly Lipschitz continuous, then it is a factor on $H^{\frac{1}{2}}(\Gamma_c)$. For more general sufficient conditions that $\mathfrak{F}$ is a factor and a rigorous treatment of the theory of factors of Sobolev spaces we refer to [89].

To give the weak formulation of the problem with Tresca friction we define for $g \in H^{-\frac{1}{2}}(\Gamma_c)$, $g \geq 0$ (i.e., $\langle g, h \rangle_{\Gamma_c} \geq 0$ for all $h \in H^{\frac{1}{2}}(\Gamma_c)$ with $h \geq 0$) the linear functional $j : \mathbf{H}^1(\Omega) \to \mathbb{R}$ by

$$j(\boldsymbol{y}) := \int_{\Gamma_c} \mathfrak{F} g \|\boldsymbol{\tau}_T \boldsymbol{y}\| \, dx. \tag{5.6}$$

Since $\boldsymbol{\tau}_T \boldsymbol{y} \in \mathbf{H}^{\frac{1}{2}}(\Gamma_c)$, the functional $j(\cdot)$ is well-defined. The contact problem with given friction can now be written as minimization of a non-smooth functional over the set of admissible deformations, i.e., as

$$\min_{\boldsymbol{y} \in \boldsymbol{d} + \mathbf{K}} J(\boldsymbol{y}) := \frac{1}{2} a(\boldsymbol{y}, \boldsymbol{y}) - L(\boldsymbol{y}) + j(\boldsymbol{y}), \tag{$\mathcal{P}$}$$

or equivalently as elliptic variational inequality [46]:

$$\begin{cases} \text{Find } \boldsymbol{y} \in \boldsymbol{d} + \mathbf{K} \text{ such that} \\ a(\boldsymbol{y}, \boldsymbol{z} - \boldsymbol{y}) + j(\boldsymbol{z}) - j(\boldsymbol{y}) \geq L(\boldsymbol{z} - \boldsymbol{y}) \ \text{ for all } \boldsymbol{z} \in \boldsymbol{d} + \mathbf{K}. \end{cases} \tag{5.7}$$

Due to the Korn inequality (4.3) the functional $J(\cdot)$ is uniformly convex, furthermore it is lower semicontinuous. Since $\mathbf{K}$ is nonempty, closed and convex, problem ($\mathcal{P}$) and equivalently problem (5.7) admit a unique solution $\bar{\boldsymbol{y}} \in \boldsymbol{d} + \mathbf{K}$.

1.3. Dual problem and extremality conditions. In this section we calculate the Fenchel dual problem corresponding to ($\mathcal{P}$). We first consider general $g \in H^{-\frac{1}{2}}(\Gamma_c)$, whereas in a second step we restrict the results to the case of a more regular given friction bound $g \in L^2(\Gamma_c)$. Then we compare the extremality conditions for the two cases and discuss the regularity of the multipliers.

The case $g \in H^{-\frac{1}{2}}(\Gamma_c)$. Let $g \in H^{-\frac{1}{2}}(\Gamma_c)$ be the given friction. To calculate the dual of ($\mathcal{P}$) we recall and extend the definitions from Chapter 4, Section 1.4. Let $\mathcal{F} : \mathbf{Y} \to \mathbb{R}$ be defined by

$$\mathcal{F}(\boldsymbol{y}) := \begin{cases} -L(\boldsymbol{y}) & \text{if } \boldsymbol{y} \in \boldsymbol{d} + \mathbf{K}, \\ \infty & \text{else,} \end{cases}$$

and further

$$\mathbf{V} := \{\underline{\boldsymbol{p}} \in (L^2(\Omega))^{n \times n} : p_{ij} = p_{ji} \text{ for all } 1 \leq i, j \leq n\}.$$

We introduce $\Lambda \in \mathcal{L}(\mathbf{Y}, \mathbf{V} \times \mathbf{H}^{\frac{1}{2}}(\Gamma_c))$ by

$$\Lambda \boldsymbol{y} := (\Lambda_1 \boldsymbol{y}, \Lambda_2 \boldsymbol{y}) = (\boldsymbol{\varepsilon}(\boldsymbol{y}), \boldsymbol{\tau}_T \boldsymbol{y}), \tag{5.8}$$

and $\mathcal{G} : \boldsymbol{V} \times \mathbf{H}^{\frac{1}{2}}(\Gamma_c) \to \mathbb{R}$ by

$$\mathcal{G}(\underline{\boldsymbol{q}}, \boldsymbol{\nu}) := \frac{1}{2} \int_\Omega \underline{\boldsymbol{q}} : \mathbb{C} \underline{\boldsymbol{q}} \, dx + \int_{\Gamma_c} \mathfrak{F} g \|\boldsymbol{\nu}\| \, dx, \tag{5.9}$$

i.e., $(\mathcal{P})$ can be expressed as

$$\min_{\boldsymbol{y}\in\mathbf{Y}}\Big\{\mathcal{F}(\boldsymbol{y})+\mathcal{G}(\Lambda\boldsymbol{y})\Big\}.$$

Endowing $\mathbf{V}\times\mathbf{H}^{\frac{1}{2}}(\Gamma_c)$ with the usual product norm, it is easy to see that $\mathcal{F}$ and $\mathcal{G}$ are convex, proper and lower semicontinuous, and that there exists $\boldsymbol{y}_0\in\mathbf{Y}$ such that $\mathcal{F}(\boldsymbol{y}_0)<\infty$, $\mathcal{G}(\Lambda\boldsymbol{y}_0)<\infty$ and that $\mathcal{G}$ is continuous at $\Lambda\boldsymbol{y}_0$. Thus, the conditions of Theorem 2.5 are satisfied implying that no duality gap occurs and the extremality conditions (2.8) hold and characterize the solutions of primal and dual problem. We now calculate the convex conjugate functions $\mathcal{F}^\star,\mathcal{G}^\star$ corresponding to $\mathcal{F},\mathcal{G}$, respectively. For this purpose let $(\underline{\boldsymbol{p}},\boldsymbol{\mu})\in\mathbf{V}^\star\times\mathbf{H}^{-\frac{1}{2}}(\Gamma_c)$, and let $\mathbf{V}$ be identified with its topological dual $\mathbf{V}^\star$. Then,

$$\begin{aligned}\mathcal{F}^\star(-\Lambda^\star(\underline{\boldsymbol{p}},\boldsymbol{\mu}))&=\sup_{\boldsymbol{y}\in\mathbf{Y}}\Big\{\big\langle-\Lambda^\star(\underline{\boldsymbol{p}},\boldsymbol{\mu}),\boldsymbol{y}\big\rangle_{\mathbf{Y}^\star,\mathbf{Y}}-\mathcal{F}(\boldsymbol{y})\Big\}\\&=\sup_{\substack{\boldsymbol{y}\in\mathbf{Y}\\ \tau_N\boldsymbol{y}-d\le 0\text{ on }\Gamma_c}}\Big\{\big\langle-\Lambda_1^\star\underline{\boldsymbol{p}},\boldsymbol{y}\big\rangle_{\mathbf{Y}^\star,\mathbf{Y}}+\big\langle-\Lambda_2^\star\boldsymbol{\mu},\boldsymbol{y}\big\rangle_{\mathbf{Y}^\star,\mathbf{Y}}+(\boldsymbol{f},\boldsymbol{y})_\Omega+(\boldsymbol{t},\boldsymbol{\tau}\boldsymbol{y})_{\Gamma_n}\Big\}.\end{aligned}$$

Proceeding the calculation analogously to Section 1.4 of Chapter 4 shows that $\mathcal{F}^\star(-\Lambda^\star(\underline{\boldsymbol{p}},\boldsymbol{\mu}))$ equals $+\infty$ unless

$$-\operatorname{Div}\underline{\boldsymbol{p}}=\boldsymbol{f}.\tag{5.10}$$

We now suppose that (5.10) holds. Then, with $\Sigma=\Gamma_c\cup\Gamma_n$ we have that

$$\begin{aligned}&\mathcal{F}^\star(-\Lambda^\star(\underline{\boldsymbol{p}},\boldsymbol{\mu}))\\&\quad=\sup_{\substack{\boldsymbol{y}\in\mathbf{Y}\\ \tau_N\boldsymbol{y}-d\le 0\text{ on }\Gamma_c}}\Big\{\big\langle-\boldsymbol{\pi}\underline{\boldsymbol{p}},\boldsymbol{\tau}\boldsymbol{y}\big\rangle_{\mathbf{H}_{00}^{-\frac{1}{2}}(\Sigma),\mathbf{H}_{00}^{\frac{1}{2}}(\Sigma)}-\big\langle\boldsymbol{\mu},\boldsymbol{\tau}_T\boldsymbol{y}\big\rangle_{\Gamma_c}+(\boldsymbol{t},\boldsymbol{\tau}_T\boldsymbol{y})_{\Gamma_n}\Big\}.\end{aligned}$$

This is $+\infty$ except that

$$\boldsymbol{\pi}\underline{\boldsymbol{p}}=\boldsymbol{t}\text{ in }\mathbf{L}^2(\Gamma_n)\quad\text{and}\quad\boldsymbol{\pi}_T\underline{\boldsymbol{p}}+\boldsymbol{\mu}=0\text{ in }\mathbf{H}^{-\frac{1}{2}}(\Gamma_c).\tag{5.11}$$

Thus,

$$\mathcal{F}^\star(-\Lambda^\star(\underline{\boldsymbol{p}},\boldsymbol{\mu}))=\begin{cases}-\big\langle\pi_N\underline{\boldsymbol{p}},d\big\rangle_{\Gamma_c} & \text{if (5.10), (5.11) hold,}\\ & \text{and }\pi_N\underline{\boldsymbol{p}}\le 0\text{ in }H^{-\frac{1}{2}}(\Gamma_c),\\ \infty & \text{else.}\end{cases}$$

We now turn to the calculation of $\mathcal{G}^\star(\underline{\boldsymbol{p}}, \boldsymbol{\mu})$.

$$\begin{aligned}
\mathcal{G}^\star(\underline{\boldsymbol{p}}, \boldsymbol{\mu}) &= \sup_{(\underline{\boldsymbol{q}},\boldsymbol{\nu})\in \mathbf{V}\times\mathbf{H}^{\frac{1}{2}}(\Gamma_c)} \Big\{ (\underline{\boldsymbol{p}}, \underline{\boldsymbol{q}})_\Omega + \big\langle \boldsymbol{\nu}, \boldsymbol{\mu}\big\rangle_{\Gamma_c} - \mathcal{G}(\underline{\boldsymbol{q}}, \boldsymbol{\nu}) \Big\} \\
&= \sup_{(\underline{\boldsymbol{q}},\boldsymbol{\nu})\in \mathbf{V}\times\mathbf{H}^{\frac{1}{2}}(\Gamma_c)} \Big\{ (\underline{\boldsymbol{p}}, \underline{\boldsymbol{q}})_\Omega + \big\langle \boldsymbol{\nu}, \boldsymbol{\mu}\big\rangle_{\Gamma_c} - \frac{1}{2}\int_\Omega \underline{\boldsymbol{q}} : \mathbb{C}\underline{\boldsymbol{q}}\, dx - \big\langle \mathfrak{F}g, \|\boldsymbol{\nu}\|\big\rangle_{\Gamma_c} \Big\} \\
&= \sup_{\underline{\boldsymbol{q}}\in\mathbf{V}} \Big\{ (\underline{\boldsymbol{p}}, \underline{\boldsymbol{q}}) - \frac{1}{2}\int_\Omega \underline{\boldsymbol{q}} : \mathbb{C}\underline{\boldsymbol{q}}\, dx \Big\} + \sup_{\boldsymbol{\nu}\in\mathbf{H}^{\frac{1}{2}}(\Gamma_c)} \Big\{ \big\langle \boldsymbol{\nu}, \boldsymbol{\mu}\big\rangle_{\Gamma_c} - \big\langle \mathfrak{F}g, \|\boldsymbol{\nu}\|\big\rangle_{\Gamma_c} \Big\}. \qquad (5.12)
\end{aligned}$$

It is easy to verify that

$$\sup_{\underline{\boldsymbol{q}}\in\mathbf{V}} \Big\{ (\underline{\boldsymbol{p}}, \underline{\boldsymbol{q}}) - \frac{1}{2}\int_\Omega \underline{\boldsymbol{q}} : \mathbb{C}\underline{\boldsymbol{q}}\, dx \Big\} = \frac{1}{2}\int_\Omega \mathbb{C}^{-1}\underline{\boldsymbol{p}} : \underline{\boldsymbol{p}}\, dx.$$

To calculate the second supremum in (5.12) we distinguish two cases. Clearly, if $\big\langle \mathfrak{F}g, \|\boldsymbol{\nu}\|\big\rangle_{\Gamma_c} - \big\langle \boldsymbol{\nu}, \boldsymbol{\mu}\big\rangle_{\Gamma_c} \geq 0$ for all $\boldsymbol{\nu} \in \mathbf{H}^{\frac{1}{2}}(\Gamma_c)$, then

$$\sup_{\boldsymbol{\nu}\in\mathbf{H}^{\frac{1}{2}}(\Gamma_c)} \Big\{ \big\langle \boldsymbol{\nu}, \boldsymbol{\mu}\big\rangle_{\Gamma_c} - \big\langle \mathfrak{F}g, \|\boldsymbol{\nu}\|\big\rangle_{\Gamma_c} \Big\} = 0.$$

However, if there exists $\boldsymbol{\nu}^* \in \mathbf{H}^{\frac{1}{2}}(\Gamma_c)$ such that $\big\langle \mathfrak{F}g, \|\boldsymbol{\nu}^*\|\big\rangle_{\Gamma_c} - \big\langle \boldsymbol{\nu}^*, \boldsymbol{\mu}\big\rangle_{\Gamma_c} < 0$, then for every $t > 0$ the element $t\boldsymbol{\nu}^* \in \mathbf{H}^{\frac{1}{2}}(\Gamma_c)$ and

$$\big\langle t\boldsymbol{\nu}^*, \boldsymbol{\mu}\big\rangle_{\Gamma_c} - \big\langle \mathfrak{F}g, \|t\boldsymbol{\nu}^*\|\big\rangle_{\Gamma_c} = t\big\langle \boldsymbol{\nu}^*, \boldsymbol{\mu}\big\rangle_{\Gamma_c} - t\big\langle \mathfrak{F}g, \|\boldsymbol{\nu}^*\|\big\rangle_{\Gamma_c},$$

and thus

$$\sup_{\boldsymbol{\nu}\in\mathbf{H}^{\frac{1}{2}}(\Gamma_c)} \Big\{ \big\langle \boldsymbol{\nu}, \boldsymbol{\mu}\big\rangle_{\Gamma_c} - \big\langle \mathfrak{F}g, \|\boldsymbol{\nu}\|\big\rangle_{\Gamma_c} \Big\} = \infty.$$

Summarizing our results we get that

$$\mathcal{G}^\star(\underline{\boldsymbol{p}}, \boldsymbol{\mu}) = \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega \mathbb{C}^{-1}\underline{\boldsymbol{p}} : \underline{\boldsymbol{p}}\, dx & \text{if } \big\langle \mathfrak{F}g, \|\boldsymbol{\nu}\|\big\rangle_{\Gamma_c} - \big\langle \boldsymbol{\nu}, \boldsymbol{\mu}\big\rangle_{\Gamma_c} \geq 0 \text{ for all } \boldsymbol{\nu} \in \mathbf{H}^{\frac{1}{2}}(\Gamma_c), \\ \infty & \text{else.} \end{cases}$$

The next lemma summarizes the above results for the dual of $(\mathcal{P})$ in the general case that $g \in H^{-\frac{1}{2}}(\Gamma_c)$.

LEMMA 5.1. *The dual problem corresponding to* $(\mathcal{P})$ *is given by*

$$(\mathcal{P}^\star) \qquad \sup_{\substack{(\underline{\boldsymbol{p}},\boldsymbol{\mu})\in\mathbf{V}\times\mathbf{H}^{-1/2}(\Gamma_c) \\ \textit{s.t. } (5.10),\ (5.11),\ \pi_N\underline{\boldsymbol{p}}\leq 0 \textit{ in } H^{-1/2}(\Gamma_c), \\ \textit{and } \langle \mathfrak{F}g,\|\boldsymbol{\nu}\|\rangle_{\Gamma_c} - \langle \boldsymbol{\nu},\boldsymbol{\mu}\rangle_{\Gamma_c} \geq 0 \\ \textit{for all } \boldsymbol{\nu}\in\mathbf{H}^{1/2}(\Gamma_c).}} -\frac{1}{2}\int_\Omega \mathbb{C}^{-1}\underline{\boldsymbol{p}} : \underline{\boldsymbol{p}}\, dx + \big\langle \pi_N\underline{\boldsymbol{p}}, d\big\rangle_{\Gamma_c}.$$

Existence of a solution to $(\mathcal{P}^\star)$ follows from standard arguments in duality theory (see Theorem 2.5). Uniqueness holds due to the uniform convexity of the cost functional in $(\mathcal{P}^\star)$.

Next we evaluate the extremality conditions for $(\mathcal{P})$ and the dual $(\mathcal{P}^\star)$. These conditions that characterize the solution variable $\boldsymbol{y}$ of the primal problem and $(\underline{\boldsymbol{p}}, \boldsymbol{\mu})$ of the dual problem also relate the primal and dual solution variables. Let us first focus on the extremality condition $-\Lambda^\star(\underline{\boldsymbol{p}}, \boldsymbol{\mu}) \in \partial\mathcal{F}(\boldsymbol{y})$. From the definition of the subdifferential one gets

$$\begin{aligned}\mathcal{F}(\boldsymbol{y}) - \mathcal{F}(\boldsymbol{z}) &\le \big\langle -\Lambda^\star(\underline{\boldsymbol{p}}, \boldsymbol{\mu}), \boldsymbol{y} - \boldsymbol{z}\big\rangle_{\mathbf{Y}^\star,\mathbf{Y}} \\ &= (\underline{\boldsymbol{p}}, -\underline{\boldsymbol{\varepsilon}}(\boldsymbol{y}-\boldsymbol{z}))_\Omega + \big\langle \boldsymbol{\mu}, -\boldsymbol{\tau}_T(\boldsymbol{y}-\boldsymbol{z})\big\rangle_{\Gamma_c}\end{aligned}$$

for all $\boldsymbol{z} \in \mathbf{Y}$. Using the definition of $\mathcal{F}$, this is equivalent to $\boldsymbol{y} \in \boldsymbol{d} + \mathbf{K}$ and

$$(\underline{\boldsymbol{p}}, \underline{\boldsymbol{\varepsilon}}(\boldsymbol{z}-\boldsymbol{y})) - L(\boldsymbol{z}-\boldsymbol{y}) + \big\langle \boldsymbol{\mu}, \boldsymbol{\tau}_T(\boldsymbol{z}-\boldsymbol{y})\big\rangle_{\Gamma_c} \ge 0 \text{ for all } \boldsymbol{z} \in \boldsymbol{d} + \mathbf{K}. \tag{5.13}$$

Next we turn to the condition $(\underline{\boldsymbol{p}}, \boldsymbol{\mu}) \in \partial\mathcal{G}(\Lambda\boldsymbol{y})$. By the definition of $\mathcal{G}$ we get

$$\begin{aligned}\mathcal{G}(\Lambda\boldsymbol{y}) - \mathcal{G}(\Lambda\boldsymbol{z}) &= \frac{1}{2}a(\boldsymbol{y},\boldsymbol{y}) - \frac{1}{2}a(\boldsymbol{z},\boldsymbol{z}) + \big\langle \mathfrak{F}g, \|\boldsymbol{\tau}_T\boldsymbol{y}\| - \|\boldsymbol{\tau}_T\boldsymbol{z}\|\big\rangle_{\Gamma_c} \\ &\le \int_\Omega \underline{\boldsymbol{p}} : \underline{\boldsymbol{\varepsilon}}(\boldsymbol{y}-\boldsymbol{z})\,dx + \big\langle \boldsymbol{\mu}, \boldsymbol{\tau}_T(\boldsymbol{y}-\boldsymbol{z})\big\rangle_{\Gamma_c}\end{aligned} \tag{5.14}$$

for all $\boldsymbol{z} \in \mathbf{Y}$. Restricting ourselves to $\boldsymbol{z} \in \mathbf{Y}$ with $\boldsymbol{\tau}_T\boldsymbol{z} = \boldsymbol{\tau}_T\boldsymbol{y}$ on Γ_c, (5.14) shows that

$$\frac{1}{2}\int_\Omega (\mathbb{C}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{y}) - 2\underline{\boldsymbol{p}}) : \underline{\boldsymbol{\varepsilon}}(y)\,dx \le \frac{1}{2}\int_\Omega (\mathbb{C}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{z}) - 2\underline{\boldsymbol{p}}) : \underline{\boldsymbol{\varepsilon}}(z)\,dx.$$

Thus, $\boldsymbol{y}$ minimizes the above functional over all $\{\boldsymbol{z} \in \mathbf{Y} : \boldsymbol{\tau}_T\boldsymbol{z} = \boldsymbol{\tau}_T\boldsymbol{y}\}$. The first-order necessary condition for this convex minimization problem implies that

$$\underline{\boldsymbol{p}} = \mathbb{C}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{y}), \tag{5.15}$$

i.e., the dual variable $\underline{\boldsymbol{p}}$ at the solution coincides with the stress tensor $\underline{\boldsymbol{\sigma}}(\boldsymbol{y})$ corresponding to the primal solution variable. Plugging (5.15) into (5.14) shows that

$$\frac{1}{2}a(\boldsymbol{y}-\boldsymbol{z}, \boldsymbol{y}-\boldsymbol{z}) - \langle \mathfrak{F}g, \|\boldsymbol{\tau}_T\boldsymbol{y}\| - \|\boldsymbol{\tau}_T\boldsymbol{z}\|\rangle_{\Gamma_c} + \big\langle \boldsymbol{\mu}, \boldsymbol{\tau}_T(\boldsymbol{y}-\boldsymbol{z})\big\rangle_{\Gamma_c} \ge 0 \tag{5.16}$$

for all $\boldsymbol{z} \in \mathbf{Y}$. We next aim to derive an interpretation for (5.16). Therefore, let us take $\boldsymbol{z}^* \in \mathbf{Y}$ arbitrarily. Setting $\boldsymbol{z} := \boldsymbol{y} + t(\boldsymbol{z}^* - \boldsymbol{y}) \in \mathbf{Y}$ for $t \in (0,1)$ in (5.16) results in

$$\begin{aligned}\frac{t^2}{2}a(\boldsymbol{y}-\boldsymbol{z}^*, \boldsymbol{y}-\boldsymbol{z}^*) - \big\langle \mathfrak{F}g, \|\boldsymbol{\tau}_T\boldsymbol{y}\| - \|\boldsymbol{\tau}_T(\boldsymbol{y} + t(\boldsymbol{z}-\boldsymbol{y}))\|\big\rangle_{\Gamma_c}& \\ + t\big\langle \boldsymbol{\mu}, \boldsymbol{\tau}_T(\boldsymbol{y}-\boldsymbol{z}^*)\big\rangle_{\Gamma_c} \ge 0.&\end{aligned} \tag{5.17}$$

Since $\mathfrak{F}g$ is a nonnegative functional we can utilize the convexity of the Euclidean norm $\|\cdot\|$ and $\boldsymbol{y}+t(\boldsymbol{z}^*-\boldsymbol{y})=(1-t)\boldsymbol{y}+t\boldsymbol{z}^*$ to obtain

$$\begin{aligned}&\big\langle \mathfrak{F}g, \|\boldsymbol{\tau}_T(\boldsymbol{y}+t(\boldsymbol{z}^*-\boldsymbol{y}))\|-\|\boldsymbol{\tau}_T\boldsymbol{y}\|\big\rangle_{\Gamma_c}\\ &\leq \big\langle \mathfrak{F}g, (1-t)\|\boldsymbol{\tau}_T\boldsymbol{y}\|+t\|\boldsymbol{\tau}_T\boldsymbol{z}^*\|-\|\boldsymbol{\tau}_T\boldsymbol{y}\|\big\rangle_{\Gamma_c}\\ &= t\big\langle \mathfrak{F}g, \|\boldsymbol{\tau}_T\boldsymbol{z}^*\|-\|\boldsymbol{\tau}_T\boldsymbol{y}\|\big\rangle_{\Gamma_c}.\end{aligned}$$

Hence, (5.17) becomes

$$\frac{t^2}{2}a(\boldsymbol{y}-\boldsymbol{z}^*,\boldsymbol{y}-\boldsymbol{z}^*)+t\big\langle \mathfrak{F}g, \|\boldsymbol{\tau}_T\boldsymbol{z}^*\|-\|\boldsymbol{\tau}_T\boldsymbol{y}\|\big\rangle_{\Gamma_c}+t\big\langle \boldsymbol{\mu}, \boldsymbol{\tau}_T(\boldsymbol{y}-\boldsymbol{z}^*)\big\rangle_{\Gamma_c}\geq 0. \tag{5.18}$$

Dividing (5.18) by t and then letting $t\to\infty$ shows that

$$\big\langle \mathfrak{F}g, \|\boldsymbol{\tau}_T\boldsymbol{y}\|\big\rangle_{\Gamma_c}-\big\langle \boldsymbol{\mu}, \boldsymbol{\tau}_T\boldsymbol{y}\big\rangle_{\Gamma_c}\leq \big\langle \mathfrak{F}g, \|\boldsymbol{\tau}_T\boldsymbol{z}\|\big\rangle_{\Gamma_c}-\big\langle \boldsymbol{\mu}, \boldsymbol{\tau}_T\boldsymbol{z}\big\rangle_{\Gamma_c} \tag{5.19}$$

for all $\boldsymbol{z}\in\mathbf{Y}$, where we return to the notation $\boldsymbol{z}$ instead of $\boldsymbol{z}^*$. Together with the constraint

$$\big\langle \mathfrak{F}g, \|\boldsymbol{\nu}\|\big\rangle_{\Gamma_c}-\big\langle \boldsymbol{\mu}, \boldsymbol{\nu}\big\rangle_{\Gamma_c}\geq 0 \text{ for all } \boldsymbol{\nu}\in\mathbf{H}^{\frac12}(\Gamma_c) \tag{5.20}$$

in $(\mathcal{P}^\star)$, inequality (5.20) implies that

$$\big\langle \mathfrak{F}g, \|\boldsymbol{\tau}_T\boldsymbol{y}\|\big\rangle_{\Gamma_c}-\big\langle \boldsymbol{\mu}, \boldsymbol{\tau}_T\boldsymbol{y}\big\rangle_{\Gamma_c}=0. \tag{5.21}$$

Thus, the unique solution $\bar{\boldsymbol{y}}$ of the primal problem $(\mathcal{P})$ and the unique solution $(\underline{\bar{\boldsymbol{p}}},\bar{\boldsymbol{\mu}})$ of the dual problem $(\mathcal{P}^\star)$ are uniquely characterized by the conditions (5.13), (5.15), (5.20), (5.21). As in the case of the contact problem without friction one can introduce a Lagrange multiplier λ for the inequality (5.13). The extremality conditions obtained with this additional multiplier are summarized in the next lemma.

LEMMA 5.2. *The solution* $\bar{\boldsymbol{y}}\in\boldsymbol{d}+\mathbf{K}$ *of* $(\mathcal{P})$ *and the solution* $(\underline{\bar{\boldsymbol{p}}},\bar{\boldsymbol{\mu}})\in\mathbf{V}\times\mathbf{H}^{-\frac12}(\Gamma_c)$ *of* $(\mathcal{P}^\star)$ *are characterized by* $\underline{\boldsymbol{\sigma}}(\bar{\boldsymbol{y}})=\underline{\boldsymbol{p}}$ *and by the existence of* $\bar\lambda\in H^{-\frac12}(\Gamma_c)$ *such that*

$$a(\bar{\boldsymbol{y}},\boldsymbol{z})-L(\boldsymbol{z})+\big\langle\bar{\boldsymbol{\mu}},\boldsymbol{\tau}_T\boldsymbol{z}\big\rangle_{\Gamma_c}+\big\langle\bar\lambda,\tau_N\boldsymbol{z}\big\rangle_{\Gamma_c}=0 \text{ for all } \boldsymbol{z}\in\mathbf{Y}, \tag{5.22a}$$

$$\big\langle\bar\lambda,\tau_N\boldsymbol{z}\big\rangle_{\Gamma_c}\leq 0 \text{ for all } \boldsymbol{z}\in\mathbf{K}, \tag{5.22b}$$

$$\big\langle\bar\lambda,\tau_N\bar{\boldsymbol{y}}-d\big\rangle_{\Gamma_c}=0, \tag{5.22c}$$

$$\big\langle \mathfrak{F}g, \|\boldsymbol{\nu}\|\big\rangle_{\Gamma_c}-\big\langle \bar{\boldsymbol{\mu}}, \boldsymbol{\nu}\big\rangle_{\Gamma_c}\geq 0 \text{ for all } \boldsymbol{\nu}\in\mathbf{H}^{\frac12}(\Gamma_c), \tag{5.22d}$$

$$\big\langle \mathfrak{F}g, \|\boldsymbol{\tau}_T\bar{\boldsymbol{y}}\|\big\rangle_{\Gamma_c}-\big\langle \bar{\boldsymbol{\mu}}, \boldsymbol{\tau}_T\bar{\boldsymbol{y}}\big\rangle_{\Gamma_c}=0. \tag{5.22e}$$

Recall that according to (5.11) and (5.15), for the multiplier $\bar{\boldsymbol{\mu}}$ corresponding to the non-differentiability of the primal functional $J(\cdot)$ we have the mechanical interpretation

$$\bar{\boldsymbol{\mu}}=-\boldsymbol{\sigma}_T\bar{\boldsymbol{y}}. \tag{5.23}$$

Similarly as in the case for the contact problem without friction one finds a mechanical interpretation for the above introduced multiplier $\bar{\lambda}$ corresponding to the contact condition. Following (4.15), the multiplier $\bar{\lambda}$ is the negative stress in normal direction, i.e.,

$$\bar{\lambda} = -\sigma_N \bar{\boldsymbol{y}}. \tag{5.24}$$

We now investigate the case that the given friction obeys more regularity.

The case $g \in L^2(\Gamma_c)$. We now comment on the dual problem for $(\mathcal{P})$ and the corresponding extremality conditions in case that the given friction coefficient $g \in L^2(\Gamma_c)$. This additional regularity allows us to replace the space $\mathbf{H}^{\frac{1}{2}}(\Gamma_c)$ in the definition of $\mathcal{G}$ and Λ in (5.8), (5.9) by $\mathbf{L}^2(\Gamma_c)$ and to replace the corresponding duality products by L^2-scalar products. We can now identify $\mathbf{L}^2(\Gamma_c)$ with its dual and follow the same arguments as before. This shows that the Lagrange multiplier $\bar{\boldsymbol{\mu}}$ belonging to the non-differentiability of the cost functional admits the higher regularity $\bar{\boldsymbol{\mu}} \in \mathbf{L}^2(\Gamma_c)$, compared to the discussion before, where $\bar{\boldsymbol{\mu}} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_c)$ only. Thus, the constraint on $\bar{\boldsymbol{\mu}} \in \mathbf{L}^2(\Gamma_c)$ in $(\mathcal{P}^\star)$ becomes

$$(\mathfrak{F}g, \|\boldsymbol{\nu}\|) - (\bar{\boldsymbol{\mu}}, \boldsymbol{\nu})_{\Gamma_c} \geq 0 \text{ for all } \boldsymbol{\nu} \in \mathbf{L}^2(\Gamma_c), \tag{5.25}$$

and can be simplified to

$$\|\bar{\boldsymbol{\mu}}\| \leq \mathfrak{F}g \text{ a.e. on } \Gamma_c. \tag{5.26}$$

To verify this equivalence we first observe for arbitrary $\boldsymbol{\nu} \in \mathbf{L}^2(\Gamma_c)$ from the inequality

$$\begin{aligned}(\mathfrak{F}g, \|\boldsymbol{\nu}\|)_{\Gamma_c} - (\bar{\boldsymbol{\mu}}, \boldsymbol{\nu})_{\Gamma_c} &\geq (\mathfrak{F}g, \|\boldsymbol{\nu}\|)_{\Gamma_c} - (\|\bar{\boldsymbol{\mu}}\|, \|\boldsymbol{\nu}\|)_{\Gamma_c} \\ &= (\mathfrak{F}g - \|\bar{\boldsymbol{\mu}}\|, \|\boldsymbol{\nu}\|)_{\Gamma_c}\end{aligned}$$

that (5.26) implies (5.25). Conversely, assume that (5.26) does not hold, i.e.,

$$S := \{x \in \Gamma_c : \mathfrak{F}g - \|\bar{\boldsymbol{\mu}}\| < 0 \text{ a.e.}\}$$

has positive measure. To infer from this assumption a contradiction to (5.25) we choose $\boldsymbol{\nu}^* \in \mathbf{L}^2(\Gamma_c)$ with

$$\boldsymbol{\nu}^*(x) := \begin{cases} \bar{\boldsymbol{\mu}}(x) & \text{on } S, \\ 0 & \text{on } \Gamma_c \setminus S. \end{cases}$$

This leads to

$$\begin{aligned}(\mathfrak{F}g, \|\boldsymbol{\nu}^*\|) - (\bar{\boldsymbol{\mu}}, \boldsymbol{\nu}^*) &= \big(\mathfrak{F}g - \|\bar{\boldsymbol{\mu}}\|, \|\bar{\boldsymbol{\mu}}\|\big)_S \\ &= -\big(\mathfrak{F}g - \|\bar{\boldsymbol{\mu}}\|, \mathfrak{F}g - \|\bar{\boldsymbol{\mu}}\|\big)_S + \big(\min(0, \mathfrak{F}g - \|\bar{\boldsymbol{\mu}}\|), \mathfrak{F}g\big)_S \\ &\leq -\int_S (\mathfrak{F}g - \|\bar{\boldsymbol{\mu}}\|)^2 \, dx < 0,\end{aligned}$$

which is a contradiction to (5.25). Hence, we have shown the equivalence between (5.25) and (5.26). This allows us to write the dual problem for $g \in L^2(\Gamma_c)$ as follows:

$$\sup_{\substack{(\underline{\boldsymbol{p}},\boldsymbol{\mu})\in \mathbf{V}\times\mathbf{L}^2(\Gamma_c)\\ \text{s.t. } (5.10),\,(5.11),\,\pi_N\underline{\boldsymbol{p}}\le 0 \text{ in } H^{-1/2}(\Gamma_c),\\ \text{and } \|\boldsymbol{\mu}\|\le\mathfrak{F}g \text{ a.e. on } \Gamma_c.}} -\frac{1}{2}\int_\Omega \mathbb{C}^{-1}\underline{\boldsymbol{p}} : \underline{\boldsymbol{p}}\,dx + \langle \pi_N\underline{\boldsymbol{p}}, d\rangle_{\Gamma_c}. \tag{5.27}$$

Utilizing the relation between primal and dual variables, in particular (5.15) and (5.24), one can transform (5.27) into

$$\begin{cases} -\min\limits_{\substack{\lambda\ge 0 \text{ in } H^{-1/2}(\Gamma_c)\\ \|\boldsymbol{\mu}\|\le\mathfrak{F}g \text{ a.e. on } \Gamma_c}} \dfrac{1}{2}a(\boldsymbol{y}_{\lambda,\boldsymbol{\mu}},\boldsymbol{y}_{\lambda,\boldsymbol{\mu}}) + \langle\lambda, d\rangle_{\Gamma_c},\\ \text{where } \boldsymbol{y}_{\lambda,\boldsymbol{\mu}} \text{ satisfies}\\ a(\boldsymbol{y}_{\lambda,\boldsymbol{\mu}},\boldsymbol{z}) - L(\boldsymbol{z}) + \langle\lambda,\tau_N\boldsymbol{z}\rangle_{\Gamma_c} + (\boldsymbol{\mu},\boldsymbol{\tau}_T\boldsymbol{z})_{\Gamma_c} = 0 \text{ for all } \boldsymbol{z}\in\mathbf{Y}. \end{cases} \tag{5.28}$$

Note that problem (5.28) is an equivalent form for the dual problem (5.27), now written in the variables λ and $\boldsymbol{\mu}$. The primal variable $\boldsymbol{y}_{\lambda,\boldsymbol{\mu}}$ appears only as auxiliary variable determined from λ and $\boldsymbol{\mu}$.

In the case that $g \in L^2(\Gamma_c)$, the extremality conditions corresponding to $(\mathcal{P})$ and (5.27) can be written more explicitly. As shown above, (5.22d) is equivalent to

$$\|\bar{\boldsymbol{\mu}}\| \le \mathfrak{F}g \text{ a.e. on } \Gamma_c. \tag{5.22d$'$}$$

Let us turn to condition (5.22e). From the inequality

$$0 = (\mathfrak{F}g, \|\boldsymbol{\tau}_T\bar{\boldsymbol{y}}\|)_{\Gamma_c} - (\bar{\boldsymbol{\mu}},\boldsymbol{\tau}_T\bar{\boldsymbol{y}})_{\Gamma_c} \ge (\mathfrak{F}g - \|\bar{\boldsymbol{\mu}}\|, \|\boldsymbol{\tau}_T\bar{\boldsymbol{y}}\|)_{\Gamma_c} \ge 0 \tag{5.29}$$

it follows that

$$\big(\mathfrak{F}g - \|\bar{\boldsymbol{\mu}}\|\big)\|\boldsymbol{\tau}_T\bar{\boldsymbol{y}}\| = 0 \text{ a.e. on } \Gamma_c, \tag{5.30}$$

and thus, for almost all $x \in \Gamma_c$ either $\boldsymbol{\tau}_T\bar{\boldsymbol{y}}(x) = 0$ or $\|\bar{\boldsymbol{\mu}}(x)\| = \mathfrak{F}(x)g(x)$. The latter equality implies together with (5.29) that

$$\|\bar{\boldsymbol{\mu}}(x)\|\|\boldsymbol{\tau}_T\bar{\boldsymbol{y}}(x)\| = \bar{\boldsymbol{\mu}}(x)^\top\boldsymbol{\tau}_T\bar{\boldsymbol{y}}(x),$$

which holds if and only if there exists $\varrho \ge 0$ such that

$$\bar{\boldsymbol{\mu}}(x) = \varrho(x)\boldsymbol{\tau}_T\bar{\boldsymbol{y}}(x).$$

Utilizing $\|\bar{\boldsymbol{\mu}}(x)\| = \mathfrak{F}(x)g(x)$ one gets that $\varrho(x) = \mathfrak{F}(x)g(x)/\|\boldsymbol{\tau}_T\bar{\boldsymbol{y}}(x)\|$. Thus, we have shown that (5.22e) is equivalent to

$$\begin{cases} \boldsymbol{\tau}_T\bar{\boldsymbol{y}} = 0 \text{ or}\\ \boldsymbol{\tau}_T\bar{\boldsymbol{y}} \ne 0 \text{ and } \bar{\boldsymbol{\mu}} = \mathfrak{F}g\frac{\boldsymbol{\tau}_T\bar{\boldsymbol{y}}}{\|\boldsymbol{\tau}_T\bar{\boldsymbol{y}}\|}. \end{cases} \tag{5.22e$'$}$$

Provided $\mathfrak{F}g > 0$, conditions (5.22d′) and (5.22e′) can equivalently be expressed as

$$\mathfrak{F}g(\sigma\bar{\boldsymbol{\mu}} + \boldsymbol{\tau}_T\bar{\boldsymbol{y}}) - \max(\mathfrak{F}g\sigma, \|\sigma\bar{\boldsymbol{\mu}} + \boldsymbol{\tau}_T\bar{\boldsymbol{y}}\|)\bar{\boldsymbol{\mu}} = 0 \text{ for arbitrary } \sigma > 0. \tag{5.31}$$

To prove the above equivalence we start with showing that (5.31) implies (5.22d′) and (5.22e′). From (5.31) it follows that

$$\bar{\boldsymbol{\mu}} = \mathfrak{F}g\frac{\sigma\bar{\boldsymbol{\mu}} + \boldsymbol{\tau}_T\bar{\boldsymbol{y}}}{\max(\mathfrak{F}g\sigma, \|\sigma\bar{\boldsymbol{\mu}} + \boldsymbol{\tau}_T\bar{\boldsymbol{y}}\|)},$$

which immediately implies (5.22d′). To prove (5.22e′) we distinguish the cases

$$\mathfrak{F}g\sigma \geq \|\sigma\bar{\boldsymbol{\mu}} + \boldsymbol{\tau}_T\bar{\boldsymbol{y}}\| \quad \text{and} \quad \mathfrak{F}g\sigma < \|\sigma\bar{\boldsymbol{\mu}} + \boldsymbol{\tau}_T\bar{\boldsymbol{y}}\|. \tag{5.32}$$

In the case that the first inequality in (5.32) holds true, we get from (5.31) that $\mathfrak{F}g(\sigma\bar{\boldsymbol{\mu}} + \boldsymbol{\tau}_T\bar{\boldsymbol{y}}) - \mathfrak{F}g\sigma\bar{\boldsymbol{\mu}} = 0$, and thus $\boldsymbol{\tau}_T\bar{\boldsymbol{y}} = 0$, i.e., we have found the upper case in (5.22e′). If we are in the case of the right inequality in (5.32), it first follows that $\boldsymbol{\tau}_T\bar{\boldsymbol{y}} \neq 0$, since assuming the opposite leads, together with the right inequality in (5.32) to $\mathfrak{F}g < \|\bar{\boldsymbol{\mu}}\|$, which is a contradiction. Furthermore, (5.31) yields

$$\mathfrak{F}g(\sigma\bar{\boldsymbol{\mu}} + \boldsymbol{\tau}_T\bar{\boldsymbol{y}}) = \|\sigma\bar{\boldsymbol{\mu}} + \boldsymbol{\tau}_T\bar{\boldsymbol{y}}\|\bar{\boldsymbol{\mu}}, \tag{5.33}$$

and it follows that

$$\mathfrak{F}g\boldsymbol{\tau}_T\bar{\boldsymbol{y}} = \left(\|\sigma\bar{\boldsymbol{\mu}} + \boldsymbol{\tau}_T\bar{\boldsymbol{y}}\| - \mathfrak{F}g\sigma\right)\bar{\boldsymbol{\mu}} = \varrho\bar{\boldsymbol{\mu}} \tag{5.34}$$

with $\varrho := \|\sigma\bar{\boldsymbol{\mu}}+\boldsymbol{\tau}_T\bar{\boldsymbol{y}}\| - \mathfrak{F}g\sigma > 0$. Considering the norms in (5.33) and (5.34), we find that $\varrho = \|\boldsymbol{\tau}_T\bar{\boldsymbol{y}}\|^{-1}\mathfrak{F}g$ and thus we are in the lower case in (5.22e′). Hence, we have shown that (5.22d′) and (5.22e′) follow from (5.31). Verifying the converse implication can be done easily by distinguishing the two cases in (5.22e′).

1.4. Weak formulation of the contact problem with Coulomb friction. Having the results from the previous section available we can now give a weak formulation of the contact problem with Coulomb friction that circumvents the problems mentioned in Chapter 1.1, namely the lack of regularity of $\boldsymbol{\sigma}_N\boldsymbol{y}$ in (5.4). The formulation given below utilizes the contact problem with given friction $g \in H^{-\frac{1}{2}}(\Gamma_c)$ and a fixed point idea. We define the cone of nonnegative functionals over $H^{\frac{1}{2}}(\Gamma_c)$:

$$H_+^{-\frac{1}{2}}(\Gamma_c) := \{\xi \in H^{-\frac{1}{2}}(\Gamma_c) : \left\langle \xi, \eta \right\rangle_{\Gamma_c} \geq 0 \text{ for all } \eta \in H^{\frac{1}{2}}(\Gamma_c), \eta \geq 0\}.$$

Then we consider the following mapping

$$\Psi : H_+^{-\frac{1}{2}}(\Gamma_c) \longrightarrow H_+^{-\frac{1}{2}}(\Gamma_c)$$

defined by

$$\Psi(g) := \lambda_g,$$

where λ_g is the unique multiplier for the contact condition in (5.22) for the problem with given friction g. Due to the fact that

$$H^{\frac{1}{2}}(\Gamma_c) = \{\tau_N \boldsymbol{z} : \boldsymbol{z} \in \mathbf{Y}\},$$

property (5.22b) implies that $\lambda_g \in H_+^{-\frac{1}{2}}(\Gamma_c)$ for every $g \in H_+^{-\frac{1}{2}}(\Gamma_c)$, which shows that Ψ is welldefined. This allows us, having (5.24) in mind, to call $\boldsymbol{y} \in \mathbf{Y}$ weak solution of the Signorini problem with Coulomb friction if its negative normal boundary stress $-\sigma_N \boldsymbol{y}$ is a fixed point of the mapping Ψ. In general, such a fixed point for the mapping Ψ does not exist, i.e., the Coulomb friction problem admits a solution only under certain conditions. In the next section we briefly summarize existence results for the static contact problem with Coulomb friction.

1.5. Existence results for the contact problem with Coulomb friction. Much effort was necessary to gain conditions that guarantee the existence of a solution to the contact problem with Coulomb friction. Generally, it can be said that the problem admits a (weak) solution if the friction coefficient $\mathfrak{F}$ is sufficiently small, see, e.g., [40, 53, 62, 72, 90], the survey article [5] and the references given therein. To be precise, in [40] it is shown that a weak solution to the Coulomb frictional contact problem exists if

$$\mathfrak{F}(x) < \begin{cases} \dfrac{\sqrt{3-4\nu}}{2-2\nu} & \text{for } n = 2, \\[2ex] \sqrt{\dfrac{3-4\nu}{4-4\nu}} & \text{for } n = 3, \end{cases} \tag{5.35}$$

for all $x \in \Gamma_c$, where ν denotes the Poisson ratio and n the dimension of Ω. Less recent contributions prove the existence of a solution under stronger conditions on the friction coefficient $\mathfrak{F}$, see [72,90]. The latter contributions apply fixed point ideas, whereas in [40] a penalization method combined with a priori estimates is used.

2. The Regularized Contact Problem with Tresca Friction

In this section we introduce and analyze a regularized version of the contact problem with given friction that allows the application of infinite-dimensional semi-smooth Newton methods. We discuss the primal and dual as well as the extremality conditions for the regularized problem. Furthermore, we shall investigate the behavior of the solutions as the regularization parameters tend to infinity.

2.1. Regularized primal and dual problems. In this section we assume that the given friction $g \in L^2(\Gamma_c)$. Motivated from the previous chapters we start our consideration with a regularized version of the dual problem (5.27) written in the form (5.28). For this purpose, we define for $\gamma_1, \gamma_2 > 0$, given $\hat{\lambda} \in L^2(\Gamma_c)$ and $\hat{\boldsymbol{\mu}} \in \mathbf{L}^2(\Gamma_c)$ the functional $J^\star_{\gamma_1,\gamma_2} : L^2(\Gamma_c) \times \mathbf{L}^2(\Gamma_c) \to \mathbb{R}$ by

$$\begin{aligned} J^\star_{\gamma_1\gamma_2}(\lambda, \boldsymbol{\mu}) :=& \frac{1}{2} a(\boldsymbol{y}_{\lambda,\boldsymbol{\mu}}, \boldsymbol{y}_{\lambda,\boldsymbol{\mu}}) + (\lambda, d)_{\Gamma_c} + \frac{1}{2\gamma_1}\|\lambda - \hat{\lambda}\|^2_{\Gamma_c} \\ &+ \frac{1}{2\gamma_2}\|\boldsymbol{\mu} - \hat{\boldsymbol{\mu}}\|^2_{\Gamma_c} - \frac{1}{2\gamma_1}\|\hat{\lambda}\|^2_{\Gamma_c} - \frac{1}{2\gamma_2}\|\hat{\boldsymbol{\mu}}\|^2_{\Gamma_c}, \end{aligned}$$

where $\boldsymbol{y}_{\lambda,\boldsymbol{\mu}} \in \mathbf{Y}$ satisfies

$$a(\boldsymbol{y}_{\lambda,\boldsymbol{\mu}}, \boldsymbol{z}) - L(\boldsymbol{z}) + (\lambda, \tau_N \boldsymbol{z})_{\Gamma_c} + (\boldsymbol{\mu}, \boldsymbol{\tau}_T \boldsymbol{z})_{\Gamma_c} = 0 \text{ for all } \boldsymbol{z} \in \mathbf{Y}. \tag{5.36}$$

Now, the regularized dual problem with given friction can be written in compact form as

$$(\mathcal{P}^\star_{\gamma_1\gamma_2}) \qquad - \min_{\lambda \geq 0,\ \|\boldsymbol{\mu}\| \leq \mathfrak{F}g \text{ a.e. on } \Gamma_c} J^\star_{\gamma_1\gamma_2}(\lambda, \boldsymbol{\mu}).$$

Obviously, the last two terms in the definition of $J^\star_{\gamma_1\gamma_2}$ are constants and can thus be neglected in the optimization problem $(\mathcal{P}^\star_{\gamma_1\gamma_2})$. However, they are introduced with regard to the primal problem corresponding to $(\mathcal{P}^\star_{\gamma_1\gamma_2})$, which we turn to next. We define the functional $J_{\gamma_1\gamma_2} : \mathbf{Y} \to \mathbb{R}$ by

$$\begin{aligned} J_{\gamma_1\gamma_2}(\boldsymbol{y}) :=& \frac{1}{2} a(\boldsymbol{y}, \boldsymbol{y}) - L(\boldsymbol{y}) \\ &+ \frac{1}{2\gamma_1}\|\max(0, \hat{\lambda} + \gamma_1(\tau_N \boldsymbol{y} - d))\|^2_{\Gamma_c} + \frac{1}{\gamma_2}\int_{\Gamma_c} h(\boldsymbol{\tau}_T \boldsymbol{y}, \hat{\boldsymbol{\mu}})\, dx, \end{aligned}$$

where $h(\cdot\,,\cdot) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is defined as follows:

$$h(x, \alpha) := \begin{cases} \mathfrak{F}g\|\gamma_2 x + \alpha\| - \frac{1}{2}\mathfrak{F}^2 g^2 & \text{if } \quad \|\gamma_2 x + \alpha\| > \mathfrak{F}g, \\ \frac{1}{2}\|\gamma_2 x + \alpha\|^2 & \text{if } \quad \|\gamma_2 x + \alpha\| \leq \mathfrak{F}g. \end{cases} \tag{5.37}$$

Then, the primal problem corresponding to $(\mathcal{P}^\star_{\gamma_1\gamma_2})$ is

$$(\mathcal{P}_{\gamma_1\gamma_2}) \qquad \min_{\boldsymbol{y} \in \boldsymbol{d} + \mathbf{K}} J_{\gamma_1\gamma_2}(\boldsymbol{y}).$$

It is easy to see that the problems $(\mathcal{P}_{\gamma_1\gamma_2})$ and $(\mathcal{P}^\star_{\gamma_1\gamma_2})$ admit unique solutions $\boldsymbol{y}_{\gamma_1\gamma_2}$ and $(\lambda_{\gamma_1\gamma_2}, \boldsymbol{\mu}_{\gamma_1\gamma_2})$, respectively. It seems noteworthy that the regularization turns the primal problem into the unconstrained minimization of a continuously differentiable functional, while the corresponding dual problem is still the constrained minimization of a quadratic functional.

To shorten notation we henceforth mark all variables of the regularized problems only by the index 'γ' instead of 'γ_1,γ_2'. Then, provided that $\mathfrak{F}g > 0$ almost everywhere, it can be shown that the extremality conditions relating $(\mathcal{P}_{\gamma_1\gamma_2})$ and

$(\mathcal{P}^\star_{\gamma_1,\gamma_2})$ are

$$a(\boldsymbol{y}_\gamma,\boldsymbol{z})-L(\boldsymbol{z})+(\boldsymbol{\mu}_\gamma,\boldsymbol{\tau}_T\boldsymbol{z})_{\Gamma_c}+(\lambda_\gamma,\tau_N\boldsymbol{z})_{\Gamma_c}=0 \text{ for all } \boldsymbol{z}\in\mathbf{Y}, \tag{5.38a}$$

$$\lambda_\gamma-\max(0,\hat\lambda+\gamma_1(\tau_N\boldsymbol{y}_\gamma-d))=0 \quad \text{on } \Gamma_c, \tag{5.38b}$$

$$\mathfrak{F}g(\gamma_2\boldsymbol{\tau}_T\boldsymbol{y}_\gamma+\hat{\boldsymbol{\mu}})-\max(\mathfrak{F}g,\|\gamma_2\boldsymbol{\tau}_T\boldsymbol{y}_\gamma+\hat{\boldsymbol{\mu}}\|)\boldsymbol{\mu}_\gamma=0 \quad \text{on } \Gamma_c. \tag{5.38c}$$

2.2. Convergence as $\gamma_1,\gamma_2\to\infty$. In this section we investigate the convergence of the primal variable $\boldsymbol{y}_\gamma$ as well as the dual variables $\lambda_\gamma,\boldsymbol{\mu}_\gamma$ as the regularization parameters γ_1,γ_2 tend to infinity. We therefore denote by $\bar{\boldsymbol{y}}$ the solution of $(\mathcal{P})$ and by $(\bar\lambda,\bar{\boldsymbol{\mu}})$ the solution to $(\mathcal{P}^\star)$.

THEOREM 5.3. *For all $\hat\lambda\in L^2(\Gamma_c)$, $\hat{\boldsymbol{\mu}}\in\mathbf{L}^2(\Gamma_c)$ and for a given friction $g\in L^2(\Gamma_c)$, the primal variable $\boldsymbol{y}_\gamma$ converges to $\bar{\boldsymbol{y}}$ strongly in $\mathbf{Y}$ and the dual variables $(\lambda_\gamma,\boldsymbol{\mu}_\gamma)$ converge to $(\bar\lambda,\bar{\boldsymbol{\mu}})$ weakly in $H^{-\frac12}(\Gamma_c)\times\mathbf{L}^2(\Gamma_c)$ as $\gamma_1\to\infty$ and $\gamma_2\to\infty$.*

PROOF. Parts of the proof for the above theorem are similar to the proof for Theorem 4.6. However, for the reader's convenience we repeat these parts in shortened form. Recall that both, $(\boldsymbol{y}_\gamma,\lambda_\gamma,\boldsymbol{\mu}_\gamma)$ and $(\bar{\boldsymbol{y}},\bar\lambda,\bar{\boldsymbol{\mu}})$ satisfy (5.22a). Plugging $\boldsymbol{z}:=\boldsymbol{y}_\gamma-\bar{\boldsymbol{y}}$ in (5.38a) gives

$$a(\boldsymbol{y}_\gamma,\boldsymbol{y}_\gamma-\bar{\boldsymbol{y}})-L(\boldsymbol{y}_\gamma-\bar{\boldsymbol{y}})+(\lambda_\gamma,\tau_N(\boldsymbol{y}_\gamma-\bar{\boldsymbol{y}}))_{\Gamma_c}+(\boldsymbol{\mu}_\gamma,\boldsymbol{\tau}_T(\boldsymbol{y}_\gamma-\bar{\boldsymbol{y}}))_{\Gamma_c}=0. \tag{5.39}$$

First we estimate the term $(\lambda_\gamma,\tau_N(\boldsymbol{y}_\gamma-\bar{\boldsymbol{y}}))_{\Gamma_c}$:

$$\begin{aligned}(\lambda_\gamma,\tau_N(\boldsymbol{y}_\gamma-\bar{\boldsymbol{y}}))_{\Gamma_c}&=(\lambda_\gamma,\tau_N\boldsymbol{y}_\gamma-d)_{\Gamma_c}-(\lambda_\gamma,\tau_N\bar{\boldsymbol{y}}-d)_{\Gamma_c}\\&\ge\gamma_1^{-1}(\lambda_\gamma,\hat\lambda+\gamma_1(\tau_N\boldsymbol{y}_\gamma-d))_{\Gamma_c}-\gamma_1^{-1}(\lambda_\gamma,\hat\lambda)_{\Gamma_c},\end{aligned}$$

where $(\lambda_\gamma,\tau_N\bar{\boldsymbol{y}}-d)_{\Gamma_c}\le0$ was used. Thus,

$$(\lambda_\gamma,\tau_N(\boldsymbol{y}_\gamma-\bar{\boldsymbol{y}}))_{\Gamma_c}\ge\gamma_1^{-1}(\lambda_\gamma,\max(0,\hat\lambda+\gamma_1(\tau_N\boldsymbol{y}_\gamma-d)))_{\Gamma_c}-\gamma_1^{-1}(\lambda_\gamma,\hat\lambda)_{\Gamma_c}$$

$$=\gamma_1^{-1}\|\lambda_\gamma\|^2_{\Gamma_c}-\gamma_1^{-1}(\lambda_\gamma,\hat\lambda)_{\Gamma_c} \tag{5.40}$$

$$=\frac{1}{2\gamma_1}\|\lambda_\gamma-\hat\lambda\|^2_{\Gamma_c}+\frac{1}{2\gamma_1}\|\lambda_\gamma\|^2_{\Gamma_c}-\frac{1}{2\gamma_1}\|\hat\lambda\|^2_{\Gamma_c}\ge-\frac{1}{2\gamma_1}\|\hat\lambda\|^2_{\Gamma_c}. \tag{5.41}$$

Next, we focus on $(\boldsymbol{\mu}_\gamma,\boldsymbol{\tau}_T(\boldsymbol{y}_\gamma-\bar{\boldsymbol{y}}))_{\Gamma_c}$:

$$\begin{aligned}(\boldsymbol{\mu}_\gamma,\boldsymbol{\tau}_T(\boldsymbol{y}_\gamma-\bar{\boldsymbol{y}}))_{\Gamma_c}&\le(\mathfrak{F}g,\|\boldsymbol{\tau}_T(\bar{\boldsymbol{y}}-\boldsymbol{y}_\gamma)\|)_{\Gamma_c}\\&\le c_1\|\mathfrak{F}g\|_{\Gamma_c}\|\bar{\boldsymbol{y}}-\boldsymbol{y}_\gamma\|_{\mathbf{Y}},\end{aligned} \tag{5.42}$$

with some $c_1>0$ from the trace theorem. Equations (5.39) and (5.40) imply that

$$\begin{aligned}&a(\boldsymbol{y}_\gamma,\boldsymbol{y}_\gamma)+\frac{1}{\gamma_1}\|\lambda_\gamma\|^2_{\Gamma_c}\\&\quad\le a(\boldsymbol{y}_\gamma,\bar{\boldsymbol{y}})+\frac{1}{\gamma_1}(\lambda_\gamma,\hat\lambda)_{\Gamma_c}+L(\boldsymbol{y}_\gamma-\bar{\boldsymbol{y}})-(\boldsymbol{\mu}_\gamma,\boldsymbol{\tau}_T(\boldsymbol{y}_\gamma-\bar{\boldsymbol{y}}))_{\Gamma_c}.\end{aligned} \tag{5.43}$$

Using (5.42), the coercivity (with constant $c > 0$) and the continuity (with a constant $C > 0$) of $a(\cdot\,,\cdot)$ in (5.43) results in

$$c\|\boldsymbol{y}_\gamma\|_{\mathbf{Y}}^2 + \frac{1}{\gamma_1}\|\lambda_\gamma\|_{\Gamma_c}^2$$
$$\leq C\|\boldsymbol{y}_\gamma\|_{\mathbf{Y}}\|\bar{\boldsymbol{y}}\|_{\mathbf{Y}} + \Big(\|L\|_{\mathcal{L}(\mathbf{Y},\mathbb{R})} + c_1\|\mathfrak{F}g\|_{\Gamma_c}\Big)\|\boldsymbol{y}_\gamma - \bar{\boldsymbol{y}}\|_{\mathbf{Y}} + \frac{1}{\gamma_1}\|\lambda_\gamma\|_{\Gamma_c}\|\hat{\lambda}\|_{\Gamma_c},$$

which shows that

$$c\|\boldsymbol{y}_\gamma\|_{\mathbf{Y}} + \frac{1}{\gamma_1}\|\lambda_\gamma\|_{\Gamma_c} \tag{5.44}$$

is uniformly bounded with respect to $\gamma_1 \geq 1$. Hence $\boldsymbol{y}_\gamma$ is bounded in $\mathbf{Y}$ and λ_γ in $H^{-\frac{1}{2}}(\Gamma_c)$ from (5.38a). Consequently, there exist $(\tilde{\boldsymbol{y}}, \tilde{\lambda}) \in \mathbf{Y} \times H^{-\frac{1}{2}}(\Gamma_c)$ and a sequence γ_k with $\lim_{k\to\infty}\gamma_k = \infty$ such that

$$\boldsymbol{y}_{\gamma_k} \rightharpoonup \tilde{\boldsymbol{y}} \text{ weakly in } \mathbf{Y} \quad\text{and}\quad \lambda_{\gamma_k} \rightharpoonup \tilde{\lambda} \text{ weakly in } H^{-\frac{1}{2}}(\Gamma_c), \tag{5.45}$$

where the latter convergence follows from (5.38a), if only those $\boldsymbol{z} \in \mathbf{Y}$ with $\boldsymbol{\tau}_T\boldsymbol{z} = 0$ are inserted. Since $\|\boldsymbol{\mu}_\gamma\| \leq \mathfrak{F}g$ a.e. on Γ_c for all $\gamma_1, \gamma_2 > 0$, there exists $\tilde{\boldsymbol{\mu}}$ and a subsequence γ_{k_l} of γ_k, such that

$$\boldsymbol{\mu}_{\gamma_{k_l}} \rightharpoonup \tilde{\boldsymbol{\mu}} \text{ weakly in } \mathbf{L}^2(\Gamma_c), \tag{5.46}$$

and, since the set $\{\boldsymbol{\xi} \in \mathbf{L}^2(\Gamma_c) : \|\boldsymbol{\xi}\| \leq \mathfrak{F}g\}$ is convex, also the weak limit $\tilde{\boldsymbol{\mu}}$ satisfies $\|\tilde{\boldsymbol{\mu}}\| \leq \mathfrak{F}g$ almost everywhere. In the sequel we dismiss the subscript k_l with γ_{k_l}. Due to the definition of λ_γ,

$$\frac{1}{\gamma_1}\|\lambda_\gamma\|_{\Gamma_c}^2 = \gamma_1\|\max(0, \tfrac{1}{\gamma_1}\hat{\lambda} + \tau_N\boldsymbol{y}_\gamma - d)\|_{\Gamma_c}^2. \tag{5.47}$$

Since the expression in (5.44) is uniformly bounded with respect to γ_1, the above inequality implies that

$$\|\max(0, \tfrac{1}{\gamma_1}\hat{\lambda} + \tau_N\boldsymbol{y}_\gamma - d)\|_{\Gamma_c}^2 \to 0 \tag{5.48}$$

as $\gamma_1 \to \infty$. Since $H^{\frac{1}{2}}(\Gamma_c)$ embeds compactly into $L^2(\Gamma_c)$, (5.45) and (5.48) show that $\tau_N\boldsymbol{y}_\gamma$ converges to $\tau_N\tilde{\boldsymbol{y}}$ almost everywhere on Γ_c. Thus, (5.48) implies that $\tau_N\tilde{\boldsymbol{y}} - d \leq 0$ almost everywhere on Γ_c, i.e., $\tilde{\boldsymbol{y}} \in \boldsymbol{d} + \mathbf{K}$.
Subtracting equation (5.22a) for $(\boldsymbol{y}_\gamma, \lambda_\gamma, \boldsymbol{\mu}_\gamma)$ from the same equation for $(\bar{\boldsymbol{y}}, \bar{\lambda}, \bar{\boldsymbol{\mu}})$ and setting $\boldsymbol{z} := \boldsymbol{y}_\gamma - \bar{\boldsymbol{y}}$ yields

$$a(\boldsymbol{y}_\gamma - \bar{\boldsymbol{y}}, \boldsymbol{y}_\gamma - \bar{\boldsymbol{y}}) = -\big\langle \lambda_\gamma - \bar{\lambda}, \tau_N(\boldsymbol{y}_\gamma - \bar{\boldsymbol{y}})\big\rangle_{\Gamma_c} - (\boldsymbol{\mu}_\gamma - \bar{\boldsymbol{\mu}}, \boldsymbol{\tau}_T(\boldsymbol{y}_\gamma - \bar{\boldsymbol{y}}))_{\Gamma_c}. \tag{5.49}$$

Let us now estimate the term

$$-(\boldsymbol{\mu}_\gamma - \bar{\boldsymbol{\mu}}, \boldsymbol{\tau}_T(\boldsymbol{y}_\gamma - \bar{\boldsymbol{y}}))_{\Gamma_c} = (\boldsymbol{\mu}_\gamma - \bar{\boldsymbol{\mu}}, \boldsymbol{\tau}_T\bar{\boldsymbol{y}})_{\Gamma_c} + (\bar{\boldsymbol{\mu}} - \boldsymbol{\mu}_\gamma, \boldsymbol{\tau}_T\boldsymbol{y}_\gamma)_{\Gamma_c}. \tag{5.50}$$

From (5.22e) one deduces

$$\begin{aligned}(\boldsymbol{\mu}_\gamma - \bar{\boldsymbol{\mu}}, \boldsymbol{\tau}_T\bar{\boldsymbol{y}})_{\Gamma_c} &= (\boldsymbol{\mu}_\gamma, \boldsymbol{\tau}_T\bar{\boldsymbol{y}})_{\Gamma_c} - (\mathfrak{F}g, \|\boldsymbol{\tau}_T\bar{\boldsymbol{y}}\|)_{\Gamma_c}\\ &\leq (\|\boldsymbol{\mu}_\gamma\| - \mathfrak{F}g, \|\boldsymbol{\tau}_T\bar{\boldsymbol{y}}\|)_{\Gamma_c} \leq 0\end{aligned}$$

as estimation for the first term on the right hand side of (5.50). To get an estimation for the last term in (5.50) we distinguish two cases. Firstly, we consider sets where

$$\|\gamma_2 \boldsymbol{\tau}_T \boldsymbol{y}_\gamma + \hat{\boldsymbol{\mu}}\| \geq \mathfrak{F}g \text{ and thus } \boldsymbol{\mu}_\gamma = \mathfrak{F}g \frac{\gamma_2 \boldsymbol{\tau}_T \boldsymbol{y}_\gamma + \hat{\boldsymbol{\mu}}}{\|\gamma_2 \boldsymbol{\tau}_T \boldsymbol{y}_\gamma + \hat{\boldsymbol{\mu}}\|}.$$

Then we have the pointwise estimates

$$\begin{aligned}
(\bar{\boldsymbol{\mu}} - \boldsymbol{\mu}_\gamma)^\top \boldsymbol{\tau}_T \boldsymbol{y}_\gamma &= \frac{1}{\gamma_2}\left(\bar{\boldsymbol{\mu}} - \frac{\mathfrak{F}g}{\|\gamma_2 \boldsymbol{\tau}_T \boldsymbol{y}_\gamma + \hat{\boldsymbol{\mu}}\|}(\gamma_2 \boldsymbol{\tau}_T \boldsymbol{y}_\gamma + \hat{\boldsymbol{\mu}})\right)^\top \left(\gamma_2 \boldsymbol{\tau}_T \boldsymbol{y}_\gamma + \hat{\boldsymbol{\mu}} - \hat{\boldsymbol{\mu}}\right) \\
&= \frac{1}{\gamma_2}\Big\{\bar{\boldsymbol{\mu}}^\top(\gamma_2 \boldsymbol{\tau}_T \boldsymbol{y}_\gamma + \hat{\boldsymbol{\mu}}) - \mathfrak{F}g\|\gamma_2 \boldsymbol{\tau}_T \boldsymbol{y}_\gamma + \hat{\boldsymbol{\mu}}\| \\
&\quad + \frac{\mathfrak{F}g}{\|\gamma_2 \boldsymbol{\tau}_T \boldsymbol{y}_\gamma + \hat{\boldsymbol{\mu}}\|}(\gamma_2 \boldsymbol{\tau}_T \boldsymbol{y}_\gamma + \hat{\boldsymbol{\mu}})^\top \hat{\boldsymbol{\mu}} - \bar{\boldsymbol{\mu}}^\top \hat{\boldsymbol{\mu}}\Big\} \\
&\leq \frac{1}{\gamma_2}\left\{\mathfrak{F}g\|\hat{\boldsymbol{\mu}}\| + \|\bar{\boldsymbol{\mu}}\|\|\hat{\boldsymbol{\mu}}\|\right\}
\end{aligned}$$

Let us turn to the case that

$$\|\gamma_2 \boldsymbol{\tau}_T \boldsymbol{y}_\gamma + \hat{\boldsymbol{\mu}}\| < \mathfrak{F}g \text{ and } \boldsymbol{\mu}_\gamma = \gamma_2 \boldsymbol{\tau}_T \boldsymbol{y}_\gamma + \hat{\boldsymbol{\mu}}.$$

Then, pointwise almost everywhere holds

$$\begin{aligned}
(\bar{\boldsymbol{\mu}} - \boldsymbol{\mu}_\gamma)^\top \boldsymbol{\tau}_T \boldsymbol{y}_\gamma &= (\bar{\boldsymbol{\mu}} - \gamma_2 \boldsymbol{\tau}_T \boldsymbol{y} - \hat{\boldsymbol{\mu}})^\top \boldsymbol{\tau}_T \boldsymbol{y}_\gamma \\
&\leq -\gamma_2\|\boldsymbol{\tau}_T \boldsymbol{y}_\gamma\| + \frac{1}{\gamma_2}\|\bar{\boldsymbol{\mu}} - \hat{\boldsymbol{\mu}}\|(\mathfrak{F}g + \|\hat{\boldsymbol{\mu}}\|).
\end{aligned}$$

Combining the above estimates shows that

$$-(\boldsymbol{\mu}_\gamma - \bar{\boldsymbol{\mu}}, \boldsymbol{\tau}_T(\boldsymbol{y}_\gamma - \bar{\boldsymbol{y}}))_{\Gamma_c} \leq \frac{1}{\gamma_2}K(\hat{\boldsymbol{\mu}}, \bar{\boldsymbol{\mu}}), \tag{5.51}$$

where $K(\hat{\boldsymbol{\mu}}, \bar{\boldsymbol{\mu}})$ is independent of γ_1, γ_2.

Using (5.41), (5.49), (5.51) and the coercivity of $a(\cdot\,,\cdot)$ imply that

$$\begin{aligned}
0 &\leq \limsup_{\gamma_1,\gamma_2\to\infty} c\|\boldsymbol{y}_\gamma - \bar{\boldsymbol{y}}\|_{\mathbf{Y}}^2 \\
&\leq \lim_{\gamma_1,\gamma_2\to\infty}\left\{\left\langle\bar{\lambda}, \tau_N(\boldsymbol{y}_\gamma - \bar{\boldsymbol{y}})\right\rangle + \frac{1}{\gamma_2}K(\hat{\boldsymbol{\mu}}, \bar{\boldsymbol{\mu}}) + \frac{1}{\gamma_1}\|\hat{\lambda}\|_{\Gamma_c}^2\right\} \\
&= \lim_{\gamma_1,\gamma_2\to\infty}\left\{\left\langle\bar{\lambda}, \tau_N\tilde{\boldsymbol{y}} - d\right\rangle - \left\langle\bar{\lambda}, \tau_N\bar{\boldsymbol{y}} - d\right\rangle\right\} \\
&= \lim_{\gamma_1,\gamma_2\to\infty}\left\langle\bar{\lambda}, \tau_N\tilde{\boldsymbol{y}} - d\right\rangle \leq 0,
\end{aligned}$$

where $\tau_N\tilde{\boldsymbol{y}} - d \leq 0$ on Γ_c was used. From the above estimate follows that $\boldsymbol{y}_\gamma \to \bar{\boldsymbol{y}}$ strongly in $\mathbf{Y}$ and thus $\tilde{\boldsymbol{y}} = \bar{\boldsymbol{y}}$.
Passing to the limit in

$$a(\boldsymbol{y}_\gamma, \boldsymbol{z}) - L(\boldsymbol{z}) + (\lambda_\gamma, \tau_N\boldsymbol{z})_{\Gamma_c} + (\boldsymbol{\mu}_\gamma, \boldsymbol{\tau}_T\boldsymbol{z})_{\Gamma_c} = 0 \text{ for all } \boldsymbol{z} \in \mathbf{Y}$$

yields

$$a(\bar{\boldsymbol{y}}, \boldsymbol{z}) - L(\boldsymbol{z}) + \left\langle \tilde{\lambda}, \tau_N \boldsymbol{z} \right\rangle_{\Gamma_c} + (\tilde{\boldsymbol{\mu}}, \boldsymbol{\tau}_T \boldsymbol{z})_{\Gamma_c} = 0 \text{ for all } \boldsymbol{z} \in \mathbf{Y}. \tag{5.52}$$

Comparing (5.52) with (5.22a) and taking into account that

$$H^{\frac{1}{2}}(\Gamma_c) = \{\tau_N \boldsymbol{z} : \boldsymbol{z} \in \mathbf{Y}\} \text{ and } \mathbf{H}^{\frac{1}{2}}(\Gamma_c) = \{\boldsymbol{\tau}_T \boldsymbol{z} : \boldsymbol{z} \in \mathbf{Y}\} \tag{5.53}$$

yields that $\tilde{\lambda} = \bar{\lambda}$ and $\tilde{\boldsymbol{\mu}} = \bar{\boldsymbol{\mu}}$. Thus, every sequence γ_n with $\gamma_n \to \infty$ for $n \to \infty$ contains a subsequence $\gamma_{n_{k_l}}$ such that

$$\boldsymbol{y}_{\gamma_{n_{k_l}}} \to \bar{\boldsymbol{y}} \text{ in } \mathbf{Y},\ \lambda_{\gamma_{n_{k_l}}} \rightharpoonup \bar{\lambda} \text{ in } H^{-\frac{1}{2}}(\Gamma_c) \text{ and } \boldsymbol{\mu}_{\gamma_{n_{k_l}}} \rightharpoonup \bar{\boldsymbol{\mu}} \text{ in } \mathbf{L}^2(\Gamma_c).$$

This implies, due to the uniqueness of the solution variables $\bar{\boldsymbol{y}}, \bar{\lambda}, \bar{\boldsymbol{\mu}}$ that the whole family $\{(\boldsymbol{y}_\gamma, \lambda_\gamma, \boldsymbol{\mu}_\gamma)\}$ converges as stated in the theorem. ☐

2.3. Discussion of (5.38c)**.** In this section we focus on equation (5.38c), i.e., on

$$\mathfrak{F}g(\gamma_2 \boldsymbol{\tau}_T \boldsymbol{y}_\gamma + \hat{\boldsymbol{\mu}}) - \max(\mathfrak{F}g, \|\gamma_2 \boldsymbol{\tau}_T \boldsymbol{y}_\gamma + \hat{\boldsymbol{\mu}}\|)\boldsymbol{\mu}_\gamma = 0. \tag{5.54}$$

Observe that this equation involves nonlinearities of rather different nature: the max-operator and the product between the Euclidean norm and $\boldsymbol{\mu}_\gamma$. In the case of plane elasticity, i.e., in the case where the dimension of Ω is 2, the function (5.54) can be simplified significantly. To be precise, we can eliminate the absolute value function and the product between $\boldsymbol{\tau}_T \boldsymbol{y}_\gamma$ and $\boldsymbol{\mu}_\gamma$. This can be seen as follows. Obviously, in plane elasticity the tangential component $\boldsymbol{\tau}_T \boldsymbol{y}_\gamma$ is always of the form $(\tau_T \boldsymbol{y}_\gamma)\boldsymbol{t}$, with $\boldsymbol{t}$ denoting the unit tangential vector (i.e., the unit outward normal vector rotated in the mathematically positive direction) Let us suppose that also $\hat{\boldsymbol{\mu}}$ is chosen such that $\hat{\boldsymbol{\mu}} = \hat{\mu}\boldsymbol{t}$ with $\hat{\mu} \in L^2(\Gamma_c)$. Then, equation (5.54) implies

$$\mathfrak{F}g(\gamma_2 \tau_T \boldsymbol{y}_\gamma + \hat{\mu})\boldsymbol{t} - \max(\mathfrak{F}g, \|\gamma_2 \tau_T \boldsymbol{y}_\gamma + \hat{\mu}\|)\boldsymbol{\mu}_\gamma = 0,$$

and thus $\boldsymbol{\mu}_\gamma = \boldsymbol{t}\mu_\gamma$ with some $\mu_\gamma \in L^2(\Gamma_c)$. This yields that

$$\mathfrak{F}g(\gamma_2 \tau_T \boldsymbol{y}_\gamma + \hat{\mu}) - \max(\mathfrak{F}g, |\gamma_2 \tau_T \boldsymbol{y}_\gamma + \hat{\mu}|)\mu_\gamma = 0. \tag{5.55}$$

Distinguishing the cases

$$|\gamma_2 \tau_T \boldsymbol{y}_\gamma + \hat{\mu}| \le \mathfrak{F}g \text{ and } \gamma_2 \tau_T \boldsymbol{y}_\gamma + \hat{\mu} \begin{cases} \ge & \mathfrak{F}g, \\ \le & -\mathfrak{F}g, \end{cases}$$

it is easy to show that (5.55) is equivalent to

$$\begin{aligned} \mathfrak{F}g(\gamma_2 \tau_T \boldsymbol{y}_\gamma + \hat{\mu} - \mu_\gamma) &- \max(0, \gamma_2 \tau_T \boldsymbol{y}_\gamma + \hat{\mu} - \mathfrak{F}g)\mu_\gamma \\ &+ \min(0, \gamma_2 \tau_T \boldsymbol{y}_\gamma + \hat{\mu} + \mathfrak{F}g)\mu_\gamma = 0, \end{aligned} \tag{5.56}$$

and, for instance by a direct calculation, one can verify that this, again, is equivalent to

$$\gamma_2 \tau_T \boldsymbol{y}_\gamma + \hat{\mu} - \mu_\gamma - \max(0, \gamma_2 \tau_T \boldsymbol{y}_\gamma + \hat{\mu} - \mathfrak{F}g) - \min(0, \gamma_2 \tau_T \boldsymbol{y}_\gamma + \hat{\mu} + \mathfrak{F}g) = 0. \tag{5.57}$$

Note that the formulation (5.57) was also used as reformulation of the extremality conditions of the simplified friction problem, see Chapter 3. The discussion for this model problem showed that, from an algorithmic point of view, it may be advantageous to introduce $\xi \in L^2(\Gamma_c)$ and consider instead of (5.57) the equivalent

$$\left\{ \begin{array}{l} \gamma_2(\xi_\gamma - \tau_T \boldsymbol{y}_\gamma) + \mu_\gamma - \hat{\mu} = 0, \\ \xi_\gamma - \max(0, \xi_\gamma + \sigma(\mu_\gamma - \mathfrak{F}g)) - \min(0, \xi_\gamma + \sigma(\mu_\gamma + \mathfrak{F}g)) = 0 \end{array} \right. \tag{5.58}$$

for arbitrary $\sigma > 0$. Recall that (5.57) results from (5.58) by means of setting $\sigma = \gamma^{-1}$ and substituting the upper line from (5.58) into the lower.

An interesting aspect of (5.57) and (5.58) compared to (5.56) is that these equations also apply in case that on some part of Γ_c holds $\mathfrak{F}g = 0$. Then, (5.57) and (5.58) enforce the correct $\mu_\gamma = 0$, which is not necessarily the case for (5.56). Another advantage of (5.57) and (5.58) compared to (5.56) is that (5.57) and (5.58) only involve one type of nonlinearity, namely the max- and min-function, whereas (5.56) involves additional nonlinearities. Thus, the formulation (5.57) and (5.58) might be advantageous for the linearization in Newton-type approaches and might lead to better numerical results.

3. Algorithms for Contact Problems with Tresca Friction

In this section we propose algorithms for the solution of the contact problem with Tresca friction ($\mathcal{P}$) (with dual ($\mathcal{P}^\star$)) and its regularized versions ($\mathcal{P}_{\gamma_1,\gamma_2}$) (with dual ($\mathcal{P}^\star_{\gamma_1,\gamma_2}$)). Our iterative methods treat both, contact and friction condition together. The approach is motivated from our investigation of the Signorini elasticity problem in Chapter 4 and our results for the simplified friction problem, see Chapter 3.

Firstly, we present a semi-smooth Newton method that applies for any dimension $n \geq 2$. Then, we focus on a semi-smooth Newton method based on a different NC-function that applies only for planar elasticity i.e., for the case $n = 2$. The latter leads to a Newton method which has the form of an active set strategy. Then, an exact as well as an inexact first-order augmented Lagrangian method for the solution of the Tresca friction contact problem are presented and analyzed.

3.1. A semi-smooth Newton method for arbitrary dimension $n \geq 2$.

In this section we derive a generalized Newton method for the solution of contact problems with given friction in arbitrary dimension $n \geq 2$. We assume the given friction g to be an element in $L^2(\Gamma_c)$ and consider the optimality system (5.38), which, for the reader's convenience, we recall below.

$$a(\boldsymbol{y}_\gamma, \boldsymbol{z}) - L(\boldsymbol{z}) + (\boldsymbol{\mu}_\gamma, \boldsymbol{\tau}_T \boldsymbol{z})_{\Gamma_c} + (\lambda_\gamma, \tau_N \boldsymbol{z})_{\Gamma_c} = 0 \text{ for all } \boldsymbol{z} \in \mathbf{Y}, \tag{5.59a}$$

$$\lambda_\gamma - \max(0, \hat{\lambda} + \gamma_1(\tau_N \boldsymbol{y}_\gamma - d)) = 0 \quad \text{on } \Gamma_c, \tag{5.59b}$$

$$\mathfrak{F}g(\gamma_2 \boldsymbol{\tau}_T \boldsymbol{y}_\gamma + \hat{\boldsymbol{\mu}}) - \max(\mathfrak{F}g, \|\gamma_2 \boldsymbol{\tau}_T \boldsymbol{y}_\gamma + \hat{\boldsymbol{\mu}}\|)\boldsymbol{\mu}_\gamma = 0 \quad \text{on } \Gamma_c. \tag{5.59c}$$

We henceforth consider a reduced version of (5.59), where the variable $\boldsymbol{y}_\gamma$ is eliminated from the system using the linear equation (5.59a). For this purpose, we introduce some notation. We denote by $\tilde{\boldsymbol{y}}$ the solution to the variational equality

$$a(\boldsymbol{y}, \boldsymbol{z}) - L(\boldsymbol{z}) = 0 \quad \text{for all } \boldsymbol{z} \in \mathbf{Y}.$$

Furthermore, we denote by $B_1^{-1} \in \mathcal{L}(L^2(\Gamma_c), \mathbf{Y})$ the solution mapping for the variational equality

$$a(\boldsymbol{y}, \boldsymbol{z}) - (\lambda, \tau_N \boldsymbol{z})_{\Gamma_c} = 0 \quad \text{for all } \boldsymbol{z} \in \mathbf{Y},$$

for given $\lambda \in L^2(\Gamma_c)$. Similarly, given $\boldsymbol{\mu} \in \mathbf{L}^2(\Gamma_c)$, we denote by $B_2^{-1}\boldsymbol{\mu}$ the solution to

$$a(\boldsymbol{y}, \boldsymbol{z}) - (\boldsymbol{\mu}, \boldsymbol{\tau}_T \boldsymbol{z})_{\Gamma_c} = 0 \quad \text{for all } \boldsymbol{z} \in \mathbf{Y},$$

i.e., $B_2^{-1} \in \mathcal{L}(\mathbf{L}^2(\Gamma_c), \mathbf{Y})$. To shorten the notation we further introduce

$$\tilde{\boldsymbol{y}}_N := \tau_N \tilde{\boldsymbol{y}}, \quad \tilde{\boldsymbol{y}}_T := \boldsymbol{\tau}_T \tilde{\boldsymbol{y}}, \tag{5.60}$$

and the linear operators

$$\begin{aligned}
C_N &:= \tau_N B_1^{-1} \in \mathcal{L}(L^2(\Gamma_c), L^2(\Gamma_c)),\\
D_N &:= \tau_N B_2^{-1} \in \mathcal{L}(L^2(\Gamma_c), L^2(\Gamma_c)),\\
C_T &:= \boldsymbol{\tau}_T B_1^{-1} \in \mathcal{L}(\mathbf{L}^2(\Gamma_c), \mathbf{L}^2(\Gamma_c)),\\
D_T &:= \boldsymbol{\tau}_T B_2^{-1} \in \mathcal{L}(\mathbf{L}^2(\Gamma_c), \mathbf{L}^2(\Gamma_c)).
\end{aligned}$$

Given $\lambda \in L^2(\Gamma_c)$ and $\boldsymbol{\mu} \in \mathbf{L}^2(\Gamma_c)$, and utilizing the notation introduced above, for given $(\lambda, \boldsymbol{\mu}) \in L^2(\Gamma_c) \times \mathbf{L}^2(\Gamma_c)$ the solution $\boldsymbol{y}_{\lambda,\boldsymbol{\mu}}$ to

$$a(\boldsymbol{y}, \boldsymbol{z}) - L(\boldsymbol{z}) + (\lambda, \tau_N \boldsymbol{z})_{\Gamma_c} + (\boldsymbol{\mu}, \boldsymbol{\tau}_T \boldsymbol{z})_{\Gamma_c} = 0 \quad \text{for all } \boldsymbol{z} \in \mathbf{Y}$$

can be written as

$$\boldsymbol{y}_{\lambda,\boldsymbol{\mu}} = \tilde{\boldsymbol{y}} - B_1^{-1}\lambda - B_2^{-1}\boldsymbol{\mu}. \tag{5.61}$$

The corresponding normal and tangential components of the trace of $\boldsymbol{y}_{\lambda,\boldsymbol{\mu}}$ are

$$\tau_N \boldsymbol{y}_{\lambda,\boldsymbol{\mu}} = \tilde{\boldsymbol{y}}_N - C_N\lambda - D_N\boldsymbol{\mu}, \tag{5.62}$$

$$\boldsymbol{\tau}_T \boldsymbol{y}_{\lambda,\boldsymbol{\mu}} = \tilde{\boldsymbol{y}}_T - C_T\lambda - D_T\boldsymbol{\mu}. \tag{5.63}$$

The next lemma summarizes basic properties of the mappings defined above.

Lemma 5.4. *The linear mappings C_N, D_N, C_T, D_T defined in* (5.62) *and* (5.63) *are selfadjoint, positive semi-definite and compact.*

PROOF. The proof for the above assertion in similar to that of Lemma 3.14 in Chapter 3. □

The above definitions allow one to state a reduced version of the optimality system (5.59). We henceforth assume that there exists a constant $g_0 > 0$ such that $\mathfrak{F}g \geq g_0$ almost everywhere. Then we can define $F : L^2(\Gamma_c) \times \mathbf{L}^2(\Gamma_c) \to L^2(\Gamma_c) \times \mathbf{L}^2(\Gamma_c)$ by

$$F(\lambda, \boldsymbol{\mu}) = \begin{pmatrix} \lambda - \max(0, \hat{\lambda} + \gamma_1(\tilde{\boldsymbol{y}}_N - C_N\lambda - D_N\boldsymbol{\mu} - d)) \\ \boldsymbol{\mu} - \mathfrak{F}g\dfrac{\gamma_2(\tilde{\boldsymbol{y}}_T - C_T\lambda - D_T\boldsymbol{\mu}) + \hat{\boldsymbol{\mu}}}{\max(\mathfrak{F}g, \|\gamma_2(\tilde{\boldsymbol{y}}_T - C_T\lambda - D_T\boldsymbol{\mu}) + \hat{\boldsymbol{\mu}}\|)} \end{pmatrix}.$$

Our goal is now to apply a generalized Newton method for the solution of the nonlinear equation $F(\lambda, \boldsymbol{\mu}) = 0$, with F defined above. For this purpose we denote by F_1 and F_2 the first and second component of F, respectively. The generalized derivative of F is denoted by $G_F = (G_{F_1}, G_{F_2})$, where as derivative for the max-function we utilize (2.10). This leads, for $(\delta_\lambda, \delta_{\boldsymbol{\mu}}) \in L^2(\Gamma_c) \times \mathbf{L}^2(\Gamma_c)$ to

$$G_{F_1}(\lambda, \boldsymbol{\mu})(\delta_\lambda, \delta_{\boldsymbol{\mu}}) = \delta_\lambda + \gamma_1\chi_{\mathcal{A}_c}(C_N\delta_\lambda + D_N\delta_{\boldsymbol{\mu}}), \tag{5.64}$$

where

$$\mathcal{A}_c = \{x \in \Gamma_c : \hat{\lambda} + \gamma_1(\tilde{\boldsymbol{y}}_N - C_N\lambda - D_N\boldsymbol{\mu} - d) > 0\} \tag{5.65}$$

and $\chi_{\mathcal{S}}$ denotes the characteristic function for a set $\mathcal{S} \subset \Gamma_c$. To derive a generalized derivative for the second component F_2 of F we use the concise $\boldsymbol{\tau}_T\boldsymbol{y}_{\lambda,\boldsymbol{\mu}}$ instead of $\tilde{\boldsymbol{y}}_T - C_T\lambda - D_T\boldsymbol{\mu}$ and again the generalized derivative of the max-function as given in (2.10). Then, one derives that

$$\begin{aligned} &G_{F_2}(\lambda, \boldsymbol{\mu})(\delta_\lambda, \delta_{\boldsymbol{\mu}}) \\ &\quad= \delta_{\boldsymbol{\mu}} + \gamma_2\frac{\mathfrak{F}g}{\max(\mathfrak{F}g, \|\gamma_2\boldsymbol{\tau}_T\boldsymbol{y}_{\lambda,\boldsymbol{\mu}} + \hat{\boldsymbol{\mu}}\|)}(C_T\delta_\lambda + D_T\delta_{\boldsymbol{\mu}}) \\ &\qquad - \gamma_2\chi_{\mathcal{A}_f}\frac{\mathfrak{F}g(\gamma_2\boldsymbol{\tau}_T\boldsymbol{y}_{\lambda,\boldsymbol{\mu}} + \hat{\boldsymbol{\mu}})^\top(C_T\delta_\lambda + D_T\delta_{\boldsymbol{\mu}})}{\|\gamma_2\boldsymbol{\tau}_T\boldsymbol{y}_{\lambda,\boldsymbol{\mu}} + \hat{\boldsymbol{\mu}}\|^3}(\gamma_2\boldsymbol{\tau}_T\boldsymbol{y}_{\lambda,\boldsymbol{\mu}} + \hat{\boldsymbol{\mu}}) \\ &\quad= \delta_{\boldsymbol{\mu}} + \gamma_2\chi_{\mathcal{I}_f}(C_T\delta_\lambda + D_T\delta_{\boldsymbol{\mu}}) + \gamma_2\chi_{\mathcal{A}_f}\frac{\mathfrak{F}g}{\|\gamma_2\boldsymbol{\tau}_T\boldsymbol{y}_{\lambda,\boldsymbol{\mu}} + \hat{\boldsymbol{\mu}}\|} \\ &\qquad \cdot \left\{ I - \frac{(\gamma_2\boldsymbol{\tau}_T\boldsymbol{y}_{\lambda,\boldsymbol{\mu}} + \hat{\boldsymbol{\mu}})(\gamma_2\boldsymbol{\tau}_T\boldsymbol{y}_{\lambda,\boldsymbol{\mu}} + \hat{\boldsymbol{\mu}})^\top}{\|\gamma_2\boldsymbol{\tau}_T\boldsymbol{y}_{\lambda,\boldsymbol{\mu}} + \hat{\boldsymbol{\mu}}\|^2} \right\}(C_T\delta_\lambda + D_T\delta_{\boldsymbol{\mu}}), \end{aligned} \tag{5.66}$$

where

$$\begin{aligned} \mathcal{A}_f &= \{x \in \Gamma_c : \|\hat{\boldsymbol{\mu}} + \gamma_2(\tilde{\boldsymbol{y}}_T - C_T\lambda - D_T\boldsymbol{\mu})\| > \mathfrak{F}g\}, \\ \mathcal{I}_f &= \{x \in \Gamma_c : \|\hat{\boldsymbol{\mu}} + \gamma_2(\tilde{\boldsymbol{y}}_T - C_T\lambda - D_T\boldsymbol{\mu})\| \leq \mathfrak{F}g\}. \end{aligned} \tag{5.67}$$

Above, I denotes the $n \times n$ identity matrix and $(\boldsymbol{a}^\top \boldsymbol{b})\boldsymbol{c} = (\boldsymbol{c}\boldsymbol{a}^\top)\boldsymbol{b}$ for all $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^n$ is used. We next summarize the semi-smooth Newton method for the regularized contact problem with Tresca friction, where here and in the sequel we dismiss the subscript 'γ' for the iterates, i.e., the k-th iterates are denoted by $\lambda^k, \boldsymbol{\mu}^k$.

Algorithm: (FC-SS)

(1) Initialize $(\lambda^0, \boldsymbol{\mu}^0) \in L^2(\Gamma_c) \times \mathbf{L}^2(\Gamma_c)$, and set $k := 0$.

(2) Determine

$$\begin{aligned}
\mathcal{A}_c^{k+1} =& \{x \in \Gamma_c : \hat{\lambda} + \gamma_1(\tilde{\boldsymbol{y}}_N - C_N\lambda^k - D_N\boldsymbol{\mu}^k - d) > 0\}, \\
\mathcal{A}_f^{k+1} =& \{x \in \Gamma_c : \|\hat{\boldsymbol{\mu}} + \gamma_2(\tilde{\boldsymbol{y}}_T - C_T\lambda^k - D_T\boldsymbol{\mu}^k)\| > \mathfrak{F}g\}. \\
\mathcal{I}_f^{k+1} =& \{x \in \Gamma_c : \|\hat{\boldsymbol{\mu}} + \gamma_2(\tilde{\boldsymbol{y}}_T - C_T\lambda^k - D_T\boldsymbol{\mu}^k)\| \leq \mathfrak{F}g\}.
\end{aligned}$$

(3) Perform a semi-smooth Newton step, i.e., solve

$$G_F(\lambda^k, \boldsymbol{\mu}^k)(\delta_\lambda^k, \delta_{\boldsymbol{\mu}}^k) = -F(\lambda^k, \boldsymbol{\mu}^k)$$

with $G_F = (G_{F_1}, G_{F_2})$ as given in (5.64) and (5.66) and the active sets from Step 2.

(4) Update

$$\begin{aligned}
\lambda^{k+1} &:= \lambda^k + \delta_\lambda^k, \\
\boldsymbol{\mu}^{k+1} &:= \boldsymbol{\mu}^k + \delta_{\boldsymbol{\mu}}^k,
\end{aligned}$$

$k := k + 1$ and, unless an appropriate stopping criterion is met, go to Step 2.

In order to address the solvability of the linearized system in Step 3 of (FC-SS), we calculate a different form for the semi-smooth Newton step. This form only involves $\tau_N \boldsymbol{y}^k, \boldsymbol{\tau}_T \boldsymbol{y}^k$ and $\tau_N \boldsymbol{y}^{k+1}, \boldsymbol{\tau}_T \boldsymbol{y}^{k+1}$ that can be uniquely derived from $\lambda^k, \boldsymbol{\mu}^k$ and $\lambda^{k+1}, \boldsymbol{\mu}^{k+1}$ by means of (5.62) and (5.63). We consider

$$G_F(\lambda^k, \boldsymbol{\mu}^k)(\delta_\lambda^k, \delta_{\boldsymbol{\mu}}^k) = -F(\lambda^k, \boldsymbol{\mu}^k) \tag{5.68}$$

with $\delta_\lambda^k = \lambda^{k+1} - \lambda^k, \delta_{\boldsymbol{\mu}}^k = \boldsymbol{\mu}^{k+1} - \boldsymbol{\mu}^k$. Similarly as in the case of pure contact we get

$$\lambda^{k+1} = \begin{cases} 0 & \text{on } \mathcal{I}_c^{k+1}, \\ \hat{\lambda} + \gamma_1(\tau_N \boldsymbol{y}^{k+1} - d) & \text{on } \mathcal{A}_c^{k+1}. \end{cases} \tag{5.69}$$

For the second component in (5.68) we find, using (5.66) that

$$\boldsymbol{\mu}^{k+1} = \begin{cases} \gamma_2 \boldsymbol{\tau}_T \boldsymbol{y}^k + \hat{\boldsymbol{\mu}} & \text{on } \mathcal{I}_f^{k+1}, \\ W(\boldsymbol{y}^k)(\gamma_2 \boldsymbol{\tau}_T \boldsymbol{y}^{k+1} + \hat{\boldsymbol{\mu}}) + w(\boldsymbol{y}^k)(\gamma_2 \boldsymbol{\tau}_T \boldsymbol{y}^k + \hat{\boldsymbol{\mu}}) & \text{on } \mathcal{A}_f^{k+1}, \end{cases} \tag{5.70}$$

where

$$W(\boldsymbol{y}) = \frac{\mathfrak{F}g}{\|\gamma_2\boldsymbol{\tau}_T\boldsymbol{y}+\hat{\boldsymbol{\mu}}\|}\left\{I - \frac{(\gamma_2\boldsymbol{\tau}_T\boldsymbol{y}+\hat{\boldsymbol{\mu}})(\gamma_2\boldsymbol{\tau}_T\boldsymbol{y}+\hat{\boldsymbol{\mu}})^\top}{\|\gamma_2\boldsymbol{\tau}_T\boldsymbol{y}+\hat{\boldsymbol{\mu}}\|^2}\right\} \tag{5.71}$$

and

$$w(\boldsymbol{y}) = \frac{\mathfrak{F}g}{\|\gamma_2\boldsymbol{\tau}_T\boldsymbol{y}+\hat{\boldsymbol{\mu}}\|}\frac{(\gamma_2\boldsymbol{\tau}_T\boldsymbol{y}+\hat{\boldsymbol{\mu}})(\gamma_2\boldsymbol{\tau}_T\boldsymbol{y}+\hat{\boldsymbol{\mu}})^\top}{\|\gamma_2\boldsymbol{\tau}_T\boldsymbol{y}+\hat{\boldsymbol{\mu}}\|^2}.$$

Equivalently to Steps 3, 4 that are written in the dual variables $\lambda, \boldsymbol{\mu}$ one can solve a variational equality in the (primal) variable $\boldsymbol{y}$ only, namely

$$a(\boldsymbol{y}^{k+1}, \boldsymbol{z}) - L(\boldsymbol{z}) + (\lambda^{k+1}, \tau_N\boldsymbol{z})_{\Gamma_c} + (\boldsymbol{\mu}^{k+1}, \boldsymbol{\tau}_T\boldsymbol{z})_{\Gamma_c} = 0 \text{ for all } \boldsymbol{z}\in\mathbf{Y}, \tag{5.72}$$

where $\lambda^{k+1}, \boldsymbol{\mu}^{k+1}$ are eliminated using the right hand side expressions in (5.69) and (5.70). The resulting equation involves as variable only $\boldsymbol{y}^{k+1}$ and, provided (5.72) admits a solution, the dual iterates λ^{k+1} and $\boldsymbol{\mu}^{k+1}$ can be uniquely derived by means of (5.69) and (5.70). The next lemma addresses the unique solvability of the system in Step 3 of (FC-SS).

LEMMA 5.5. *For all iterates* $(\lambda^k, \boldsymbol{\mu}^k) \in L^2(\Gamma_c)\times\mathbf{L}^2(\Gamma_c)$ *the system in Step 3 of* (FC-SS) *admits a unique solution* $(\delta_\lambda^k, \delta_{\boldsymbol{\mu}}^k)$.

PROOF. Following the above discussion it remains to show that (5.72) admits a unique solution. Fur this purpose, we characterize $\boldsymbol{y}^{k+1}$ as solution of an unconstrained uniformly convex minimization problem. This implies that $\boldsymbol{y}^{k+1}$ always exists and is unique, which carries over to λ^{k+1} and $\boldsymbol{\mu}^{k+1}$ by means of (5.69) and (5.70), yielding the assertion of the lemma.

We start by showing that $W(\boldsymbol{y})$ is symmetric and positive semi-definite for each $\boldsymbol{y}\in\mathbf{Y}$ on the subset of Γ_c, where $\|\gamma_2\boldsymbol{\tau}_T\boldsymbol{y}+\hat{\boldsymbol{\mu}}\| > 0$. Symmetry is clear and since the first multiplicative term in $W(\boldsymbol{y})$ is a positive constant, we only have to consider for arbitrary $\boldsymbol{z}\in\mathbb{R}^n$

$$\begin{aligned}
&\boldsymbol{z}^\top\left\{I - \frac{(\gamma_2\boldsymbol{\tau}_T\boldsymbol{y}(x)+\hat{\boldsymbol{\mu}})(\gamma_2\boldsymbol{\tau}_T\boldsymbol{y}(x)+\hat{\boldsymbol{\mu}})^\top}{\|\gamma_2\boldsymbol{\tau}_T\boldsymbol{y}+\hat{\boldsymbol{\mu}}\|^2}\right\}\boldsymbol{z}\\
&\geq \|\boldsymbol{z}\|^2 - \frac{\|(\gamma_2\boldsymbol{\tau}_T\boldsymbol{y}+\hat{\boldsymbol{\mu}})^\top\boldsymbol{z}\|^2}{\|\gamma_2\boldsymbol{\tau}_T\boldsymbol{y}+\hat{\boldsymbol{\mu}}\|^2}\\
&\geq 0.
\end{aligned}$$

Hence, $W(\boldsymbol{y})$ is positive semi-definite for all $\boldsymbol{y}\in\mathbf{Y}$ on the subset of Γ_c, where $\|\gamma_2\boldsymbol{\tau}_T\boldsymbol{y}+\hat{\mu}\| > 0$ holds. It follows that for every $\boldsymbol{y}\in\mathbf{Y}$ there exists $W_1(\boldsymbol{y})$ such that $W_1(\boldsymbol{y})^\top W_1(\boldsymbol{y}) = W(\boldsymbol{y})$. We can now consider the following auxiliary problem:

$$\begin{aligned}
\min_{\boldsymbol{y}\in\mathbf{Y}} \frac{1}{2}a(\boldsymbol{y},\boldsymbol{y}) - L(\boldsymbol{y}) + \big(w(\boldsymbol{y}^k), \boldsymbol{\tau}_T\boldsymbol{y}\big)_{\mathcal{A}_f} + \frac{1}{2\gamma_1}\|\hat{\lambda}+\gamma_1(\tau_N\boldsymbol{y}-d)\|^2_{\mathcal{A}_c}\\
+ \frac{1}{2\gamma_2}\|W_1(\boldsymbol{y}^k)(\gamma_2\boldsymbol{\tau}_T\boldsymbol{y}+\hat{\boldsymbol{\mu}})\|^2_{\mathcal{A}_f} + \frac{1}{2\gamma_2}\|\gamma_2\boldsymbol{\tau}_T\boldsymbol{y}+\hat{\boldsymbol{\mu}}\|^2_{\mathcal{I}_f}.
\end{aligned} \tag{5.73}$$

Clearly, the above minimization problem admits a unique solution. Calculating the corresponding first-order necessary optimality conditions leads to the variational equality

$$a(\boldsymbol{y},\boldsymbol{z}) - L(\boldsymbol{z}) + (\lambda^{k+1},\tau_N \boldsymbol{z})_{\Gamma_c} + (\boldsymbol{\mu}^{k+1},\boldsymbol{\tau}_T \boldsymbol{z})_{\Gamma_c} = 0 \text{ for all } \boldsymbol{z} \in \mathbf{Y}, \tag{5.74}$$

with λ^{k+1} and $\boldsymbol{\mu}^{k+1}$ as defined in (5.69) and (5.70). The equations (5.74), (5.69) and (5.70) also characterize $\boldsymbol{y}^{k+1}$ as solution of (5.73). This shows that $\boldsymbol{y}^{k+1}$ is uniquely determined. By means of (5.69) and (5.70) this also holds true for λ^{k+1} and $\boldsymbol{\mu}^{k+1}$ and ends the proof. $\square$

Next we turn to the convergence analysis for algorithm (FC-SS). Exploiting the properties of semi-smooth Newton methods results in the following local convergence result.

THEOREM 5.6. *Suppose that $\|\lambda^0 - \lambda_\gamma\|_{\Gamma_c}$ and $\|\boldsymbol{\mu}^0 - \boldsymbol{\mu}_\gamma\|_{\Gamma_c}$ are sufficiently small. Then, for all $\hat{\lambda}, \hat{\boldsymbol{\mu}} \in L^2(\Gamma_c) \times \mathbf{L}^2(\Gamma_c)$ the iterates $(\lambda^k, \boldsymbol{\mu}^k)$ of* (FC-SS) *converge superlinearly to $(\lambda_\gamma, \boldsymbol{\mu}_\gamma)$ in $L^2(\Gamma_c) \times \mathbf{L}^2(\Gamma_c)$.*

PROOF. To prove local superlinear convergence we use properties of semi-smooth Newton methods. First, note that the mapping F_1 is Newton differentiable, since inside the max-function the variables λ and $\boldsymbol{\mu}$ only appear under smoothing operators, namely under C_N and D_N. To be precise, using embedding theorems for Sobolev spaces [1] implies the continuous embeddings

$$C_N(L^2(\Gamma_c)) \hookrightarrow L^q(\Gamma_c) \text{ and } C_T(\mathbf{L}^2(\Gamma_c)) \hookrightarrow \mathbf{L}^q(\Gamma_c)$$

for any $q < \infty$ if the dimension n of Ω is 2 and for $q = 2(n-1)/(n-2) > 2$ if $n \geq 3$. This guarantees the norm gap required for Newton differentiability of the max-function. Thus, F_1 is Newton differentiable.

To argue Newton differentiability of F_2 we introduce some notation. We denote $b := \hat{\boldsymbol{\mu}} + \gamma_2 \tilde{\boldsymbol{y}}_T \in \mathbf{L}^2(\Gamma_c)$, $C := \gamma_2 C_T$ and $D := \gamma_2 D_T$. Moreover, we introduce the mapping

$$\Theta : \begin{cases} L^2(\Gamma_c) \times \mathbf{L}^2(\Gamma_c) & \longrightarrow \quad L^2(\Gamma_c), \\ (\lambda, \boldsymbol{\mu}) & \mapsto \quad \max(\mathfrak{F}g, \|b - C\lambda - D\boldsymbol{\mu}\|). \end{cases}$$

Note that $\Theta(\lambda, \boldsymbol{\mu}) \geq g_0$ for all $(\lambda, \boldsymbol{\mu}) \in L^2(\Gamma_c) \times \mathbf{L}^2(\Gamma_c)$. From the same arguments as for F_1 it follows that Θ is Newton differentiable and we denote its generalized derivative by G_Θ. Next we show that

$$\Upsilon : \begin{cases} L^2(\Gamma_c) \times \mathbf{L}^2(\Gamma_c) & \longrightarrow \quad \mathbf{L}^2(\Gamma_c), \\ (\lambda, \boldsymbol{\mu}) & \mapsto \quad \dfrac{b - C\lambda - D\boldsymbol{\mu}}{\max(\mathfrak{F}g, \|b - C\lambda - D\boldsymbol{\mu}\|)} \end{cases}$$

is Newton differentiable as well. For this purpose we consider for $\lambda, h \in L^2(\Gamma_c)$ and $\boldsymbol{\mu}, \boldsymbol{k} \in \mathbf{L}^2(\Gamma_c)$

$$
\begin{aligned}
&\Xi(\lambda, h, \boldsymbol{\mu}, \boldsymbol{k}) \\
&:= \frac{b - C(\lambda + h) - D(\boldsymbol{\mu} + \boldsymbol{k})}{\Theta(\lambda + h, \boldsymbol{\mu} + \boldsymbol{k})} - \frac{b - C\lambda - D\boldsymbol{\mu}}{\Theta(\lambda, \boldsymbol{\mu})} \\
&\quad + \frac{G_\Theta(\lambda + h, \boldsymbol{\mu} + \boldsymbol{k})(h, k)(b - C\lambda - D\boldsymbol{\mu})}{\Theta(\lambda, \boldsymbol{\mu})^2} - \frac{-Ch - D\boldsymbol{k}}{\Theta(\lambda, \boldsymbol{\mu})} \\
&= (b - C\lambda - D\boldsymbol{\mu}) \frac{\Theta(\lambda, \boldsymbol{\mu}) - \Theta(\lambda + h, \boldsymbol{\mu} + \boldsymbol{k}) + G_\Theta(\lambda + h, \boldsymbol{\mu} + \boldsymbol{k})(h, \boldsymbol{k})}{\Theta(\lambda, \boldsymbol{\mu})\Theta(\lambda + h, \boldsymbol{\mu} + \boldsymbol{k})} \\
&\quad + (b - C\lambda - D\boldsymbol{\mu}) \frac{G_\Theta(\lambda + h, \boldsymbol{\mu} + \boldsymbol{k})(h, \boldsymbol{k})\big\{\Theta(\lambda + h, \boldsymbol{\mu} + \boldsymbol{k}) - \Theta(\lambda, \boldsymbol{\mu})\big\}}{\Theta(\lambda, \boldsymbol{\mu})^2\Theta(\lambda + h, \boldsymbol{\mu} + \boldsymbol{k})} \\
&\quad + (Ch + D\boldsymbol{k}) \frac{\Theta(\lambda + h, \boldsymbol{\mu} + \boldsymbol{k}) - \Theta(\lambda, \boldsymbol{\mu})}{\Theta(\lambda, \boldsymbol{\mu})\Theta(\lambda + h, \boldsymbol{\mu} + \boldsymbol{k})} \\
&=: \text{(I)} + \text{(II)} + \text{(III)}.
\end{aligned}
$$

We now derive the following estimates for the $\mathbf{L}^2$-norms of the above introduced expressions (I), (II) and (III):

$$
\begin{aligned}
\|\text{(I)}\|_{\Gamma_c} &\le \frac{1}{g_0} \big\|\Theta(\lambda, \boldsymbol{\mu}) - \Theta(\lambda + h, \boldsymbol{\mu} + \boldsymbol{k}) + G_\Theta(\lambda + h, \boldsymbol{\mu} + \boldsymbol{k})(h, \boldsymbol{k})\big\|_{\Gamma_c} \\
&\le a(h, \boldsymbol{k})\big(\|h\|_{\Gamma_c} + \|\boldsymbol{k}\|_{\Gamma_c}\big),
\end{aligned}
$$

with $a(h, \boldsymbol{k}) \to 0$ as $\|h\|_{\Gamma_c} + \|\boldsymbol{k}\|_{\Gamma_c} \to 0$. Here, the Newton differentiability of Θ is used. For (II) we find

$$
\begin{aligned}
\|\text{(II)}\|_{\Gamma_c} &\le \frac{1}{g_0^2} \big\|G_\Theta(\lambda + h, \boldsymbol{\mu} + \boldsymbol{k})(h, \boldsymbol{k})\big\{\Theta(\lambda + h, \boldsymbol{\mu} + \boldsymbol{k}) - \Theta(\lambda, \boldsymbol{\mu})\big\}\big\|_{\Gamma_c} \\
&\le b(h, \boldsymbol{k})\big(\|h\|_{\Gamma_c} + \|\boldsymbol{k}\|_{\Gamma_c}\big),
\end{aligned}
$$

with $b(h, \boldsymbol{k}) \to 0$ as $\|h\|_{\Gamma_c} + \|\boldsymbol{k}\|_{\Gamma_c} \to 0$, where $\|G_\Theta(\lambda, \boldsymbol{\mu})\|_{\mathcal{L}(L(\Gamma_c) \times \mathbf{L}(\Gamma_c), \mathbf{L}(\Gamma_c))} \le 1$ for all $(\lambda, \boldsymbol{\mu}) \in L^2(\Gamma_c) \times \mathbf{L}^2(\Gamma_c)$ was used. Finally,

$$
\begin{aligned}
\|\text{(III)}\|_{\Gamma_c} &\le \frac{1}{g_0^2} \|\big\{\Theta(\lambda + h, \boldsymbol{\mu} + \boldsymbol{k}) - \Theta(\lambda, \boldsymbol{\mu})\big\}\|_{\Gamma_c} \|Ch + D\boldsymbol{k}\|_{\Gamma_c} \\
&\le c(h, \boldsymbol{k})\big(\|h\|_{\Gamma_c} + \|\boldsymbol{k}\|_{\Gamma_c}\big),
\end{aligned}
$$

where $c(h, \boldsymbol{k}) \to 0$ as $\|h\|_{\Gamma_c} + \|\boldsymbol{k}\|_{\Gamma_c} \to 0$. This proves Newton differentiability of Υ, and thus of F_2.

In order to apply Theorem 2.9 that assures superlinear convergence of the iterates, it remains to show that the family $\{G_F(\lambda, \boldsymbol{\mu}) : (\lambda, \boldsymbol{\mu}) \in L^2(\Gamma_c) \times \mathbf{L}^2(\Gamma_c)\}$ has uniformly bounded inverses. For this purpose we choose arbitrary $(h, \boldsymbol{k}) \in$

$L^2(\Gamma_c) \times \mathbf{L}^2(\Gamma_c)$ and consider the equation

$$G_F(\lambda, \boldsymbol{\mu})(\delta_\lambda, \delta_{\boldsymbol{\mu}}) = (h, \boldsymbol{k}).$$

Similarly as in the proof of Lemma 5.5 one can introduce an auxiliary variable $\delta_{\boldsymbol{y}}$ corresponding to $\delta_\lambda, \delta_{\boldsymbol{\mu}}$ by $\delta_{\boldsymbol{y}} := -B_1^{-1}\delta_\lambda - B_2^{-1}\delta_{\boldsymbol{\mu}}$. Using this new variable, one derives from (5.64) and (5.66) that

$$\begin{aligned} \delta_\lambda &= h + \gamma_1 \chi_{\mathcal{A}_c} \tau_N \delta_{\boldsymbol{\mu}}, \\ \delta_{\boldsymbol{\mu}} &= \boldsymbol{k} + \gamma_2 \chi_{\mathcal{I}_f} \delta_{\boldsymbol{y}} + \gamma_2 \chi_{\mathcal{A}_f} W(\boldsymbol{y}) \delta_{\boldsymbol{y}}, \end{aligned} \tag{5.75}$$

where $\boldsymbol{y} = \tilde{\boldsymbol{y}} - B_1^{-1}\lambda - B_2^{-1}\boldsymbol{\mu}$, $W(\boldsymbol{y})$ is defined as in (5.71) and the active and inactive sets are those from (5.65) and (5.67). Using (5.75) and $\delta_{\boldsymbol{y}} := -B_1^{-1}\delta_\lambda - B_2^{-1}\delta_{\boldsymbol{\mu}}$, it is not difficult to verify that $\delta_{\boldsymbol{y}}$ is the unique solution to

$$\begin{aligned} \min_{\boldsymbol{\delta} \in \mathbf{Y}} \tilde{J}(\boldsymbol{\delta}) := \frac{1}{2} a(\boldsymbol{\delta}, \boldsymbol{\delta}) + (h, \tau_N \boldsymbol{\delta})_{\Gamma_c} + (\boldsymbol{k}, \boldsymbol{\tau}_T \boldsymbol{\delta})_{\Gamma_c} + \frac{\gamma_1}{2} \|\tau_N \boldsymbol{\delta}\|^2_{\mathcal{A}_c} \\ + \frac{\gamma_2}{2} \|\boldsymbol{\tau}_T \boldsymbol{\delta}\|^2_{\mathcal{I}_f} + \frac{\gamma_2}{2} \|W_1(\boldsymbol{y}) \boldsymbol{\tau}_T \boldsymbol{\delta}\|^2_{\mathcal{A}_f}, \end{aligned}$$

where W_1 is defined as for (5.73). From the coercivity of $a(\cdot\,,\cdot)$ and the fact that $\tilde{J}(\delta_{\boldsymbol{y}}) \le 0$ we conclude that

$$c\|\delta_{\boldsymbol{y}}\|^2_{\mathbf{Y}} \le \|h\|_{\Gamma_c} \|\delta_{\boldsymbol{y}}\|_{\mathbf{Y}} + \|\boldsymbol{k}\|_{\Gamma_c} \|\delta_{\boldsymbol{y}}\|_{\mathbf{Y}}.$$

Combining this with (5.75) shows that there exists a constant $C > 0$ such that

$$\|\delta_\lambda\|_{\Gamma_c} + \|\delta_{\boldsymbol{\mu}}\|_{\Gamma_c} \le C\big(\|h\|_{\Gamma_c} + \|\boldsymbol{k}\|_{\Gamma_c}\big).$$

This yields that $\|G_F(\lambda, \boldsymbol{\mu})^{-1}\|_{\mathcal{L}(L^2(\Gamma_c) \times \mathbf{L}^2(\Gamma_c), \mathbf{L}^2(\Gamma_c))} \le C$ for all $(\lambda, \boldsymbol{\mu}) \in L^2(\Gamma_c) \times \mathbf{L}^2(\Gamma_c)$ and ends the proof.

□

Let us point out that, differently from the generalized Newton methods for the solution of the Signorini contact problem, the function F involves two non-linearities. The max-function, which corresponds to an active set strategy in the generalized Newton method, and the quotient between a vector and its norm, which entails that the Newton method cannot fully be interpreted as active set strategy as for the pure contact problem. In the case of planar elasticity (i.e., for $n = 2$) this can be overcome by exchanging the complementarity function (5.59c). Doing so leads to a semi-smooth Newton method that can be seen from an active set perspective. We point out that (FC-SS) applies for arbitrary dimension, i.e., it can also be used for frictional contact problems in planar elasticity.

3.2. A different semi-smooth Newton method for plane elasticity. Here we present a slightly different generalized Newton method for the solution of the problems $(\mathcal{P}_{\gamma_1,\gamma_2})$ and $(\mathcal{P}^\star_{\gamma_1,\gamma_2})$ in case of plane elasticity, i.e., in case that $n = 2$. As observed in Section 2.3, in the 2D–case one can replace (5.59c) by a different function that involves as nonlinearities only the min- and max- operator. Aside from the fact that this can be advantageous for a numerical algorithm, another

advantage of this formulation (and the resulting algorithm) is that it can handle the case that $\mathfrak{F}g = 0$ on some set of positive measure as well. This case causes difficulties for (FC-SS), for this reason we had to assume that $\mathfrak{F}g \geq g_0 > 0$. The application of the semi-smooth Newton method to the modified NC-function leads to an iterative strategy that allows an interpretation as active set strategy. For the close relationship between active set strategies and a specific application of the semi-smooth Newton method see also [58].
According to the discussion in Section 2.3, in particular to equation (5.58), in the case of planar elasticity we are concerned with the solution of the system

$$a(\boldsymbol{y}_\gamma, \boldsymbol{z}) - L(\boldsymbol{z}) + (\mu_\gamma \boldsymbol{t}, \boldsymbol{\tau}_T \boldsymbol{z})_{\Gamma_c} + (\lambda_\gamma, \tau_N \boldsymbol{z})_{\Gamma_c} = 0 \text{ for all } \boldsymbol{z} \in \mathbf{Y}, \tag{5.76a}$$

$$\lambda_\gamma - \max(0, \hat{\lambda} + \gamma_1(\tau_N \boldsymbol{y}_\gamma - d)) = 0 \text{ on } \Gamma_c, \tag{5.76b}$$

$$\left\{ \begin{array}{l} \gamma_2(\xi_\gamma - \tau_T \boldsymbol{y}_\gamma) + \mu_\gamma - \hat{\mu} = 0, \\ \xi_\gamma - \max(0, \xi_\gamma + \sigma(\mu_\gamma - \mathfrak{F}g)) - \min(0, \xi_\gamma + \sigma(\mu_\gamma + \mathfrak{F}g)) = 0 \end{array} \right. \tag{5.76c}$$

on Γ_c, where $\xi_\gamma \in L^2(\Gamma_c)$, and $(\tau_T \boldsymbol{y}_\gamma)\boldsymbol{t} = \boldsymbol{\tau}_T \boldsymbol{y}_\gamma$ ($\boldsymbol{t}$ denoting the unit outward tangential vector along Γ_c), furthermore $\gamma_1, \gamma_2 > 0$, $\hat{\lambda}, \hat{\mu} \in L^2(\Gamma_c)$ are given and $\sigma > 0$ is arbitrary. Compared to the Newton method from the previous section that applies in the general case $n \geq 2$, the algorithm below is related closer to the methods investigated for the Signorini contact problem (Chapter 4) and the simplified friction problem (Chapter 3). We shall now present our algorithm, where for the iterates we again drop the index 'γ'.

Algorithm: (FC-SS2D)

(1) Choose $(\lambda^0, \xi^0, \mu^0, \boldsymbol{y}^0) \in L^2(\Gamma_c) \times L^2(\Gamma_c) \times L^2(\Gamma_c) \times \mathbf{Y}$, $\sigma > 0$ and set $k := 0$.
(2) Determine

$$\begin{aligned}
\mathcal{A}_c^{k+1} &= \{x \in \Gamma_c : \hat{\lambda} + \gamma_1(\tau_N \boldsymbol{y}^k - d) > 0\}, \\
\mathcal{I}_c^{k+1} &= \Gamma_c \setminus \mathcal{A}_c^{k+1}, \\
\mathcal{A}_{f,-}^{k+1} &= \{x \in \Gamma_c : \xi^k + \sigma(\mu^k + \mathfrak{F}g) < 0\}, \\
\mathcal{A}_{f,+}^{k+1} &= \{x \in \Gamma_c : \xi^k + \sigma(\mu^k - \mathfrak{F}g) > 0\}, \\
\mathcal{I}_f^{k+1} &= \Gamma_c \setminus (\mathcal{A}_{f,-}^{k+1} \cup \mathcal{A}_{f,+}^{k+1}).
\end{aligned}$$

(3) If $k \geq 1$, $\mathcal{A}_c^{k+1} = \mathcal{A}_c^k$, $\mathcal{A}_{f,-}^{k+1} = \mathcal{A}_{f,-}^k$ and $\mathcal{A}_{f,+}^{k+1} = \mathcal{A}_{f,+}^k$ stop, else

(4) Solve

$$
\begin{aligned}
&a(\boldsymbol{y}^{k+1},\boldsymbol{z})-L(\boldsymbol{z})+(\mu^{k+1}\boldsymbol{t},\boldsymbol{\tau}_T\boldsymbol{z})_{\Gamma_c}+(\lambda^{k+1},\tau_N\boldsymbol{z})_{\Gamma_c}=0 \text{ for all } \boldsymbol{z}\in\mathbf{Y},\\
&\lambda^{k+1}=0 \text{ on } \mathcal{I}_c^{k+1},\ \ \lambda^{k+1}=\hat{\lambda}+\gamma_1(\tau_N\boldsymbol{y}^{k+1}-d) \text{ on } \mathcal{A}_c^{k+1},\\
&\mu^{k+1}-\hat{\mu}-\gamma_2\tau_T\boldsymbol{y}^{k+1}=0 \text{ on } \mathcal{I}_f^{k+1},\\
&\mu^{k+1}=-\mathfrak{F}g \text{ on } \mathcal{A}_{f,-}^{k+1},\ \ \mu^{k+1}=\mathfrak{F}g \text{ on } \mathcal{A}_{f,+}^{k+1}.
\end{aligned}
$$

(5) Set

$$
\begin{aligned}
\lambda^{k+1}&:=\hat{\lambda}+\gamma_1(\tau_N\boldsymbol{y}^{k+1}-d) \text{ on } \mathcal{A}_c^{k+1},\\
\mu^{k+1}&:=\hat{\mu}+\gamma_2\tau_T\boldsymbol{y}^{k+1} \ \text{ on } \mathcal{I}_f^{k+1},\\
\xi^{k+1}&:=\begin{cases}\tau_T\boldsymbol{y}^{k+1}+\gamma_2^{-1}(\hat{\mu}+\mathfrak{F}g) & \text{on } \mathcal{A}_{f,-}^{k+1},\\ \tau_T\boldsymbol{y}^{k+1}+\gamma_2^{-1}(\hat{\mu}-\mathfrak{F}g) & \text{on } \mathcal{A}_{f,+}^{k+1},\\ 0 & \text{on } \mathcal{I}_f^{k+1},\end{cases}
\end{aligned}
$$

$k:=k+1$ and go to Step 2.

Note that a solution to the system in Step 4 exists and is unique, since it represents the necessary and sufficient optimality conditions for the equality constrained auxiliary problem

$$
\min_{\substack{\lambda=0 \text{ on } \mathcal{I}_c^{k+1},\\ \mu=-\mathfrak{F}g \text{ on } \mathcal{A}_{f,-}^{k+1},\ \mu=\mathfrak{F}g \text{ on } \mathcal{A}_{f,+}^{k+1}}} J^\star_{\gamma_1,\gamma_2}(\lambda,\mu\boldsymbol{t}),
$$

with $J^\star_{\gamma_1,\gamma_2}$ as defined in (5.36). One can prove the following lemma that justifies the stopping criterion in Step 3 of Algorithm (FC-SS2D).

LEMMA 5.7. *If Algorithm* (FC-SS2D) *stops, the last iterate $\boldsymbol{y}^k$ is the solution to* $(\mathcal{P}_{\gamma_1\gamma_2})$ *and the pair* $(\lambda^k,\mu^k\boldsymbol{t})$ *solves* $(\mathcal{P}^\star_{\gamma_1\gamma_2})$.

PROOF. The proof relies on the fact that, if the active sets coincide for two consecutive iterations, the iterates satisfy the complementarity conditions (5.76b) and (5.76c). Since the proof is very similar to those for Lemma 3.12 and Lemma 4.7, we omit the details. □

Provided we choose $\sigma=\gamma_2^{-1}$, the above algorithm can be interpreted as a semi-smooth Newton method in infinite-dimensional spaces. To prove this assertion, we consider a reduced system instead of (5.76). Thereby, as in the dual problem $(\mathcal{P}^\star_{\gamma_1\gamma_2})$, the primal variable $\boldsymbol{y}$ only acts as an auxiliary variable that depends on the dual variables (λ,μ). The reduction to the variables $\lambda,\boldsymbol{\mu}$ can be done in a similar fashion as in the previous section, therefore we omit the details.

We introduce the mapping $F : L^2(\Gamma_c) \times L^2(\Gamma_c) \to L^2(\Gamma_c) \times L^2(\Gamma_c)$ by

$$F(\lambda,\mu) = \begin{pmatrix} \lambda - \max(0, \hat{\lambda} + \gamma_1(\tau_N \boldsymbol{y}_{\lambda,\mu} - d)) \\ \gamma_2 \tau_T \boldsymbol{y}_{\lambda,\mu} + \hat{\mu} - \mu - \max(0, \gamma_2 \tau_T \boldsymbol{y}_{\lambda,\mu} + \hat{\mu} - \mathfrak{F}g) \dots \\ \qquad \dots - \min(0, \gamma_2 \tau_T \boldsymbol{y}_{\lambda,\mu} + \hat{\mu} + \mathfrak{F}g) \end{pmatrix}, \tag{5.77}$$

where, for given λ and μ we denote by $\boldsymbol{y}_{\lambda,\mu}$ the solution to

$$a(\boldsymbol{y},\boldsymbol{z}) - L(\boldsymbol{z}) + (\mu \boldsymbol{t}, \boldsymbol{\tau}_T \boldsymbol{z})_{\Gamma_c} + (\lambda, \tau_N \boldsymbol{z})_{\Gamma_c} = 0 \text{ for all } \boldsymbol{z} \in \mathbf{Y}. \tag{5.78}$$

Since $\tau_N \boldsymbol{y} \in H^{\frac{1}{2}}(\Gamma_c)$ and $\boldsymbol{\tau}_T \boldsymbol{y} \in \mathbf{H}^{\frac{1}{2}}(\Gamma_c)$, we observe the norm gap required for Newton differentiability of the max- and min- functional in (5.77). Thus, we can apply the semi-smooth Newton method from Section 2, Chapter 2 to the equation $F(\lambda,\mu) = 0$. Calculating the explicit form of the Newton step leads to Algorithm (FC-SS2D) with $\sigma = \gamma_2^{-1}$. This close relationship between active set strategies and a specific application of semi-smooth Newton methods, first observed in [58], leads to the following local convergence result for Algorithm (FC-SS2D).

THEOREM 5.8. *Suppose that there exists a constant $g_0 > 0$ such that $\mathfrak{F}g \geq g_0$, further that $\sigma \geq \gamma_2^{-1}$ and that $\|\lambda^0 - \lambda_\gamma\|_{\Gamma_c}$, $\|\boldsymbol{\mu}^0 - \boldsymbol{\mu}_\gamma\|_{\Gamma_c}$ are sufficiently small. Then the iterates $(\lambda^k, \xi^k, \mu^k, \boldsymbol{y}^k)$ of* (FC-SS2D) *converge superlinearly to $(\lambda_\gamma, \xi_\gamma, \mu_\gamma, \boldsymbol{y}_\gamma)$ in $L^2(\Gamma_c) \times L^2(\Gamma_c) \times L^2(\Gamma_c) \times \mathbf{Y}$.*

PROOF. The proof consists of two steps: First we prove the assertion for $\sigma = \gamma_2^{-1}$ and then we utilize this result for the general case $\sigma \geq \gamma_2^{-1}$.
Step 1: For $\sigma = \gamma_2^{-1}$ (FC-SS2D) can be interpreted as semi-smooth Newton method for the equation $F(\lambda,\mu) = 0$ (F as defined in (5.77)). We already argued Newton differentiability of F. To apply Theorem 2.9 it remains to show that the generalized derivatives have uniformly bounded inverses. This can be done in a similar fashion as in the proof of Theorem 3.15 and the proof for Theorem 2.2 in [69]. Hence, we can apply Theorem 2.9 to obtain superlinear convergence of the iterates (λ^k, μ^k) in $L^2(\Gamma_c) \times L^2(\Gamma_c)$. Clearly, this convergence carries over to the variables μ^k and $\boldsymbol{y}^k$.
Step 2: For $\sigma > \gamma_2^{-1}$ we cannot use the above argument directly. Nevertheless, one can prove superlinear convergence of the iterates by showing that in a neighborhood of the solution the iterates of (FC-SS2D) with $\sigma > \gamma_2^{-1}$ coincide with those of (FC-SS2D) with $\sigma = \gamma_2^{-1}$. The argumentation for this fact exploits the smoothing properties of the Neumann-to-Dirichlet mapping for the elasticity equation: First, we again consider the case $\sigma = \gamma_2^{-1}$. Clearly, for all $k \geq 1$ we have $\lambda^k - \lambda^{k-1} \in L^2(\Gamma_c)$ and $\mu^k - \mu^{k-1} \in L^2(\Gamma_c)$. The corresponding difference $\boldsymbol{y}^k - \boldsymbol{y}^{k-1}$ of the primal variables satisfies

$$a(\boldsymbol{y}^k - \boldsymbol{y}^{k-1}, \boldsymbol{z}) + \big((\mu^k - \mu^{k-1})\boldsymbol{t}, \boldsymbol{\tau}_T \boldsymbol{z}\big)_{\Gamma_c} + (\lambda^k - \lambda^{k-1}, \tau_N \boldsymbol{z})_{\Gamma_c} = 0 \ \text{ for all } \boldsymbol{z} \in \mathbf{Y}.$$

From regularity results for systems of elliptic variational equalities (see [13]) it follows that there exists a constant $C > 0$ such that

$$\|y^k - y^{k-1}\|_{C^0(\Gamma_c)} \le C\big(\|\lambda^k - \lambda^{k-1}\|_{\Gamma_c} + \|\mu^k - \mu^{k-1}\|_{\Gamma_c}\big). \tag{5.79}$$

We now show that (5.79) implies

$$\mathcal{A}^k_{f,-} \cap \mathcal{A}^{k+1}_{f,+} = \mathcal{A}^k_{f,+} \cap \mathcal{A}^{k+1}_{f,-} = \emptyset \tag{5.80}$$

provided that $\|\lambda^0 - \lambda_\gamma\|_{\Gamma_c}$ and $\|\mu^0 - \mu_\gamma\|_{\Gamma_c}$ are sufficiently small. If $\mathcal{B} := \mathcal{A}^k_{f,-} \cap \mathcal{A}^{k+1}_{f,+} \neq \emptyset$, then it follows that $y^{k-1} + \gamma_2^{-1}(\hat\mu + \mathfrak{F}g) < 0$ and $y^k + \gamma_2^{-1}(\hat\mu - \mathfrak{F}g) > 0$ on $\mathcal{B}$, which implies that $y^k - y^{k-1} > 2\gamma_2^{-1}\mathfrak{F}g \ge 2\gamma_2^{-1}\mathfrak{F}g_0 > 0$ on $\mathcal{B}$. This contradicts (5.79) provided that $\|\lambda^0 - \lambda_\gamma\|_{\Gamma_c}$ and $\|\mu^0 - \mu_\gamma\|_{\Gamma_c}$ are sufficiently small. Analogously, one can show that $\mathcal{A}^k_{f,+} \cap \mathcal{A}^{k+1}_{f,-} = \emptyset$.

We now choose an arbitrary $\sigma \ge \gamma_2^{-1}$ and assume that (5.80) holds for (FC-SS2D) if σ was chosen γ_2^{-1}. Then we can argue that in a neighborhood of the solution the iterates of (FC-SS2D) are independent of $\sigma \ge \gamma_2^{-1}$. To verify this assertion we separately consider the sets $\mathcal{I}^k_f, \mathcal{A}^k_{f,-}$ and $\mathcal{A}^k_{f,+}$. On $\mathcal{I}^k_f$ we have that $\xi^k = 0$ and thus σ has no influence when determining the new active and inactive sets. On the set $\mathcal{A}^k_{f,-}$ we have that $\mu^k = -\mathfrak{F}g$. Here, we consider two types of sets: Firstly, sets where $\xi^k < 0$ belong to $\mathcal{A}^{k+1}_{f,-}$ for the next iteration independently from σ. And, secondly, if $\xi^k > 0$ we have

$$\xi^k + \sigma(\mu^k - \mathfrak{F}g) = \xi^k - 2\sigma\mathfrak{F}g.$$

Sets where $\xi^k - 2\sigma\mathfrak{F}g \le 0$ are transfered to $\mathcal{I}^{k+1}_f$, and those where $0 < \xi^k - 2\sigma\mathfrak{F}g \le \xi^k - 2\gamma_2^{-1}\mathfrak{F}g$ belong to $\mathcal{A}^{k+1}_{f,+}$ for the next iteration. However, the case that $x \in \mathcal{A}^k_{f,-} \cap \mathcal{A}^{k+1}_{f,+}$ cannot occur for $\sigma \ge \gamma_2^{-1}$, since it is already ruled out by (5.80) for $\sigma = \gamma_2^{-1}$.

This shows that in a neighborhood of the solution the iterates are the same for all $\sigma \ge \gamma_2^{-1}$, and thus the superlinear convergence result from Step 1 carries over to the general case $\sigma \ge \gamma_2^{-1}$, which ends the proof. □

In the numerical realization of the algorithm (FC-SS2D) it turns out that choosing small values for σ may not be optimal, since it can lead to the following behavior: Points that are active with respect to the upper bound become active with respect to the lower bound in the next iteration, and vice versa. This may lead to cycling of the iterates. Such undesired behavior can be overcome by choosing larger values for σ, e.g., $\sigma = 1$. The above theorem proves local superlinear convergence of the iterates for all $\sigma \ge \gamma_2^{-1}$, though only the choice $\sigma = \gamma_2^{-1}$ allows interpretation as semi-smooth Newton method in infinite dimensions.

3.3. Exact and inexact augmented Lagrangian methods. There are several possibilities to utilize a sequence of solutions for the regularized problem with Tresca friction ($\mathcal{P}_{\gamma_1\gamma_2}$) for the solution of the original problem with Tresca friction ($\mathcal{P}$). Clearly, one approach is, letting the regularization parameters tend to infinity. Then, Theorem 5.3 guarantees that the regularized solution variables converge to the solution of the original problem. One disadvantage of this method is that large regularization parameters may lead to illconditioned systems. A different approach utilizes the first-order augmented Lagrangian method that is based on an update strategy for the dual variables. This method is related to the Uzawa algorithm, a commonly used method for the treatment of saddle point problems. The augmented Lagrangian approach applies for the solution of contact problems with given friction in infinite dimensions provided that the solution variables $(\bar{\lambda}, \bar{\boldsymbol{\mu}}) \in L^2(\Gamma_c) \times \mathbf{L}^2(\Gamma_c)$. Assuming that this regularity holds, we present an exact as well as an inexact augmented Lagrangian method for the solution of $(\mathcal{P})$ and $(\mathcal{P}^\star)$. For the application of the method we can exploit the fact that the auxiliary problems in every iteration step of the augmented Lagrangian method coincide with the regularized problems discussed in the previous sections. The main advantage of the augmented Lagrangian compared to the pure penalty approach is that the penalty parameters can be chosen small, nevertheless the method converges to the solution of the original problem. This circumvents the usage of large penalty parameters and may thus be advantageous in numerical practice. The methods presented in this section apply for the solution of the contact problem with Tresca friction regardlessly of the dimension of Ω.

Augmented Lagrangian method. In this and the following section we assume that $\bar{\lambda} \in L^2(\Gamma_c)$, i.e., the multiplier corresponding to the contact condition in $(\mathcal{P})$ is square integrable. For the pure contact problem Section 4.1 gives conditions that guarantee this regularity. Note that in Section 1.3 it is shown that L^2-regularity always holds true for $\bar{\boldsymbol{\mu}}$ provided $g \in L^2(\Gamma_c)$, which we henceforth assume.

As the Uzawa algorithm, the augmented Lagrangian method is an update strategy for the dual variables in $(\mathcal{P})$ and $(\mathcal{P}^\star)$. It can be considered as an implicit version of Uzawa's algorithm (cp. [67]). Its main advantage compared to the latter strategy is its unconditional convergence for all penalty (or regularization) parameters $\gamma_1, \gamma_2 > 0$, whereas Uzawa's method only converges for sufficiently small (and possibly very small) penalty parameters, which may lead to extremely slow convergence. However, a drawback of the methods presented here is that every iteration step requires to solve a nonlinear problem compared to the linear auxiliary problem in the Uzawa algorithm. Since this nonlinear problem is exactly of the form $(\mathcal{P}^\star_{\gamma_1,\gamma_2})$, we can use the strategies presented in the previous sections for the solution of this auxiliary problem. The first-order augmented Lagrangian method is specified next.

Algorithm: (FC-ALM)

(1) Choose $(\lambda^0, \boldsymbol{\mu}^0) \in L^2(\Gamma_c) \times \mathbf{L}^2(\Gamma_c)$ and set $l := 0$.

(2) Choose $\gamma_1^{l+1}, \gamma_2^{l+1} > 0$, and solve $(\mathcal{P}_{\gamma_1\gamma_2})$ with $\hat{\lambda} := \lambda^l$ and $\hat{\boldsymbol{\mu}} = \boldsymbol{\mu}^l$, i.e., determine $(\boldsymbol{y}^{l+1}, \lambda^{l+1}, \boldsymbol{\mu}^{l+1}) \in \mathbf{Y} \times L^2(\Gamma_c) \times \mathbf{L}^2(\Gamma_c)$ such that

$$a(\boldsymbol{y}^{l+1}, \boldsymbol{z}) - L(\boldsymbol{z}) + (\lambda^{l+1}, \tau_N \boldsymbol{z})_{\Gamma_c} + (\boldsymbol{\mu}^{l+1}, \boldsymbol{\tau}_T \boldsymbol{z})_{\Gamma_c} = 0 \ \text{ for all } \boldsymbol{z} \in \mathbf{Y},$$

$$\lambda^{l+1} = \max(0, \lambda^l + \gamma^{l+1}(\tau_N \boldsymbol{y}^{l+1} - d)) \ \text{ on } \Gamma_c,$$

$$\mathfrak{F}g(\gamma_2 \boldsymbol{\tau}_T \boldsymbol{y}^{l+1} + \boldsymbol{\mu}^l) - \max(\mathfrak{F}g, \|\gamma_2 \boldsymbol{\tau}_T \boldsymbol{y}^{l+1} + \boldsymbol{\mu}^l\|)\boldsymbol{\mu}^{l+1} = 0 \ \text{ on } \Gamma_c.$$

(3) Update $l := l + 1$ and go to Step 2.

For the following convergence proof we denote by $\bar{\boldsymbol{y}}$ and $(\bar{\lambda}, \bar{\boldsymbol{\mu}})$ the solutions of $(\mathcal{P})$ and $(\mathcal{P}^\star)$, respectively. Then the following global convergence result (i.e., convergence from arbitrary initialization) holds true.

THEOREM 5.9. *For every choice of parameters* $0 < \gamma_1^0 \le \gamma_1^1 \le \gamma_1^2 \le \dots$ *and* $0 < \gamma_2^0 \le \gamma_2^1 \le \gamma_2^2 \le \dots$ *the iterates* $(\lambda^l, \boldsymbol{\mu}^l)$ *of* (FC-ALM) *converge weakly to* $(\bar{\lambda}, \bar{\boldsymbol{\mu}})$ *in* $L^2(\Gamma_c) \times \mathbf{L}^2(\Gamma_c)$. *In addition, the corresponding iterates* $\boldsymbol{y}^l$ *converge strongly to* $\bar{\boldsymbol{y}}$ *in* $\mathbf{Y}$.

PROOF. The proof follows from results of the next section. □

Let us briefly comment on the role of the parameters γ_1^l, γ_2^l in (FC-ALM). One may start with moderate values in Step 2 of (FC-ALM) and increase these values during the iteration. However, the iterates of (FC-ALM) converge without requiring that γ_1^l, γ_2^l tend to infinity which is a strong difference to penalty methods. Next we turn to a generalized version of (FC-ALM) that allows a certain inexactness in the bilinear from $a(\cdot\,,\cdot)$.

Inexact augmented Lagrangian methods. The inexact version of the augmented Lagrangian method presented in this chapter is motivated from inexact versions of the Uzawa algorithm [19, 25–27]. On the one hand it provides a convergence result for a preconditioned version of the augmented Lagrangian method, on the other hand it is of interest with respect to the stability of numerical implementations of the first-order augmented Lagrangian method.
Let $\tilde{a}(\cdot\,,\cdot)$ be a scalar product that approximates $a(\cdot\,,\cdot)$ in the sense that there is an $\omega \ge 1$ such that

$$a(\boldsymbol{z}, \boldsymbol{z}) \le \tilde{a}(\boldsymbol{z}, \boldsymbol{z}) \le \omega a(\boldsymbol{z}, \boldsymbol{z}) \text{ for all } \boldsymbol{z} \in \mathbf{Y}. \tag{5.81}$$

This approximation property can be discussed analogously as for the pure contact problem, see page 70. We now specify the inexact augmented Lagrangian method for the contact problem with Tresca friction.

Algorithm: (FC-IALM)

(1) Choose $(\lambda^0, \boldsymbol{\mu}^0) \in L^2(\Gamma_c) \times \mathbf{L}^2(\Gamma_c)$ and set $l := 0$.

(2) Choose $\gamma_1^{l+1}, \gamma_2^{l+1} > 0$ and determine $(\boldsymbol{y}^{l+1}, \lambda^{l+1}, \boldsymbol{\mu}^{l+1})$ such that

$$\begin{aligned}
&\tilde{a}(\boldsymbol{y}^{l+1}, \boldsymbol{z}) - \tilde{a}(\boldsymbol{y}^{l}, \boldsymbol{z}) \\
&+ \left\{a(\boldsymbol{y}^{l}, \boldsymbol{z}) - L(\boldsymbol{z}) + (\lambda^{l+1}, \tau_N \boldsymbol{z})_{\Gamma_c} + (\boldsymbol{\mu}^{l+1}, \boldsymbol{\tau}_T \boldsymbol{z})_{\Gamma_c}\right\} = 0 \text{ for all } \boldsymbol{z} \in \mathbf{Y}, \\
&\lambda^{l+1} = \max(0, \lambda^l + \gamma_1^{l+1}(\tau_N \boldsymbol{y}^{l+1} - d)) \;\text{ on } \Gamma_c, \\
&\mathfrak{F}g(\gamma_2 \boldsymbol{\tau}_T \boldsymbol{y}^{l+1} + \boldsymbol{\mu}^l) - \max(\mathfrak{F}g, \|\gamma_2 \boldsymbol{\tau}_T \boldsymbol{y}^{l+1} + \boldsymbol{\mu}^l\|)\boldsymbol{\mu}^{l+1} = 0 \;\text{ on } \Gamma_c.
\end{aligned}$$

(3) Update $l := l + 1$ and go to Step 2.

Provided the scalar product $\tilde{a}(\cdot\,,\cdot)$ is a sufficiently good approximation for $a(\cdot\,,\cdot)$, we can prove convergence of the iterates of (FC-IALM) from any initialization. Observe that for $\tilde{a}(\cdot\,,\cdot) = a(\cdot\,,\cdot)$, the above algorithm coincides with (FC-ALM). Hence, by proving the next theorem we are also proving Theorem 5.9. Again we denote by $\bar{\boldsymbol{y}}$ and $(\bar{\lambda}, \bar{\boldsymbol{\mu}})$ the solutions of $(\mathcal{P})$ and $(\mathcal{P}^\star)$, respectively.

THEOREM 5.10. *Suppose that* (5.81) *holds for some* $\omega < 5$. *Then for every choice of parameters* $0 < \gamma_1^0 \le \gamma_1^1 \le \gamma_1^2 \le \dots$ *and* $0 < \gamma_2^0 \le \gamma_2^1 \le \gamma_2^2 \le \dots$ *the iterates* $(\lambda^l, \boldsymbol{\mu}^l)$ *of* (FC-IALM) *converge weakly to* $(\bar{\lambda}, \bar{\boldsymbol{\mu}})$ *in* $L^2(\Gamma_c) \times \mathbf{L}^2(\Gamma_c)$. *Moreover, the corresponding iterates* $\boldsymbol{y}^l$ *converge strongly to* $\bar{\boldsymbol{y}}$ *in* $\mathbf{Y}$.

PROOF. Let us denote $\delta_y^l := \boldsymbol{y}^l - \bar{\boldsymbol{y}} \in \mathbf{Y}$, $\delta_\lambda^l := \lambda^l - \bar{\lambda} \in L^2(\Gamma_c)$ and $\delta_\mu^l := \boldsymbol{\mu}^l - \bar{\boldsymbol{\mu}} \in \mathbf{L}^2(\Gamma_c)$, where $\bar{\boldsymbol{y}}$ and $(\bar{\lambda}, \bar{\boldsymbol{\mu}})$ denote the solution variables of $(\mathcal{P})$ and $(\mathcal{P}^\star)$, respectively. Since $(\bar{\boldsymbol{y}}, \bar{\lambda}, \bar{\boldsymbol{\mu}})$ satisfies (5.22a), we have for $l \ge 1$ that

$$a(\bar{\boldsymbol{y}}, \delta_y^{l+1}) - L(\delta_y^{l+1}) + (\bar{\lambda}, \tau_N \delta_y^{l+1})_{\Gamma_c} + (\bar{\boldsymbol{\mu}}, \boldsymbol{\tau}_T \delta_y^{l+1})_{\Gamma_c} = 0, \tag{5.82}$$

and from step 2 in (FC-IALM) we obtain

$$\begin{aligned}\tilde{a}(\boldsymbol{y}^{l+1} - \boldsymbol{y}^l, \delta_y^{l+1}) &+ a(\boldsymbol{y}^l, \delta_y^{l+1}) - L(\delta_y^{l+1}) \\ &+ (\lambda^{l+1}, \tau_N \delta_y^{l+1})_{\Gamma_c} + (\boldsymbol{\mu}^{l+1}, \boldsymbol{\tau}_T \delta_y^{l+1})_{\Gamma_c} = 0.\end{aligned} \tag{5.83}$$

Subtracting (5.82) from (5.83) results in

$$\begin{aligned}
0 &= \tilde{a}(\boldsymbol{y}^{l+1} - \boldsymbol{y}^l, \delta_y^{l+1}) + a(\boldsymbol{y}^l - \bar{\boldsymbol{y}}, \delta_y^{l+1}) \\
&\quad + (\lambda^{l+1} - \bar{\lambda}, \tau_N \delta_y^{l+1})_{\Gamma_c} + (\boldsymbol{\mu}^{l+1} - \bar{\boldsymbol{\mu}}, \boldsymbol{\tau}_T \delta_y^{l+1})_{\Gamma_c} \\
&= \tilde{a}(\delta_y^{l+1} - \delta_y^l, \delta_y^{l+1}) + a(\delta_y^l, \delta_y^{l+1}) + (\delta_\lambda^{l+1}, \tau_N \delta_y^{l+1})_{\Gamma_c} + (\delta_\mu^{l+1}, \boldsymbol{\tau}_T \delta_y^{l+1})_{\Gamma_c}.
\end{aligned} \tag{5.84}$$

Due to Step (2) of (FC-IALM), (5.22b) and (5.22c) we have

$$\lambda^{l+1} = P_1(\lambda^l + \gamma_1^{l+1}(\tau_N \boldsymbol{y}^{l+1} - d)),\; \bar{\lambda} = P_1(\bar{\lambda} + \gamma_1^{l+1}(\tau_N \bar{\boldsymbol{y}} - d)), \tag{5.85}$$

$$\boldsymbol{\mu}^{l+1} = P_2(\boldsymbol{\mu}^l + \gamma_2^{l+1} \boldsymbol{\tau}_T \boldsymbol{y}^{l+1}) \;\text{ and }\; \bar{\boldsymbol{\mu}} = P_2(\bar{\boldsymbol{\mu}} + \gamma_2^{l+1} \boldsymbol{\tau}_T \bar{\boldsymbol{y}}), \tag{5.86}$$

where $P_1 : L^2(\Gamma_c) \to L^2(\Gamma_c)$ denotes the pointwise projection onto the convex set $K_1 = \{\xi \in L^2(\Gamma_c) : \xi \ge 0 \text{ a.e.}\}$, and $P_2 : \mathbf{L}^2(\Gamma_c) \to \mathbf{L}^2(\Gamma_c)$ the projection onto the convex $\mathbf{L}^2$-ball $K_2 = \{\boldsymbol{\xi} \in \mathbf{L}^2(\Gamma_c) : \|\boldsymbol{\xi}\| \leq \mathfrak{F}g \text{ a.e.}\}$. From the properties of projections onto convex sets, in particular (2.2), one gets

$$\big(\lambda^{l+1} - \bar{\lambda}, (\lambda^l + \gamma_1^{l+1}(\tau_N \boldsymbol{y}^{l+1} - d)) - \lambda^{l+1}) - (\bar{\lambda} + \gamma_1^{l+1}(\tau_N \bar{\boldsymbol{y}} - d) - \bar{\lambda})\big)_{\Gamma_c} \geq 0,$$

$$\big(\boldsymbol{\mu}^{l+1} - \bar{\boldsymbol{\mu}}, (\boldsymbol{\mu}^l + \gamma_2^{l+1}\boldsymbol{\tau}_T \boldsymbol{y}^{l+1} - \boldsymbol{\mu}^{l+1}) - (\bar{\boldsymbol{\mu}} + \gamma_2^{l+1}\boldsymbol{\tau}_T \bar{\boldsymbol{y}} - \bar{\boldsymbol{\mu}})\big)_{\Gamma_c} \geq 0.$$

Thus, we obtain

$$\begin{aligned}
&(\delta_\lambda^{l+1}, \tau_N \delta_y^{l+1})_{\Gamma_c} + (\delta_\mu^{l+1}, \boldsymbol{\tau}_T \delta_y^{l+1})_{\Gamma_c} \\
&= (\gamma_1^{l+1})^{-1} \big(\lambda^{l+1} - \bar{\lambda}, (\lambda^l + \gamma_1^{l+1}(\tau_N \boldsymbol{y}^{l+1} - d)) - (\bar{\lambda} + \gamma_1^{l+1}(\tau_N \bar{\boldsymbol{y}} - d)\big)_{\Gamma_c} \\
&\quad + (\gamma_2^{l+1})^{-1} \big(\boldsymbol{\mu}^{l+1} - \bar{\boldsymbol{\mu}}, (\boldsymbol{\mu}^l + \gamma_2^{l+1}\boldsymbol{\tau}_T \boldsymbol{y}^{l+1}) - (\bar{\boldsymbol{\mu}} + \gamma_2^{l+1}\boldsymbol{\tau}_T \bar{\boldsymbol{y}}\big)_{\Gamma_c} \\
&\quad - (\gamma_1^{l+1})^{-1}(\lambda^{l+1} - \bar{\lambda}, \lambda^l - \bar{\lambda})_{\Gamma_c} - (\gamma_2^{l+1})^{-1}(\boldsymbol{\mu}^{l+1} - \bar{\boldsymbol{\mu}}, \boldsymbol{\mu}^l - \bar{\boldsymbol{\mu}})_{\Gamma_c} \\
&\geq (\gamma_1^{l+1})^{-1}\|\lambda^{l+1} - \bar{\lambda}\|_{\Gamma_c}^2 - (\gamma_1^{l+1})^{-1}(\lambda^{l+1} - \bar{\lambda}, \lambda^l - \bar{\lambda})_{\Gamma_c} \\
&\quad + (\gamma_2^{l+1})^{-1}\|\boldsymbol{\mu}^{l+1} - \bar{\boldsymbol{\mu}}\|_{\Gamma_c}^2 - (\gamma_2^{l+1})^{-1}(\boldsymbol{\mu}^{l+1} - \bar{\boldsymbol{\mu}}, \boldsymbol{\mu}^l - \bar{\boldsymbol{\mu}})_{\Gamma_c}
\end{aligned}$$

$$\geq \frac{1}{2\gamma_1^{l+1}}\|\delta_\lambda^{l+1}\|_{\Gamma_c}^2 - \frac{1}{2\gamma_1^{l+1}}\|\delta_\lambda^l\|_{\Gamma_c}^2 + \frac{1}{2\gamma_2^{l+1}}\|\delta_\mu^{l+1}\|_{\Gamma_c}^2 - \frac{1}{2\gamma_2^{l+1}}\|\delta_\mu^l\|_{\Gamma_c}^2. \tag{5.87}$$

Let us now turn to the estimation of $\tilde{a}(\delta_y^{l+1} - \delta_y^l, \delta_y^{l+1}) + a(\delta_y^l, \delta_y^{l+1})$.

$$\begin{aligned}
&\tilde{a}(\delta_y^{l+1} - \delta_y^l, \delta_y^{l+1}) + a(\delta_y^l, \delta_y^{l+1}) \\
&= \tilde{a}(\tfrac{1}{2}\delta_y^l - \delta_y^{l+1}, \tfrac{1}{2}\delta_y^l - \delta_y^{l+1}) - \frac{1}{4}\tilde{a}(\delta_y^l, \delta_y^l) + a(\delta_y^l, \delta_y^{l+1}) \\
&\geq a(\tfrac{1}{2}\delta_y^l - \delta_y^{l+1}, \tfrac{1}{2}\delta_y^l - \delta_y^{l+1}) - \frac{\omega}{4}a(\delta_y^l, \delta_y^l) + a(\delta_y^l, \delta_y^{l+1}) \\
&= \frac{1}{4}a(\delta_y^l, \delta_y^l) + a(\delta_y^{l+1}, \delta_y^{l+1}) - \frac{\omega}{4}a(\delta_y^l, \delta_y^l) \\
&= \frac{1}{4}(1 - \omega)a(\delta_y^l, \delta_y^l) + a(\delta_y^{l+1}, \delta_y^{l+1}).
\end{aligned}$$

Utilizing the above estimate, (5.84) and (5.87) yield that

$$\begin{aligned}
&\frac{1}{2\gamma_1^{l+1}}\|\delta_\lambda^{l+1}\|_{\Gamma_c}^2 + \frac{1}{2\gamma_2^{l+1}}\|\delta_\mu^{l+1}\|_{\Gamma_c}^2 \\
\leq &\frac{1}{2\gamma_1^{l+1}}\|\delta_\lambda^l\|_{\Gamma_c}^2 + \frac{1}{2\gamma_2^{l+1}}\|\delta_\mu^l\|_{\Gamma_c}^2 + \frac{1}{4}(\omega - 1)a(\delta_y^l, \delta_y^l) - a(\delta_y^{l+1}, \delta_y^{l+1}).
\end{aligned}$$

Introducing the auxiliary variable

$$\kappa^l := \frac{1}{2\gamma_1^l}\|\delta_\lambda^l\|_{\Gamma_c}^2 + \frac{1}{2\gamma_2^l}\|\delta_\mu^l\|_{\Gamma_c}^2 + a(\delta_y^l, \delta_y^l),$$

the above estimate implies

$$\begin{aligned}\kappa^{l+1} &\le \frac{1}{2\gamma_1^{l+1}}\|\delta_\lambda^l\|_{\Gamma_c}^2 + \frac{1}{2\gamma_2^{l+1}}\|\delta_\mu^l\|_{\Gamma_c}^2 + \frac{1}{4}(\omega-1)a(\delta_y^l,\delta_y^l)\\ &\le \kappa^l - a(\delta_y^l,\delta_y^l) + \frac{1}{4}(\omega-1)a(\delta_y^l,\delta_y^l)\\ &= \kappa^l + \frac{1}{4}(\omega-5)a(\delta_y^l,\delta_y^l),\end{aligned}$$

where $\gamma_1^l \le \gamma_1^{l+1}$ and $\gamma_2^l \le \gamma_2^{l+1}$ were used. Since $1 \le \omega < 5$, the sequence κ^l is monotonically decreasing, it is obviously bounded from below, and thus convergent. Hence,

$$\lim_{l\to\infty} a(\delta_y^l,\delta_y^l) = 0, \tag{5.88}$$

resulting in $\boldsymbol{y}^l \to \bar{\boldsymbol{y}}$ strongly in $\mathbf{Y}$. From Step 2 of (FC-IALM) follows that $\lambda^l \to \bar\lambda$ weakly in $L^2(\Gamma_c)$ and $\boldsymbol{\mu}^l \to \bar{\boldsymbol{\mu}}$ weakly in $\mathbf{L}^2(\Gamma_c)$. □

4. The Contact Problem with Coulomb Friction

In this section we presents results on the contact problem with Coulomb friction. Firstly, a regularized Coulomb friction problem is discussed, and existence of a solution for this problem is proved. Then, possible generalizations of this approach for the Coulomb friction problem without regularization are discussed. In the preceding section we briefly state the fixed point algorithm for the numerical realization of the regularized Coulomb friction problem that makes use of solutions of the contact problem with Tresca friction. Moreover, an algorithm that combines the fixed point idea and the augmented Lagrangian update for the solution of the Coulomb friction problem is proposed.

4.1. A regularized Coulomb friction problem. We now state a regularized version of the contact problem with Coulomb friction. The regularization in the Coulomb problem corresponds to the regularization in $(\mathcal{P}_{\gamma_1,\gamma_2})$ and $(\mathcal{P}^\star_{\gamma_1,\gamma_2})$ for the problem with given friction. We first formulate the smoothed Coulomb friction problem as variational inequality and derive equivalent ways to state this problem. The smoothing leads to a welldefined problem that does not face the difficulties of (5.5), we refer to the discussion at the end of Section 1.1. The variational formulation of the problem to be considered is

$$\begin{aligned}&a(\boldsymbol{y},\boldsymbol{z}-\boldsymbol{y}) + \big(\max(0,\hat\lambda+\gamma_1(\tau_N\boldsymbol{y}-d)),\tau_N(\boldsymbol{z}-\boldsymbol{y})\big)_{\Gamma_c} - L(\boldsymbol{z}-\boldsymbol{y})\\ &\quad+\int_{\Gamma_c}\mathfrak{F}\max(0,\hat\lambda+\gamma_1(\tau_N\boldsymbol{y}-d))\,\{h(\boldsymbol{\tau}_T\boldsymbol{z},\hat{\boldsymbol{\mu}})-h(\boldsymbol{\tau}_T\boldsymbol{y},\hat{\boldsymbol{\mu}})\}\,dx \ge 0\end{aligned} \tag{5.89}$$

for all $\boldsymbol{z}\in\mathbf{Y}$, with $h(\cdot\,,\cdot)$ as defined in (5.37). The variational inequality (5.89) shows, that $\boldsymbol{v}=\boldsymbol{y}$ is the solution to the problem

$$\min_{\boldsymbol{v}\in\mathbf{Y}} a(\boldsymbol{y},\boldsymbol{v}) + \big(\lambda_\gamma,\tau_N\boldsymbol{v}\big)_{\Gamma_c} - L(\boldsymbol{v}) + \int_{\Gamma_c}\mathfrak{F}\lambda_\gamma\,h(\boldsymbol{\tau}_T\boldsymbol{v},\hat{\boldsymbol{\mu}})\,dx, \tag{5.90}$$

where $\lambda_\gamma = \max(0, \hat\lambda + \gamma_1(\tau_N \boldsymbol{y} - d))$. Due to the differentiability of $h(\cdot\,,\cdot)$ with respect to the first variable (we denote the derivative by $h'(\cdot\,,\cdot)$), the first-order necessary conditions for the minimization problem (5.90) lead to a variational equality, namely to

$$\begin{aligned} a(\boldsymbol{y},\boldsymbol{z}) + \big(\max(0,\hat\lambda + \gamma_1(\tau_N\boldsymbol{y} - d)), \tau_N \boldsymbol{z}\big)_{\Gamma_c} - L(\boldsymbol{z}) \\ + \int_{\Gamma_c} \mathfrak{F} \max(0, \hat\lambda + \gamma_1(\tau_N \boldsymbol{y} - d)) h'(\boldsymbol{\tau}_T \boldsymbol{z}, \hat{\boldsymbol{\mu}})\, dx = 0. \end{aligned} \tag{5.91}$$

Thus, we have found an equivalent way to characterize the solution of (5.89). In what follows, we derive a third possibility to characterize the solution of the regularized Coulomb friction contact problem. Similarly as in Section 1.4 for the original Coulomb friction problem, one can obtain the solution for the regularized Coulomb friction problem by means of a sequence of regularized Tresca friction problems. We introduce the cone of nonnegative L^2-functions

$$L^2_+(\Gamma_c) := \{\xi \in L^2(\Gamma_c) : \xi \geq 0 \text{ a.e.}\}$$

and the mapping

$$\Psi_\gamma : L^2_+(\Gamma_c) \longrightarrow L^2_+(\Gamma_c)$$

defined by

$$\Psi_\gamma(g) := \lambda_\gamma,$$

where λ_γ is given by

$$\lambda_\gamma = \max(0, \hat\lambda + \gamma_1(\tau_N \boldsymbol{y}_\gamma - d))$$

with $\boldsymbol{y}_\gamma$ denoting the unique solution of the regularized contact problem with friction $g \in L^2_+(\Gamma_c)$. Then, it can be verified easily that the variable $\boldsymbol{y}_\gamma$ corresponding to a fixed point of the mapping Ψ_γ solves (5.89).

It is the aim of this section to prove that the regularized problem with Coulomb friction (5.89) admits a solution. This is achieved by characterizing the solution as fixed point of the mapping Ψ_γ. In a first step, we investigate the mapping

$$\Phi_\gamma : L^2_+(\Gamma_c) \longrightarrow \mathbf{Y}, \tag{5.92}$$

that maps a given friction $g \in L^2_+(\Gamma_c)$ to the corresponding solution $\boldsymbol{y}_\gamma$ of $(\mathcal{P}_{\gamma_1,\gamma_2})$. In the next lemma we show that the mapping Φ_γ is Lipschitz-continuous.

Lemma 5.11. *For every $\gamma_1, \gamma_2 > 0$ and $\hat\lambda \in L^2(\Gamma_c), \hat{\boldsymbol{\mu}} \in \mathbf{L}^2(\Gamma_c)$ the mapping Φ_γ defined above is Lipschitz-continuous with constant*

$$\mathfrak{L} = \frac{\|\mathfrak{F}\|_\infty c_1}{\kappa}, \tag{5.93}$$

where $\|\mathfrak{F}\|_\infty$ denotes the essential supremum of $\mathfrak{F}$, κ the coercivity constant of $a(\cdot\,,\cdot)$ and c_1 a constant from a trace theorem. In particular, the Lipschitz constant $\mathfrak{L}$ does not depend on the regularization parameters γ_1, γ_2.

PROOF. Let us fix $\gamma_1, \gamma_2 > 0$ and choose $g, \tilde{g} \in L^2_+(\Gamma_c)$. We denote the solution variables belonging to g and $\tilde{g}$ by $(\boldsymbol{y}, \boldsymbol{\mu}, \lambda)$ and $(\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{\mu}}, \tilde{\lambda})$, respectively, i.e., for simplicity of the notation we omit the index 'γ' that indicates that we are dealing with the solution of the regularized problem with Tresca friction. Subtracting equation (5.38a) for $(\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{\mu}}, \tilde{\lambda})$ from the same equation for $(\boldsymbol{y}, \boldsymbol{\mu}, \lambda)$ and setting $\boldsymbol{z} := \boldsymbol{y} - \tilde{\boldsymbol{y}}$ yields

$$a(\boldsymbol{y} - \tilde{\boldsymbol{y}}, \boldsymbol{y} - \tilde{\boldsymbol{y}}) + (\boldsymbol{\mu} - \tilde{\boldsymbol{\mu}}, \boldsymbol{\tau}_T(\boldsymbol{y} - \tilde{\boldsymbol{y}}))_{\Gamma_c} + (\lambda - \tilde{\lambda}, \tau_N(\boldsymbol{y} - \tilde{\boldsymbol{y}}))_{\Gamma_c} = 0. \tag{5.94}$$

Utilizing (5.38b) one can interpret λ as orthogonal projection of $\hat{\lambda} + \gamma_1(\tau_N \boldsymbol{y} - d)$ onto $L^2_+(\Gamma_c)$, and similarly for $\tilde{\lambda}$. This implies, using (2.2), that

$$(\lambda - \tilde{\lambda}, \tau_N(\boldsymbol{y} - \tilde{\boldsymbol{y}}))_{\Gamma_c} = \frac{1}{\gamma_1}\Big(\lambda - \tilde{\lambda}, (\hat{\lambda} + \gamma_1(\tau_N \boldsymbol{y} - d)) - (\hat{\lambda} + \gamma_1(\tau_N \tilde{\boldsymbol{y}} - d))\Big)_{\Gamma_c} \geq 0.$$

Using the above estimate and (5.94), one deduces that

$$a(\boldsymbol{y} - \tilde{\boldsymbol{y}}, \boldsymbol{y} - \tilde{\boldsymbol{y}}) \leq (\boldsymbol{\mu} - \tilde{\boldsymbol{\mu}}, \boldsymbol{\tau}_T(\tilde{\boldsymbol{y}} - \boldsymbol{y}))_{\Gamma_c}. \tag{5.95}$$

In the remaining part of this proof we establish pointwise almost everywhere estimates for the right hand side in (5.95). We introduce the notation $\xi := \gamma_2 \boldsymbol{\tau}_T \boldsymbol{y} + \hat{\boldsymbol{\mu}}$ and $\tilde{\xi} := \gamma_2 \boldsymbol{\tau}_T \tilde{\boldsymbol{y}} + \hat{\boldsymbol{\mu}}$ that leads to $\boldsymbol{\tau}_T(\tilde{\boldsymbol{y}} - \boldsymbol{y}) = \gamma_2^{-1}(\tilde{\xi} - \xi)$. We now distinguish several cases that originate from the max-function in (5.38c).

(1) $\|\xi\| \leq \mathfrak{F}g$, $\|\tilde{\xi}\| \leq \mathfrak{F}g$: It follows that

$$(\boldsymbol{\mu} - \tilde{\boldsymbol{\mu}})^\top \boldsymbol{\tau}_T(\tilde{\boldsymbol{y}} - \boldsymbol{y}) = \frac{1}{\gamma_2}(\xi - \tilde{\xi})^\top(\tilde{\xi} - \xi) \leq 0.$$

(2) $\|\xi\| \geq \mathfrak{F}g$, $\|\tilde{\xi}\| \geq \mathfrak{F}g$: In this case we get

$$\begin{aligned}
&(\boldsymbol{\mu} - \tilde{\boldsymbol{\mu}})^\top \boldsymbol{\tau}_T(\tilde{\boldsymbol{y}} - \boldsymbol{y}) \\
&= \frac{1}{\gamma_2}\Big(\mathfrak{F}g\frac{\xi}{\|\xi\|} - \mathfrak{F}\tilde{g}\frac{\tilde{\xi}}{\|\tilde{\xi}\|}\Big)^\top(\tilde{\xi} - \xi) \\
&= \frac{1}{\gamma_2}\left(\mathfrak{F}(g - \tilde{g})\frac{\xi}{\|\xi\|}\right)^\top(\tilde{\xi} - \xi) + \frac{1}{\gamma_2}\left(\mathfrak{F}\tilde{g}\Big(\frac{\xi}{\|\xi\|} - \frac{\tilde{\xi}}{\|\tilde{\xi}\|}\Big)\right)^\top(\tilde{\xi} - \xi) \\
&\leq \frac{1}{\gamma_2}\mathfrak{F}|g - \tilde{g}|\|\tilde{\xi} - \xi\| \\
&= \mathfrak{F}|g - \tilde{g}|\|\boldsymbol{\tau}_T(\boldsymbol{y} - \tilde{\boldsymbol{y}})\|.
\end{aligned}$$

(3) $\|\xi\| \leq \mathfrak{F}g$, $\|\tilde{\xi}\| \geq \mathfrak{F}\tilde{g}$. We start our estimates for this case with

$$\begin{aligned}
(\boldsymbol{\mu} - \tilde{\boldsymbol{\mu}})^\top \boldsymbol{\tau}_T(\tilde{\boldsymbol{y}} - \boldsymbol{y}) &= \frac{1}{\gamma_2}\left(\xi - \mathfrak{F}\tilde{g}\frac{\tilde{\xi}}{\|\tilde{\xi}\|}\right)^\top(\tilde{\xi} - \xi) \\
&\leq \frac{1}{\gamma_2}\Big(\|\xi\|\|\tilde{\xi}\| - \|\xi\|^2 - \mathfrak{F}\tilde{g}\|\tilde{\xi}\| + \mathfrak{F}\tilde{g}\|\xi\|\Big),
\end{aligned} \tag{5.96}$$

and continue separately with four cases.

(a) $\|\xi\| \le \mathfrak{F}\tilde{g}$, $\|\tilde{\xi}\| \le \mathfrak{F}g$. The above inequalities imply that

$$\|\xi\| \le \mathfrak{F}\tilde{g} \le \|\tilde{\xi}\| \le \mathfrak{F}g. \tag{5.97}$$

By (5.97) we obtain

$$\mathcal{Z} := (\|\tilde{\xi}\| - \|\xi\|)^2 + (\|\xi\| - \mathfrak{F}\tilde{g})^2 - (\|\tilde{\xi}\| - \mathfrak{F}\tilde{g})^2 \ge 0.$$

Utilizing $\mathcal{Z} \ge 0$ and (5.96), we obtain after a small calculation that

$$\begin{aligned}
&(\boldsymbol{\mu} - \tilde{\boldsymbol{\mu}})^\top \boldsymbol{\tau}_T(\tilde{\boldsymbol{y}} - \boldsymbol{y}) \\
&\quad \le \frac{1}{\gamma_2}\left(\|\xi\|\|\tilde{\xi}\| - \|\xi\|^2 - \mathfrak{F}\tilde{g}\|\tilde{\xi}\| + \mathfrak{F}\tilde{g}\|\xi\| + \frac{\mathcal{Z}}{2}\right) \\
&\quad = 0.
\end{aligned}$$

(b) $\|\xi\| \ge \mathfrak{F}\tilde{g}$, $\|\tilde{\xi}\| \ge \mathfrak{F}g$. In this case we obtain the following chain of inequalities:

$$\mathfrak{F}\tilde{g} \le \|\xi\| \le \mathfrak{F}g \le \|\tilde{\xi}\|,$$

and thus

$$\mathcal{Z} := (\|\tilde{\xi}\| - \|\xi\|)^2 + (\|\xi\| - \mathfrak{F}g)^2 - (\|\tilde{\xi}\| - \mathfrak{F}g)^2 \ge 0.$$

Similarly to the case discussed above we obtain

$$\begin{aligned}
&(\boldsymbol{\mu} - \tilde{\boldsymbol{\mu}})^\top \boldsymbol{\tau}_T(\tilde{\boldsymbol{y}} - \boldsymbol{y}) \\
&\quad \le \frac{1}{\gamma_2}\left(\|\xi\|\|\tilde{\xi}\| - \|\xi\|^2 - \mathfrak{F}\tilde{g}\|\tilde{\xi}\| + \mathfrak{F}\tilde{g}\|\xi\| + \frac{\mathcal{Z}}{2}\right) \\
&\quad = \frac{1}{\gamma_2}\mathfrak{F}(g - \tilde{g})(\|\tilde{\xi}\| - \|\xi\|) \\
&\quad \le \frac{1}{\gamma_2}\mathfrak{F}(g - \tilde{g})\|\xi - \tilde{\xi}\| \\
&\quad = \mathfrak{F}|g - \tilde{g}|\|\boldsymbol{\tau}_T(\boldsymbol{y} - \tilde{\boldsymbol{y}})\|.
\end{aligned}$$

(c) $\|\xi\| \ge \mathfrak{F}\tilde{g}$, $\|\tilde{\xi}\| \le \mathfrak{F}g$. These inequalities imply, together with those in Case (3) that

$$\mathfrak{F}\tilde{g} \le \min(\|\xi\|, \|\tilde{\xi}\|), \ \max(\|\xi\|, \|\tilde{\xi}\| \le \mathfrak{F}g. \tag{5.98}$$

Utilizing $\|\xi\|\|\tilde{\xi}\| \leq \frac{1}{2}(\|\xi\|^2 + \|\tilde{\xi}\|^2)$ one obtains

$$\begin{aligned}
&(\boldsymbol{\mu} - \tilde{\boldsymbol{\mu}})^\top \boldsymbol{\tau}_T(\tilde{\boldsymbol{y}} - \boldsymbol{y}) \\
&\qquad \leq \frac{1}{\gamma_2}\left(\frac{1}{2}\|\xi\|^2 - \frac{1}{2}\|\tilde{\xi}\|^2 - \mathfrak{F}\tilde{g}\|\tilde{\xi}\| + \mathfrak{F}\tilde{g}\|\xi\|\right) \\
&\qquad \leq \frac{1}{2\gamma_2}(\|\xi\| - \|\tilde{\xi}\|)(\|\xi\| - \mathfrak{F}\tilde{g} + \|\tilde{\xi}\| - \mathfrak{F}\tilde{g}) \\
&\qquad \leq \frac{1}{\gamma_2}\|\xi - \tilde{\xi}\|\mathfrak{F}(g - \tilde{g}) \\
&\qquad = \mathfrak{F}|g - \tilde{g}|\|\boldsymbol{\tau}_T(\boldsymbol{y} - \tilde{\boldsymbol{y}})\|.
\end{aligned}$$

(d) $\|\xi\| \leq \mathfrak{F}\tilde{g}$, $\|\tilde{\xi}\| \geq \mathfrak{F}g$. In this last case we obtain the following relations:

$$\|\xi\| \leq \min(\mathfrak{F}g, \mathfrak{F}\tilde{g}),\ \max(\mathfrak{F}g, \mathfrak{F}\tilde{g}) \leq \|\tilde{\xi}\|.$$

This shows that

$$\mathcal{Z} := (\|\tilde{\xi}\| - \|\xi\|)^2 + (\|\xi\| - \mathfrak{F}\tilde{g})^2 - (\|\tilde{\xi}\| - \mathfrak{F}g)^2 \geq 0.$$

Thus, we get

$$\begin{aligned}
&(\boldsymbol{\mu} - \tilde{\boldsymbol{\mu}})^\top \boldsymbol{\tau}_T(\tilde{\boldsymbol{y}} - \boldsymbol{y}) \\
&\qquad \leq \frac{1}{\gamma_2}\left(\|\xi\|\|\tilde{\xi}\| - \|\xi\|^2 - \mathfrak{F}\tilde{g}\|\tilde{\xi}\| + \mathfrak{F}\tilde{g}\|\xi\| + \frac{\mathcal{Z}}{2}\right) \\
&\qquad \leq \frac{1}{\gamma_2}\mathfrak{F}|g - \tilde{g}|\left|\|\tilde{\xi}\| - \|\xi\|\right| \\
&\qquad \leq \mathfrak{F}|g - \tilde{g}|\|\boldsymbol{\tau}_T(\boldsymbol{y} - \tilde{\boldsymbol{y}})\|.
\end{aligned}$$

(4) $\|\xi\| \geq \mathfrak{F}g$, $\|\tilde{\xi}\| \leq \mathfrak{F}\tilde{g}$. This case is analogous to Case (3).

The above pointwise estimates show that

$$(\boldsymbol{\mu} - \tilde{\boldsymbol{\mu}}, \boldsymbol{\tau}_T(\tilde{\boldsymbol{y}} - \boldsymbol{y}))_{\Gamma_c} \leq \int_{\Gamma_c} \mathfrak{F}|g - \tilde{g}|\, \|\boldsymbol{\tau}_T(\boldsymbol{y} - \tilde{\boldsymbol{y}})\|\, dx,$$

and furthermore that

$$\begin{aligned}
(\boldsymbol{\mu} - \tilde{\boldsymbol{\mu}}, \boldsymbol{\tau}_T(\tilde{\boldsymbol{y}} - \boldsymbol{y}))_{\Gamma_c} &\leq \|\mathfrak{F}(g - \tilde{g})\|_{\Gamma_c}\|\boldsymbol{\tau}_T(\boldsymbol{y} - \tilde{\boldsymbol{y}})\|_{\Gamma_c} \\
&\leq c_1\|\mathfrak{F}\|_\infty\|g - \tilde{g}\|_{\Gamma_c}\|\boldsymbol{y} - \tilde{\boldsymbol{y}}\|_{\mathbf{Y}}.
\end{aligned}$$

Utilizing (5.95) and the coercivity of $a(\cdot\,,\cdot)$, this shows

$$\kappa\|\boldsymbol{y} - \tilde{\boldsymbol{y}}\|_{\mathbf{Y}}^2 \leq a(\boldsymbol{y} - \tilde{\boldsymbol{y}}, \boldsymbol{y} - \tilde{\boldsymbol{y}}) \leq c_1\|\mathfrak{F}\|_\infty\|g - \tilde{g}\|_{\Gamma_c}\|\boldsymbol{y} - \tilde{\boldsymbol{y}}\|_{\mathbf{Y}} \tag{5.99}$$

and concludes the proof of the lemma. □

In the next lemma we address properties of the mapping Ψ_γ.

LEMMA 5.12. *For every* $\gamma_1, \gamma_2 > 0$ *and* $\hat{\lambda} \in L^2(\Gamma_c), \hat{\boldsymbol{\mu}} \in \mathbf{L}^2(\Gamma_c)$ *the mapping* $\Psi_\gamma : L^2_+(\Gamma_c) \to L^2_+(\Gamma_c)$ *is compact and Lipschitz-continuous with constant*

$$\mathfrak{L} = \frac{c\gamma}{\kappa}\|\mathfrak{F}\|_\infty,$$

where c *is a constant resulting from trace theorems.*

PROOF. We consider the following chain of mappings.

$$\begin{array}{ccccccc} L^2_+(\Gamma_c) & \xrightarrow{\Phi_\gamma} & \mathbf{Y} & \xrightarrow{\Theta} & L^2(\Gamma_c) & \xrightarrow{\Upsilon} & L^2_+(\Gamma_c) \\ g & \mapsto & \boldsymbol{y} & \mapsto & \tau_N \boldsymbol{y} & \mapsto & \max(0, \hat{\lambda} + \gamma_1(\tau_N \boldsymbol{y} - d)) \end{array} \tag{5.100}$$

From Lemma 5.11 we already know that Φ_γ is Lipschitz-continuous. The mapping Θ consists of the linear trace mapping from $\mathbf{Y}$ into $H^{\frac{1}{2}}(\Gamma_c)$ and the compact embedding of this space into $L^2(\Gamma_c)$. Therefore, it is compact and linear, in particular Lipschitz-continuous with a constant we denote by $c_2 > 0$. Finally, since

$$\|\max(0, \hat{\lambda} + \gamma_1(\xi - d)) - \max(0, \hat{\lambda} + \gamma_1(\tilde{\xi} - d))\|_{\Gamma_c} \le \gamma_1 \|\xi - \tilde{\xi}\|_{\Gamma_c} \tag{5.101}$$

for all $\xi, \tilde{\xi} \in L^2(\Gamma_c)$, the mapping Υ is Lipschitz-continuous with constant γ_1. From the fact that Ψ_γ is the composition of the mappings $\Upsilon, \Theta, \Phi_\gamma$, namely

$$\Psi_\gamma = \Upsilon \circ \Theta \circ \Phi_\gamma,$$

we can conclude that Ψ_γ is Lipschitz-continuous with constant

$$\mathfrak{L} = \frac{c_1 c_2 \gamma_1}{\kappa}\|\mathfrak{F}\|_\infty,$$

where c_1, c_2 are constants from trace theorems. Concerning the compactness, we clearly have that the composition of Θ and Φ_γ is compact. It remains to show that L^2-convergent sequences remain L^2-convergent under the mapping Υ. This follows immediately from (5.101), which ends the proof. $\square$

We now easily obtain an existence result for the contact problem with Coulomb friction.

THEOREM 5.13. *The mapping* Ψ_γ *admits at least one fixed point, i.e., the regularized Coulomb friction problem* (5.89) *or equivalently* (5.91) *admits a solution. If* $\|\mathfrak{F}\|_\infty$ *is sufficiently small, the solution is unique.*

PROOF. We apply the Leray-Schauder fixed point theorem (see [45, p. 222]) to the mapping $\Psi_\gamma : L^2(\Gamma_c) \to L^2(\Gamma_c)$. Using Lemma 5.12 it suffices to show that λ is bounded in $L^2(\Gamma_c)$ independently of g. This is clear taking into account the dual problem $(\mathcal{P}^\star_{\gamma_1\gamma_2})$. Indeed,

$$\min_{\lambda \ge 0,\ \|\boldsymbol{\mu}\| \le \mathfrak{F}g \text{ a.e. on } \Gamma_c} J^\star_{\gamma_1,\gamma_2}(\lambda, \boldsymbol{\mu}) \le \min_{\lambda \ge 0} J^\star_{\gamma_1,\gamma_2}(\lambda, 0) < \infty,$$

since the neglect of the constraint on $\boldsymbol{\mu}$ leads to the contact problem without friction, which admits a solution independent from $\mathfrak{F}g$. Hence, the Leray-Schauder theorem guarantees the existence of a solution to the regularized Coulomb friction problem. Uniqueness of the solution holds if $\mathfrak{F}$ is such that

$$\mathfrak{L} := \frac{c_1 c_2 \gamma_1}{\kappa} \|\mathfrak{F}\|_\infty < 1$$

due to the fact that in this case Ψ_γ is a contraction. □

In the next section we comment on the methods utilized in this section for proving existence of a solution for the Coulomb frictional contact problem without regularization.

4.2. Remarks on the Coulomb friction problem without regularization. In this section we briefly address the question, which methods utilized in the previous section also apply in the case of Coulomb friction without regularization and which do not. Firstly, we are interested in an analogue to Lemma 5.11 in the case without regularization. As observed above, the Lipschitz-constant in Lemma 5.11 is independent of γ_1, γ_2, which suggests that the result does not require any regularizing term. To verify this conjecture we define the mapping

$$\Phi : L^2_+(\Gamma_c) \longrightarrow \mathbf{Y}, \tag{5.102}$$

that maps a given friction $g \in L^2_+(\Gamma_c)$ to the corresponding solution $\boldsymbol{y}$ of $(\mathcal{P})$. In the next lemma we show that the mapping Φ is Lipschitz-continuous.

LEMMA 5.14. *The mapping Φ defined above is Lipschitz-continuous with constant*

$$\mathfrak{L} = \frac{\|\mathfrak{F}\|_\infty c_1}{\kappa},$$

where $\|\mathfrak{F}\|_\infty$ denotes the essential supremum of $\mathfrak{F}$, κ the coercivity constant of $a(\cdot\,,\cdot)$ and c_1 a constant from a trace theorem.

PROOF. The proof is similar to the one for (5.11), however, it turns out to be significantly shorter, since we can spare considering the different cases for the pointwise estimates. In the sketch that we give here the notation from the proof for Lemma 5.11 is used. Similarly as done there, we derive from (5.22a) that

$$a(\boldsymbol{y}-\tilde{\boldsymbol{y}}, \boldsymbol{y}-\tilde{\boldsymbol{y}}) + (\boldsymbol{\mu}-\tilde{\boldsymbol{\mu}}, \boldsymbol{\tau}_T(\boldsymbol{y}-\tilde{\boldsymbol{y}}))_{\Gamma_c} + \big\langle \lambda-\tilde{\lambda}, \tau_N(\boldsymbol{y}-\tilde{\boldsymbol{y}})\big\rangle_{\Gamma_c} = 0, \tag{5.103}$$

where $\langle\cdot\,,\cdot\rangle_{\Gamma_c}$ denotes the duality product. With $\boldsymbol{y}$ and $\tilde{\boldsymbol{y}}$ we again denote the solutions corresponding to g and $\tilde{g}$, respectively. Using (5.22b) and (5.22c) we derive that

$$\begin{aligned}\big\langle \lambda-\tilde{\lambda}, \tau_N(\boldsymbol{y}-\tilde{\boldsymbol{y}})\big\rangle_{\Gamma_c} &= \big\langle \lambda, (\tau_N\boldsymbol{y}-d)-(\tau_N\tilde{\boldsymbol{y}}-d)\big\rangle_{\Gamma_c} \\ &\quad - \big\langle \tilde{\lambda}, (\tau_N\boldsymbol{y}-d)-(\tau_N\tilde{\boldsymbol{y}}-d)\big\rangle_{\Gamma_c} \\ &= -\big\langle \lambda, \tau_N\tilde{\boldsymbol{y}}-d\big\rangle_{\Gamma_c} - \big\langle \tilde{\lambda}, \tau_N\boldsymbol{y}-d)\big\rangle_{\Gamma_c} \geq 0.\end{aligned}$$

From this estimate and from (5.103), one deduces that

$$a(\boldsymbol{y}-\tilde{\boldsymbol{y}}, \boldsymbol{y}-\tilde{\boldsymbol{y}}) \leq (\boldsymbol{\mu}-\tilde{\boldsymbol{\mu}}, \boldsymbol{\tau}_T(\tilde{\boldsymbol{y}}-\boldsymbol{y}))_{\Gamma_c}. \tag{5.104}$$

It remains to establish pointwise estimates for the right hand side in (5.104). Recall that the equations (5.22d′) and (5.22e′) hold for both, $(\boldsymbol{y}, \boldsymbol{\mu})$ and $(\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{\mu}})$. To estimate $(\boldsymbol{\mu} - \tilde{\boldsymbol{\mu}})^\top \boldsymbol{\tau}_T(\tilde{\boldsymbol{y}} - \boldsymbol{y})$ we first consider the case that $\boldsymbol{\tau}_T \boldsymbol{y} \neq 0$ and $\boldsymbol{\tau}_T \tilde{\boldsymbol{y}} \neq 0$:

$$
\begin{aligned}
(\boldsymbol{\mu} - \tilde{\boldsymbol{\mu}})^\top \boldsymbol{\tau}_T(\tilde{\boldsymbol{y}} - \boldsymbol{y}) &= \left(\mathfrak{F} g \frac{\boldsymbol{\tau}_T \boldsymbol{y}}{\|\boldsymbol{\tau}_T \boldsymbol{y}\|} \right)^\top \boldsymbol{\tau}_T(\tilde{\boldsymbol{y}} - \boldsymbol{y}) - \left(\mathfrak{F} \tilde{g} \frac{\boldsymbol{\tau}_T \tilde{\boldsymbol{y}}}{\|\boldsymbol{\tau}_T \tilde{\boldsymbol{y}}\|} \right)^\top \boldsymbol{\tau}_T(\tilde{\boldsymbol{y}} - \boldsymbol{y}) \\
&\leq \mathfrak{F} g \|\boldsymbol{\tau}_T \tilde{\boldsymbol{y}}\| - \mathfrak{F} g \|\boldsymbol{\tau}_T \boldsymbol{y}\| - \mathfrak{F} \tilde{g} \|\boldsymbol{\tau}_T \tilde{\boldsymbol{y}}\| + \mathfrak{F} \tilde{g} \|\boldsymbol{\tau}_T \boldsymbol{y}\| \\
&\leq \mathfrak{F} |g - \tilde{g}| \|\boldsymbol{\tau}_T(\boldsymbol{y} - \tilde{\boldsymbol{y}})\|.
\end{aligned}
$$

Let us next consider the case that $\boldsymbol{\tau}_T \boldsymbol{y} = 0$ and $\boldsymbol{\tau}_T \tilde{\boldsymbol{y}} \neq 0$. Here we get

$$
\begin{aligned}
(\boldsymbol{\mu} - \tilde{\boldsymbol{\mu}})^\top \boldsymbol{\tau}_T(\tilde{\boldsymbol{y}} - \boldsymbol{y}) &= \left(\boldsymbol{\mu} - \mathfrak{F} \tilde{g} \frac{\boldsymbol{\tau}_T \tilde{\boldsymbol{y}}}{\|\boldsymbol{\tau}_T \tilde{\boldsymbol{y}}\|} \right)^\top \boldsymbol{\tau}_T \tilde{\boldsymbol{y}} \\
&\leq \|\boldsymbol{\mu}\| \|\boldsymbol{\tau}_T \tilde{\boldsymbol{y}}\| - \mathfrak{F} \tilde{g} \|\boldsymbol{\tau}_T \tilde{\boldsymbol{y}}\| \\
&\leq \mathfrak{F}(g - \tilde{g}) \|\boldsymbol{\tau}_T(\boldsymbol{y} - \tilde{\boldsymbol{y}})\|,
\end{aligned}
$$

where we used $\|\boldsymbol{\mu}\| \leq \mathfrak{F} g$. Obviously, for the case $\boldsymbol{\tau}_T \boldsymbol{y} \neq 0$ and $\boldsymbol{\tau}_T \tilde{\boldsymbol{y}} = 0$ one obtains the same estimate. The remaining case $\boldsymbol{\tau}_T \boldsymbol{y} = 0$ and $\boldsymbol{\tau}_T \tilde{\boldsymbol{y}} = 0$ is trivial, such that we obtain

$$
(\boldsymbol{\mu} - \tilde{\boldsymbol{\mu}}, \boldsymbol{\tau}_T(\tilde{\boldsymbol{y}} - \boldsymbol{y}))_{\Gamma_c} \leq \int_{\Gamma_c} \mathfrak{F} |g - \tilde{g}| \, \|\boldsymbol{\tau}_T(\boldsymbol{y} - \tilde{\boldsymbol{y}})\| \, dx,
$$

and can proceed as in the proof of Lemma 5.11 to finish the proof. □

We can now discuss, which parts of Lemma 5.14 and Theorem 5.13 require a regularization and which do not. Clearly, the mapping Θ that appears in the proof of Lemma 5.14 is independent of any regularization. Let us now turn to Υ. This mapping is Lipschitz-continuous with constant γ_1, which already indicates that the regularization of the contact condition is required to obtain Lipschitz-continuity. Indeed, for the problem $(\mathcal{P})$, i.e., the problem without regularization, we cannot even expect the multiplier $\bar{\lambda}$ to be an element in $L^2(\Gamma_c)$. This shows that for the assertion of Lemma 5.12 the regularization with respect to the contact condition is essential. Actually, this is not the case for the regularization of the friction condition, since neither the mapping Φ, nor Θ and Υ involve γ_2. Using this observation and Lemma 5.14, one can verify that both, Lemma 5.12 and Theorem 5.13 also hold true for the case that we apply a regularization only for the contact condition, while leaving the non-differentiability in $(\mathcal{P})$ unchanged.

4.3. Algorithm for the solution of the regularized Coulomb friction problem. As mentioned before, the fixed point idea presented in Section 4.1 can be exploited numerically for the solution of the regularized Coulomb friction problem. This idea and slight modifications of this idea are commonly used to calculate the solution of the Coulomb friction problem by a sequence of Tresca friction problems (see, e.g., [54,55,80]). We now specify the fixed point algorithm.

Algorithm: (CFC-FP)

(1) Choose $\gamma_1, \gamma_2 > 0$, $\hat{\lambda}$ and $\hat{\boldsymbol{\mu}}$. Initialize $g^0 \in L^2(\Gamma_c)$, and set $m := 0$.
(2) Determine the solution $(\lambda^m, \boldsymbol{\mu}^m)$ to problem $(\mathcal{P}^\star_{\gamma_1,\gamma_2})$ with given friction g^m.
(3) Update $g^{m+1} := \lambda^m$, $m := m + 1$ and, unless an appropriate stopping criterion is met, go to Step 2.

Provided that $\|\mathfrak{F}\|_\infty$ is sufficiently small, Lemma 5.12 leads to a convergence result for the above algorithm.

THEOREM 5.15. *Suppose that $\|\mathfrak{F}\|_\infty$ is sufficiently small, then Algorithm* (CFC-FP) *converges regardless of the initialization.*

PROOF. The proof follows immediately from the fact that, provided $\|\mathfrak{F}\|_\infty$ is sufficiently small, the Lipschitz constant of the mapping Ψ_γ is smaller that 1. □

One can consider various modifications of Algorithm (CFC-FP). In [54] a splitting type algorithm, based on a finite dimensional dual formulation of the Tresca friction problem is presented and the numerical performance is tested. This approach is generalized to 3D in [55]. In [80] a Gauß-Seidel-like generalization of (CFC-FP) in the framework of monotone multigrid methods is proposed, and the author reports on favorable numerical results.

Here we propose a modification of (CFC-FP) that combines both the first-order augmented Lagrangian update and the fixed point idea of Algorithm (CFC-FP).

Algorithm: (CFC-ALM-FP)

(1) Initialize $\gamma_1^1, \gamma_2^1 > 0$, $(\hat{\lambda}^1, \hat{\boldsymbol{\mu}}^1) \in L^2(\Gamma_c) \times \mathbf{L}^2(\Gamma_c)$ and $g^0 \in L^2(\Gamma_c)$, $m := 0$.
(2) Choose $\gamma_1^m, \gamma_2^m > 0$ and determine the solution $(\lambda^m, \boldsymbol{\mu}^m)$ to problem $(\mathcal{P}^\star_{\gamma_1,\gamma_2})$ with given friction g^m and $\hat{\lambda} := \hat{\lambda}^m$, $\hat{\boldsymbol{\mu}} := \hat{\boldsymbol{\mu}}^m$.
(3) Update $g^{m+1} := \lambda^m$, $\hat{\lambda}^{m+1} := \lambda^m$, $\hat{\boldsymbol{\mu}}^{m+1} := \boldsymbol{\mu}^m$ and $m := m + 1$. Unless an appropriate stopping criterion is met, go to Step 2.

For the following brief discussion of the above algorithm, we assume that a solution to the Coulomb friction problem as defined in Section 1.4 exists and that the solution variables are sufficiently smooth. To be precise, we assume that the fixed point λ^* of the mapping Ψ (for its definition see page 93) is in $L^2(\Gamma_c)$. The variables $\boldsymbol{y}^* \in \mathbf{Y}$ and $\boldsymbol{\mu}^* \in \mathbf{L}^2(\Gamma_c)$ corresponding to λ^* satisfy

$$a(\boldsymbol{y}^*, \boldsymbol{z}) - L(\boldsymbol{z}) + (\lambda^*, \tau_N \boldsymbol{z})_{\Gamma_c} + (\boldsymbol{\mu}^*, \boldsymbol{\tau}_T \boldsymbol{z})_{\Gamma_c} = 0 \text{ for all } \boldsymbol{z} \in \mathbf{Y},$$

$$\lambda^* = \max(0, \lambda^* + \gamma_1(\tau_N \boldsymbol{y}^* - d)) \text{ on } \Gamma_c,$$

$$\mathfrak{F}\lambda^*(\gamma_2 \boldsymbol{\tau}_T \boldsymbol{y}^* + \boldsymbol{\mu}^*) - \max(\mathfrak{F}\lambda^*, \|\gamma_2 \boldsymbol{\tau}_T \boldsymbol{y}^* + \boldsymbol{\mu}^*\|)\boldsymbol{\mu}^* = 0 \text{ on } \Gamma_c,$$

where $\gamma_1, \gamma_2 > 0$. As can be seen easily, these conditions are also satisfied by a fixed point of (CFC-ALM-FP), i.e., provided convergence of (CFC-ALM-FP) the limit variables are the solution to the original (i.e., non-regularized) contact problem with Coulomb friction. We point out that this convergence is achieved without necessarily letting the penalty parameters tend to infinity. Choosing moderate values for the parameters γ_1, γ_2 avoids ill-conditioning of the systems one has to solve.

Though at the moment we do not have any convergence results for algorithm (CFC-ALM-FP), in numerical practice this method turns out to be reliable and efficient, see Section 6.

5. Numerical Results for Contact with Tresca Friction

In this section we summarize our numerical findings for the contact problem with given friction. Thereby we restrict ourselves to the case of planar elasticity. After presenting our test examples and describing the implementation, we report on tests, where, among others, the influence of regularization and grid are addressed.

5.1. Examples and implementation.

Example 1. The geometry for this example is shown in Figure 1, where for reasons of graphical presentation the gap function d was multiplied by a factor of 20. The data are as follows: $\Omega = [0,3] \times [0,1], \Gamma_c = [0,3] \times \{0\}, \Gamma_n = [0,3] \times \{1\}$ and the elastic body is subject to homogeneous Dirichlet conditions with respect to the horizontal displacement and homogeneous stress-free conditions with respect to the vertical displacement along $\{0\} \times [0,1] \cup \{3\} \times [0,1]$. Furthermore,

$$\boldsymbol{g} = \begin{cases} \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \text{on } [0,1] \times \{1\} \cup [2,3] \times \{1\}, \\ \begin{pmatrix} 0 \\ -20 \end{pmatrix} & \text{on } [1,2] \times \{1\}. \end{cases}$$

For our test runs we choose $E = 5000$ and $\nu = 0.4$. The distance towards the rigid foundation is given by $d(x_1) = 0.003(x_1 - 1.5)^2 + 0.001$ and the given friction g is $10\exp(-20(x_1 - 1.5)^2)$.

This example is interesting for several reasons. Firstly, due to the fact that $\pm\mathfrak{F}g$ act as bilateral constraints for the dual variable μ, the problem is a bottleneck problem: Close to $x_1 = 0$ (and $x_1 = 3$) the distance between upper and lower bound is nearly zero. Secondly, the geometry allows rigid motions of the complete body in vertical direction, since the elastic body is nowhere fixed with respect to the horizontal direction. Nevertheless, from the geometry it is clear that these motions are excluded if the elastic body is in contact with the foundation on some part of Γ_c. To discuss the performance of our algorithms for this example we vary, among others, the values for γ_1 and γ_2 and report on the number of iterations of the semi-smooth Newton method.

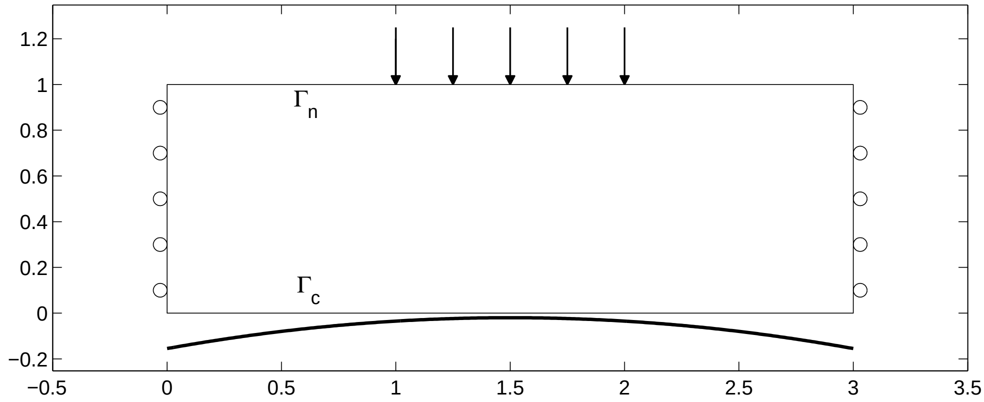


FIGURE 1. *Geometry for Example 1.*

Example 2. For this example we choose the same geometry and data as for Example 1, but we take as given friction either $g \equiv 1.5$ or $g \equiv 2.5$. Note that, compared to the previous example, the bounds for this example are not bottleneck-like. In a first series of tests we report on the number of iterations for various regularization parameters $\gamma_1, \gamma_2 > 0$ and $\nu = 0.4, 0.49$, i.e., we focus on the convergence behavior of the algorithm in the case that ν is close to 0.5. Values close to $\nu = 0.5$ corresponds to an almost incompressible material. Recall that for $\nu = 0.5$ the problem does not admit a solution. We focus on the local convergence properties of the method for $\nu = 0.4$.

Example 3. The aim of this example is to investigate the influence of the given friction on the deformation of the elastic body. The elastic body and the rigid foundation are shown in Figure 2, where for reasons of graphical presentation the gap function $d = \max(0.0015, 0.003(x_1 - 1.5)^2 + 0.001)$ was multiplied by 20. We choose the material parameters $E = 10000$ and $\nu = 0.45$. In this example we do not apply a traction force, rather we prescribe a nonzero deformation along the Dirichlet part of the boundary. As before we have as contact region $\Gamma_c := [0, 3] \times \{0\}$ and on $\Gamma_n := [0, 3] \times \{1\} \cup \{0\} \times [0, 0.2] \cup \{3\} \times [0, 0.2]$ we assume traction free, i.e., homogeneous Neumann boundary conditions. On $\Gamma_d := \{0\} \times [0.2, 1] \cup \{3\} \times [0.2, 1]$ we prescribe the deformation as follows:

$$\boldsymbol{\tau y} = \begin{cases} \begin{pmatrix} 0.003(1 - x_2) \\ -0.004 \end{pmatrix} & \text{on } \{0\} \times [0.2, 1], \\ \begin{pmatrix} -0.003(1 - x_2) \\ -0.004 \end{pmatrix} & \text{on } \{3\} \times [0.2, 1]. \end{cases}$$

With this example we investigate the influence of the friction on the deformation of the elastic body.

Implementation and setting of the parameters. For the discretization of the elasticity problems P_1 and Q_1 finite elements are used. The whole implementations is done in MATLAB on basis of the code published in [2]. This code

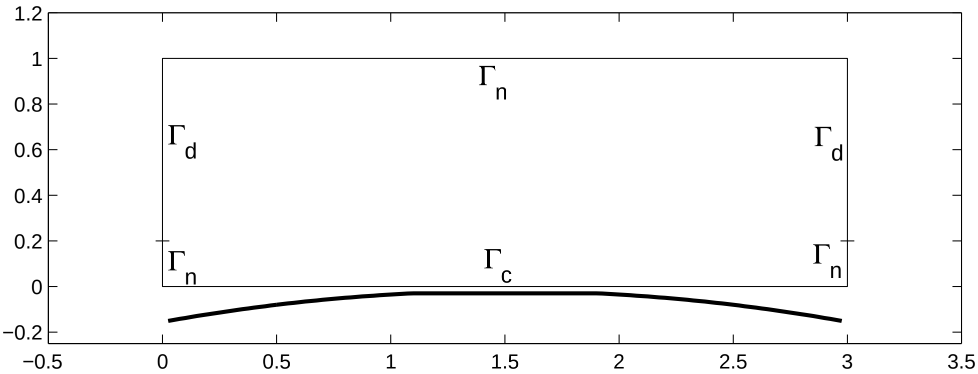


FIGURE 2. *Geometry for Example 3.*

is modified and extended in such a way that it can also be applied for contact problems with Tresca friction. The generalization to these problems is done such that only in the initialization step the stiffness matrix and the right hand side have to be assembled. Both, Dirichlet conditions and the constraints due to the changing active sets in the iteration process are realized by adding or removing constraints to the free system. Thus, the solution of a linear saddle point problem is, by far, the most time consuming part per iteration. The pointwise inequality constraints are enforced on the nodes of the finite element mesh, which corresponds to the choice of a Dirac-like basis for the discretization of the dual variables (see also [54]). For discretizations utilizing mortar finite elements we refer to [11, 65, 66, 106].

Unless otherwise specified the linear systems are solved exactly using MATLAB's backslash that makes use of the properties of sparse, symmetric matrices.

In the semi-smooth Newton method (FC-SS2D) we always choose $\sigma = 1$ and, unless otherwise specified, the method is initialized with the solution of the unconstrained dual problem, i.e., the solution of $(\mathcal{P}^\star_{\gamma_1,\gamma_2})$ neglecting the constraints. Unless otherwise specified we use $\hat{\lambda} = 0$ and $\hat{\mu} = 0$ for (FC-SS2D) and initialize the augmented Lagrangian methods with $\lambda^0 = 0$ and $\boldsymbol{\mu}^0 = 0$. For the problems with given friction, the friction coefficient can be incorporated into the given friction, thus we always choose $\mathfrak{F} \equiv 1$ in our examples.

5.2. Numerical results.

5.2.1. *Results for Example 1.* We now summarize the results of our testing for Example 1 on a mesh of 120×40 finite elements. In a first attempt we choose $\gamma_1 = \gamma_2 = 10^8$. The solution, obtained after 9 iterations, is depicted in Figure 3, where we show the deformed finite element mesh and the rigid foundation. Here, we utilize gray tones to visualize the elastic shear energy density. The dual variable μ_γ, the corresponding bounds $\pm\mathfrak{F}g$ and the tangential displacement are shown on the left of Figure 4. By inspection of the graphs one can verify that the complementarity conditions hold. Recall that active sets correspond to

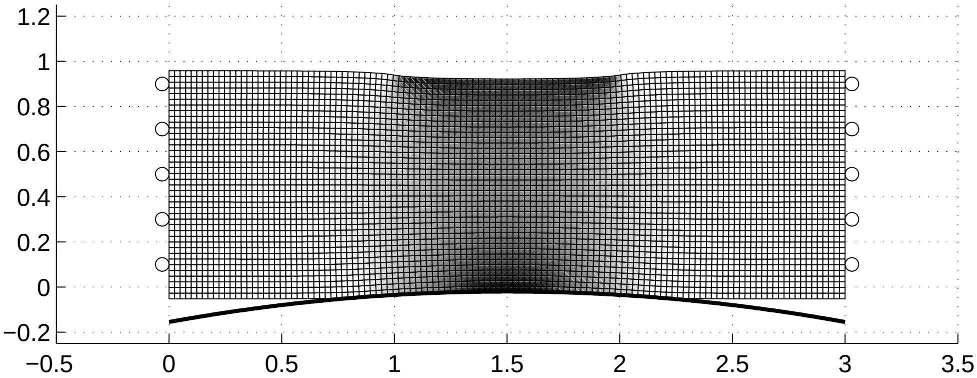


FIGURE 3. *Example 1: Deformed mesh (deformation multiplied by 20), gray tones visualize the elastic shear energy density, $\gamma_1 = \gamma_2 = 10^8$.*

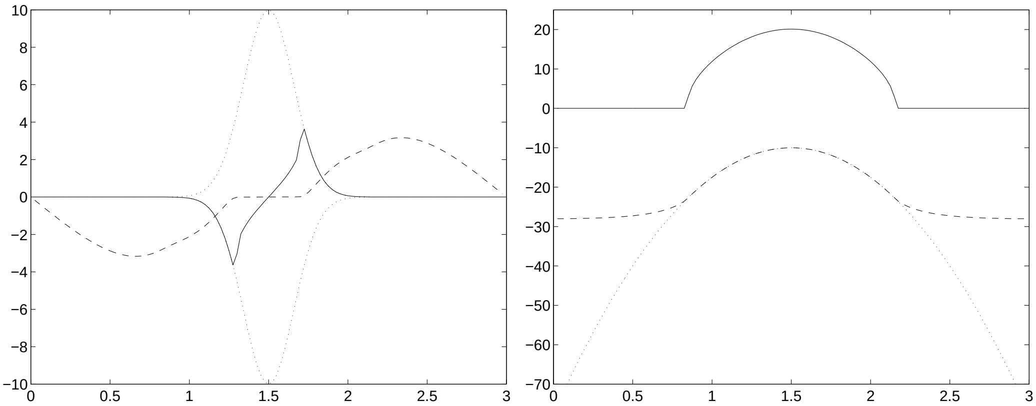


FIGURE 4. *Example 1: Left figure: dual variable μ_γ (solid) with bounds $\pm\mathfrak{F}g$ (dotted) and tangential displacement $\boldsymbol{\tau}_T y_\gamma$ (multiplied by 10^4, dashed). Right figure: Multiplier λ_γ (solid), rigid foundation (multiplied by 10^4, dotted) and normal displacement $\tau_N \boldsymbol{y}_\gamma$ (multiplied by 10^4, dashed), $\gamma_1 = \gamma_2 = 10^8$.*

parts of the boundary, where the elastic body is sliding in tangential direction, while inactive sets correspond to sticky regions, i.e., to sets where $\tau_T y_\gamma \cong 0$. The main disadvantage of utilizing the Tresca friction law is, that stick may occur on sets where the elastic body is not in contact with the rigid foundation, which is clearly undesirable in modeling realistic physical phenomena. The right hand side plot in Figure 4 depicts the rigid foundation, the displacement $\tau_N \boldsymbol{y}_\gamma$ and the corresponding multiplier λ_γ.

Let us now comment on the performance of the algorithm. To obtain convergence of the method independently of the initialization, it seems to be crucial to choose the parameter σ (i.e., the parameter in the estimation of the active sets corresponding to the friction condition) large enough, for instance, $\sigma = 1$ (which we use for our calculations). Setting $\sigma = \gamma_2^{-1}$ as suggested by the interpretation of the algorithm as infinite-dimensional semi-smooth Newton method, leads to problems in the convergence unless the initialization is already close to the solution. Using $\sigma = 1$, the method converges for all initializations and, furthermore, we observe locally fast convergence.

Since the elastic body is only fixed in horizontal direction, we cannot initialize the algorithm with the solution of the state equation (5.38a) for $\mu_\gamma = 0, \lambda_\gamma = 0$. Doing so would lead to a problem where both, friction and contact conditions, are neglected. Since in the example under consideration the bilinear form $a(\cdot\,,\cdot)$ is not coercive over the set of admissible deformations, the problem without constraints does not admit a solution. Therefore, we initialize the algorithm with the solution of the problem without friction, but with forcing that $\tau_N \boldsymbol{y}_\gamma - d = 0$ in the interval $[1.25, 1.75]$. We tested several other initializations including $\tau_N \boldsymbol{y}_\gamma - d = 0$ on all of Γ_c. For all these initializations the method converges after at most 12 iterations. The fact that the bounds for μ_γ behave bottleneck-like does not have any negative influence on the performance of the algorithm. This remarkable stability is also of interest when solving problems involving Coulomb friction by means of the fixed point methods discussed in Section 4.3.

In the iteration process we usually observe a monotonicity of the active sets for the contact condition, namely, beginning from the second iteration, the new estimate for the active set is contained in the previous one. We do not observe a similar behavior for the active sets corresponding to the friction condition. Generally, in this example the contact condition strongly dominates the convergence behavior, in particular, the contact-active sets significantly influence the estimated inactive and active sets for the friction condition. The converse influence seems to be rather small.

5.2.2. *Results for Example 2.* For this example we document on the behavior of the algorithm for various regularization parameters. For all tested material and regularization parameters the algorithm converged. The dual solution variables for $\gamma_1 = \gamma_2 = 10^8$ are depicted in Figure 5. The deformed mesh looks very similar to the one for Example 1 (see Figure 3). Table 1 displays the number of iterations for $\nu = 0.4$, $g \equiv 1.5$ and for various γ_1, γ_2. As can be seen from our tests, the number of iterations depends only weakly on γ_1, γ_2. The same test runs are also performed for $\nu = 0.49$ and $g \equiv 2.5$. The corresponding numbers of iterations are shown in Table 2. Again we observe that the algorithm requires more iterations for larger γ_1, γ_2 and also slightly more iterations as for the parameter values discussed before.

Next we investigate the speed of convergence for $\nu = 0.4$ and $g \equiv 1.5$. For this purpose we report for $k = 1, 2, \dots$ on the discrete analogue of

$$q^k := \frac{a(\boldsymbol{y}^{k+1} - \boldsymbol{y}_\gamma, \boldsymbol{y}^{k+1} - \boldsymbol{y}_\gamma)^{\frac{1}{2}}}{a(\boldsymbol{y}^{k} - \boldsymbol{y}_\gamma, \boldsymbol{y}^{k} - \boldsymbol{y}_\gamma)^{\frac{1}{2}}} + \frac{\|\lambda^{k+1} - \lambda_\gamma\|_{\Gamma_c}}{\|\lambda^{k} - \lambda_\gamma\|_{\Gamma_c}} + \frac{\|\mu^{k+1} - \mu_\gamma\|_{\Gamma_c}}{\|\mu^{k} - \mu_\gamma\|_{\Gamma_c}},$$

TABLE 1. *Example 2: Number of iterations for different values of* γ_1 *and* γ_2, $\nu = 0.4$, $g = 1.5$.

		γ_2					
		10^4	10^5	10^6	10^7	10^8	10^9
γ_1	10^4	6	7	8	8	8	8
	10^5	6	8	8	9	9	9
	10^6	7	8	12	12	13	13
	10^7	7	8	10	11	11	11
	10^8	7	8	10	11	11	11
	10^9	7	8	10	11	11	11

TABLE 2. *Same as Table 1, but with* $\nu = 0.49$, $g = 2.5$.

		γ_2					
		10^4	10^5	10^6	10^7	10^8	10^9
γ_1	10^4	5	7	8	10	14	14
	10^5	6	7	9	14	14	14
	10^6	7	8	9	15	15	15
	10^7	8	8	9	14	13	13
	10^8	8	8	9	14	13	13
	10^9	8	8	9	14	13	13

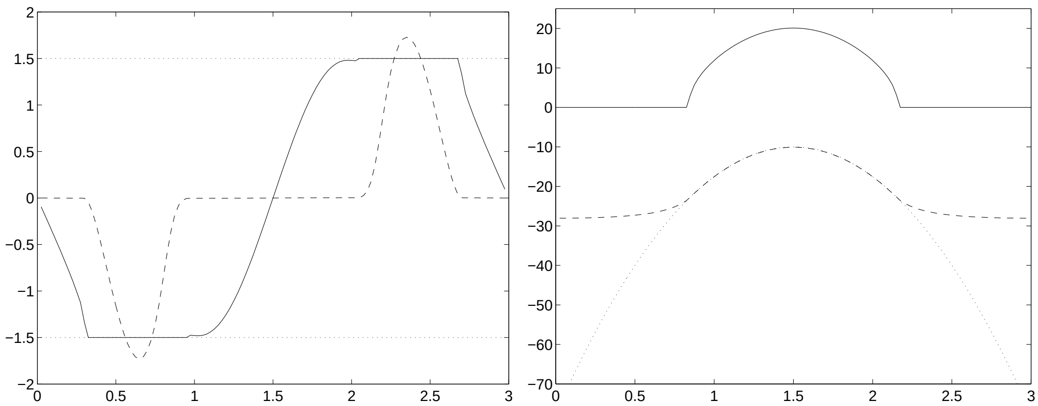


FIGURE 5. *Example 2: Left figure: Dual variable* μ_γ *(solid) with bounds* $\pm\mathfrak{F}g$ *(dotted) and tangential displacement* y_γ *(multiplied by* 10^4*, dashed). Right figure: multiplier* λ_γ *(solid), rigid foundation (multiplied by* 10^4*, dotted) and normal displacement* $\tau_N \boldsymbol{y}_\gamma$ *(multiplied by* 10^4*, dashed),* $\gamma_1 = \gamma_2 = 10^8$, $\nu = 0.4$.

TABLE 3. *Example 2: Values for q^k for $\gamma_1 = \gamma_2 = 10^4$, $\nu = 0.4$, $g \equiv 1.5$.*

k	1	2	3	4
q^k	1.11e-0	8.64e-1	3.00e-1	0

TABLE 4. *Example 2: Number of iterations on different grids and for various regularization parameters for symmetric initialization and nonsymmetric initialization (number in parentheses).*

	$\gamma_1 = \gamma_2$				
grid	10^5	10^6	10^7	10^8	10^9
60×20	6(8)	7(9)	9(12)	9(12)	9(12)
120×40	7(8)	9(11)	14(16)	13(18)	13(18)
240×80	7(9)	9(11)	16(18)	17(21)	17(21)

where $(\boldsymbol{y}_\gamma, \lambda_\gamma, \mu_\gamma)$ denote the solution variables and $(\boldsymbol{y}^k, \lambda^k, \mu^k)$ the iterates. The result is shown in Table 3, where we observe that q^k is monotonously decreasing, which indicates superlinear convergence of the iterates. We also investigate the rate of convergence of the iterates for other regularization parameters, and observed that q^k is more likely to decrease through the whole iteration process for smaller γ_1, γ_2, i.e., for more regularized problems.

We now report on the number of iterations of (FC-SS2D) for differently fine discretizations. For this purpose we consider the case $g \equiv 2.5$ and $\nu = 0.49$. Table 4 shows the number of iterations for various regularization parameters on three different grids. Thereby, the first number corresponds to the standard initialization of the algorithm, i.e., setting $\tau_N \boldsymbol{y}^0 = d$ on $[1.25, 1.75]$. The number in the parentheses corresponds to the unfavorable nonsymmetric initialization $\tau_N \boldsymbol{y}^0 = d$ on $[0, 0.5]$. One observes that the algorithm behaves only moderately mesh-dependent and that for all grids the number of iterations increases as γ_1, γ_2 increase. Furthermore, for $\gamma_1 = \gamma_2 = 10^5, 10^6$ we observe an almost mesh independent behavior. A possible explanation for this remarkable result is that for small regularization parameters the convergence region of the continuous method is large, such that we can observe mesh-independence of the semi-smooth Newton method (as analyzed [56, 61]).

The above results motivate the application of a continuation procedure with respect to the regularization parameter to reduce the overall number of iterations. In this strategy we calculate the solution for moderately large regularization parameters and utilize the obtained solution as initialization for larger regularization parameters. The outcome for our problems is reported in Table 5. We observe that continuation with respect to γ_1 and γ_2 can be used to reduce the overall number of iterations on fine grids. Moreover, one observes that the number of iterations is even more reduced for the case that a nonsymmetric initialization for (FC-SS2D) is used.

TABLE 5. *Example 2: Number of iterations on different grids using continuation w.r.t. to γ_1, γ_2 (2nd and 3th column), the resulting overall number of iterations ($\sum$) and the number without continuation strategy (last column). The numbers in parentheses correspond to a nonsymmetric initialization.*

	$\gamma_1 = \gamma_2$			$\gamma_1 = \gamma_2$
grid	10^5	$\to 10^9$	$\sum$	10^9
60×20	6(8)	+4	10(11)	9(12)
120×40	7(8)	+5	12(13)	13(18)
240×80	7(9)	+7	14(16)	17(21)

Finally, we report on tests of the augmented Lagrangian method for this problem. We terminate the iteration if

$$d^l_{\lambda,\mu} := \|\bar{\lambda} - \lambda^l\|_{\Gamma_c} + \|\bar{\boldsymbol{\mu}} - \boldsymbol{\mu}^l\|_{\Gamma_c} < 10^{-4}, \tag{5.106}$$

where for $\bar{\lambda}$ and $\bar{\boldsymbol{\mu}}$ we take the solution for the regularized problem with $\gamma_1 = \gamma_2 = 10^{10}$. Each iteration of (FC-ALM) is initialized with the solution variables of the previous iteration. Applying the exact version of the method with $\gamma_1 = \gamma_2 = 10^6$, 12 iterations of (FC-ALM) and overall 36 steps of (FC-SS2D) are necessary, until the method stops at $d^{12}_{\lambda,\mu}$ =9.87e-5. For $\gamma_1 = \gamma_2 = 10^7$ only 2 (FC-ALM)-iterations and 19 system solved are needed to get $d_{\lambda,\mu}$ =6.23e-5. To accelerate the convergence, one can increase the regularization parameters by a factor of, e.g., 5 in every (FC-ALM)-iteration. When started with $\gamma_1 = \gamma_2 = 10^6$, this procedure stops after 3 steps of (FC-ALM) and overall 19 linear solves at $d^3_{\lambda,\mu}$ =5.08e-6. If γ_1, γ_2 are increased by 10 per iteration, the algorithm terminates after 2 (FC-ALM)-iterations and requires overall 16 linear solves until it terminates with $d^2_{\lambda,\mu}$ =2.71e-5.

Next, we test an inexact version of the above method, namely we allow a maximum of 4 (FC-SS2D) iterations in the inner loop of (FC-ALM), before we update λ and μ. Applying this inexact strategy for the inner problem in (FC-ALM) may lead to iterates λ^l, μ^l, which do not satisfy $\lambda^l \geq 0$ and $|\mu^l| \leq \mathfrak{F}g$. Thus, in the inexact strategy, we apply a projection of the dual variables in order to make them admissible, before we perform the augmented Lagrangian update. The results for the resulting test runs are summarized in Table 6, where we also document the values for $d^l_{\lambda,\mu}$. It can be observed that for the first iterates of (FC-ALM) with inexact solve of the auxiliary problem smaller regularization parameters lead to better results, which is due to the smaller illconditioning of the auxiliary problem. Thus, additionally to the augmented Lagrangian update we increase γ_1 and γ_2, which leads to a fast convergence to the solution of the non-regularized contact problem with Tresca friction, see the last row of Table 6.

5.2.3. *Results for Example 3.* In the examples discussed so far the frictional behavior only has a minor influence onto the deformation, in particular onto the actual contact zone. Varying the function g we often could not observe any

TABLE 6. *Example 2: Tests for (FC-ALM) with inexact solve of the auxiliary problem, #iter denotes the number of inner iteration.*

	l	1	2	3	4	5	6
$\gamma_1 = \gamma_2 = 10^6$	$\#iter_{SS}$	4	4	3	3	2	3
	$d^m_{\lambda,\mu}$	6.85e-1	1.45e-1	2.73e-2	1.28e-2	7.85e-3	5.24e-3
$\gamma_1 = \gamma_2 = 10^7$	$\#iter_{SS}$	4	4	4	4	2	2
	$d^m_{\lambda,\mu}$	7.27e-1	1.80e-1	5.26e-2	1.52e-2	1.74e-3	3.39e-4
$\gamma_1 = \gamma_2 = 10^{4+l}$	$\#iter_{SS}$	4	4	4	2	2	
	$d^m_{\lambda,\mu}$	5.22e-1	1.43e-1	2.23e-3	3.15e-5	8.64e-8	

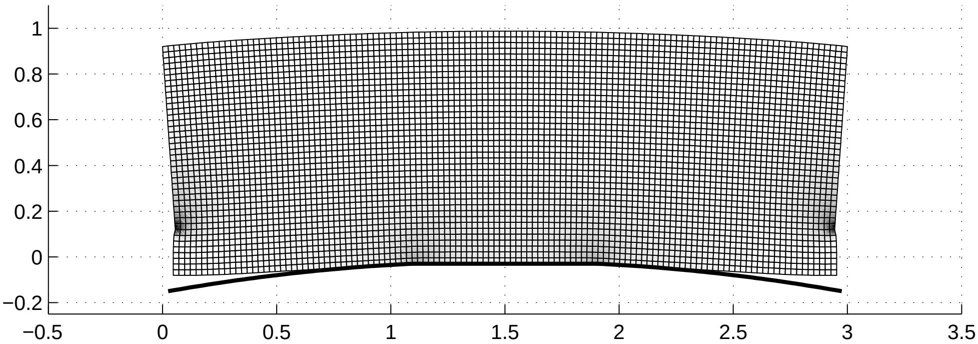


FIGURE 6. *Example 3: Deformed mesh, gray tones visualize the elastic shear energy density, $g \equiv 1$.*

influence on the (discrete) active set $\mathcal{A}_c$ at all. This example has been constructed in order to investigate the convergence behavior of the algorithms for examples where g significantly influences the deformation of the elastic body.

For $\gamma_1 = \gamma_2 = 10^8$ the semi-smooth Newton method detects the solution after 7 iterations. The deformed mesh and the elastic shear energy density for given friction of $g \equiv 1$ are shown in Figure 6. As expected, in a neighborhood of the points $(0, 0.2)$ and $(3, 0.2)$, i.e., the points where the boundary conditions change from Neumann to Dirichlet, we observe a stress concentration due to a local singularity of the solution. We also observe a (smaller) stress concentration close to the points where the rigid foundation has the kinks. On the left hand side of Figure 7 we depict the rigid foundation, the normal displacement and the corresponding multiplier. Magnifying the contact zone (see the plot in the middle of Figure 7), one can observe that the body is not in contact with the rigid foundation in the interval $[1.3, 1.7]$, i.e., here the constraint on the normal deformation is inactive, which is also reflected in the fact that the corresponding dual variable λ_γ is zero in this interval. On the right hand side of Figure 7 we show the tangential displacement, the multiplier μ_γ and the bounds $\pm\mathfrak{F}g$.

Next we investigate the influence of the given friction onto the deformation of the elastic body. Note that g directly influences μ_γ, while its influence on $\tau_N \boldsymbol{y}_\gamma$

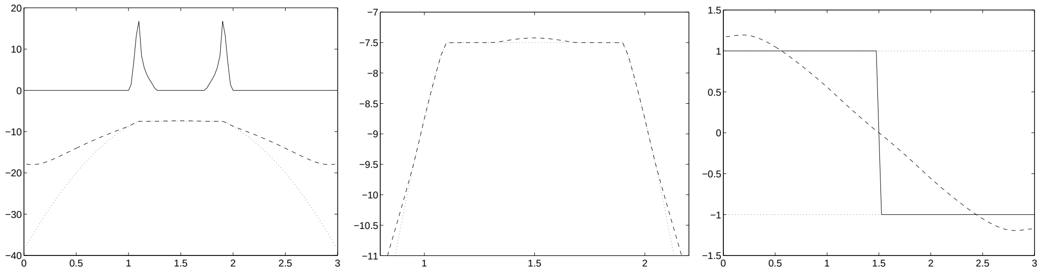


FIGURE 7. *Example 3: Left figure: Multiplier λ_γ (solid), rigid foundation (multiplied by $5 \cdot 10^3$, dotted) and normal displacement $\tau_N \boldsymbol{y}_\gamma$ (multiplied by $5 \cdot 10^3$, dashed), Middle figure: Detail of the left plot, Right figure: Dual variable μ_γ (solid) with bounds $\pm\mathfrak{F}g$ (dotted) and tangential displacement y_γ (multiplied by $5 \cdot 10^3$, dashed).*

and λ_γ is only due to the connection of the variables by means of the elasticity equation. We compare the normal displacement and the corresponding multiplier λ_γ for $g \equiv 0, 1, 3, 5, 20, 100$. For $g \equiv 0$ we get the pure contact problem (since then $\mu_\gamma \equiv 0$), i.e., the tangential displacement of the elastic body is unrestricted. Choosing $g \equiv 100$ the solution turns out to be fixed in tangential direction on all of Γ_c. In Figure 8 we show the rigid foundation, the normal displacement and the corresponding multipliers for various constant values of g. A magnification of the contact zone for $g \equiv 0, 1, 5$ can be found in Figure 9. The table in Figure 9 displays the number of active points corresponding to $g \equiv 0, 1, 3, 5, 20, 100$ for both the contact as well as the friction condition. Clearly, due to the symmetry of the geometry and data the number of points where $\mu_\gamma = -g$ and where $\mu_\gamma = g$ coincide (i.e., $\#\mathcal{A}_{f,-} = \#\mathcal{A}_{f,+}$). For all values of g the algorithm finds the solution after 7 or 8 iterations, which reflects a remarkable stability of the semi-smooth Newton method with respect to the bounds $\pm\mathfrak{F}g$ for the dual variable μ_γ.

5.3. Summary of the numerical results. In our numerical tests we observe a remarkable efficiency and reliability of the algorithms discussed in this chapter for the solution of 2D contact problems with Tresca friction. Let us first focus on the semi-smooth Newton method (FC-SS2D). This algorithm that can be seen as an active set strategy always detects the exact discrete solution after only a few iterations. We observe — beginning from the second iteration — a monotonicity of the active sets corresponding to the contact condition, namely, the new active set is always a subset of the previous active set. For the active sets for the friction condition we cannot observe a similar monotonicity.

The number of iterations of (FC-SS2D) turns out to be stable with respect to the material parameters, the given friction, the mesh and the initialization. Moreover, the regularization parameters γ_1, γ_2 in $(\mathcal{P}^\star_{\gamma_1,\gamma_2})$ only have a minor influence on the number of iterations required to detect the solution. Usually, from one iteration step to the next many points are changed from active to inactive and vice versa. Only for a nearly incompressible material (i.e., ν is close to the

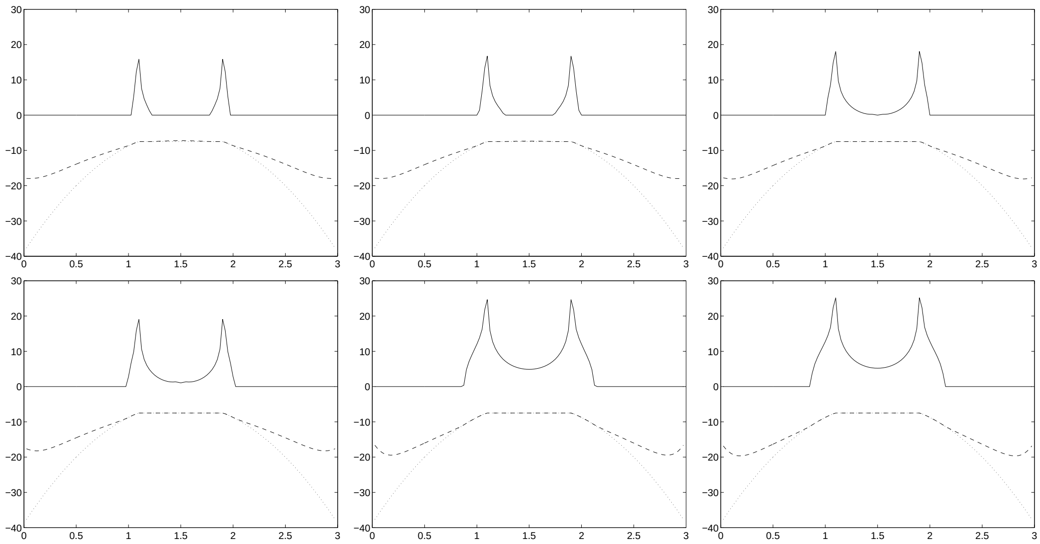
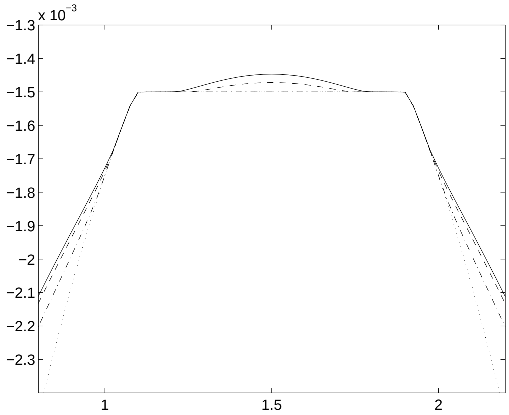


FIGURE 8. *Example 3: Rigid foundation, normal displacement $\tau_N \boldsymbol{y}_\gamma$ (both multiplied by $5 \cdot 10^3$) and corresponding multiplier λ_γ for various given friction. Upper row, from left to right: $g \equiv 0, 1, 3$, lower row, from left to right: $g \equiv 5, 20, 100$.*



g	$\#\mathcal{A}_c$	$\#\mathcal{A}_{f,+}$	$\#\mathcal{A}_{f,-}$
0	14	59	59
1	20	59	59
3	39	59	59
5	41	58	58
20	51	29	29
100	51	0	0

FIGURE 9. *Example 3: Left: Magnification of the rigid foundation (dotted) and $\tau_N \boldsymbol{y}_\gamma$ for the problem without friction (solid), $g \equiv 1$ (dashed) and $g \equiv 5$ (dashdot) for Example 3. Right: Number of active contact and friction points for various values of g.*

threshold 0.5) and for large regularization parameters and fine grids this change may become slow leading to a larger number of iteration. This can be efficiently overcome by using a continuation procedure with respect to the regularization parameters.

While the convergence speed of standard algorithms for the solution of bound constrained optimization problems (e.g., projected gradient or Newton algorithms) decreases for bottleneck-like bounds, this does not affect the semi-smooth Newton method (FC-SS2D) that can be seen to converge at a superlinear rate.

The efficiency of (FC-SS2D) is also interesting with respect to first-order augmented Lagrangian methods. These methods are update methods for the multipliers (or equivalently the dual variables), and require to solve an auxiliary problem that is of the form of our regularized problem in every iteration step. If this auxiliary problem in (FC-ALM) is solved exactly utilizing (FC-SS2D), the overall number of linear solves in higher than if (FC-SS2D) is applied with large penalty parameters. In case the auxiliary problem is only solved approximately, the overall number of system solves reduces significantly and turns out to be rather the same as for the semi-smooth Newton method with large penalty parameters. However, the iterates of (ALM) and its inexact versions converge to the solution of the original dual problem without requiring that $\gamma_1, \gamma_2 \to \infty$.

6. Numerical Results for Contact with Coulomb Friction

In this section we summarize the numerical tests of our algorithms for contact problems with Coulomb friction. Therefore, we reuse the test examples from Section 5.1. In the experiments we address, among others, the influence of the friction coefficient $\mathfrak{F}$ on the solution and compare the performance of the algorithms (CFC-FP) and (CFC-ALM-FP).

6.1. Example 1. Here we use the same geometry and data as in Example 2 of Section 5.1 with $\nu = 0.4$ and report on tests performed for Algorithm (CFC-FP) and (CFC-ALM-FP). These fixed-point-like algorithms are initialized with the solution of the pure contact problem (and we set $g^0 :\equiv 0$). As in [80, 85] the outer (i.e., the fixed point) iteration is terminated if

$$d_g^m := \frac{\|g^m - g^{m-1}\|_{\Gamma_c}}{\|g_m\|_{\Gamma_c}} \leq \varepsilon, \tag{5.107}$$

where in our tests we utilize $\varepsilon := 10^{-7}$. The solution variables for the Coulomb friction problem with $\mathfrak{F} = 0.3$ are shown in Figure 10. Observe that the bounds for the variable μ_γ in Coulomb's friction problem are given by $\mathfrak{F}\lambda_\gamma$, meaning that these bounds are not independent of the solution, as in the case of Tresca friction. Table 7 summarizes convergence results for (CFC-FP) with $\mathfrak{F} = 0.3$. One observes that the fixed point iteration converges relatively fast.

Next we report on tests for (CFC-ALM-FP). Thereby, both the variables $(\hat{\lambda}, \hat{\boldsymbol{\mu}})$ and the given friction g are updated. In every iteration step we only have to solve a regularized problem with Tresca friction. In our tests we always observed convergence of both (CFC-FP) and (CFC-ALM-FP). Recall that, provided regularity, the iterates of (CFC-ALM-FP) converge to the solution of the original Coulomb friction problem, while the iterates of (CFC-FP) converge to the solution of the regularized Coulomb friction problem. For both strategies

TABLE 7. *Example 1: Tests for (CFC-FP): for outer iteration m, we show the number of inner iterations #iter for $\gamma_1 = \gamma_2$ and the value for d_g^m as defined in* (5.107), $\mathfrak{F} = 0.3$, $\nu = 0.4$.

m	$\gamma_1 = \gamma_2$	#iter	d_g^m
0	10^9	7	1.00e0
1	10^9	6	6.12e-3
2	10^9	2	8.26e-5
3	10^9	2	1.22e-6
4	10^9	2	3.78e-8

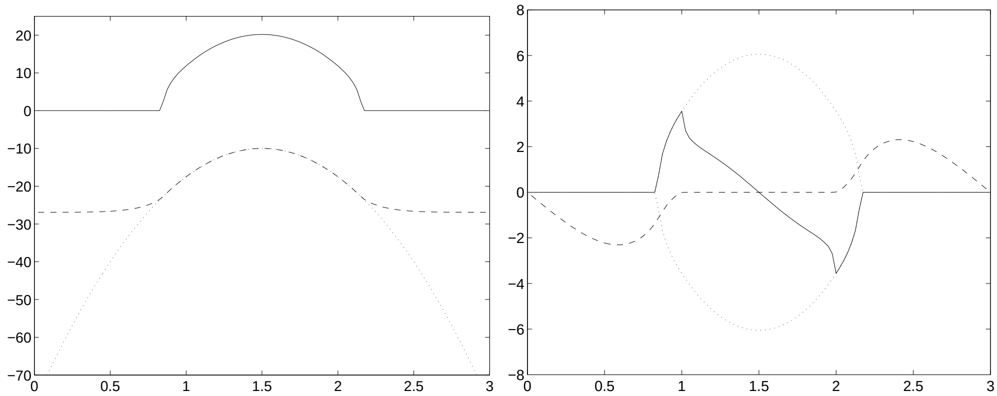


FIGURE 10. *Example 1: Left: Multiplier λ_γ (solid), rigid foundation (multiplied by 10^4, dotted) and normal displacement $\tau_N \boldsymbol{y}_\gamma$ (multiplied by 10^4, dashed), right: Variable μ_γ (solid) with bounds $\pm 0.3\lambda_\gamma$ (dotted) and tangential displacement y_γ (multiplied by 10^4, dashed).*

TABLE 8. *Same as Table 7, but for (CFC-ALM-FP) and increasing $\gamma_1 = \gamma_2$.*

m	$\gamma_1 = \gamma_2$	#iter	d_g^m
0	10^5	6	1.00e0
1	10^6	6	5.96e-2
2	10^7	2	4.43e-3
3	10^8	2	1.16e-4
4	10^9	2	2.28e-6
5	10^{10}	2	6.97e-8

overall about 20 linear solves were necessary, that is, the computational effort for solving the Coulomb friction problem is comparable to the effort for the Tresca friction problem.

TABLE 9. *Same as Table 7 but for* $\nu = 0.49$.

m	$\gamma_1 = \gamma_2$	$\#iter$	d_g^m
0	10^9	8	1.00e0
1	10^9	12	2.89e-2
2	10^9	2	6.82e-4
3	10^9	2	1.87e-5
4	10^9	2	6.32e-7
5	10^9	2	5.07e-8

TABLE 10. *Same as Table 8, but for* $\nu = 0.49$.

m	$\gamma_1 = \gamma_2$	$\#iter$	d_g^m
0	10^5	6	1.00e0
1	10^6	8	8.17e-2
2	10^7	3	7.62e-3
3	10^8	2	1.49e-4
4	10^9	2	3.41e-6
5	10^{10}	2	1.20e-7
6	10^{11}	2	7.27e-8

We now perform the same test of (CFC-FP) and (CFC-ALM-FP) as above, but with $\nu = 0.49$. The results of these test runs are shown in the Tables 9 and 10. It can be seen that in the case of nearly incompressible material the Algorithm (CFC-ALM-FP) gives slightly better results than the pure fixed point iteration (CFC-FP).

Finally, we show results obtained for two different friction coefficients, namely for $\mathfrak{F} = 0.1$ and $\mathfrak{F} = 1$. The dual solution variable and the corresponding bounds for these different values of $\mathfrak{F}$ are shown in Figure 11.

6.2. Example 2. This example continues our tests concerning the influence of the friction onto the deformation. The data are taken from Example 3, Section 5.1. In Figure 12 we depict the normal and the tangential displacement with corresponding multipliers for $\mathfrak{F} = 2, 5, 10$. One observes that in this example the friction coefficient significantly influences the deformation. For instance, in the case $\mathfrak{F} = 2$ the elastic body is in contact with the foundation in the interval $[1.4, 1.6]$, but it is not for $\mathfrak{F} = 5$ and $\mathfrak{F} = 10$.

The solutions were obtained using the Algorithms (CFC-FP) and (CFC-ALM-FP) with the stopping criterion (5.107) with $\varepsilon = 10^{-7}$. The methods perform comparably well and require overall between 20 and 25 linear solves to stop with $d_g^m \leq \varepsilon$.

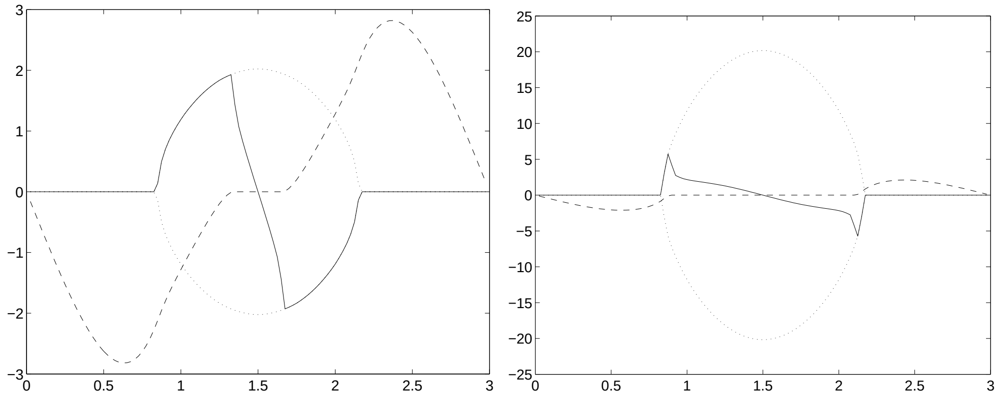


FIGURE 11. *Example 1: Left: variables μ_γ (solid) with bounds $\mathfrak{F}\lambda_\gamma$ (dotted) and tangential displacement y_γ (multiplied by 10^4, dashed) $\mathfrak{F} = 0.1$, right: same as left, but with $\mathfrak{F} = 1$.*

6.3. Summary of the numerical results. The efficiency of the algorithms for the 2D contact problem with Tresca friction carries over to contact problems with Coulomb friction. To accelerate the convergence of the standard fixed point iteration we combine the fixed point with the augmented Lagrangian update. For our test examples, the pure fixed point iteration (CFC-FP) and the combined method (CFC-ALM-FP) show a similar convergence behavior. However, the advantage of (CFC-ALM-FP) compared to (CFC-FP) is that its iterates converge to the solution of the original contact problem with Coulomb friction. We also perform tests where the friction coefficient has a rather large influence onto the deformation of the elastic body. Also for the numerical realization of these examples our algorithms present a powerful tool.

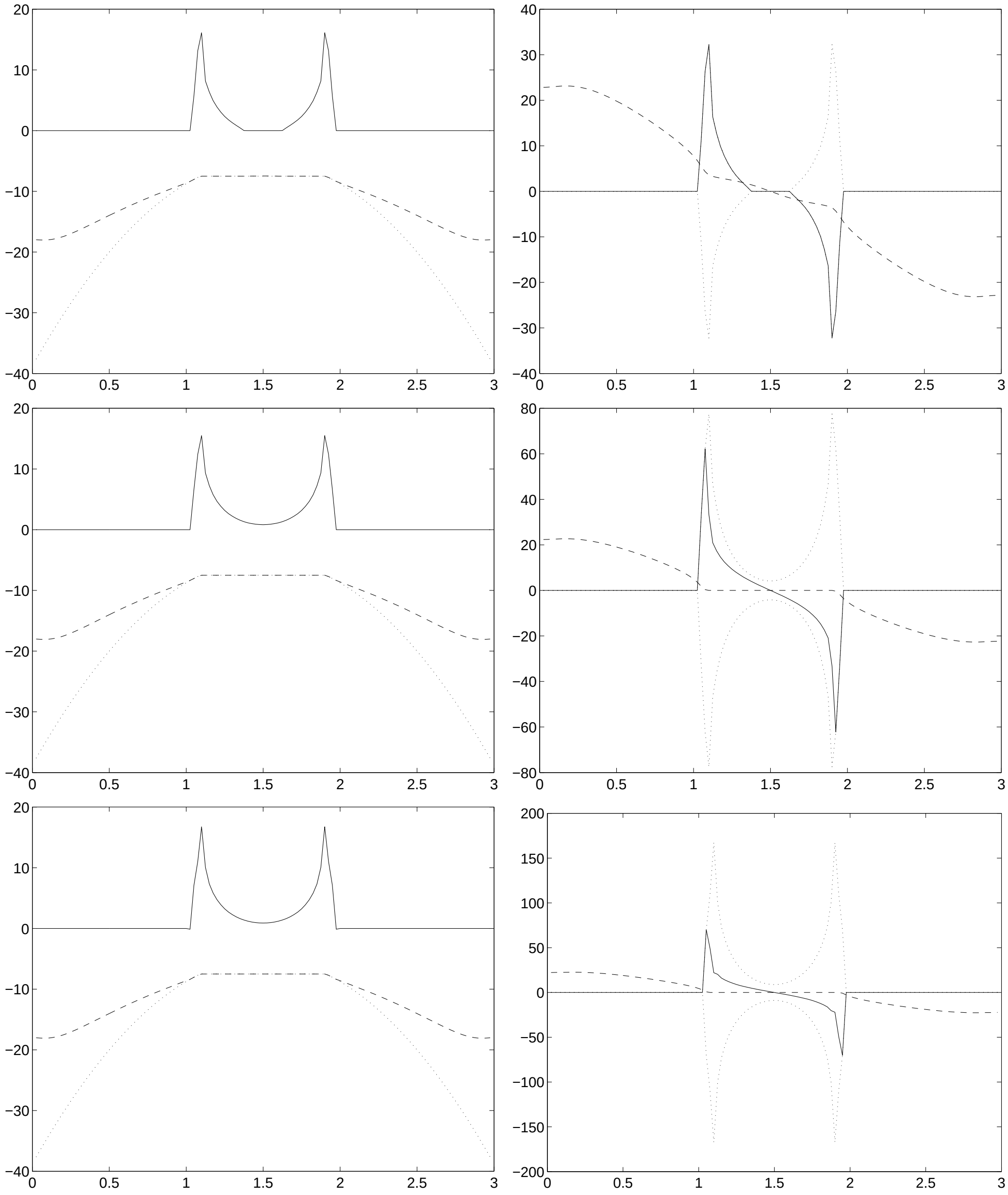


FIGURE 12. *Example 2: First row, left figure: multiplier λ_γ (solid), rigid foundation (multiplied by $5 \cdot 10^3$, dotted) and normal displacement $\tau_N \boldsymbol{y}_\gamma$ (multiplied by $5 \cdot 10^3$, dashed), Right figure: dual variable μ_γ (solid) with bounds $\pm\mathfrak{F}\lambda_\gamma$ (dotted) and tangential displacement y_γ (multiplied by $5 \cdot 10^3$, dashed) for $\mathfrak{F} = 2$. Second row: same as first, but with $\mathfrak{F} = 5$. Third row: same as first, but with $\mathfrak{F} = 10$.*

CHAPTER 6

Conclusions and Outlook

In the present work, we have analyzed the application of Newton-type methods to friction and contact problems in infinite-dimensional Hilbert spaces. Applying the Fenchel duality theorem we derived dual formulations of the problems under consideration. Using nonlinear complementarity (NC) functions the optimality systems can be written as non-differentiable operator equations. A regularization procedure that is closely related to augmented Lagrangians allows us to apply and to analyze the semi-smooth Newton method for contact and friction problems in a Hilbert space framework.

For the simplified friction problem, the Signorini problem without friction and the 2D-Signorini problem with Tresca friction, the usage of certain NC-functions leads to a generalized Newton iteration that allows interpretation as an active set algorithm. For the 3D-Signorini problem with Tresca friction we propose a generalized Newton algorithm as well. In this case, the NC-function involves additional nonlinearities. In the corresponding iterative algorithm, this implies that the iterates enter the derivatives explicitly, and not only by means of estimating active sets.

Usually, in related work the system of PDEs together with the complementarity conditions are taken as starting points for the development and analysis of algorithms for contact problems with and without friction. Here, whenever convenient, primal and dual problem formulations are also used aside from the original PDE system. This often enhances physical and mathematical insight into the problem.

The tradeoff in formulating and analyzing all algorithms in infinite dimensions is that one has to consider a family of regularized problems instead of dealing only with the original problem. Smoothing properties of the solution mapping for the elasticity equations can be exploited to obtain generalized differentiability of the max- and min- operator in an infinite dimensional Hilbert space framework. Combining the regularized problems with a first-order augmented Lagrangian method results in the convergence of the solutions to the original problem.

We carried out comprehensive tests of our algorithms for contact problems with and without friction in 2D. It turns out that semi-smooth Newton, possibly combined with augmented Lagrangian methods, yield a remarkable efficiency and reliability for the numerical realization of contact and friction problems in elasticity. In our numerical tests, we discussed among others the dependence on the regularization, on material parameters and the mesh. Furthermore, we could also confirm our theoretical findings such as superlinear convergence of the iterates. For the simplified friction problem (Chapter 3) a time-dependent version was briefly presented. Numerical tests confirmed that the algorithms can also be utilized for the efficient solution of time-dependent problems.

We also proposed and analyzed generalized Newton methods for contact with Tresca and Coulomb friction in 3D (or arbitrary dimension). From our knowledge on the application of these methods in 2D, we expect favorable numerical results for arbitrary dimension as well. Thus, it seems worthwhile to investigate the numerical performance of these methods in future research.

The results obtained in this work for the minimization of non–differentiable functionals involving the norm (or absolute value) functional may also be of interest for other non-differentiable optimization problems, e.g., for total variation-based (TV-based) image enhancement [95, 105], Bingham fluids [39, 46] and the minimization of a sum of Euclidean norms [3].

CHAPTER 7

Summary

In this thesis, friction and contact in elasticity are investigated and efficient numerical methods for these phenomena are developed. The main difficulty of these problems lies in the contact and friction conditions, which are inherently nonlinear and require the treatment of non-differentiable functions; this makes both theoretical analysis as well as reliable numerical realization truly challenging.

While in the engineering community finite-dimensional discretizations of contact and friction problems are usually studied, this work focuses on their infinite-dimensional counterparts. We apply a recently developed generalized differentiability concept in function spaces for the soution of these problems. Such an infinite-dimensional approach often provides insight into the problem structure. This is not only of theoretical interest but also of significant practical importance since the performance of a numerical algorithm is closely related to the infinite-dimensional problem structure.

The approach taken here is to a large extent based on writing the problems under consideration as optimization problems. We derive the Fenchel dual problem (see [42]), which allows us to transform a non-differentiable minimization problem into an inequality constrained maximization of a smooth functional. Whenever possible, we also see the problems from the optimizational point of view, i.e., aside from using just the first-order necessary conditions of the optimization problem, which are usually the starting points of the analysis, we additionally use for our investigation alternately the primal and dual formulations of the problem. Another important aspect of this work is the use of certain nonlinear complementarity (NC) functions. These allow one to write complementarity conditions as non-smooth operator equations and motivate the application of a generalized Newton method in infinite-dimensional function spaces (see [58,104]).

This thesis consists of three main parts: In the first part (Chapter 3), a scalar simplified friction problem is used as a model problem to investigate the phenomenon of friction. In the second part (Chapter 4), the Signorini contact problem in linear elasticity is analyzed, where the frictional behavior in the contact zone is neglected. In the third part (Chapter 5), the contact problem including friction is discussed, taking into consideration both the Tresca as well as the Coulomb friction law.

We start with a summary of the results obtained for the simplified friction problem. This problem can be stated as minimization of the non-differentiable functional

$$
(7.1) \qquad \begin{cases} J(y) := \dfrac{1}{2}\|\nabla y\|_\Omega^2 + \dfrac{\mu}{2}\|y\|_\Omega^2 - (f,y)_\Omega + g\displaystyle\int_{\Gamma_f} |\tau y(x)|\,dx \\ \text{over the set } \; Y := \{y \in H^1(\Omega) : \tau y = 0 \; \text{ a.e. on } \; \Gamma_0\}, \end{cases}
$$

where $\Omega \subset \mathbb{R}^n$, $\Gamma_0 \subset \Gamma := \partial\Omega$ is a possibly empty open set, $\Gamma_f := \Gamma \setminus \Gamma_0$, $g > 0, \mu \geq 0$, $f \in L^2(\Omega)$, τ denotes the trace operator, and $(\cdot\,,\cdot)_\Omega$ and $\|\cdot\|_\Omega$ denote the scalar product and norm in $L^2(\Omega)$, respectively. The problem can equivalently be formulated as an elliptic variational inequality of the second kind. We show by means of the Fenchel duality theorem that the dual problem of (7.1) is the inequality-constrained maximization of a smooth functional, namely

$$
(7.2) \qquad \begin{cases} \displaystyle\sup_{|\lambda| \leq g \text{ a.e. on } \Gamma_f} J^\star(\lambda) := -\tfrac{1}{2}\|\nabla y(\lambda)\|_\Omega^2 - \tfrac{\mu}{2}\|y(\lambda)\|_\Omega^2, \\ \text{where } y(\lambda) \text{ satisfies} \\ a(y(\lambda), v) - (f,v)_\Omega + (\lambda, \tau_f v)_{\Gamma_f} = 0 \;\; \text{for all } v \in Y. \end{cases}
$$

This dual formulation motivates the application of a semi-smooth Newton method for the solution of (7.1). This method is based on a recent generalized differentiability concept (see [58,104]) in function spaces. So far, it has only been applied to constrained optimization problems such as (7.2) and not to optimization problems involving a non-differentiability (such as (7.1)).

To allow the proper statement and analysis of the method in infinite dimensions, we introduce a global regularization into the dual problem which, for instance, can be motivated by augmented Lagrangians. In the corresponding Fenchel primal problem the regularization results in a local smoothing of the non-differentiability in the functional. The resulting set of optimality conditions involves both primal and dual variables and can be reformulated using nonlinear complementarity functions that involve the pointwise max- and min- operators. Since the expressions under the max- and min- functional involve a smoothing operator, we have the norm gap required for semi-smoothness of these nonlinearities (see [58, 104]). We propose a primal-dual active set strategy as well as the semi-smooth Newton method for the solution of these equations and discuss the close relationship between these two approaches. We show local superlinear convergence of both methods and conditional global convergence of the semi-smooth Newton method. Applying a first-order augmented Lagrangian method we obtain convergence to the solution of the original problems (7.1) and (7.2). This dual method converges regardless of its initialization. By means of several numerical tests, we discuss the influence of certain parameters and of the mesh on the algorithm's performance and compare the results with those obtained by the Uzawa algorithm. The proposed methods yield a remarkable efficiency (usually only 2–7 iterations are needed). Finally, the approach is also successfully applied for a dynamical version of the simplified friction problem.

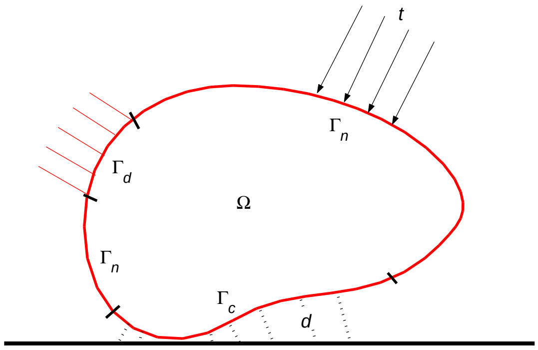


FIGURE 1. *Elastic body with rigid obstacle.*

In the second part of this thesis we develop efficient primal-dual methods for the Signorini contact problem in linear elasticity. In these problems, one is concerned with the deformation of an elastic body whose surface or boundary possibly hits a rigid foundation (see Figure 1). It is not known in advance which part of the body's surface will be in contact with the foundation. The main difficulty in Signorini problems is to identify this contact zone. Then — provided the material law is linear — the problem reduces to a linear one. The problems discussed do not involve friction in the contact zone, i.e., the deformation in the tangential direction is unrestricted.

The Signorini contact problem can be written as an inequality-constrained optimization problem, namely as

$$\min_{\substack{\boldsymbol{\tau y}=0 \text{ on } \Gamma_d,\\ \tau_N \boldsymbol{y} \le d \text{ on } \Gamma_c}} \frac{1}{2}\int_\Omega (\boldsymbol{\sigma y}) : (\boldsymbol{\varepsilon y})\,dx - \int_\Omega \boldsymbol{f y}\,dx - \int_{\Gamma_n} \boldsymbol{t \tau y}\,dx, \tag{7.3}$$

where $\Omega \subset \mathbb{R}^n$ is the bounded domain occupied by the elastic body in its initial configuration and $\Gamma_n, \Gamma_d, \Gamma_c$ are disjoint parts of the boundary (see Figure 1). Furthermore, $\boldsymbol{f}$ and $\boldsymbol{t}$ denote an inner and outer force acting on the elastic body, $\tau_N \boldsymbol{y}$ is the normal component of the trace $\boldsymbol{\tau}$ of the unknown deformation $\boldsymbol{y} \in (H^1(\Omega))^n$ and d denotes the gap between elastic body and rigid foundation. Moreover, $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$ denote the linear strain and stress tensors.

This problem can be written as an variational inequality of the first kind. We derive the Fenchel dual of (7.3) and introduce a regularization that allows algorithm statement and analysis in infinite dimensions. We present and analyze a locally superlinearly convergent semi-smooth Newton method for the problem and show that this method can be written as active set strategy. We combine the approach with a first-order augmented Lagrangian method which results in convergence to the solution of the original problem. The same holds true for an inexact version of this method that is motivated by inexact Uzawa algorithms. By means of several numerical examples in two dimensions we investigate the influence of material parameters and the mesh on the algorithm's performance.

It turns out that the algorithms always detect the solution after few (usually 4–10) iterations and that the method shows a monotone convergence behavior.

Finally, we generalize our results obtained so far to the Signorini problem with friction, where both the Tresca and Coulomb friction laws are considered. While the Tresca law leads to a variational inequality, the Coulomb law results in a quasi-variational inequality. This makes proving theoretical results difficult (e.g., a solution to the problem only exists if the friction coefficient is sufficiently small [40, 53]). Tresca friction is important in its own right; in addition, it is the essential ingredient for an iterative realization of nonlinear friction laws such as the Coulomb law. The contact problem with Tresca friction is obtained from (7.3) by adding the non-differentiable term

$$\int_{\Gamma_c} \mathfrak{F}g\|\boldsymbol{\tau}_T\boldsymbol{y}\|\,dx \tag{7.4}$$

to the functional in (7.3), where $\|\cdot\|$ denotes the Euclidean norm, $\mathfrak{F}$ the friction coefficient and g a given friction. By means of the Fenchel duality theorem we transform the resulting problem into an inequality-constrained minimization of a smooth functional, namely into

$$\begin{cases} \sup\limits_{\substack{\lambda\geq 0 \text{ in } H^{1/2}(\Gamma_c)\\ \boldsymbol{\mu}\in(L^2(\Gamma_c))^n,\,\|\boldsymbol{\mu}\|\leq\mathfrak{F}g \text{ a.e.}}} -\dfrac{1}{2}\displaystyle\int_\Omega(\boldsymbol{\sigma}\boldsymbol{y}_{\lambda,\boldsymbol{\mu}}):(\boldsymbol{\varepsilon}\boldsymbol{y}_{\lambda,\boldsymbol{\mu}})\,dx-\int_{\Gamma_c}\lambda d\,dx,\\ \text{where } \boldsymbol{y}_{\lambda,\boldsymbol{\mu}} \text{ satisfies}\\ a(\boldsymbol{y}_{\lambda,\boldsymbol{\mu}},\boldsymbol{z})-L(\boldsymbol{z})+\left\langle\lambda,\tau_N\boldsymbol{z}\right\rangle_{\Gamma_c}+(\boldsymbol{\mu},\boldsymbol{\tau}_T\boldsymbol{z})_{\Gamma_c}=0 \text{ for all } \boldsymbol{z}\in\mathbf{Y}. \end{cases} \tag{7.5}$$

We intend to apply a semi-smooth Newton method to solve (7.5), or rather to solve a regularized version of (7.5). In *two dimensions*, we can find an NC-function that only involves the pointwise max- and min-function. Thus, many techniques developed for the simplified friction problem and the pure contact problem also apply for the 2D-Tresca friction problem. In *arbitrary dimension*, the NC-function involves additional nonlinearities. Thus, the generalized Newton method cannot be fully interpreted as active set strategy. However, we can prove that it converges locally superlinear as well. Combining the algorithms with an exact or inexact first-order augmented Lagrangian method results in global convergence to the solution of the Tresca frictional contact problem without regularization.

The results for the Tresca friction problem extend to the problem with Coulomb friction. We prove existence of a solution to the (smoothed) problem with Coulomb friction, uniqueness of the solution for small friction coefficients, and discuss possible generalizations to the case without regularization. We state a fixed point as well as a combined fixed point and augmented Lagrangian algorithm for the numerical realization of the Signorini problem with Coulomb friction. The fixed point algorithm is proved to converge provided the frictional coefficient is small.

Finally, using three numerical examples in two dimensions we investigate the performance of the methods for various material parameters and grids. Furthermore, the influence of the frictional behavior on the deformation of the elastic body is investigated. The semi-smooth Newton method always converges after few (usually below 10) iterations and at a locally superlinear rate. We observe monotonicity of the active sets for the contact condition, and an almost mesh-independent behavior of the algorithm for moderate regularization parameters. Using the augmented Lagrangian method with an inexact solution of the auxiliary problem, we obtain fast convergence to the solution of the non-regularized problem with Tresca friction. The efficiency of the methods for the contact problem with Tresca friction carries over to the problem with Coulomb friction. The proposed approach has proven to be highly efficient for the numerical realization of 2D-contact problems with friction and one can expect favorable numerical results in arbitrary dimension as well.

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