Introduction to PDEs 2018, second assignment, due Monday September 24

1) Consider the 1-D gas dynamics equations (conservation of mass, momentum and energy)

$$\begin{aligned} \rho_t + (\rho\, u)_x &= 0 \\ (\rho\, u)_t + \left(\rho\, u^2 + P\right)_x &= 0 \\ \left(e + \rho\, u^2/2\right)_t + \left(e\, u + \rho\, u^3/2 + P\, u\right)_x &= 0 \end{aligned}$$

for an ideal gas, where the pressure P and internal energy e are given by

$$P = R\,\rho\, T$$

and

$$e = c_v T$$

(T stands for temperature; R and c_v are constants.)

a) Rewrite the equations (where the solution is smooth) in the form

$$U_t + A(U) U_x = 0,$$

where

$$U = \begin{pmatrix} \rho \\ T \\ u \end{pmatrix}$$

b) Show that the speed u is an eigenvalue of A; find the corresponding left eigenvector l, and an integrating factor $\mu(U)$ so that

$$\mu(U) \sum_{i=1}^{3} l_i dU_i = ds(U).$$

The Riemann invariant $s(U)$ that you have found (the only one for this system) is the gas *entropy* (or a function of it, depending on your choice for the integrating factor.) It satisfies the characteristic equation $s_t + u\, s_x = 0$. If s is initially uniform, say $s = s_0$, it follows that it will stay uniform at least until shocks form. This gives rise to the isentropic equations discussed in the notes (Riemann's mistake in his determination of the speed of sound was to think that the temperature T could be assumed uniform instead. He obviously had not paid enough attention to the Riemann invariants of the system!)

2) (Page 26 in the notes.) Prove that, when a system has a complete set of Riemann invariants, a characterization of simple waves equivalent to the one in the notes is as solutions where all but one of the n Riemann invariants are uniform in space and time.

3) (Page 29 in the notes.) Solve the Riemann problem when $F(U)$ is a linear function: $F = A\,U$. For concreteness, you may assume that U has dimension 3.