Introduction to PDEs 2018, eighth assignment, due Monday November 12th

1) (page 65 in the notes.) We have seen, via the Cole-Hopf transformation, that the solution to the initial-value problem for the Burgers equation

$$u_t + uu_x = \nu u_{xx}, \quad u(x,0) = u_0(x)$$

adopts the form

$$u(x,t) = \frac{\int \frac{x-y}{t} e^{-\frac{\Gamma}{2\nu}}\, dy}{\int e^{-\frac{\Gamma}{2\nu}}\, dy},$$

where

$$\Gamma(x,y,t) = \int_0^y u_0(z)\, dz + \frac{(x-y)^2}{2t}.$$

Using Laplace's method to characterize this solution in the limit as $\nu \to 0$, we found that $u(x,t)$ is constant along characteristic lines satisfying

$$\frac{dx}{dt} = u.$$

Show that the jump condition for shocks,

$$\frac{dx}{dt} = \frac{u^- + u^+}{2},$$

also follows from the asymptotics of the exact solution as $\nu \to 0$.