Introduction to PDEs 2018, fifth assignment, due Monday October 15

The effect of a fixed point source of sound with frequency ω in three dimensional space can be modeled by the forced wave equation

$$u_{tt} - \Delta u = \delta(x) e^{-i\omega t}\,, \tag{1}$$

where the real part of u determines the physical solution, and the speed of sound has been normalized to one. If the source is moving with constant speed v, on the other hand, the equation becomes

$$u_{tt} - \Delta u = \delta(x - vt) e^{-i\omega t}\,. \tag{2}$$

The goal of this exercise is to compare the solutions to these two equations (i.e. to observe the Doppler effect: the siren's change of pitch as a fire truck passes by.) In principle, both equations can be solved using Duhamel's principle. Yet dealing with the δ functions is a subtle matter, so we will make sure that everything is going fine by first solving the problem with a fixed source in a simpler fashion, then using this solution to check the one through Duhamel's principle, and finally changing the auxiliary function U in Duhamel's principle to account for a moving source. Good luck!

a) For equation (1), propose a solution of the form

$$u(x,t) = p(x) e^{-i\omega t}.$$

The resulting elliptic equation, a forced reduced wave equation, is

$$\Delta p + \omega^2 p = -\delta(x).$$

In a language to be introduced later in the class, we are looking for a fundamental solution to the reduced wave operator. Find such solution, using radial symmetry (in polar coordinates, nothing in the problem depends on the angles), and proposing a solution of the form

$$p = c\frac{e^{i\lambda r}}{r}$$

where $r = \|x\|$, and λ and c are constants.

A few hints: Solve the problem first away from the origin, where the right-hand side vanishes. There will be two undetermined constants (two possible values of c, each corresponding to a value of λ.) Keep only one invoking causality: waves should move away from the source at $r = 0$, not converge to it from infinity. Then determine its value by integrating the equation over a small sphere around the origin (integration is what δ functions are good for) and applying the divergence theorem to the left hand side.

b) Now solve again equation (1), this time using Duhamel's principle, and check that your solution agrees with that of part a). (The way we posed the problem, the source has been there forever, so the integration must start from $t = -\infty$.) In the process, you will have determined the auxiliary function $U(x, t, s)$.

c) Clearly, the auxiliary functions $U(x, t, s)$ for equations (1) and (2) are related, since for each value of s, the corresponding auxiliary problems are the same, only displaced in x. Use this fact to derive $U(x, t, s)$ for equation (2), and then use $U(x, t, s)$ to compute $u(x, t)$ (you may assume that $v < 1$).

d) Finally, compare the two solutions. In particular, look at the solution to (2) at a fixed point (i.e. adopt the position of an observer looking at the moving signal). Is the frequency of the received signal a constant? Try to argue a qualitative reason for the observed behavior.