6 The Burgers equation

In this chapter, we take a brief detour from the classical theory of PDEs, and study the Burgers equation,

$$u_t + uu_x = \nu u_{xx}, \tag{143}$$

which combines the effects of two prior topics: on the left, the nonlinear advection associated with conservation laws and, on the right, the diffusion associated with the heat equation. These two effects often conflict: while nonlinearity would steepen fronts into discontinuous shocks, diffusion would smooth them away. A motivation for our study is to see how these two effects end up balancing each other. In particular, we would like to see whether and how the limit of small diffusivity ν agrees with the theory of scalar conservation laws. This will bring us to introduce an important tool in applied mathematics, the method of *steepest descent*. A further motivation is that it turns out that the initial value problem for (143) can be solved exactly. This discovery by Coles and Hopf led to further research in the theory of *completely integrable systems*, with mathematical gems such as the inverse scattering technique and the description of solitary waves. Finally, the Burgers equation is arguably the simplest of a family of "canonical" equations describing different aspects of nonlinear wave motion. Other members are the Kortewegde Vries, the Benjamin-Ono and the nonlinear Schroedinger equations, just to name a few. Thus studying Burgers represents a first excursion into this rich territory.

6.1 Some physical instances

Burgers proposed equation (143) as a made-up, toy model for turbulence. It mimics the Navier-Stokes equations of fluid motion through its fluid-like expressions for nonlinear advection and diffusion, yet it is only one-dimensional and it lacks a pressure gradient driving the flow. If we think of u as a velocity, then the left-hand side of (143) represents the momentum being advected by the deterministic component of the flow, while the right-hand side models its diffusion through thermal fluctuations.

We can also derive Burgers as an extension of our model (5) for traffic flow, if one considers a situation in which a car's speed depends not only on the local car density ρ, but also on the driver's perception on how rapidly the traffic is changing ahead. A natural flux function, similar to (6) but including such dependence, is

$$Q = \rho\,(1-\rho) - \nu\rho_x \ : \tag{144}$$

a more congested traffic ahead yields a smaller car flux. Plugging this constitutive relation into (2) yields

$$\rho_t + c(\rho)\rho_x = \nu\rho_{xx},$$

where c is the characteristic speed

$$c = 1 - 2\rho.$$

Multiplying thorough by $c'(\rho) = -2$ yields the more universal form (143) of the Burgers equation for c:

$$c_t + cc_x = \nu c_{xx}.$$

An entirely similar derivation of (143) applies to flood waves, if one considers a quadratic hydrological law $Q(S)$, and adds to it a linear dependence on the slope S_x.

6.2 Viscous shocks

Without the diffusive term on the right hand side, equation (143) admits traveling shock-wave solutions, where two states $u = u^{\pm}$, with $u^- > u^+$, are separated by a discontinuity moving at speed $c = \frac{u^- + u^+}{2}$. To investigate whether there is an analogue of this for the full equation, we can plug into (143) the traveling wave ansatz

$$u(x,t) = U(x - ct), \quad U(\pm\infty) = u^{\pm}, \tag{145}$$

yielding

$$(U - c)U'(\xi) = U''(\xi),$$

where $\xi = x - ct$. This can be readily integrated once into

$$\frac{U^2}{2} - cU = \nu U' + D.$$

The boundary conditions yield

$$c = \frac{u^- + u^+}{2},$$

as in the inviscid case, and

$$D = -\frac{u^- u^+}{2}.$$

The solution so far,

$$(U - u^-)(U - u^+) = 2\nu U',$$

integrates into

$$\xi - \xi_0 = \frac{2\nu}{u^+ - u^-} \log\left(\frac{u^- - U}{U - u^+}\right).$$

The constant of integration ξ_0 is an arbitrary initial position of the wave; we'll adopt $\xi_0 = 0$. Solving for U yields

$$U = u^+ + \frac{u^- - u^+}{1 + e^{(u^- - u^+)\frac{\xi}{2\nu}}}, \tag{146}$$

a *viscous shock* solution displayed in figure (11) with $u^{\pm} = \mp 1$, $\nu = 0.05$. Notice that the lengthscale of the solution is proportional to the diffusivity ν divided by the shock's *strength*, $u^- - u^+$. The latter is the difference of characteristic speeds on the left and right, acting to compress the wave into a shock, while the diffusivity acts to smear it away. The balance between these two opposite tendencies yields the shock's "width".

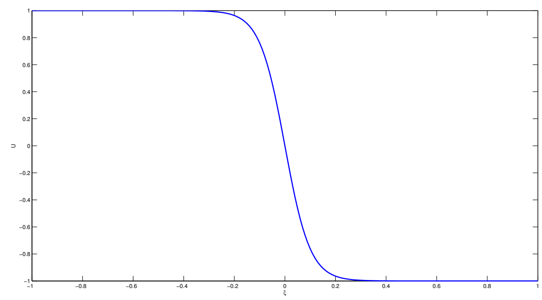

Figure 11: A viscous shock

6.3 The Cole-Hopf transformation

The initial-value problem for the Burgers equation

$$u_t + uu_x = \nu u_{xx}, \quad u(x, 0) = u_0(x) \tag{147}$$

can be solved in closed form by a non-linear change of variables, due to Cole and Hopf, that turns it into the initial-value problem for the heat equation. First we introduce a potential $\phi(x, t)$ such that

$$u = \phi_x.$$

The resulting equation

$$\phi_{xt} + \phi_x \phi_{xx} = \nu \phi_{xxx}$$

can be integrated once into

$$\phi_t + \frac{\phi_x^2}{2} = \nu \phi_{xx}.$$

The magic realization here is that there exists a nonlinear transformation that eliminates the nonlinear term in the equation, leaving behind just the other two, and thus yielding the heat equation. This magical transformation is

$$\phi = -2\nu \log(\psi),$$

and it works because

$$\phi_t = -\frac{2\nu}{\psi}\psi_t, \quad \phi_x = -\frac{2\nu}{\psi}\psi_x \quad \text{and} \quad \phi_{xx} = -\frac{2\nu}{\psi}\psi_{xx} + 2\nu\left(\frac{\psi_x}{\psi}\right)^2,$$

yielding

$$\psi_t = \nu \psi_{xx}.$$

Since $\psi(x,t) = e^{-\frac{\phi(x,t)}{2\nu}} = e^{-\frac{1}{2\nu}\int_0^x u(z,t)\,dz}$, the initial data for ψ is given by

$$\psi(x,0) = \psi_0(x) = e^{-\frac{1}{2\nu}\int_0^x u_0(z)\,dz}.$$

Then, from the solution to the heat equation on the real line, we have that

$$\psi(x,t) = \frac{1}{\sqrt{4\pi\nu t}} \int \psi_0(y) e^{-\frac{(x-y)^2}{4\nu t}}\ dy = \frac{1}{\sqrt{4\pi\nu t}} \int e^{-\frac{\Gamma}{2\nu}}\ dy,$$

where

$$\Gamma(x,y,t) = \int_0^y u_0(z)\ dz + \frac{(x-y)^2}{2t}. \tag{148}$$

The original unknown u is thus given by

$$u(x,t) = \phi_x = -2\nu\frac{\psi_x}{\psi} = \frac{\int \frac{x-y}{t} e^{-\frac{\Gamma}{2\nu}}\ dy}{\int e^{-\frac{\Gamma}{2\nu}}\ dy}, \tag{149}$$

an explicit solution to (147).

6.4 The inviscid limit and steepest descent

The reason that we kept the viscosity ν in the Burgers equation (143), breaking with our tradition of mathematical disdain for all externally provided constants, is that we would like to take the limit as $\nu \to 0$, and see how our old solution in terms of characteristics and shocks shows up as a limit of the more explicit –but also more convoluted– solution (149).

All the integrals in (149) are of the form

$$I = \int f(y) e^{-\frac{\Gamma(y)}{2\nu}}\ dy, \tag{150}$$

where the dependence of f and Γ on x and t does not matter from the viewpoint of evaluating the integral. The asymptotic estimation of such integrals for small values of ν uses *Laplace's method*, a particular instance of steepest descent (the more general methodology applies to similar integrals but with complex functions in the exponential; *stationary phase* is another particular instance, when Γ is purely imaginary.) The idea is quite simple: for small values of ν, the integral (150) is dominated by contributions near the minimum of $\Gamma(y)$ [3]. If $\Gamma(y)$ achieves its minimal value at $y = y_0$, then it can be approximated locally by its truncated Taylor expansion

$$\Gamma(y) = \Gamma(y_0) + \Gamma''(y_0)\frac{(y-y_0)^2}{2}.$$

[3] The fact that a sum or integral of exponentials is dominated by contributions near the maximum value of the exponent is at the heart of much of our quantitative understanding of the world, providing for instance the basis for statistical physics and for the theory of large deviations and rare events.

Then, because of the exponential dominance as $\nu \to 0$, we have that

$$I \approx \int f(y_0) e^{-\frac{1}{2\nu}\left(\Gamma(y_0)+\Gamma''(y_0)\frac{(y-y_0)^2}{2}\right)}\, dy,$$

a Gaussian integral that we can compute exactly:

$$I \approx f(y_0)\sqrt{\frac{4\pi\nu}{\Gamma''(y_0)}}\, e^{-\frac{\Gamma(y_0)}{2\nu}}. \tag{151}$$

In order to apply this result to (149), we need to find the location of the minimum of Γ from (148). Since Γ is continuous and it grows quadratically with y as $|y| \to \infty$, it must achieve a minimum at some finite value of y. To find it, we differentiate (148):

$$\frac{\partial \Gamma}{\partial y} = u_0(y) - \frac{x-y}{t} = 0 \Rightarrow y = x - u_0(y)t,$$

exactly the location that we would have found from tracing the characteristic $\frac{dx}{dt} = u$ back to time $t = 0$. Replacing this, through (151), in the numerator and denominator of (149), we obtain

$$u(x,t) \approx u_0(y), \quad y = x - u_0(y)t,$$

the same result that we obtained using characteristics in the inviscid Burgers equation.

What about shocks? We know that the characteristic equations give rise to multivalued solutions once characteristics cross. In our steepest descent procedure, this corresponds to finding, for a single pair (x,t), two candidate locations, y_1 and y_2 that could dominate the integrals in (149). It is clear though which one we must choose: the one providing the smallest value of Γ. The location of a shock corresponds to the point where the choice switches, i.e., where $\Gamma(x, y_1, t) = \Gamma(x, y_2, t)$. Then

$$\int_0^{y_1} u_0(z)\, dz + \frac{(x-y_1)^2}{2t} = \int_0^{y_2} u_0(z)\, dz + \frac{(x-y_2)^2}{2t}.$$

Problem: Prove that this, together with the characteristic equations defining y_1 and y_2 implicitly in terms of x and t, is equivalent to the jump conditions for the inviscid Burgers equation,

$$\frac{dx}{dt} = \frac{u_0(y_1) + u_0(y_2)}{2}.$$