Reduced MHD in Nearly Potential Magnetic Fields

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Abstract

Reduced, approximate MHD equations are derived for the case that the magnetic field is close to a potential field. The potential field can have an arbitrary three dimensional structure, as long as it is non vanishing. Finite current and pressure effects are included.

Introduction

Reduced MHD equations (Strauss, 1976, 1977, 1980, 1982, 1983, 1984) have been useful for numerical solution, (Dahlburg *et al.*1985, 1986; Strauss and Hossain, 1990) as well as being convenient for linear and nonlinear analysis. An important feature of reduced MHD systems is that they eliminate the fast compressional Alfvén wave, leaving the shear Alfvén branch and slow magnetosonic waves.

All reduced sets of equations have essentially the same structure. The main time dependent variables are the magnetic field components transverse to the background, vacuum magnetic field, as well as velocity components transverse to the vacuum field. These transverse fields are approximately incompressible, and hence are conveniently represented by streamfunctions: the poloidal magnetic flux ψ, and the electric potential ϕ. Additional equations may be used for the density ρ, the parallel velocity $v_{||}$, and the pressure p, or temperature.

Several versions of reduced MHD were developed, appropriate to various magnetic field geometries. These versions are appropriate for tokamaks (Strauss 1977, 1983), stellarators (Strauss, 1980), mirror machines (Strauss, 1982), and reverse field pinches (Strauss, 1984). Several of these can be unified by the assumptions that (1) the magnetic field is close to a vacuum field, and (2) the perturbations of interest are nearly resonant with the magnetic field.

The first assumption implies weak currents and low β. This is the case in many magnetic fusion experiments, including tokamaks, stellarators, and tandem mirrors. It is also the case in solar and stellar coronal magnetic fields, and in regions of the geomagnetic field.

A deficiency of some reduced models is that they do not include flow along the magnetic field. As pointed out by Montgomery (1992), this precludes equilibria with helical flows. This has been corrected in extensions of the reduced MHD model, such as the four field model (Hazeltine *et al.*, 1985). A better approximation can be based on low β, and small ratio of poloidal to toroidal magnetic field (ϵ/q) (Strauss, 1983; Izzo *et al.*, 1985). As well as including parallel flow, this approach yields the exact two dimensional Grad Shafranov equilibrium equation, as well as the correct Mercier interchange stability criterion. A similar two fluid model was developed by Drake and Antonsen (1984).

Another weakness of some reduced models is that they conserve a nonstandard energy. The equations presented here conserve the usual energy.

MHD Equations and Ordering

The starting point is the usual MHD equations, which may be expressed in the form

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{1}$$

$$\mathbf{J} = \nabla \times \mathbf{B} \tag{2}$$

$$\frac{\partial \mathbf{A}}{\partial t} = \mathbf{v} \times \mathbf{B} + \nabla \phi \tag{3}$$

$$\rho \frac{d\mathbf{v}}{dt} = \mathbf{J} \times \mathbf{B} - \nabla p \tag{4}$$

$$\frac{\partial \rho}{dt} = -\nabla \cdot (\rho \mathbf{v}) \tag{5}$$

The pressure is assumed to be adiabatic,

$$p = c\rho^{\gamma} \tag{6}$$

where c is a constant. It is straightforward to use a more general energy equation. This system conserves energy $\mathcal{E}$, given by

$$\mathcal{E} = \frac{1}{2} \int d^3x \left[\rho \mathbf{v}^2 + \mathbf{B}_v^2 + 2\frac{p}{\gamma - 1} \right] \tag{7}$$

assuming that the normal components of $\mathbf{B}$ and $\mathbf{v}$ vanish on the boundary of the domain of integration.

In the following reduced model, the magnetic field is close to a potential or "vacuum" field $\mathbf{B}_v$, which satisfies

$$\nabla \cdot \mathbf{B}_v = \nabla \times \mathbf{B}_v = 0 \tag{8}$$

Hence $\mathbf{B}_v$ can be expressed in the form

$$\mathbf{B}_v = \nabla \chi \tag{9}$$

where χ is the magnetic potential. The total magnetic field deviates from a potential field by

$$\mathbf{B} = \mathbf{B}_v + \mathbf{B}_1 \tag{10}$$

The ratio

$$\lambda = \frac{B_1}{B_v} \tag{11}$$

will be used as a small expansion parameter, where $B_v = |\mathbf{B}_v|, B_1 = |\mathbf{B}_1|$.

We now introduce an ordering in λ. The potential field strength B_v, the density, and the basic length scale L are of order λ^0. The Alfvén speed $v_A = B_v/(\rho)^{1/2}$ and Alfvén time L/v_A are also λ^0.

The fundamental assumption is that the relevant scale length along $\mathbf{B}_v$ is $L_{||} \sim L/\lambda$, so that

$$\mathbf{B}_v \cdot \nabla \sim \lambda \frac{B_v}{L} \sim \frac{|\mathbf{B}_1|}{L} \tag{12}$$

If we do not make this assumption, we will only find linear shear Alfvén and slow magnetosonic waves. It is only when the gradient length along the vacuum field is long, that the relatively weaker nonpotential magnetic field effects can be important. This occurs near resonances of the magnetic field lines. Resonant perturbations satisfy the magnetic differential equation

$$\mathbf{B} \cdot \nabla \phi = 0, \tag{13}$$

on a resonant field line. Since any field line is, in effect, close to a resonance, we adopt a maximal ordering valid for both resonant and non resonant field lines.

Having introduced a longer length scale parallel to $\mathbf{B}_v$ we also have a longer time scale, $L/(\lambda v_A)$. Thus the time derivatives will be taken of order λ

$$\frac{\partial}{\partial t} = \lambda \frac{\partial}{\partial t_1}. \tag{14}$$

In order that the convective terms be of the same order, as the time derivatives, the velocity should be of no higher than order λ.

$$\mathbf{v} = \lambda \mathbf{v}_1 \tag{15}$$

Finally, the pressure p will be second order,

$$p = \lambda^2 p_2. \tag{16}$$

so that $\beta = 2p/B_v^2 \sim \lambda^2$.

From the low β and low current assumptions, the force must vanish in order λ. This implies that the components of the current perpedicular to $\mathbf{B}_v$ should be of order λ^2. This requires that the magnetic vector potential is parallel to $\mathbf{B}_v$ in leading order,

$$\mathbf{A}_1 = \psi \mathbf{B}_v \tag{17}$$

From (8) and (17), $\mathbf{B}_1$ may be expressed as

$$\mathbf{B}_1 = \nabla\psi \times \mathbf{B}_v. \tag{18}$$

Reduced MHD Equations

Taking the interior product of (3) with $\mathbf{B}$ yields in order λ^2,

$$\frac{\partial \psi}{\partial t} = \frac{1}{B_v^2}\mathbf{B} \cdot \nabla\phi = \frac{\mathbf{B}_v}{B_v^2} \cdot (\nabla\phi + \nabla\phi \times \nabla\psi) \tag{19}$$

Next taking the exterior product of (3) with $\mathbf{B}_v/B_v^2$ gives

$$\mathbf{v}_\perp = \nabla\phi \times \frac{\mathbf{B}_v}{B_v^2} \tag{20}$$

There is also an order λ^2 component of $\mathbf{v}$ parallel to $\mathbf{B}$, which does not contribute to $\mathbf{v} \times \mathbf{B}$ in (3). The total velocity field will be given by:

$$\mathbf{v} = \mathbf{v}_\perp + \frac{v_\|}{B_v}\mathbf{B} \tag{21}$$

The reason the last term in (21) is taken proportional to $\mathbf{B}$ rather than just $\mathbf{B}_v$ is so that the convective derivative is

$$d/dt = \partial/\partial t + \mathbf{v}_\perp \cdot \nabla + \frac{v_\|}{B_v}\mathbf{B} \cdot \nabla \tag{22}$$

and (12) is assumed.

Similarly, (4) can be solved for the current $\mathbf{J}$,

$$\mathbf{J} = (\rho\frac{d\mathbf{v}}{dt} + \nabla p) \times \frac{\mathbf{B}_v}{B_v^2} + \frac{\mathbf{J}\cdot\mathbf{B}_v}{B_v^2}\mathbf{B} \tag{23}$$

Taking the curl of $\mathbf{B}_1$ and the interior product with $\mathbf{B}_v$ yields the parallel current

$$\frac{\mathbf{B}_v \cdot \nabla \times \mathbf{B}_1}{B_v^2} = \Delta^*\psi = \frac{1}{B_v^2}\nabla_\perp \cdot B_v^2 \nabla_\perp \psi. \tag{24}$$

where $\nabla_\perp$ is given by

$$\nabla_\perp = \nabla - \frac{\mathbf{B}_v\mathbf{B}_v}{B_v^2}\cdot\nabla = -\frac{\mathbf{B}_v}{B_v^2} \times (\mathbf{B}_v \times \nabla) \tag{25}$$

From (2), the divergence of (23) is zero. Taking the divergence of (23) and using (22) and (24) yields

$$\nabla_\perp \cdot \frac{\rho}{B_v^2}\nabla_\perp\frac{\partial\phi}{\partial t} = \mathbf{B}\cdot\nabla\Delta^*\psi - \frac{1}{B_v^4}\nabla B_v^2 \times \nabla p \cdot \mathbf{B}_v + \nabla \times \left(\frac{\rho}{B_v^2}\mathbf{v}\cdot\nabla\mathbf{v}_\perp\right)\cdot\mathbf{B}_v \tag{26}$$

Taking the $\mathbf{B}$ component of (4) yields

$$\rho\frac{dv_\|}{dt} = -\frac{1}{B_v}\mathbf{B}\cdot\nabla p \tag{27}$$

Now the continuity equation (5) has to be evolved,

$$\frac{\partial\rho}{\partial t} = -\nabla\frac{\rho}{B_v^2} \times \nabla\phi\cdot\mathbf{B}_v - \mathbf{B}\cdot\nabla\left(\frac{\rho v_\|}{B}\right). \tag{28}$$

This completes the reduced system. These equations conserve the correct energy $\mathcal{E}$, given by

$$\mathcal{E} = \frac{1}{2}\int d^3x \left[\rho\left(\frac{\nabla\phi\times\mathbf{B}_v}{B_v^2}\right)^2 + \rho v_\|^2 + (\nabla\psi\times\mathbf{B}_v)^2 + 2\frac{p}{\gamma-1}\right] \tag{29}$$

This can be verified by multiplying (26) by ϕ, integrating over volume, integrating by parts and using the flux equation (19), the continuity equation (28), and pressure equation (6). This assumes the same boundary conditions as in (7). Comparing with (7), note that the vacuum magnetic field energy is separately conserved and does not appear. This is clear physically, since the only free energy is stored in the non potential part of the field.

The boundary conditions are straightforward. At flux surfaces, which are tangent to the magnetic field, the variables ψ, ϕ, ρ, p and $v_{||}$ are all constant (or zero). Where the magnetic field intersects a material surface, there are other possibilities. Both ψ and ϕ cannot vanish, because of (19). If perturbations of ϕ vanish, one has a line tying condition, which is strongly stabilizing. If ψ perturbations vanish, one has an insulating boundary.

Conclusion

A general set of reduced MHD equations has been derived, consisting of equations (19), (21), (26), (27), (28), and (6). The equations conserves energy correctly. The reduced system is appropriate when the magnetic field is nearly a potential field, with small current and pressure effects. It should be useful for low aspect ratio tokamaks, stellarators and torsatrons, mirrors, solar and stellar coronae, and magnetospheric magnetohydrodynamics. The validity requirements are that the deviation from a potential field is sufficiently small, and that the nonlinear motions of interest are resonant or nearly resonant with the magnetic field lines.

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