Introduction to PDEs 2018, sixth assignment, due Monday October 29

1) Find the fundamental solution to the modified heat equation in n dimensions with a diffusivity that grows linearly with time:

$$u_t = t\Delta u, \quad u(x,0) = \delta(x),$$

a) through scaling, self-similarity arguments and

b) using the Fourier transform. Check that your two answers agree.

2) In which sense does the ground below keep memory of the thermal history of days and nights past? To explore this, solve the one-dimensional heat equation

$$T_t = \mu T_{zz} \text{ in } z < 0$$

($\mu > 0$), with data on the surface representing the diurnal heating cycle by the sun:

$$T(0,t) = \sin(2\pi t),$$

where the time t is measured in days. Assume that there have been days and nights for so long that any initial data has been forgotten, a quite realistic hypothesis. Plot the solution, and discuss how the vertical distance corresponding to a day scales with the soil's thermal conductivity μ.

Hints: Since the problem is linear, you may want to consider complex solutions and then keep only their real or imaginary part. With constant coefficients, complex exponentials are good candidates; moreover, the boundary condition is of that form. As in last week homework's problem about the Doppler effect, you will need to invoke causality when you factor the time dependence away (In the context of the heat equation, causality is often associated with growth or decay at infinity, unlike the wave scenario, where it selects a direction of propagation for the waves.)