9 Water waves

As an application of some of the ideas developed in this class, we now consider water waves, such as those seen on the surface of the Hudson river. We will consider various limiting situations, giving rise to linear waves, deep and shallow water equations and various canonical equations such as Burgers, KdV and nonlinear Schrödinger.

9.1 Incompressible and irrotational flows

We have seen long ago the equation modeling conservation of mass for a continuous fluid:

$$\rho_t + \nabla \cdot (\rho u) = 0, \tag{166}$$

where $\rho(x,t)$ represents the fluid's density (i.e. mass per unit volume) and $u(x,t)$ its three-dimensional velocity field.

Unlike air, water is hard to compress. Thus in many situations, water can be modeled as a fluid of constant density ρ_0, independent of both space and time (not always though, as salinity, temperature and even pressure all affect the water's density to different degrees.) We will choose units so that $\rho_0 = 1$. Then the conservation of mass equation (166) reduces to

$$\nabla \cdot u = 0, \tag{167}$$

i.e. the velocity field is divergence free.

For inviscid fluids of constant density, the equations for conservation of momentum adopt the form

$$u_t + (u \cdot \nabla)u + \nabla P = -gk, \tag{168}$$

where $P(x,t)$ represents the pressure, g the gravity constant and k a unit vector pointing in the vertical direction. If at any time t the velocity field u were irrotational, i.e.

$$\nabla \times u = 0, \tag{169}$$

then it could be represented as the gradient of a potential field:

$$u = \nabla \phi. \tag{170}$$

Then

$$(u \cdot \nabla)u = (\nabla\phi \cdot \nabla)\nabla\phi = \nabla\left(\frac{1}{2}\|\nabla\phi\|^2\right),$$

and it follows from (168) that

$$u_t = -\nabla\left(\frac{1}{2}\|\nabla\phi\|^2 + P + gz\right), \tag{171}$$

i.e. the time derivative of u is a gradient, and hence curl-free. Thus if u is irrotational at any time, it will continue to be so at all later times. This applies, in particular, to flows that are initially at rest, such as is the case, to leading order, for water in a lake or the sea before a storm. For this reason, it is usefule to model flows that are both incompressible and irrotational, and hence satisfy

$$\nabla \cdot u = 0, \quad \nabla \times u = 0.$$

Invoking (170), these reduce to Laplace's equation for the potential ϕ

$$u = \nabla \phi, \quad \Delta \phi = 0 \tag{172}$$

in the interior of the fluid. Next we consider what happens at the fluid's boundaries.

9.2 Boundary conditions

We will study surface waves, considering a mass of water bounded by a horizontal impermeable surface at the bottom and by its interface with the atmosphere at the top. We will use z for the vertical independent coordinate, $u_h = (u, v)$ for the horizontal components of the velocity field and w for its vertical component. We will position the bottom at $z = -H$, where H is the mean depth of the flow, and its upper interface at $z = \eta(x, y, t)$, with $\int \eta \, dx \, dy = 0$ (neglecting rain and evaporation.)

The condition that there is no water flow through the bottom is

$$w(x, y, -H, t) = 0,$$

or, in terms of the potential ϕ,

$$\phi_z = 0 \quad \text{at} \quad z = -H. \tag{173}$$

Similarly, the condition that there is no flow through the free surface at the top implies that the vertical velocity w should agree with the rate of change of the height η following a particle:

$$\eta_t + u_h \cdot \nabla \eta = w \quad \text{at} \quad z = \eta(x, y, t).$$

In terms of the potential ϕ,

$$\eta_t + \nabla_h \phi \cdot \nabla \eta = \phi_z \quad \text{at} \quad z = \eta(x, y, t), \tag{174}$$

where ∇_h stands for the horizontal components of the gradient (the ∇ affecting η is also horizontal, since η is not a function of z.)

The conditions in (173, 174) are called *kinematic* boundary conditions, as they are concerned with the fluid movement but not with its causes, which are the subject of *dynamics*. For Laplace's equation, one boundary condition at the bottom and one at the top should suffice. However, we have one extra unknown, η, the position of the boundary itself, i.e. we are dealing with a *free-boundary problem*. The extra condition that we need to impose is dynamic:

that the pressure just below the water surface matches that of air above (we are neglecting surface tension effects, the fact that the surface of water acts, to a certain degree, as an elastic membrane.) Since the density of air is much smaller that that or water, we will make the approximation that the pressure at the surface is constant, and we will set this constant arbitrarily to zero (exploiting the gauge invariance that a constant added to the pressure does not affect its gradient.)

Rewriting (171) in terms of the potential ϕ yields Bernoulli's equation

$$\nabla \left(\phi_t + \frac{1}{2} \|\nabla \phi\|^2 + P + gz \right) = 0.$$

Hence the condition that the pressure should vanish at the surface reads

$$\phi_t + \frac{1}{2} \|\nabla \phi\|^2 + g\eta = 0 \quad \text{at} \quad z = \eta(x, y, t), \tag{175}$$

the required dynamic boundary condition. In principle, we should have on the right-hand side not zero but an arbitrary function of t. We eliminate this using again a gauge invariance: any such function could be absorbed into ϕ_t without affecting the velocity field, that depends only on its spatial derivatives.

To summarize, our water wave problem admits the formulation

$$u = \nabla \phi,$$

where ϕ satisfies Laplace's equation

$$\Delta \phi = 0$$

in the fluid interior, and the boundary conditions

$$\phi_z = 0 \quad \text{at} \quad z = -H,$$

$$\eta_t + \nabla_h \phi \cdot \nabla \eta = \phi_z, \quad \phi_t + \frac{1}{2} \|\nabla \phi\|^2 + g\eta = 0 \quad \text{at} \quad z = \eta(x, y, t).$$

9.3 Linear waves

The problem above does not seem easy to solve: Laplace's equation in the interior of a domain with a boundary that depends on time and whose determination is part of the problem. Moreover, the boundary conditions are nonlinear. In order to move forward, we will consider waves of small amplitude, i.e. velocity fields that are small enough that we can neglect the squared term in the dynamic boundary condition, which thus becomes

$$\phi_t + g\eta = 0 \quad \text{at} \quad z = \eta(x, y, t).$$

Now, it follows from this equation that η is of the same order of ϕ, which we assumed was small. Then we can also neglect the quadratic term in the kinematic condition at the free surface, which simplifies into

$$\eta_t = \phi_z \quad \text{at} \quad z = \eta(x, y, t).$$

In these equations, η is independent of z, but ϕ is not. Since η is small, we can expand ϕ at $z=0$, which yields

$$\phi_t(x,y,\eta,t) \approx \phi_t(x,y,0,t) + \eta\phi_{tz}(x,y,0,t),$$
$$\phi_z(x,y,\eta,t) \approx \phi_z(x,y,0,t) + \eta\phi_{zz}(x,y,0,t).$$

It follows that the difference between evaluating the boundary conditions at the free boundary η or simply at $z=0$ is quadratic in ϕ and η, and thus can also be neglected in our linear approximation! Then our linear water wave problem loses its free boundary, and becomes

$$\Delta\phi = 0 \quad \text{for} \quad -H \le z \le 0,$$

with boundary conditions

$$\phi_z = 0 \quad \text{at} \quad z = -H,$$
$$\eta_t = \phi_z, \quad \phi_t + g\eta = 0 \quad \text{at} \quad z = 0.$$

We shall now proceed to solve this problem.

Consider first Laplace's equation. It admits separated solutions of the form

$$\phi(x,y,z,t) = \left[A_k(t)e^{\|k\|z} + B_k(t)e^{-\|k\|z}\right]e^{ik\cdot x},$$

where x stands for the horizontal coordinates (x,y) and k for an arbitrary horizontal wave vector. Applying the boundary condition that ϕ_z should vanish at $z=-H$ reduces the two constants A_k and B_k to just one, that we can call $\phi^k(t)$:

$$\phi(x,y,z,t) = \phi^k(t)\cosh\left(\|k\|(z+H)\right)e^{ik\cdot x}.$$

Plugging this solution into the two boundary conditions at $z=0$ yields, after proposing that $\eta(x,t) = \eta^k(t)e^{ik\cdot x}$,

$$\eta^k_t = \|k\|\phi^k\sinh\left(\|k\|H\right), \quad \phi^k_t\cosh\left(\|k\|H\right) + g\eta^k = 0.$$

Then

$$\cosh\left(\|k\|H\right)\phi^k_{tt} + g\|k\|\sinh\left(\|k\|H\right)\phi^k = 0,$$

with solution

$$\phi^k(t) = e^{i\omega t}, \quad \omega^2 = g\|k\|\tanh\left(\|k\|H\right). \tag{176}$$

and similarly for $\eta^k(t)$. Thus we have derived the dispersion relation for water waves that we quoted in the chapter on dispersive waves. In order to solve the initial value problem with

$$\eta(x,0) = f(x), \quad \eta_t(x,0) = g(x),$$

one simply writes

$$\eta = \int\left(a_k e^{i\omega_k t} + b_k e^{-i\omega_k t}\right)e^{ik\cdot x}dk, \quad \omega_k = \sqrt{g\|k\|\tanh\left(\|k\|H\right)},$$

where a_k and b_k are linear combinations of the Fourier transforms of f and g.

More will follow, these notes will be updated in the next few days.