Introduction to PDEs 2018, first assignment, due Monday September 17

1) Solve analytically the inviscid Burgers equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0$$

with initial data

$$u(x,0) = \begin{cases} -1 & \text{for } x < -1 \\ x & \text{for } -1 < x < 1 \\ 1 & \text{for } x > 1 \end{cases}$$

and plot the solution at time $t = 2$.

2) Modify the matlab code traffic.m in the notes so that it solves the inviscid Burgers equation in a finite segment $[x_1, x_2]$, with fluxes prescribed at the two ends: $Q(x_1^-, t)$ and $Q(x_2^+, t)$ given. Use it to check your solution to problem 1, and to solve the initial-boundary problem

$$u_t + \left(\frac{u^2}{2}\right)_x = 0,$$

$$u(x,0) = -\sin(x),$$

$$u\left(-\frac{\pi}{2}^-, t\right) = 1, \quad u\left(\frac{\pi}{2}^+, t\right) = -1.$$

Verify that the breaking time agrees with the one predicted by the theory. What is the solution for long times, and at what time does it stop changing?

Comment: In this, as in all problems involving nonlinear characteristics, the boundary conditions are to be interpreted as applicable only when the information they provide travels toward the inside of the domain. Otherwise, they "wait" outside until their information becomes relevant for the inside problem again. This is what is meant by the superscripts $-$ and $+$ in the boundary conditions.