Introduction to PDEs 2018, ninth assignment, due Monday November 19th

1) (page 67 of the notes.) For functions $u(x)$ defined on the segment (x_l, x_r), let $I(u)$ denote the squared norm of its derivative:

$$I(u) = \int_{x_l}^{x_r} \left(u'(x)\right)^2 \, dx.$$

a) Show by induction on the cardinality n of the partition of (x_l, x_r) into sub-segments, $(x_l, x_1, \ldots x_n, x_r)$, that the straight line

$$u(x) = u_l + (u_r - u_l)\frac{x - x_l}{x_r - x_l}$$

minimizes $I(u)$ over all continuous piecewise linear functions $u(x)$ satisfying the boundary conditions

$$u(x_l) = u_l, \quad u(x_r) = u_r.$$

b) Prove that the discrete version of the 1d Laplace's equation in a segment,

$$u_{j+1} - 2u_j + u_{j-1} = 0, \quad u_0 = u_l, \ u_n = u_r$$

is equivalent to the minimization of the discretized version of $I(u)$:

$$I_d(u) = \sum_{j=0}^{n-1} \left(u_{j+1} - u_j\right)^2.$$

Prove this in two ways: using your result from part a) (i.e. comparing the actual solution to the discrete set of equations and the minimizer of I_d), and by mimicking, in the discrete scenario, the calculus of variations (you will have to introduce a "discrete variation" η_j, and perform a "summation by parts".)

2) Using Green's functions, solve the problem

$$\frac{d^2u}{dx^2} = \begin{cases} 1 \text{ for } x \le \frac{1}{2} \\ 0 \text{ for } x > \frac{1}{2} \end{cases}, \quad u(0) = u(1) = 0.$$

3) Develop the appropriate Green's functions and use them to solve the problem

$$\frac{d^2u}{dx^2} = \begin{cases} 1 \text{ for } x \le \frac{1}{2} \\ 0 \text{ for } x > \frac{1}{2} \end{cases}, \quad u(0) = \frac{du}{dx}(1) = 0.$$

4) Verify your answers to the 2 prior questions by solving the same problems again simply by integrating twice and fitting the integration constants to the boundary conditions.

5) A function $u(x, y)$ satisfies the equation

$$\Delta u = e^{-2y} \sin(x)$$

in the strip

$$0 < x < \pi\,, \qquad y > 0\,,$$

and the boundary conditions

$$u(0, y) = u(\pi, y) = 0\,, \qquad u(x, 0) = \sin(3x)\,, \lim_{y\to\infty} u = 0.$$

Find $u_y(x, 0)$.