Optimization under rare chance constraints

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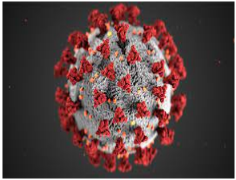
Outline

What are rare events?







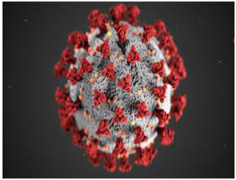
Examples:

- Material failure (bridge/tool/plane stress fractures)
- Extreme weather (tornadoes, hurricanes, heat waves)
- Rogue waves, tsunamis, earthquakes
- Financial sector/bank/company collapse
- Pandemics

Why study rare events?







Common to all:

- Rare but high impact/cost/damage

Control and mitigation:

- Design engineering structures $\Longrightarrow$ control material failure
- Design portfolio $\Longrightarrow$ control risk of investments

Optimization under rare chance constraints

$$\begin{aligned} \underset{u\in\mathcal{U}}{\text{minimize}} \quad & J(u) \\ \text{subject to} \quad & \mathbb{P}(F(u,\xi) \geq z) \leq \alpha \quad \text{for some fixed } \alpha \ll 1 \end{aligned}$$

- $F : \mathcal{U} \times \Xi \to \mathbb{R}$ . . . parameter-to-observation map, can involve PDEs
- $J : \mathcal{U} \to \mathbb{R}$ . . . cost
- $\Xi \subseteq \mathbb{R}^n$ . . . random space of uncertain parameters ξ with measure $\mathcal{P}$
- $\mathcal{U} \subseteq \mathbb{R}^m$ . . . domain for control or decision variable u
- z . . . specified upper bound
- $\alpha \in (0,1)$ . . . risk threshold

Challenges for rare chance constraints

Sampling-based methods:

$$\xi^i \overset{\text{i.i.d.}}{\sim} \mathcal{P}$$

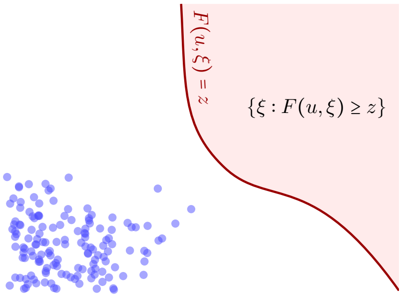
- Indicator approximation:

$$\mathbb{P}(F(u,\xi) \geq z) \approx \frac{1}{N}\sum_{i=1}^{N} \mathbb{1}_{[z,\infty)}\left(F(u,\xi^i)\right)$$

- Scenario approach:

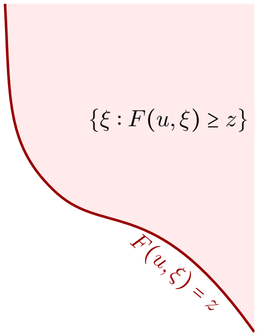
$$F(u,\xi^i) \leq z, \; i = 1,\ldots,N$$

- Spherical radial decomposition



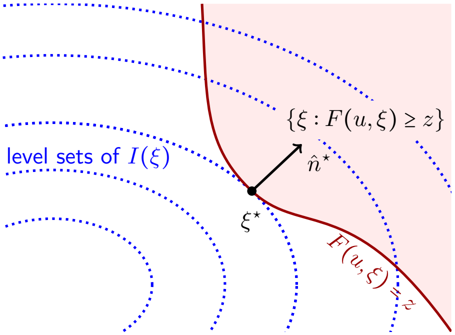
Need $N = O(\alpha^{-1})$ samples, each adds new variables and constraints; intractable for complicated F

Overview of large deviation theory-based method



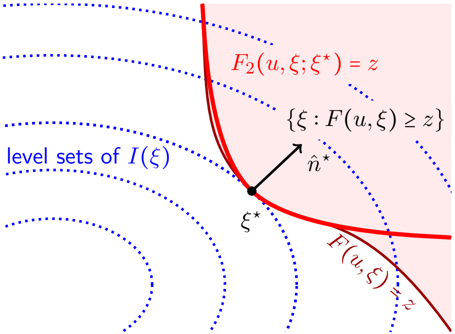
Overview of large deviation theory-based method

1. Find LDT optimizer $\xi^\star$



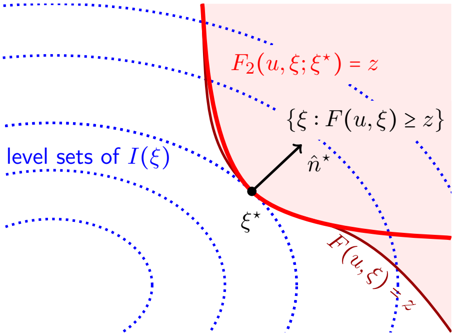
Overview of large deviation theory-based method

1. Find LDT optimizer $\xi^\star$
2. Explicit formula for geometric approx.



Overview of large deviation theory-based method

1. Find LDT optimizer $\xi^\star$
2. Explicit formula for geometric approx.
3. Bilevel reformulation



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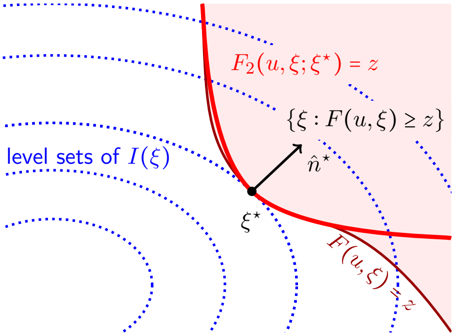


LDT-based methods:

- sampling-free
- insensitive to extremeness
- work for expensive parameter-to-observation maps
- solvable by off-the-shelf solvers

Outline

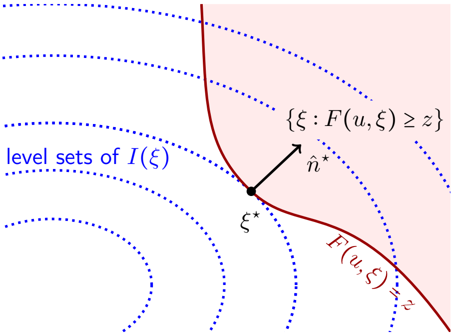
Large deviation theory

$$\mathbb{P}\left(F(u,\xi) \geq z\right) \asymp \exp\left(-I(\xi^\star(u,z))\right) \text{ as } z \to \infty$$

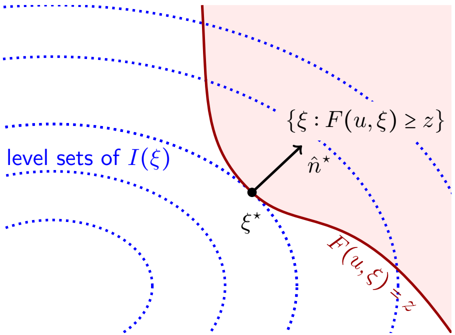
LDT optimizer:

$$\xi^\star(u,z) \coloneqq \operatorname*{argmin}_{\xi \in \Xi} \{I(\xi) : F(u,\xi) \geq z\}$$

- $I(\cdot)$: rate function
- depends only on distribution of ξ
- Legendre transform of cumulant generating fct
- Gaussian $\xi \sim \mathcal{N}(\mu, \Sigma)$: $I(\xi) = \frac{1}{2}\|\xi - \mu\|^2_{\Sigma^{-1}}$



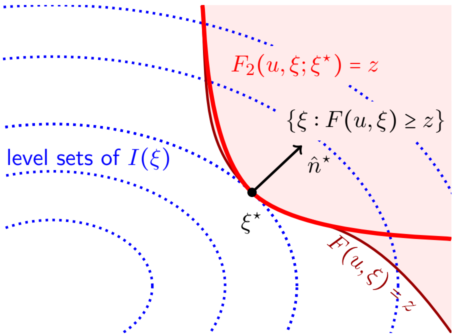
Geometric approximation of rare event probability



Geometric approximation of rare event probability

Geometric approx: measure bounded by second-order Taylor expansion

$$\mathbb{P}(F(u,\xi) \geq z) \approx P_2(u,z,\xi^\star) = \mathcal{P}\left(\{\xi \in \Xi : F_2(u,\xi;\xi^\star) \geq z\}\right)$$



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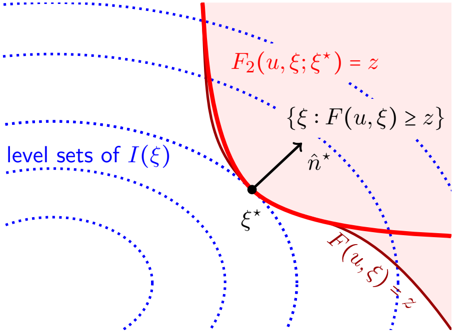
- Gaussian mixture model

$$\xi \sim \sum_{i=1}^{M} w_i \mathcal{N}(\mu_i, \Sigma_i), \quad \sum_{i=1}^{M} w_i = 1$$

- Explicit algebraic formula
- Only require local derivative info
- Universal approximation



$$P_2(u,z,\xi^\star) = \sum_{i=1}^{M} w_i \Phi(-\|\tilde{\xi}_i - \mu_i\|_{\Sigma_i^{-1}}) \det{}_{\perp \tilde{n}_i}\left(\mathrm{I}_n - \frac{\|\tilde{\xi}_i - \mu_i\|_{\Sigma_i^{-1}}}{\|\Sigma_i^{\frac{1}{2}} \nabla_\xi F_2(u,\tilde{\xi}_i;\xi^\star)\|} \Sigma_i^{\frac{1}{2}} \nabla_\xi^2 F(u,\xi^\star) \Sigma_i^{\frac{1}{2}}\right)^{-\frac{1}{2}}$$

Bilevel Reformulation

Optimization under rare chance constraints

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Bilevel opt: Approximate rare event probability with LDT estimates

$$\begin{aligned} \underset{u\in\mathcal{U}}{\text{minimize}} \quad & J(u) \\ \text{subject to} \quad & P_2(u,z,\xi^\star) \leq \alpha, \quad \xi^\star \in \underset{\xi\in\Xi}{\operatorname{argmin}} \{I(\xi) : F(u,\xi) \geq z\} \end{aligned}$$

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Single-level opt: Replace lower-level with first-order optimality conditions

$$\underset{u \in \mathcal{U}, \xi^\star \in \Xi, \lambda \in \mathbb{R}_+}{\text{minimize}} \quad J(u)$$
$$\text{subject to} \quad P_2(u,z,\xi^\star) \leq \alpha, \; F(u,\xi^\star) = z, \; \nabla I(\xi^\star) = \lambda \nabla_\xi F(u,\xi^\star)$$

Bilevel Reformulation

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Single-level opt: Replace lower-level with first-order optimality conditions

$$\begin{aligned} \underset{u \in \mathcal{U}, \xi^\star \in \Xi, \lambda \in \mathbb{R}_+}{\text{minimize}} \quad & J(u) \qquad \text{solvable by off-the-shelf solvers, e.g. Ipopt} \\ \text{subject to} \quad & P_2(u, z, \xi^\star) \leq \alpha, \; F(u, \xi^\star) = z, \; \nabla I(\xi^\star) = \lambda \nabla_\xi F(u, \xi^\star) \end{aligned}$$

Outline

Short column design

Find width and height $u = [w, h]$ to minimize area wh of rectangular cross section, avoiding material failure with probability $1 - \alpha$, with uncertain material parameter $\xi = [\xi_F, \xi_M, \xi_Y]$

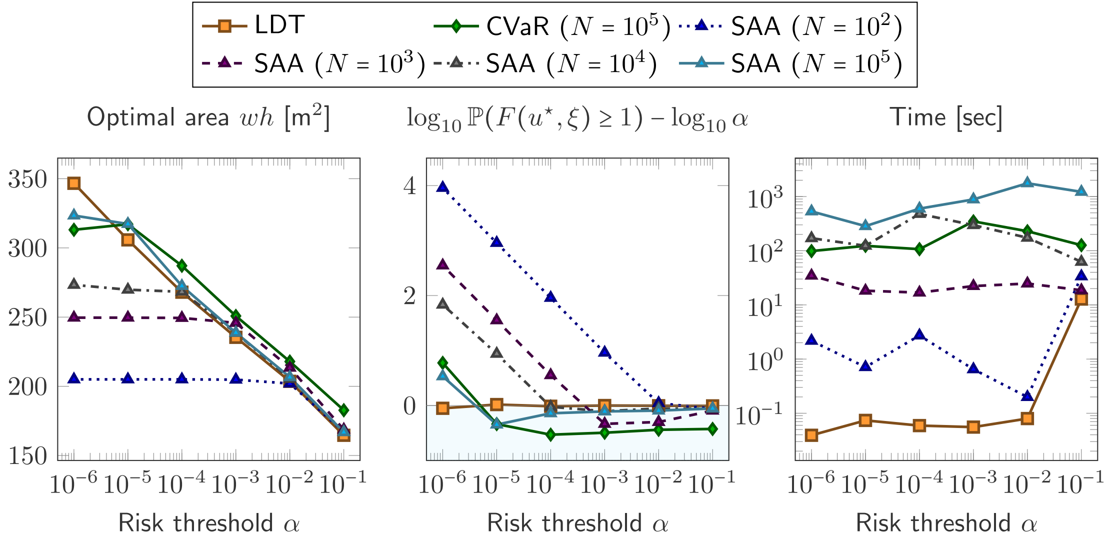
$$\begin{aligned} \underset{u=[w,h]\in\mathbb{R}^2}{\text{minimize}} \quad & wh \\ \text{subject to} \quad & L_w \le w \le U_w, \\ & L_h \le h \le U_h, \\ & \mathbb{P}(F(u,\xi) \ge 1) \le \alpha, \quad \text{for fixed } \alpha \ll 1 \end{aligned}$$

From elastic-plastic constitutive law

$$F(u,\xi) := \frac{4\xi_M}{wh^2 \exp(\xi_Y)} + \frac{\xi_F^2}{w^2 h^2 \exp(2\xi_Y)}$$

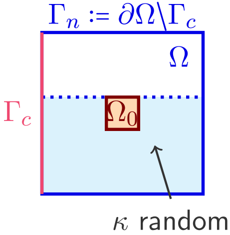
Short column design

Compare LDT-based methods with sampling-based methods (SAA, CVaR)



Optimal PDE boundary control

Control temperature at boundary Γ_c as close to 0 as possible, while ensuring average temperature in Ω_0 bounded by z with high probability



$$\begin{aligned} \underset{u}{\text{minimize}} \quad & \frac{1}{2}\int_{\Gamma_c} u^2(x)dx, \\ \text{subject to} \quad & \mathbb{P}(F(u,\xi) \geq z) \leq \alpha \end{aligned}$$

$$F(u,\xi) := \frac{1}{|\Omega_0|}\int_{\Omega_0} y(x;u,\xi)\mathrm{d}x$$

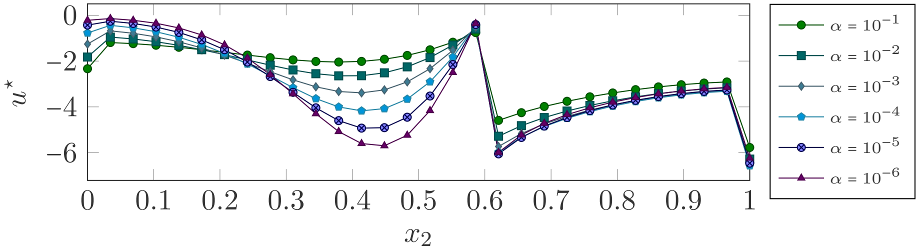
$y(x;u,\xi)$ is solution of 2D steady-state advection-diffusion equation

$$\begin{aligned} -\nabla\cdot(\kappa(x,\xi)\nabla y(x)) + w(x)\cdot\nabla y(x) &= f(x,\xi), && x \in \Omega, \\ (\kappa(x,\xi)\nabla y(x))\cdot n(x) &= \frac{1}{\epsilon_0}\left(u(x) - y(x)\right), && \text{on } \Gamma_c, \\ (\kappa(x,\xi)\nabla y(x))\cdot n(x) &= 0, && \text{on } \Gamma_n. \end{aligned}$$

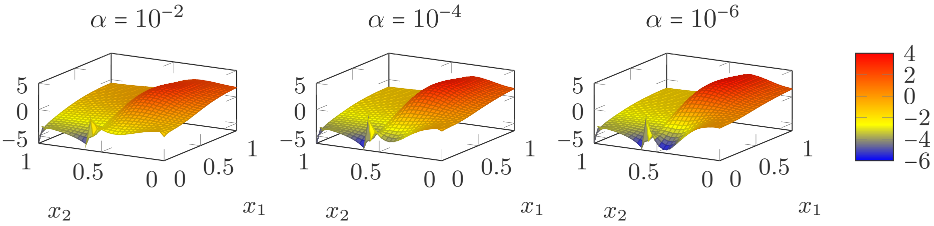
with random diffusion coefficient $\kappa(x,\xi)$ and random source $f(x,\xi)$

Optimal PDE boundary control

Sampling-based methods are untenable



Optimal boundary condition $u^\star$ with $z = 1$ for different α using LDT



Temperature y under $u^\star$ and $\xi^\star$ with $z = 1$ for different α using LDT

Outline

Main takeaways

- Large deviation theory for rare chance constraints
- Explicit algebraic formulations for Gaussian mixture
- Methods for optimization under rare chance constraints:
  - sampling-free
  - solvable by off-the-shelf solvers
  - insensitive to extremeness
- Examples on nonlinear and PDE-constrained optimization problems

Main takeaways

- Large deviation theory for rare chance constraints
- Explicit algebraic formulations for Gaussian mixture
- Methods for optimization under rare chance constraints:
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- Examples on nonlinear and PDE-constrained optimization problems

Future directions

- Refine asymptotic approx
- Generalize to more distributions
- Release some regularity assumptions
- SunA7 (March 24, 8:30 - 10:00am): Anirudh Subramanyam
  Self-Structured Importance Sampling for Chance-Constrained Optimization
- Improve efficiency for large-scale problems: SDEs/SPDEs (ongoing)

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