Foliations in moduli spaces of abelian varieties and dimension of leaves

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dedicated to Yuri Manin on his seventieth birthday

Introduction

In the theory of foliations in moduli spaces of abelian varieties, as developed in [32], we study *central leaves*. Consider a p-divisible group X_0 over a field K, and let $\mathrm{Def}(X_0) = \mathrm{Spf}(\Gamma)$ and $D(X_0) = \mathrm{Spec}(\Gamma)$. Consider $g \in \mathbb{Z}_{>0}$ and consider $\mathcal{A}_g \otimes \mathbb{F}_p$, the moduli space of polarized abelian varieties (in this paper to be denoted by $\mathcal{A}_g$); choose $[(A, \lambda)] = x \in \mathcal{A}$ and $(A, \lambda)[p^\infty] = (X, \lambda)$. Here is the central question of this paper: determine

$$\text{unpolarized case: } \dim(\mathcal{C}_{X_0}(D(X_0))) = ?; \text{polarized case: } \dim(\mathcal{C}_{(X,\lambda)}(\mathcal{A})) = ?.$$

For the notation $\mathcal{C}_-(-)$ see 1.7. We give a combinatorial description of certain numbers associated with a Newton polygon, such as "dim$(-)$", "sdim$(-)$", "cdu$(-)$", "cdp$(-)$". We show these give the dimension of a stratum or a leaf, in the unpolarized and in the principally polarized case. We give 3 different proofs that these formulas for the dimension of a central leaf are correct:

$$\dim(\mathcal{C}_Y(D(X))) = \mathrm{cdu}(\beta), \quad \beta := \mathcal{N}(Y), \quad \text{see Theorem 4.5 and}$$

$$\dim\left(\mathcal{C}_{(X,\lambda)}(\mathcal{A}_g)\right) = \mathrm{cdp}(\xi), \quad \xi := \mathcal{N}(X), \quad \text{see Theorem 5.4;}$$

One proof is based on the theory of minimal p-divisible groups, as developed in [36], together with a result by T. Wedhorn, see [42], [43]; this was the proof I first had in mind, written up in the summer of 2002.

The second proof is based on the theory of Chai about Serre-Tate coordinates, a generalization from the ordinary case to central leaves in an arbitrary Newton polygon stratum, see [2]. This generalization was partly stimulated by the first proof, and the question to "explain" the dimension formula which came out of my computations.

A third proof, in the unpolarized case and in the polarized case ($p > 2$), is based on recent work by E. Viehmann, see [40], [41], where the dimension of Rapoport-Zink spaces, and hence the dimension of isogeny leaves is computed in the (un)polarized case; the almost product structure of an open Newton polygon stratum by central and isogeny leaves, as in [32], see 7.16, finishes a proof of the results.

These results enable us to answer a question, settle a conjecture, about bounds of the dimension of components of a Newton polygon stratum, see Section 6.

These results find their natural place in joint work with Ching-Li Chai, which we expect finally to appear in [5]. I thank Chai for the beautiful things I learned from him, in particular for his elegant generalization of Serre-Tate canonical coordinates used in the present paper.

The results of this paper were already announced earlier, e.g. see [32] 3.17, [1] 7.10, 7.12.

Historical remarks. Moduli for polarized abelian varieites in positive characteristic were studied in the fundamental work by Yuri Manin, see [21]. That paper was and is a great source of inspiration.

In summer 2000 I gave a talk in Oberwolfach on foliations in moduli spaces of abelian varieties. After my talk, in the evening of Friday 4-VIII-2000 Bjorn Poonen asked me several questions, especially related to the problem I raised to determine the dimensions of central leaves. Our discussion resulted in Problem 21 in [8]. His expectations coincided with computations I had made of these dimensions for small values of g. Then I jumped to the conclusion what those dimensions for an arbitrary Newton polygon could be; that is what was proved later, and reported on here, see 4.5, 5.4. I thank Bjorn Poonen for his interesting questions; our discussion was valuable for me.

A suggestion to the reader. The results of this paper are in Sections 4, 5 and 6; we refer to the introductions of those sections. The reader could start reading those sections and refer to other sections whenever definitions or results are needed. In Section 1 we explain some of the concepts used in this paper. In the Sections 2 and 3 we describe preliminary results used in the proofs. In Section 7 we list some of the well-known methods and results we need for our proofs.

Various strata NP - EO - Fol. Here is a short survey of strata and foliations, to be defined, explained and studied below. For an abelian variety A, with a polarization (sometimes supposed to be principal) we can study the following objects:

NP $A \quad \mapsto \quad A[p^\infty] \quad \mapsto \quad A[p^\infty]/\sim_k$ over an algebraically closed field: the isogeny class of its p-divisible group; by the Dieudonné - Manin theorem, see 7.2, we can identify this isogeny class of p-divisible groups with the Newton polygon of A. We obtain the Newton polygon strata, see 1.4 and 7.8.

EO $(A,\lambda) \quad \mapsto \quad (A,\lambda)[p] \quad \mapsto \quad (A,\lambda)[p]/\cong_k$ over an algebraically closed field:
we obtain EO-strata; see [30], see 1.6. Important feature (Kraft, Oort): the number of geometric isomorphism classes of group schemes of a given rank annihilated by p is *finite*.

Fol $(A,\lambda) \quad \mapsto \quad (A,\lambda)[p^\infty] \mapsto (A,\lambda)[p^\infty]/\cong_k$ over an algebraically closed field:
we obtain a foliation of an open Newton polygon stratum; see [32] and 1.7. Note that for $f < g-1$ the number of (central) leaves is *infinite*.

Note: $X \cong Y \quad \Rightarrow \quad \mathcal{N}(X) = \mathcal{N}(Y)$; conclusion: every central leaf in **Fol** is contained in exactly one Newton polygon stratum in **NP**.
Note: $X \cong Y \quad \Rightarrow \quad X[p] = Y[p]$; conclusion: every central leaf in **Fol** is contained in exactly EO-stratum in **EO**.

However, a NP-stratum may contain many EO-strata, an EO-stratum may intersect several NP-strata, see 8.6. Whether an EO-stratum equals a central leaf is studied and answered in the theory of minimal p-divisible groups, see 1.5 and 7.5.

Isogeny correpondences are finite-to-finite above central leaves, but may blow up and down subsets of isogeny leaves; see 7.22 and Section 6.

1 Notations

We fix a prime number p. All base schemes and base fields will be in characteristic p. We write K for a field, and we write k and Ω for algebraically closed fields of characteristic p.

We study the (coarse) moduli scheme $\mathcal{A}_g$ of polarized abelian varieties of dimension g in characteristic p; this notation is used instead of $\mathcal{A}_g \otimes \mathbb{F}_p$. We write $\mathcal{A}_{g,1}$ for the moduli scheme of principally polarized abelian varieties of dimension g in characteristic p. We will use letters like A, B to denote abelian varieties.

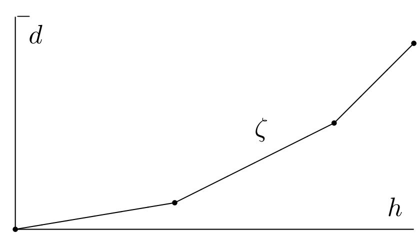
For the notion of a p-divisible group we refer to the literature, e.g. [13]; also see [3], 1.18. Instead of the term p-divisible group the equivalent notion “Barsotti-Tate group” is used. We will use letter like X, Y to denote a p-divisible group. For an abelian variety A, or an abelian scheme, and a prime number p we write $A[p^\infty] = \cup_i A[p^i] = X$ for its p-divisible group.

For finite group schemes and for p-divisible groups over a perfect field in characteristic p we use the theory of *covariant* Dieudonné modules. In [21] the contravariant theory was developed. However it turned out that the covariant theory was easier to handle in deformation theory; see [30], 15.3 for references.

A warning and a remark on notation. Under the *covariant* Dieudonné module theory the Frobenius morphism on a group scheme is transformed into the Verschiebung homomorphism on its Dieudonné module; this homomorphism is denoted by $\mathcal{V}$; the analogous statement for V being transformed into $\mathcal{F}$; in shorthand notation $\mathbb{D}(F) = \mathcal{V}$ and $\mathbb{D}(V) = \mathcal{F}$, see [30], 15.3. In order not to confuse F on group schemes and the Frobenius on modules we have chosen the notation $\mathcal{F}$ and $\mathcal{V}$. An example: for an abelian variety A over a perfect field, writing $\mathbb{D}(A[p^\infty]) = M$ we have $\mathbb{D}(A[F]) = M/\mathcal{V}M$.

1.1. Newton polygons. Suppose given integers $h, d \in \mathbb{Z}_{\geq 0}$; here $h =$ "height", $d =$ "dimension". In case of abelian varieties we will choose $h = 2g$, and $d = g$. A Newton polygon γ (related to h and d) is a polygon $\gamma \subset \mathbb{Q} \times \mathbb{Q}$ (or, if you wish in $\mathbb{R} \times \mathbb{R}$), such that:

- γ starts at $(0,0)$ and ends at (h,d);
- γ is lower convex;
- any slope β of γ has the property $0 \leq \beta \leq 1$;
- the breakpoints of γ are in $\mathbb{Z} \times \mathbb{Z}$; hence $\beta \in \mathbb{Q}$.



Note that a Newton polygon determines (and is determined by)

$$\beta_1, \cdots, \beta_h \in \mathbb{Q} \text{ with } 0 \leq \beta_1 \leq \cdots \leq \beta_h \leq 1 \quad \leftrightarrow \quad \zeta.$$

Sometimes we will give a Newton polygon by data $\sum_i (m_i, n_i)$; here $m_i, n_i \in \mathbb{Z}_{\geq 0}$, with $\gcd(m_i, n_i) = 1$, and $m_i/(m_i + n_i) \leq m_j/(m_j + n_j)$ for $i \leq j$, and $h = \sum_i (m_i + n_i)$, $d = \sum_i m_i$. From these data we construct the related Newton polygon by choosing the slopes $m_i/(m_i+n_i)$ with multiplicities $h_i = m_i + n_i$. Conversely clearly any Newton polygon can be encoded in a unique way in such a form.

Let ζ be a Newton polygon. Suppose that the slopes of ζ are $1 \geq \beta_1 \geq \cdots \geq \beta_h \geq 0$; this polygon has slopes $\beta_h, \cdots, \beta_1$ (non-decreasing order), and it is *lower convex*. We write ζ^* for the polygon starting at $(0,0)$ constructed using the slopes $\beta_1, \cdots, \beta_h$ (non-increasing order); note that ζ^* is *upper convex*, and that the beginning and end point of ζ and of ζ^* coincide. Note that $\zeta = \zeta^*$ iff ζ is isoclinic (i.e. there is only one slope).

We say that ζ is *symmetric* if $h = 2g$ is even, and the slopes $1 \geq \beta_1 \geq \cdots \geq \beta_h \geq 0$ satisfy $\beta_i = 1 - \beta_{h-i+1}$ for $1 \leq i \leq h$. We say that ζ is supersingular, and we write $\zeta = \sigma$, if all slopes are equal to $1/2$. A symmetric Newton polygon is isoclinic this is the case iff the Newton polygon is supersingular.

1.2. We will associate to a p-divisible group X over a field K its Newton polygon $\mathcal{N}(X)$. This will be the "Newton polygon of the characteristic polynomial of Frobenius on X"; this terminology is incorrect in case K is not the prime field $\mathbb{F}_p$. Here is a precise definition.

Let $m, n \in \mathbb{Z}_{\geq 0}$; we are going to define a p-divisble group $G_{m,n}$. We write $G_{1,0} = \mathbb{G}_m[p^\infty]$ and $G_{0,1} = \underline{\mathbb{Q}_p/\mathbb{Z}_p}$. For positive, coprime values of m and n we choose a perfect field K, we write $M_{m,n} = R_K/R_K(\mathcal{V}^n - \mathcal{F}^m)$, where R_K is the Dieudonné ring. We define $G_{m,n}$ by $\mathbb{D}(G_{m,n}) = M_{m,n}$. Note that this works over any perfect field. This p-divisible group is defined over $\mathbb{F}_p$ and we will use the same notation over any field K, instead of writing $(G_{m,n})_K = (G_{m,n})_{\mathbb{F}_p} \otimes K$. Note that $M_{m,n}/\mathcal{V}{\cdot}M_{m,n}$ is a K-vector space of dimension m. Hence the dimension of $G_{m.n}$ is m. We see that the height of $G_{m,n}$ is $h = m + n$. We can show that under Serre-duality we have $G^t_{m,n} = G_{n,m}$.

We define $\mathcal{N}(G_{m,n})$ as the polygon which has slope $m/(m+n)$ with multiplicity $h = m + n$. Note : this is the F-slope on $G_{m,n}$, and it is the $\mathcal{V}$-slope on $M_{m,n}$. Indeed over $\mathbb{F}_p$ the Frobenius $F : G_{m,n} \to G_{m,n}$ has the property $F^{m+n} = F^m V^m = p^m$.

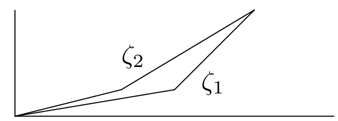
Let X be a p-divisble group over a field K. Choose an algebraic closure $K \subset k$. Choose an isogeny $X_k \sim \quad \Pi_i\ (G_{m_i,n_i})$ see 7.1 and 7.2. We define $\mathcal{N}(X)$ as the "union" of these $\mathcal{N}(G_{m_i,n_i})$, i.e. take the slopes of these isogeny factors, and order all slopes in non-decreasing order. By the Dieudonné-Manin theorem we know that *over an algebraically closed field there is a bijective correspondence between isogeny classes p-divisible groups on the one hand and, and Newton polygons on the other hand,* see 7.2. For an abelian variety A we write $\mathcal{N}(A)$ instead of $\mathcal{N}(A[p^\infty])$.

For a commutative group scheme G over a field K we define the number $f = f(G)$ by: $\mathrm{Hom}(\mu_p, G_k) \cong (\mathbb{Z}/p)^f$, where k is an algebraically closed field. For a p-divisble group X, respectively an abelian variety A the number $f(X)$, respectively $f(A)$ is called the p-rank. Note that in these cases this number is the multiplicity of the slope equal to one in the Newton polygon.

For an abelian variety A its Newton polygon ξ is *symmetric*; by definition this means that the multiplicity of the slope β in ξ is the same as the multiplicity of the slope $1-\beta$. This was proved by Manin over finite fields. The general case follows from the duality theorem [28] 19.1; we see that $A^t[p^\infty] = A[p^\infty]^t$; using moreover $(G_{m,n})^t = G_{n,m}$ and the definition of the Newton polygon of a p-divisible group we conclude that $\mathcal{N}(A)$ is symmetric.

1.3. The graph of Newton polygons. For Newton polygons we introduce a partial ordering.

We write $\zeta_1 \succ \zeta_2$ if ζ_1 is "below" ζ_2,
i.e. if no point of ζ_1 is strictly above ζ_2.



1.4. Newton polygon strata. If S is a base scheme, and $\mathcal{X} \to S$ is a p-divisible group over S we write

$$\mathcal{W}_\zeta(S) = \{s \in S \mid \mathcal{N}(\mathcal{X}_s) \prec \zeta\} \subset S$$

and

$$\mathcal{W}^0_\zeta(S) = \{s \in S \mid \mathcal{N}(\mathcal{X}_s) = \zeta\} \subset S.$$

Grothendieck showed in his Montreal notes [11] that "Newton polygons go up under specialization". The proof was worked out by Katz, see 7.8.

1.5. Minimal p-divisible groups. See [36] and [37]. In the isogeny class of $G_{m,n}$ we single out one p-divisible group $H_{m,n}$ specifically; for a description see [15], 5.3 - 5.7; the p-divisible group $H_{m,n}$ is defined over $\mathbb{F}_p$, it is isogenous with $G_{m,n}$, and

the endomorphism ring $\mathrm{End}(H_{m,n} \otimes k)$ *is the maximal order in the endomorphism algebra* $\mathrm{End}(H_{m,n} \otimes k) \otimes \mathbb{Q}$;

these conditions determine $H_{m,n} \otimes \overline{\mathbb{F}_p}$ up to isomorphism. This p-divisible group $H_{m,n}$ is called *minimal*.

One can define $H_{m,n}$ over $\mathbb{F}_p$ by defining its (covariant) Dieudonné module by: $\mathbb{D}(H_{(m,n),\mathbb{F}_p}) = M_{(m,n),\mathbb{F}_p}$, this module has a basis as free module over $W = W_\infty(\mathbb{F}_p)$ given by $\{e_0, \cdots, e_{h-1}\}$, where $h = m+n$, write $p{\cdot}e_i = e_{i+h}$ inductively for all $i \geq 0$, there is an endomorphism $\pi \in \mathrm{End}(H_{(m,n),\mathbb{F}_p})$ with $\pi(e_i) = e_{i+1}$, and $\pi^n = \mathcal{F} \in \mathrm{End}(H_{(m,n),\mathbb{F}_p})$ and $\pi^m = \mathcal{V} \in \mathrm{End}(H_{(m,n),\mathbb{F}_p})$, hence $\pi^h = p \in \mathrm{End}(H_{(m,n),\mathbb{F}_p})$.

If $\zeta = \sum_i (m_i, n_i)$ we write $H(\zeta) := \sum_i H_{m_i,n_i}$, the minimal p-divisible group with Newton polygon equal to ζ. We write $G(\zeta) = H(\zeta)[p]$, the minimal BT_1 group scheme attached to ζ.

In case $\mu \in \mathbb{Z}_{>0}$ we write

$$H_{d,c} = (H_{m,n})^{\mu}, \quad \text{where} \quad d := \mu m, \quad c := \mu n, \quad \gcd(m,n) = 1.$$

For further information see 7.3.

1.6. Basic reference: [30]. We say that G a BT_1 group scheme is, or, a p-divisible group truncated at level one, if is is annihilated by p, and the image of V and the kernel of F are equal; for more information see [13], 1.1 Let $X \to S$ be a p-divisble group over a base S (in characteristic p). We write

$$\mathcal{S}_G(S) := \{s \in S \mid \exists \Omega \quad X_s[p] \otimes \Omega \cong G \otimes \Omega\}.$$

This is called the Ekedahl-Oort stratum defined by X/S. This is a locally closed subset in S. Polarizations can be considered, but are not taken into account in the definition of $\mathcal{S}_G(-)$. See [30], Section 9 for the case of principal polarizations.

Let G be a BT_1 group scheme over an algebraically closed field which is symmetric in the sense of [30], 5.1, i.e. there is an isomorphism $G \cong G^D$. To G we attached in [30], 5.6 an elementary sequence, denoted by $\mathrm{ES}(G)$. A important point is the fact (not easy in case $p = 2$) that a "principally polarized" BT_1 group scheme over an algebraically closed field is uniquely determined by this sequence; this was proved in [30], Section 9; in case $p > 2$ the proof is much easier, and the fact holds in a much more general situation, see [26], Section 5, in particular Coroll. 5.4.

1.7. Basic reference: [32]. Let X be a p-divisible group over a field K and let $\mathcal{Y} \to S$ be a p-divisible group over a base scheme S. We write

$$\mathcal{C}_X(S) = \{s \in S \mid \exists \Omega, \exists \mathcal{Y}_s \otimes \Omega \cong X \otimes \Omega\};$$

here Ω is an algebraically closed field containing $\kappa(s)$ and K. Consider a quasi-polarized p-divisible group (X, λ) over a field. Let $(\mathcal{Y}, \mu) \to S$ be a quasi-polarized p-divisible group over a base scheme S. We write

$$\mathcal{C}_{(X,\lambda)}(S) = \{s \in S \mid \exists \Omega, \exists (\mathcal{Y}, \mu)_s \otimes \Omega \cong (X, \lambda) \otimes \Omega\}.$$

See 7.12 for the fact that *any central leaf is closed in an open Newton polygon stratum.*

We write $\mathcal{I}_X(S)$ and $\mathcal{I}_{(X,\lambda)}(S)$ for the notion of isogeny leaves introduced in [32], Section 4, see 4.10 and 4.11. We recall the definition in the polarized case $S = \mathcal{A}_g \otimes \mathbb{F}_p$. Let $x = [(X, \lambda)]$ be given over a perfect field. Write $\mathcal{H}_{\alpha}(x)$ for the set of points in $\mathcal{A}_g \otimes \mathbb{F}_p$ connected to x by iterated α_p-isogenies (over extension fields). In general this is not a closed subset of $\mathcal{A}_g \otimes \mathbb{F}_p$. However the union of all irreducible components of $\mathcal{H}_{\alpha}(x)$ containing x is a closed subset; this subset with the induced reduced scheme structure is denoted by $\mathcal{I}_{(X,\lambda)}(\mathcal{A}_g \otimes \mathbb{F}_p)$; for the definition in the general (un)polarized case, and for existence theorems, see [32], Section 4. Note that formal completion of

$\mathcal{I}_{(X,\lambda)}(\mathcal{A}_g \otimes \mathbb{F}_p)$ at the point x is the reduced, reduction mod p of the related Rapoport-Zink space; an analogous statement holds for the unpolarized case; for the definition of these spaces see [39], Section 2 for the unpolarized case and Chapter 3 for the polarized case.

1.8. Suppose $X \to S$ and $Y \to T$ are p-divisible groups. Consider triples $(f : U \to S, \quad g : U \to T, \quad \psi : X_f \to Y_g)$, where $f : U \to X$ and $g : U \to T$ are morphisms, and where $\psi : X_f = X \times_U S \to Y_g = Y \times_U S$ is an isogeny. An object representing such triples in the category of schemes over $S \times T$ is called an isogeny correspondence.

Consider polarized abelian schemes $(A, \mu) \to S$ and $(B, \nu) \to T$. Triples $(f : U \to X, \quad g : U \to T, \quad \psi : A_f \to B_g)$ such that $f^*(\mu) = g^*(\nu)$ define isogeny correspondences between families of polarized abelian varieites. These are also called Hecke correspondences. See [9], VII.3 for a slightly more general notion. See [3] for a discussion.

One important feature in our discussion is the fact that isogeny correspondence are finite-to-finite above central leaves. But note that isogeny correspondences in general blow up and down as correspondences in $(\mathcal{A}_g \otimes \mathbb{F}_p) \times (\mathcal{A}_g \otimes \mathbb{F}_p)$.

1.9. Let X_0 be a p-divisible group over a field K. We write $\mathrm{Def}(X_0)$ for the local deformation space in characteristic p of X_0. By this we mean the following. Consider all local Artin rings R with a residue class homomorphism $R \to K$ such that $p{\cdot}1 = 0$ in R. Consider all p-divisble groups X over $\mathrm{Spec}(R)$ plus an identification $X \otimes_R K = X_0$. This functor on the category of such algebras is prorepresentable. The prorepresenting formal scheme is denoted by $\mathrm{Def}(X_0)$.

The prorepresenting formal p-divisible group can be written as $\mathcal{X} \to \mathrm{Def}(X_0) = \mathrm{Spf}(\Gamma)$. This affine formal scheme comes from a p-divisible group over $\mathrm{Spec}(\Gamma)$, e.g. see [14], 2.4.4. This object wil be denoted by $X \to \mathrm{Spec}(\Gamma) =: D(X_0)$.

An analogous definition can be given for the local deformation space $\mathrm{Def}(X_0, \mu_0) = \mathrm{Spf}(\Gamma)$ of a quasi-polarized p-divisible group. In this case we will write $D(X_0, \mu_0) = \mathrm{Spec}((\Gamma)$.

Consider the local deformation space $\mathrm{Def}(A_0, \mu_0)$ of a polarized abelian variety (A_0, μ_0). By the Chow-Grothendieck algebraization theory, see [10], III^1.5.4, we know that there exists a polarized abelian scheme $(A, \mu) \to D(A_0, \mu_0) := \mathrm{Spec}(\Gamma)$ of which the corresponding formal scheme is the prorepresenting object of this deformation functor.

2 Computation of the dimension of automorphism schemes

Consider minimal p-divisible groups as in 1.5, and their BT_1 group schemes $H_{d,c}[p]$. Consider homomorphism group schemes between such, automorphism group schemes and their dimensions. Automorphism group schemes as defined in [42], 5.7, and the analogous definition for homomorphism group schemes. *In this section we compute the dimension of Hom-schemes and of Aut-schemes.* In order to compute these dimensions it suffices to compute the dimension of such schemes of homomorphisms and automorphisms between Dieudonné modules, as explained in [42], 5.7. These computations use methods of proof, as in [23], Sections 4 and 5, [25], [37], 2.4. We carry out the proof of the first proposition, and leave the proof of the second, which is also a direct verification, to a future publication.

2.1. Proposition. *Suppose $a, b, d, c \in \mathbb{Z}_{\geq 0}$; assume that $a/(a+b) \geq d/(d+c)$. Then:*

$$\dim\left(\underline{\mathrm{Hom}}(H_{a,b}[p], H_{d,c}[p])\right) = bd = \dim\left(\underline{\mathrm{Hom}}(H_{d,c}[p], H_{a,b}[p])\right);$$

$$\dim\left(\underline{\mathrm{Aut}}(H_{d,c}[p])\right) = dc.$$

In fact, much more is true in case of minimal p-divisible groups. For $I, J \in \mathbb{Z}_{>0}$ we have

$$\dim(\underline{\mathrm{Hom}}(H_{a,b}[p^I], H_{d,c}[p^J])) \quad = \quad \dim(\underline{\mathrm{Hom}}(H_{a,b}[p], H_{d,c}[p])).$$

Proof. If $a' = \mu{\cdot}a$ and $b' = \mu{\cdot}b$, we have $H_{a',b'} \cong (H_{a,b})^{\mu}$. Hence it suffices to compute these dimensions in case $\gcd(a, b) = 1 = \gcd(d, c)$. From now on we suppose we are in this case. We distinguish three possibilities:

(1) $1/2 \geq a/(a+b)$;
(2) $a/(a+b) \geq 1/2 d/(d+c)$:
(3) $a/(a+b) \geq d/(d+c) \geq 1/2$.

We will see that a proof of (2) is easy. Note that once (1) is proved, (3) follows by duality; indeed, $(H_{a,b})D = H_{b,a}$. Most of the work will be devoted to proving the case (1).

We remind the reader of some notation introduced in [37]. Finite words with letters $\mathcal{F}$ and $\mathcal{V}$ are considered. They are treated in a cyclic way, finite cyclic words repeat itself infinitely often. For such a word w a finite BT_1 group scheme G_w over a perfect field K is constructed by taking a basis for $\mathbb{D}(G_w) = \sum_{a \leq i \leq h} K.z_i$ of the same cardinality as the number h of letters in w. For $w = L_1 \cdots L_h$ we define:

$$L_i = \mathcal{F} \quad \Rightarrow \quad \mathcal{F}z_i = z_{i+1}, \quad \mathcal{V}z_{i+1} = 0;$$

$$L_i = \mathcal{V} \quad \Rightarrow \quad \mathcal{V}z_{i+1} = z_i, \quad \mathcal{F}z_i = 0;$$

i.e. the $L_i = \mathcal{F}$ acting clock-wise in the circular set $\{z_i, \cdots z_h\}$ and $\mathcal{V}$ acting anti-clockwise; see [37], page 282. A circular word w defines in this way a (finite) BT_1 group scheme. Moreover over k a word w is indecomposable iff G_w is indecomposable, see [37] , 1.5. By a theorem of Kraft, see [37] , 1.5, this classifies all BT_1 group schemes over an algebraically closed field.

We define a *finite string* $\sigma : w' \to w$ between words as a pair $((\mathcal{V}s\mathcal{F}), (\mathcal{F}s\mathcal{V}))$ (see [37] page 283), where s is a finite non-cyclic word, $(\mathcal{V}s\mathcal{F})$ is contained in w' and $(\mathcal{F}s\mathcal{V})$ is contained in w; note that "contained in w" means that it is a subword of $\cdots www \cdots$. In [37], 2.4 we see that for indecomposable words w', w a k-basis for $\mathrm{Hom}(G_{w'}, G_w)$ can be given by the set of strings from w' to w. From this we conclude:

$$\dim\left(\underline{\mathrm{Hom}}(G_{w'}, G_w)\right) \quad \textit{equals the number of strings from } w' \textit{ to } w.$$

For $G_{w'} = H_{a,b}$ we write $\mathbb{D}(H_{a,b}) = W{\cdot}e_0 \oplus \cdots \oplus W{\cdot}e_{a+b-1}$, with $\mathcal{F}e_i = e_{i+b}$ and $\mathcal{V}e_i = e_{i+a}$. For $G_{w'=} = H_{d,c}$ we write $\mathbb{D}(H_{a,b}) = W{\cdot}f_0 \oplus \cdots \oplus W{\cdot}f_{d+c-1}$, $\mathcal{F}e_i = e_{i+c}$, $\quad \mathcal{V}e_i = e_{i+d}$. The number of symbols $\mathcal{V}$ in w' equals b; we choose some numbering $\{\mathcal{V} \mid \mathcal{V} \text{ in } w'\} = \{\nu_1, \cdots, \nu_b\}$. Also we choose $\{\mathcal{F} \mid \mathcal{F} \text{ in } w\} = \{\varphi_1, \cdots, \varphi_d\}$.

Claim. *For indices $1 \le i \le b$ and $1 \le j \le d$ there exists a unique non-cyclic finite word s such that $((\nu_i\ s\ \mathcal{F}), (\varphi_j\ s\ \mathcal{V}))$ is a string from w' to w. This gives a bijective map*

$$\{\nu_1, \cdots, \nu_b\} \times \{\varphi_1, \cdots, \varphi_d\} \quad \longrightarrow \quad \{\textit{string} \quad w' \to w\}.$$

Note that the claim proves the first equality in 2.1.

Proof of the Claim, case (2). In this case $b \ge a$ and $d \ge c$. We see that every $\mathcal{F}$ in w' is between letters $\mathcal{V}$, and every $\mathcal{V}$ in w is is between letters $\mathcal{F}$. This shows that a string $((\mathcal{V}s\mathcal{F}), (\mathcal{F}s\mathcal{V}))$ can only appear in this case with the empty word s, and that any $(\nu_i\ \ \mathcal{F})$ and any j gives rise to a unique string $((\nu_i\ \ \mathcal{F}), (\varphi_j\ \ \mathcal{V}))$. Hence the claim follows in this case.

Proof of the Claim, case (1). First we note that for a finite word t of length at least the greatest common divisor C of $a+b$ and $d+c$ there is no string $((\mathcal{V}t\mathcal{F}), (\mathcal{F}t\mathcal{V}))$ from w' to w. Indeed, after applying the first letter, and then C letters in t we should obtain the *same* action on the starting base elements of the string in $\mathbb{D}(G_{w'})$ and in $\mathbb{D}(G_w)$, a contradiction with $\mathcal{V} \neq \mathcal{F}$.

We start with some $\mathcal{V}$ in w' and some $\mathcal{F}$ in w and inductively consider words t such that $(\mathcal{V}t)$ is a subword of w'. We check whether $(\mathcal{F}t)$ is a subword of w. We know that this process stops. Let s be the last word for which $\mathcal{F}s$ is a subword of w. We are going to show that under these conditions $((\mathcal{V}s\mathcal{F}), (\mathcal{F}s\mathcal{V}))$ is a string from w' to w. Indeed, we claim:

(1a) *If* $(\mathcal{V}t\mathcal{V})$ *is contained in* w' *and* $(\mathcal{F}t)$ *is contained in* w *then* $(\mathcal{F}t\mathcal{V})$ *is contained in* w.

Note that this fact implies the claim; indeed, the first time the inductive process stops it is at $(\mathcal{V}s\mathcal{F})$ in w' and $(\mathcal{F}s\mathcal{V})$ in w.

Suppose that in (1a) the letter $\mathcal{F}$ appears γ times in t and $\mathcal{V}$ appears δ times in t. We see:

$$\mathcal{V}(e_x)t\mathcal{V} = e_N \quad \Longrightarrow \quad N = x - 2a + \gamma b - \delta a \geq 0.$$

Let us write

$$\mathcal{F}(f_y)t = f_M; \quad \text{hence} \quad M = y + c + \gamma c - \delta d.$$

We show:

$$N \geq 0 \quad \& \quad \frac{d}{c} \geq \frac{a}{b} \quad \Longrightarrow \quad M > d.$$

Indeed, as $x \leq a + b - 1$ we see: $N \geq 0 \Rightarrow$

$$a+b-1-2a+\gamma b-\delta a \geq 0 \quad \Rightarrow \quad (\gamma+1)b \geq (\delta+1)a \quad \Rightarrow \quad \frac{d}{c} \leq \frac{a}{b} < \frac{\gamma+1}{\delta+1}.$$

Hence

$$M = y + (\gamma+1)c - \delta a \geq (\gamma+1)c - \delta a > d.$$

We see that $\mathcal{F}(f_M)$ is not defined; as $(\mathcal{F}t)$ is contained in w, say $\mathcal{F}(f_z)t = f_y$, we see that $\mathcal{F}(f_z)t\mathcal{V}$ is defined, i.e. $(\mathcal{F}t\mathcal{V})$ is contained in w. We see that claim (1a) follows. This ends the proof of first equality in all cases.

For the proof of the second equality we choose number the symbols $\mathcal{F}$ in w', number the symbols $\mathcal{V}$ in w, and perform a proof analogous tho the proof of the first equality. This shows the second equality.

For the third equality we observe that $\dim(\underline{\mathrm{Aut}}(H_{d,c}[p]))$ equals the number of finite strings involved, and the result follows. This ends the proof of the proposition.

$\square$

2.2. Proposition. *Suppose* $d, c \in \mathbb{Z}_{\geq 0}$ *with* $d > c$. *Let* λ *be a principal quasi-polarization on* $H_{d,c} \times H_{c,d}$. *Then:*

$$\dim(\underline{\mathrm{Aut}}((H_{d,c} \times H_{c,d}, \lambda)[p])) = c(c+1) + dc.$$

Moreover:

$$\dim(\underline{\mathrm{Aut}}(((H_{1,1})^r, \lambda)[p])) = \frac{1}{2}\cdot r(r+1)$$

for a principal quasi-polarization λ.

The proof is a direct verification, with methods as in [23], Sections 4 and 5, [25], [37], 2.4. $\square$

3 Serre-Tate coordinates, see [2], see [1], §7

For moduli of ordinary abelian varieties there exist canonical Serre-Tate parameters. Ching-Li Chai showed how to generalize that concept from the ordinary case to Serre-Tate parameters on a central leaf in $\mathcal{A}_{g,1}$. Results in this section are due to Chai.

3.1. The Serre - Tate theorem. Let A_0 be an abelian variety, and $X_0 = A_0[p^\infty]$. We obtain a natural morphism

$$\mathrm{Def}(A_0) \xrightarrow{\sim} \mathrm{Def}(X_0), \quad A \mapsto A[p^\infty];$$

a basic theorem of Serre and Tate says that this is an *isomorphism.* An analogous statement holds for (polarized abelian variety) $\mapsto$ (quasi-polarized p-divisible group). See [20], 6.ii; a proof first appeared in print in [22]; also see [7], [16]. See [3], Section 2.

3.2. Let (A,λ) be an *ordinary* principally polarized abelian variety; write $(X,\lambda) = (A,\lambda)[p^\infty]$. Deformations of (A,λ) are described by extensions of $(X,\lambda)_{\mathrm{et}}$ by $(X,\lambda)_{\mathrm{loc}}$. This shows that $\mathrm{Def}(X,\lambda)$ has the structure of a formal group. Let $n \in \mathbb{Z}_{\geq 3}$ be not divisible by p and let $[(A,\lambda,\gamma)] = a \in \mathcal{A}_{g,1,n} \otimes \mathbb{F}_p$. Write $(\mathcal{A}_{g,1,n} \otimes \mathbb{F}_p)^{/a}$ for the formal completion at a. Using the Serre-Tate theorem, see 3.1, we see that we have an isomorphism:

$$(\mathcal{A}_{g,1,n} \otimes \mathbb{F}_p)^{/a} \quad \cong \quad (\mathbb{G}_m[p^\infty])^{g(g+1)/2},$$

canonically up to $\mathbb{Z}_p$-linear transformations: *the Serre-Tate canonical coordinates*; see [18]; see [24], Introduction.

Discussion. One can try to formulate an analogous result around a non-ordinary point. Generalizations of Serre-Tate coordinates run into several difficulties. In an arbitrary deformation there is no reason that the slope filtration on the p-divisible group remains constant (as it does in the ordinary case). Even supposing that the slope filtration remains constant or supposing that the slope subfactors remain constant does not give the desired generalization. However it turns out that if we suppose that *under deformation the geometric isomorphism type of the p-divisible group remains geometrically constant*, the slope filtration exists and is constant. Describing extensions Chai arrives at a satisfactory generalization of Serre-Tate coordinates. Note that for the ordinary case and for $f = g-1$ the leaf is the whole open Newton polygon stratum; however for p-rank $= f < g-1$, the inclusion $C(x) \subset W_\xi$ is proper; this can be seen by observing that in these cases isogeny leaves are positive dimensional, or by using the computation of dimensions we carry out in this paper.

The input for this generalization is precisely the tool provided by the theory of *central leaves* as in [32]. We follow ideas basically due to Ching-Li Chai: we extract from [2], and from [1], §7 the information we need here.

Let Z be a p-divisible group, $\mathrm{Def}(Z) = \mathrm{Spf}(\Gamma)$ and $D(Z) = \mathrm{Spec}(\Gamma)$. Suppose that $Z = X_1 \times \cdots \times X_u$, where the summands are isoclinic of slopes $\nu_1 > \cdots > \nu_u$. Write $Z_{i,j} = X_i \times X_j$.

3.3. Proposition.

$$\dim\left(\mathcal{C}_Z(D(Z))\right) \quad = \quad \sum_{1\le i<j\le u} \dim\left(\mathcal{C}_{Z_{i,j}}(D(Z_{(i,j)}))\right).$$

Note that the "group-like structure" on the formal completion at a point of the leaf $\mathcal{C}_Z(D)$ can be described using the notion of "cascades" as in [24], 0.4.

Let (Z,λ) be a principally quasi-polarized p-divisible group, and consider $D = D(Z,\lambda)$. Suppose that $Z = X_1 \times \cdots \times X_u$, where the summands are isoclinic of slopes $\nu_1 > \cdots > \nu_u$. Then the heights of X_i and X_{u+1-i} are equal and $\nu_i = 1 - \nu_{u+1-i}$. We have the following pairs of summands:

$X_i + X_j$, with $1 \le i < j < u+1-i$ and $X_{u+1-j} + X_{u+1-i}$, and
$X_i + X_{u+1-i}$ for $1 \le i \le t/2$.

In this ways all pairs are described. Note that

$Z_{i,j} := X_i + X_j + X_{u+1-j} + X_{u+1-i}$ for $1 \le i < j < u+1-i$, and
$S_i := X_i + X_{u+1-i}$ for $1 \le i \le u/2$, and $S_{(u+1)/2}$ if u is odd

are principally quasi-polarized p-divisible groups (write the induced polarization again by λ on each of them).

3.4. Proposition.

$$\dim\left(\mathcal{C}_{(Z,\lambda)}(D(Z,\lambda))\right) =$$
$$= \sum_{1\le i<j<u+1-i} \dim\left(\mathcal{C}_{Z_{(i,j)}}(D(Z_{(i,j)})\right) + \sum_{1\le i\le u/2} \dim\left(\mathcal{C}_{(S_i,\lambda)}(D(S_i,\lambda))\right).$$

Note that

$$\{(i,j) \mid 1 \le i < j < u{+}1{-}i\} \xrightarrow{\sim} \{(I,J) \mid 1 \le I < J \quad \text{and} \quad u{+}1{-}I < J \le u\}$$

is a bijection under the map $(i,j) \mapsto (I = u+1-j, J = u+1-i)$. Indeed $i < j$ implies $I < J$ and $j < u+1-i$ gives $j = u+1-I < J = u+1-i$. In this case λ gives an isomorphism $X_i \times X_j \xrightarrow{\sim} X_J \times X_I$.

An example. The group structure on a leaf can be easily understood in the case of *two slopes*. This was the starting point for Chai to describe the relevant generalization of Serre-Tate coordinates from the ordinary case to the arbitrary case.

3.5. Theorem (Chai). *Let X be isoclinic of slope ν_X, height h_X and Y isoclinic of slope ν_Y and height h_Y. Suppose $\nu_Y > \nu_X$. Write $Z = Y \times X$. At every point of the central leaf $C = \mathcal{C}_Z(D(Z))$ the formal completion has the structure of a p-divisible group, isoclinic of slope $\nu_Y - \nu_X$, of height $h_X{\cdot}h_Y$, and*

$$\dim\left(\mathcal{C}_Z(D(Z))\right) \quad = \quad (\nu_Y - \nu_X){\cdot}h_X{\cdot}h_Y.$$

Suppose moreover there exists a principal quasi-polarization λ on Z; this implies $h_X = h_Y$ and $\nu_X = 1 - \nu_Y$. The central leaf $\mathcal{C}_{(Z,\lambda)}(\mathrm{Def}(Z,\lambda))$ has the structure of a p-divisible group, isoclinic of slope $\nu_Y - \nu_X$, of height $h_X{\cdot}(h_X+1)/2$, and

$$\dim\left(\mathcal{C}_{(Z,\lambda)}(D(Z,\lambda)\right) \quad = \quad \frac{1}{2}(\nu_Y - \nu_X){\cdot}h_X{\cdot}(h_X+1).$$

See [1], 7.5.2.

3.6. Let Z be an *isoclinic* p-divisible group. Then $\dim\left(\mathcal{C}_Z(D(Z))\right) = 0$. This can also be seen from a generalization of the previous theorem: take $\nu_Y = \nu_X$. This fact was already known as the isogeny theorem, see [15], 2.17.

4 The dimension of central leaves, the unpolarized case

In this section we compute the dimension of a central leaf in the local deformation space of an (unpolarized) p-divisible group.

4.1. Notation. Let ζ be a Newton polygon, and $(x,y) \in \mathbb{Q} \times \mathbb{Q}$. We write:
$(x,y) \prec \zeta$ if (x,y) is on or above ζ,
$(x,y) \precneqq \zeta$ if (x,y) is strictly above ζ,
$(x,y) \succ \zeta$ if (x,y) is on or below ζ,
$(x,y) \succneqq \zeta$ if (x,y) is strictly below ζ.

4.2. Notation. We fix integers $h \geq d \geq 0$, and we write $c := h - d$. We consider Newton polygons ending at (h,d). For such a Newton polygon ζ we write:

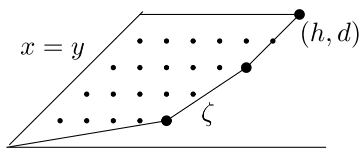
$$\Diamond(\zeta) = \{(x,y) \in \mathbb{Z} \times \mathbb{Z} \mid y < d, \quad y < x, \quad (x,y) \prec \zeta\},$$

and we write

$$\boxed{\dim(\zeta) := \#(\Diamond(\zeta)).}$$

See 7.10 for an explanation why we did choose this terminology.

Example:



$\zeta = 2 \times (1, 0) + (2, 1) + (1, 5) =$

$= 6 \times \frac{1}{6} + 3 \times \frac{2}{3} + 2 \times \frac{1}{1}; \quad h = 11.$

Here $\dim(\zeta) = \#(\Diamond(\zeta)) = 22$.

See 7.10.

4.3. Notation. We write:

$\Diamond(\zeta;\zeta^*) := \{(x,y) \in \mathbb{Z}\times\mathbb{Z} \mid (x,y) \prec \zeta,\ (x,y) \not\precneqq \zeta^*\}$, $\boxed{\mathrm{cdu}(\zeta) := \#\left(\Diamond(\zeta;\zeta^*)\right)}$;

"cdu" = <u>d</u>imension of <u>c</u>entral leaf, <u>u</u>npolarized case; see 4.5 for an explanation.

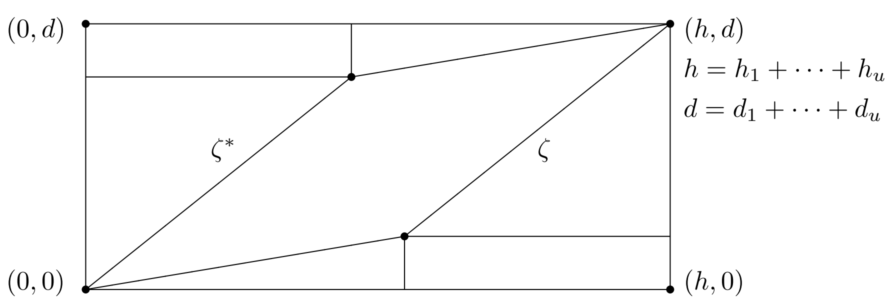
We suppose $\zeta = \sum_{1\leq i\leq u} \mu_i{\cdot}(m_i, n_i)$, written in such a way that $\gcd(m_i, n_i) = 1$ for all i, and $\mu_i \in \mathbb{Z}_{>0}$, and $i < j \Rightarrow (m_i/(m_i+n_i)) > (m_j/(m_j+n_j))$. Write $d_i = \mu_i{\cdot}m_i$ and $c_i = \mu_i{\cdot}n_i$ and $h_i = \mu_i{\cdot}(m_i + n_i)$; write $\nu_i = m_i/(m_i + n_i) = d_i/(d_i + c_i)$ for $1 \leq i \leq u$. Note that the slope $\nu_i = \mathrm{slope}(G_{m_i,n_i}) = m_i/(m_i + n_i) = d_i/h_i$: this Newton polygon is the "Frobenius-slopes" Newton polygon of $\sum (G_{m_i,n_i})^{\mu_i}$. Note that the slope ν_i appears h_i times; these slopes with these multiplicities give the set $\{\beta_j \mid 1 \leq j \leq h := h_1 + \cdots + h_u\}$ of all slopes of ζ.

4.4. Combinatorial Lemma, the unpolarized case. *The following numbers are equal*

$$\#\left(\Diamond(\zeta;\zeta^*)\right) =: \mathrm{cdu}(\zeta) \quad = \quad \sum_{i=1}^{i=h} \ (\zeta^*(i) - \zeta(i)) =$$

$$= \sum_{1\leq i<j\leq u} (d_i c_j - d_j c_i) = \sum_{1\leq i<j\leq u} (d_i h_j - d_j h_i) = \sum_{1\leq i<j\leq u} h_j{\cdot}h_i{\cdot}(\nu_i - \nu_j).$$

Example:

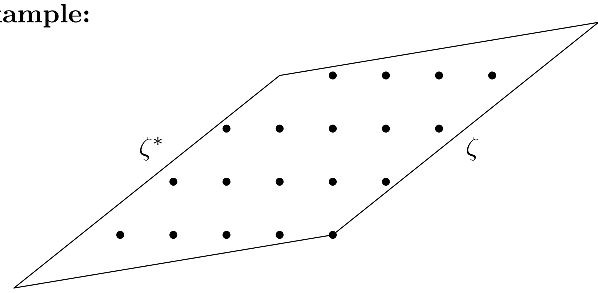


A **proof** for this lemma is not difficult. The equality $\mathrm{cdu}(\zeta) = \sum(\zeta^*(i) - \zeta(i))$ can be seen as follows. From every break point of ζ^* draw a vertical line up, and a horizontal line to the left; from every break point of ζ draw a vertical line down and a horizontal line to the right. This divides the remaining space of the $h \times d$ rectangle into triangles and rectangles. Pair opposite triangles to a rectangle. In each of these take lattice points, in the interior and in the lower or right hand sides; in this way all lattice points in the large rectangle belong to precisely one of the subspaces; for each of the subspaces we have the formula that the number of such lattice points is the total length of vertical lines. This proves the desired equality for $\mathrm{cdu}(\zeta)$. The other equalities follow by a straightforward computation. $\square$

4.5. Theorem. (Dimension formula, the unpolarized case.) *Let X_0 be a p-divisible group, $D = D(X_0)$; let $y \in D$, let Y be the p-divisible group given by y with $\beta = \mathcal{N}(Y) \succ \mathcal{N}(X_0)$;*

$$\dim(\mathcal{C}_Y(D)) \quad = \quad \mathrm{cdu}(\beta).$$

Example:



$\dim(\mathcal{C}_X(D)) = \#((\lozenge(\zeta;\zeta^*)).$
$(\frac{4}{5} - \frac{1}{6})\cdot 5\cdot 6 = 19,$
$d_1h_2 - d_2h_1 = 4\cdot 6 - 1\cdot 5 = 19;$
$d_1c_2 - d_2c_1 = 4\cdot 5 - 1\cdot 1 = 19.$

First proof. It suffices to prove this theorem in case $Y = X_0$. Write $\mathcal{N}(Y) = \zeta$. By 7.19 it suffices to prove this theorem in case $Y = H(\zeta)$. By 7.4, see 7.5, we know that in this case $\mathcal{C}_Y(D) = S_{Y[p]}(D)$. Let $\beta = \sum \mu_i\cdot(m_i, n_i)$; we suppose that $i < j \Rightarrow m_i/(m_i+n_i) > m_j/(m_j+n_j)$; write $d_i = \mu_i\cdot m_i$ and $d = \sum d_i$; write $c_i = \mu_i\cdot n_i$ and $c = \sum c_i$. We know: $\dim(\mathrm{Def}(Y)) = \dim Y\cdot\dim Y^t = dc$. By 7.27, using 2.1 and 4.4, we conclude:

$$\dim(\mathcal{C}_Y(D)) = \dim(S_{H(\beta)[p]}(D)) = \dim(\mathrm{Def}(Y)) - \dim(\underline{\mathrm{Aut}}(H(\beta)[p])) =$$

$$= (\sum d_i)(\sum c_i) - (\sum_i d_i\cdot c_i) - 2\cdot\sum_{i<j}(c_i\cdot d_j) = \sum_{i<j}(d_ih_j - d_jh_i) = \mathrm{cdu}(\beta).$$

$\square$4.5

Second proof. Assume, as above, that $Y = X_0 = H_\beta$. Write $Z_{i,j} = H_{d_i,c_i} \times H_{d_j,c_j}$. A **proof** of 4.5 follows from 3.5 using 3.3 and 7.19:

$$\dim(\mathcal{C}_Y(D)) = \sum_{i<j}\dim(\mathcal{C}_{Z_{i,j}}(D(Z_{i,j}))) = \sum_{i<j} h_j\cdot h_i\cdot(\nu_i - \nu_j),$$

where $\nu_i = d_i/(d_i + c_i) = m_i/(m_i + n_i)$ and $h_i = d_i + c_i$. Conclude by using 3.3. $\square$4.5

4.6. Remark. A variant of the first proof can be given as follows. At first prove 4.5 in the case of two slopes, as was done above. Then conclude using 3.3.

4.7. Remark: a third proof. We use a recent result by Eva Viehmann, see [40]. Write

$$\zeta = \sum_j (m_j, n_j), \quad \gcd(m_i, n_j) = 1, \quad h_j = m_j + n_j,$$

$$\lambda_j = m_j/h_j, \quad d = \sum m_j, \quad c = \sum n_j, \quad j < s \Rightarrow \lambda_j \geq \lambda_s.$$

We write $\mathrm{idu}(\zeta)$ for the dimension of the isogeny leaf, as in [32], of $Y = X_0$ in $D = D(X_0)$. By the theory of Rapoport-Zink spaces, as in [39], we see that the reduction modulo p completed at a point gives an isogeny leaf completed at that point. Hence $\mathrm{idu}(\zeta)$ is also the dimension of that Rapoport-Zink space modulo p defined by X_0. This dimension is computed in [40] Theorem B:

$$\mathrm{idu}(\zeta) = \sum_i (m_i - 1)(n_i - 1)/2 \quad + \sum_{i>j} m_i n_j.$$

Let ρ be the ordinary Newton polygon, equal to $d(1, 0) + c(0, 1)$ in the case studied here. Note that

$$\{(x, y) \mid \rho^* \precneqq (x, y) \prec \zeta^*\} \cup \{(x, y) \mid \zeta^* \precneqq (x, y) \prec \zeta\} =$$

$$= \{(x, y) \mid \rho^* \precneqq (x, y) \prec \zeta\}.$$

We know that $\dim(\zeta) = \mathrm{cdu}(\zeta) + \mathrm{idu}(\zeta)$ by the "almost product structure" on Newton polygon strata, see 7.16. By the computation of Viehmann we see that

$$\mathrm{idu}(\zeta) = \#\left(\{(x, y) \mid \rho^* \precneqq (x, y) \prec \zeta^*\}\right).$$

Hence the dimension of the central leaf in this case equals

$$\#\left(\{(x, y) \mid \zeta^* \precneqq (x, y) \prec \zeta\}\right).$$

This proves the Theorem 4.5. $\square$

5 The dimension of central leaves, the polarized case

In this section we compute the dimension of a central leaf in the local deformation space of a polarized p-divisible group, and in the moduli space of polarized abelian varieties.

5.1. Notation. We fix an integer g. For every *symmetric* Newton polygon ξ of height $2g$ we define:

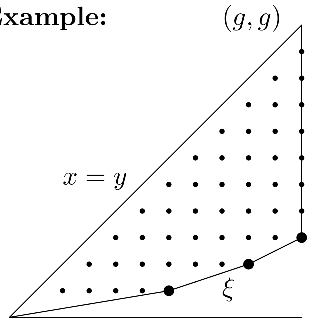
$$\triangle(\xi) = \{(x,y) \in \mathbb{Z}\times\mathbb{Z} \mid y < x \leq g, \quad (x,y) \ \text{ on or above } \ \xi\},$$

and we write

$$\boxed{\operatorname{sdim}(\xi) := \#(\triangle(\xi)).}$$

See 7.11 for explanation of notation.

Example:



$\dim(\mathcal{W}_\xi(\mathcal{A}_{g,1}\otimes\mathbb{F}_p)) = \#(\triangle(\xi))$

$\xi = (5,1) + (2,1) + 2{\cdot}(1,1) + (1,2) + (1,5)$,

g=11; slopes: $\{6\times\frac{5}{6}, 3\times\frac{2}{3}, 4\times\frac{1}{2}, 3\times\frac{1}{3}, 6\times\frac{1}{6}\}$.

This case: $\dim(\mathcal{W}_\xi(\mathcal{A}_{g,1}\otimes\mathbb{F}_p)) = \operatorname{sdim}(\xi) = 48$.

See 7.11.

5.2. Let ξ be a symmetric Newton polygon. For convenience we adapt notation to the symmetric situation:

$$\xi = \mu_1{\cdot}(m_1,n_1) + \cdots + \mu_s{\cdot}(m_s,n_s) + r{\cdot}(1,1) + \mu_s{\cdot}(n_s,m_s) + \cdots + \mu_1{\cdot}(n_1,m_1)$$

with:

$m_i > n_i$ and $\gcd(m_i,n_i) = 1$ for all i,
$1 \leq i < j \leq s \Rightarrow (m_i/(m_i+n_i)) > (m_j/(m_j+n_j))$,
$r \geq 0$ and $s \geq 0$.

We write $d_i = \mu_i{\cdot}m_i$, and $c_i = \mu_i{\cdot}n_i$, and $h_i = d_i + c_i$. Write $g := \left(\sum_{1\leq i\leq s}(d_i+c_i)\right) + r$ and write $u = 2s+1$.

We write:

$$\triangle(\xi;\xi^*) := \{(x,y)\in\mathbb{Z}\times\mathbb{Z} \mid (x,y) \prec \xi, \quad (x,y) \gneqq \xi^*, \quad x \leq g\},$$

$$\boxed{\operatorname{cdp}(\xi) := \#\left(\triangle(\xi;\xi^*)\right);}$$

"cdp" = <u>d</u>imension of <u>c</u>entral leaf, <u>p</u>olarized case.

Write $\xi = \sum_{1\leq i\leq u}\mu_i{\cdot}(m_i,n_i)$, i.e. $(m_j,n_j) = (n_{u+1-j}, m_{u+1-j})$ for $s < j \leq u$ and $r(1,1) = \mu_{s+1}(m_{s+1},n_{s+1})$. Write $\nu_i = m_i/(m_i+n_i)$ for $1 \leq i \leq u$; hence $\nu_i = 1 - \nu_{u+1-i}$ for all i.

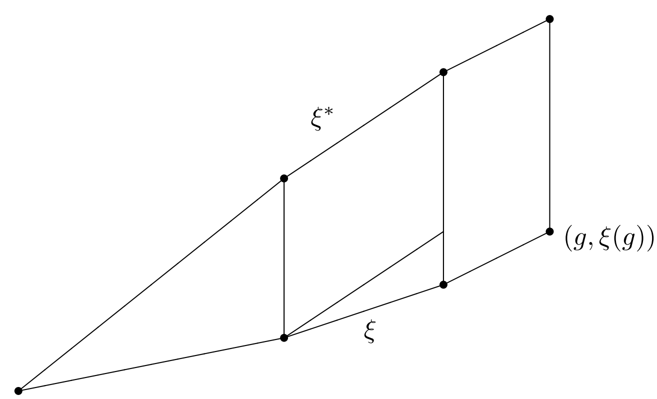
5.3. Combinatorial Lemma, the polarized case. *The following numbers are equal*

$$\#\,(\triangle(\xi;\xi^*)) =: \mathrm{cdp}(\xi) = \frac{1}{2}\mathrm{cdu}(\xi) + \frac{1}{2}(\xi^*(g) - \xi(g)) = \sum_{1\leq j\leq g}(\xi^*(j) - \xi(j)) =$$

$$= \sum_{1\leq i\leq s}\left(\frac{1}{2}\cdot d_i(d_i+1) - \frac{1}{2}\cdot c_i(c_i+1)\right) + \sum_{1\leq i<j}^{j\leq s}(d_i - c_i)h_j + \left(\sum_{i=1}^{i=s}(d_i - c_i)\right)\cdot r =$$

$$= \frac{1}{2}\sum_{1\leq i\leq s}(2\nu_i - 1)h_i(h_i+1) + \frac{1}{2}\sum_{1\leq i<j\neq u+1-i}(\nu_i - \nu_j)h_ih_j.$$

Example:



A **proof** of this lemma is not difficult. The first equalities follow from the unpolarized lemma, and from the definitions of cdu(-) and cdp(-). For a proof of the one but last equality draw vertical lines connecting breakpoints, and then draw lines from the breakpoints of ξ with slopes and lengths as in ξ^*; this divides $\triangle(\xi;\xi^*)$ into subspaces, where lattice points are considered in the interior, and on lower and right hand sides of the triangles and parallelograms created. Counting points in each of these give all summands of the right hand side of the last equality.

For the last equality:

$$\frac{1}{2}\cdot d_i(d_i+1) - \frac{1}{2}\cdot c_i(c_i+1) = \frac{1}{2}(d_i - c_i)(d_i + c_i + 1) = \frac{1}{2}(2\nu_i - 1)h_i(h_i+1);$$

for $1 \leq i \leq s$:

$$2\cdot(d_i - c_i)\cdot r = \left((\nu_i - \frac{1}{2}) + (\frac{1}{2} - \nu_{u+1-i})\right)\cdot h_i\cdot 2r =$$

$$= (\nu_i - \nu_{s+1})h_ih_{s+1} + (\nu_{s+1} - \nu_{u+i-1})h_{s+1}h_i;$$

for $1 \leq i < j \leq s$ we have:

$$2(d_i - c_i)h_j = 2((d_ic_j - c_id_j) + (d_id_j - c_ic_j)) =$$

$$= (\nu_i - \nu_j)h_ih_j + (\nu_i - \nu_{u+1-j})h_ih_j + (\nu_j - \nu_{u+1-i})h_ih_j + (\nu_{u+1-j} - \nu_{u+1-i})h_ih_j;$$

this shows

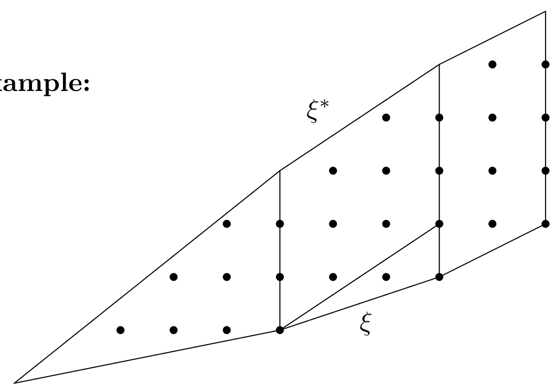
$$\sum_{1\leq i<j}^{j\leq s}(d_i - c_i)h_j = \frac{1}{2}\sum_{1\leq i<j\neq u+1-i,\ \ i\neq s+1,\ \ j\neq s+1} (\nu_i - \nu_j)h_ih_j.$$

Hence the last equality is proved. □

5.4. (Dimension formula, the polarized case.) *Let (A,λ) be a polarized abelian variety. Let $(X,\lambda) = (A,\lambda)[p^\infty]$; write $\xi = \mathcal{N}(A)$; then*

$$\dim\left(\mathcal{C}_{(X,\lambda)}(\mathcal{A}\otimes\mathbb{F}_p)\right) \quad = \quad \mathrm{cdp}(\xi).$$

Example:



$\dim(\mathcal{C}_{(A,\lambda)}(\mathcal{A}_g\otimes\mathbb{F}_p)) = \sum_{0<i\leq g}(\xi^*(i) - \xi(i))$.
slopes $1/5, 4/5, h = 5$: $\frac{1}{2}4{\cdot}5 - \frac{1}{2}1{\cdot}2 = 9$,
slopes $1/3, 2/3, h = 3$: $\frac{1}{2}2{\cdot}3 - \frac{1}{2}1{\cdot}2 = 2$,
$(d_1 - c_1)h_2 = 3{\cdot}3 = 9$,
$(d_1 + d_2 - c_1 - c_2)r = 4{\cdot}2 = 8$,
$\dim(\mathcal{C}_{(A,\lambda)}(\mathcal{A}_g\otimes\mathbb{F}_p)) = \#\left(\triangle(\zeta;\zeta^*)\right) = 28$.

5.5. Notation used in the proof of 5.4. Using 7.7 and 7.21 we only need to prove Theorem 5.4 in case λ is a principal polarization on

$$A[p^\infty] = H(\xi) =: Z = Z_1\times\cdots\times Z_s\times Z_{s+1}\times Z_{s+2}\times\cdots\times Z_u,$$

$Y_i := Z_i = H_{d_i,c_i}$, $Z_{s+1} = S_{s+1} = (H_{1,1})^r$, $X_i := Z_{u+1-i} = H_{c_i,d_i}$ $1\leq i\leq s{+}1$.

Write $S_i = H_{d_i,c_i}\times H_{c_i,d_i}$ for $i\leq s$, and we write λ for the induced quasi-polarization on S_i for $1\leq i\leq s+1$; note that $r\geq 0$. We have:

$$Z = Y_1\times\cdots\times Y_s\times Z_{s+1}\times X_s\times\cdots\times X_1.$$

First proof. We have:

$$\dim(\underline{\mathrm{Aut}}((Z,\lambda)[p])) =$$
$$\textstyle\sum_{i\leq s+1}\dim(\underline{\mathrm{Aut}}((S_i,\lambda)[p])) + \frac{1}{2}\cdot\sum_{i\neq j\neq u+1-i}\ \dim(\underline{\mathrm{Hom}}(Z_i,Z_j)) =$$
$$= \textstyle\sum_{i\leq s+1}\dim(\underline{\mathrm{Aut}}((S_i,\lambda)[p])) \quad + \quad \sum_{1\leq i<j\neq u+1-i}\ \dim(\underline{\mathrm{Hom}}(Z_i,Z_j)).$$

Using 2.1 and 2.2 and using the notation introduced, a computation shows:

$$\mathrm{cdp}(\xi) \quad + \quad \dim(\underline{\mathrm{Aut}}((Z,\lambda)[p])) = \frac{1}{2}\cdot g(g+1).$$

Indeed, write

$$I = \sum_{1\leq i\leq s}\left(\frac{1}{2}\cdot d_i(d_i+1) - \frac{1}{2}\cdot c_i(c_i+1)\right),$$

$$II = \sum_{1\leq i<j}^{j\leq s}(d_i-c_i)h_j, \quad III = \left(\sum_{i=1}^{i=s}(d_i-c_i)\right)\cdot r.$$

Note that:

$$1\leq i<j\leq s: \quad \dim(\underline{\mathrm{Hom}}(Y_i,Y_j)) = c_i\cdot d_j,$$

$$1\leq i<j=s+1: \quad \dim(\underline{\mathrm{Hom}}(Y_i,Z_{s+1})) = c_i\cdot r,$$

$$1\leq i<s<j: \quad \dim(\underline{\mathrm{Hom}}(Y_i,Z_j)) = c_i\cdot c_{u+1-j}, \quad Z_j = X_{u+1-j},$$

$$i=s+1<j: \quad \dim(\underline{\mathrm{Hom}}(Z_{s+1},Z_j)) = r\cdot c_{u+1-j},$$

$$s<i<j: \quad \dim(\underline{\mathrm{Hom}}(Z_i,Z_j)) = d_{u+1-i}\cdot c_{u+1-j}.$$

Direct verification gives:

$$I + II + III + \quad \sum_{i\leq s}(d_ic_i + c_i(c_i+1)) \quad + \quad \frac{1}{2}\cdot r(r+1) \quad +$$
$$+\frac{1}{2}\cdot\sum_{i\neq j\neq u+1-i}\dim(\underline{\mathrm{Hom}}(Z_i,Z_j)) =$$
$$= (d_1+\cdots+d_s+r+c_s+\cdots+c_1)(d_1+\cdots+d_s+r+c_s+\cdots+c_1+1)/2.$$

First we suppose $p>2$, and prove the theorem in this case. Indeed, using 7.6 and 7.28 we see:

$$\dim\left(\mathcal{C}_{(X,\lambda)}(\mathcal{A}\otimes\mathbb{F}_p)\right) = \dim(\mathcal{A}\otimes\mathbb{F}_p) - \dim(\underline{\mathrm{Aut}}((Z,\lambda)[p])) = \mathrm{cdp}(\xi).$$

Hence Theorem 5.4 is proved in case $p>2$.

Let p and q be prime numbers, let ξ be a symmetric Newton polygon, and let $H^{(p)}(\xi)$ respectively $H^{(q)}(\xi)$ be this minimal p-divisible group in characteristic p, respectively in characteristic q, both with a principal quasi-polarization λ. Their elementary sequences as defined in [30] are equal:

Claim.

$$\varphi((H^{(p)}(\xi),\lambda)[p]) = \varphi((H^{(q)}(\xi),\lambda)[q]).$$

The proof is a direct verification: in the process of constructing the canonical filtration, the characteristic plays no role.

In order to conclude the proof of Theorem 5.4 in case $p = 2$ we can follow two different roads. One is by using 7.6 and 7.28 we see:

$$\dim\left(\mathcal{C}_{(X,\lambda)}(\mathcal{A}\otimes\mathbb{F}_p)\right) = \dim(\mathcal{A}\otimes\mathbb{F}_p) - \dim(\underline{\mathrm{Aut}}((Z,\lambda)[p])) = \mathrm{cdp}(\xi).$$

This argument in the proof of 5.4 works in all characteristics by the generalization of Wedhorn's 7.28, see 7.29, see [43]; QED for 5.4.

One can also show that once 5.4 is proved in one characteristic, it follows in every characteristic. Here is the argument.
Next we assume $p > 2$ and $q = 2$, and we prove the theorem in characteristic $q = 2$. We have seen that the theorem holds in the case $p > 2$. In that case we know, using 7.6 and [30], Theorem 1.2, that

$$\mathrm{cdp}(\xi) = \dim\left(\mathcal{C}_{(X,\lambda)}(\mathcal{A}\otimes\mathbb{F}_p)\right) = \dim\left(S_{(X,\lambda)[p]}(\mathcal{A}\otimes\mathbb{F}_p)\right) = \mid \mathrm{ES}((H^{(p)}(\xi),\lambda)[p]) \mid .$$

Hence

$$\dim\left(\mathcal{C}_{(X,\lambda)}(\mathcal{A}\otimes\mathbb{F}_q)\right) = \varphi((H^{(q)}(\xi),\lambda)[q]) = \varphi((H^{(p)}(\xi),\lambda)[p]) = \mathrm{cdp}(\xi).$$

This ends the first proof of Theorem 5.4.

5.6. (A proof of 5.4 in the case of two slopes). *Suppose $\xi = (d,c)+(c,d)$ with $d > c$, i.e. $s = 1$ and $r = 0$ in the notation used above,* i.e. the case of a symmetric Newton polygon with only two different slopes. Write $g = d + c$. *In this case*

$$\mathrm{cdp}(\xi) = \frac{1}{2}d(d+1) - \frac{1}{2}c(c+1) = \frac{1}{2}\cdot(d-c)(d+c+1) = (1+\cdots+g)\left(\frac{d}{d+c} - \frac{c}{d+c}\right).$$

We choose $X = H_{d,c} \times H_{c,d}$, and $G = X[p]$; let λ be the principal quasi-polarization on X over k. Note this is unique up to isomorphism, see [32], Proposition 3.7. Let $\varphi(G) = \mathrm{ES}(G)$ be the elementary sequence of G, in the notation and terminology as in [30]. Then

$$\varphi = \{0,\cdots,\varphi(c) = 0, 1, 2, \cdots, \varphi(d) = d-c, d-c, \cdots, d-c\}.$$

Hence in this case

$$\dim\left(\mathcal{C}_{(X,\lambda)}(\mathcal{A}\otimes\mathbb{F}_p)\right) = c{\cdot}0 + (1+\cdots+d-c) + c{\cdot}(d-c) \quad = \quad \mathrm{cdp}(\xi).$$

Proof. In order to write down a final sequence for $(H_{m,n}\times H_{n,m})^{\mu}$ it suffices to know a canonical filtration for $Z = (H_{m,n}\times H_{n,m})$. Write $\mathbb{D}(H_{m,n}) = M_{m,n}$, the covariant Dieudonné module; there is a W-basis $M_{m,n} = W{\cdot}e_0 \oplus \cdots \oplus W{\cdot}e_{m+n-1}$, and $\mathcal{F}(e_i) = e_{i+n}$, and $\mathcal{V}(e_i) = e_{i+m}$, with the convention $e_{j+m+n} = pe_j$. Also we have $\mathbb{D}(H_{n,m}) = M_{n,m} = W{\cdot}f_0 \oplus \cdots \oplus W{\cdot}f_{m+n-1}$, and $\mathcal{F}(f_j) = f_{j+n}$ and $\mathcal{V}(f_j) = f_{i+m}$; the quasi-polarization can be given by $< e_i, f_j >= \delta_{i,m+n-1-j}$. Consider the k-basis for $\mathcal{V}(\mathbb{D}(Z[p]))$ given by

$$\{x_1 = e_{m+n-1}, \cdots, x_n = e_m, x_{n+1} = e_{m-1}, \cdots, x_m = e_n,$$

$$x_{m+1} = f_{m+n-1}, \cdots, x_{m+n} = f_m\};$$

this can be completed to a symplectic basis for $\mathbb{D}(Z[p])$; write $N_j = k{\cdot}x_1\oplus\cdots\oplus k{\cdot}x_j$ for $1 \leq j \leq m+n$. Direct verification shows that $0 \subset N_1 \subset \cdots \subset N_{m+n}$ plus the symplectic dual filtration is a final filtration of $\mathbb{D}(Z[p])$. From this we compute φ as indicated, and the result for $H_{m,n}\times H_{n,m}$ follows. This proves the lemma. □

Remark. It seems attractive to prove 5.4 in the general case along these lines by computing $\mid \varphi \mid$. There is an algorithm for determining the canonical filtration in general, but I do not know a closed formula in ξ for computing $\mid \varphi \mid$, with $\varphi = \mathrm{ES}(H(\xi))$. Therefore, in the previous proof of 5.4 we made a detour via 7.27.

5.7. Lemma. *Let $(Z = Y\times X, \lambda)$ be a principally quasi-polarized p-divisible group, where X is isoclinic of slope ν_X, height h_X, and Y isoclinic of slope ν_Y and height h_Y. Suppose $1 \geq \nu_Y > \frac{1}{2} > \nu_X = 1-\nu_Y \geq 0$. Write $d_x = h_X{\cdot}\nu_X$, and $\nu_X = d_X/c_X$; analogous notation for d_Y and c_Y; write $d = d_Y = c_X$, and $c = c_Y = d_X$ and $g = d+c$. Then:*

$$\dim\left(\mathcal{C}_{(Z,\lambda)}(D(Z,\lambda))\right) \quad = \quad \frac{1}{2}(\nu_Y-\nu_X){\cdot}h_X{\cdot}(h_X+1) = \frac{1}{2}d(d+1) - \frac{1}{2}c(c+1).$$

First proof. By 7.21 it suffices to prove this lemma in case $X = H_{c,d}$ and $Y = H_{d,c}$. By 5.6 the result follows.
Second proof. The result follows from 3.5. □5.7

5.8. Second proof. This **proof** of 5.4 follows from 3.4 using Lemma 4.5 and Lemma 5.7.

□5.4

5.9. Remark. Third proof in the polarized case; $p > 2$. In [41] the dimension of Rapoport-Zink spaces in the polarized case is computed. Here $p > 2$. Using our computation of $\mathrm{cdp}(-)$, analogous to 4.7, a proof of 5.4 follows from this result by Viehmann.

6 The dimension of Newton polygon strata

The dimension of a Newton polygon stratum in $\mathcal{A}_{g,1}$ is known, see 7.26. However it was unclear what the possible dimensions of Newton polygon strata in the non-principally polarized case could be. In this section we settle this question, partly solving an earlier conjecture.

6.1. We know that $\dim(\mathcal{W}_\xi(\mathcal{A}_{g,1})) = \mathrm{sdim}(\xi)$, see 5.1 and 7.11. We like to know what the dimension could be of an irreducible component of $\mathcal{W}^0_\xi(\mathcal{A}_g)$. Note that isogeny correspondences blow up and down in general, hence various dimensions a priori can appear.

Write $\mathcal{V}_f(\mathcal{A}_g)$ for the moduli space of polarized abelian varieties having p-rank at most f; this is a closed subset, and we give it the induced reduced scheme structure. By [27], Th. 4.1 we know that every irreducible component of this space has dimension exactly equal to $(g(g+1)/2)-(g-f) = ((g-1)g/2)+f$ (it seems a miracle that under blowing up and down this locus after all has only components exactly of this dimension).

Let ξ be a symmetric Newton polygon. Let its p-rank be $f = f(\xi)$. This is the multiplicity of the slope 1 in ξ; for a symmetric Newton polygon it is also the multiplicity of the slope 0. Clearly

$$\mathcal{W}^0_\xi(\mathcal{A}_g) \quad \subset \quad \mathcal{V}_{f(\xi)}(\mathcal{A}_g).$$

Hence for every irreducible component

$$T \subset \mathcal{W}^0_\xi(\mathcal{A}_g) \quad \text{we have} \quad \dim(T) \leq \frac{1}{2}(g-1)g + f.$$

In [31], 5.8 we find the conjecture that

for any ξ we expected there would be an irreducible component T of $\mathcal{W}^0_\xi(\mathcal{A}_g)$ with $\dim(T) = ((g-1)g/2) + f(\xi)$?

In this section we settle this question completely by showing that this is true for many Newton polygons, but not true for all. The result is that a component can have the maximal possible (expected) dimension: for many symmetric Newton polygons the conjecture is correct (for those when $\delta(\xi) = 0$, for notation see below), but for every $g > 4$ there exists a ξ for which the conjecture fails (those with $\delta(\xi) > 0$); see 6.3 for the exact statement.

6.2. Notation. Consider $\mathcal{W}^0_\xi(\mathcal{A}_g)$ and consider every irreducible component of this locus; let $\mathrm{minsd}(\xi)$ be the minimum of $\dim(T)$ where T ranges through the set of such irreducible components of $\mathcal{W}^0_\xi(\mathcal{A}_g)$, and let $\mathrm{maxsd}(\xi)$ be the maximum. Write

$$\delta = \delta(\xi) := \lceil(\xi(g))\rceil - \#\left(\{(x,y) \in \mathbb{Z}\times\mathbb{Z} \mid f(\xi) < x < g, \quad (x,y) \in \xi\}\right) - 1,$$

where $\lceil b \rceil$ is the smallest integer not smaller than b. Note that $\xi(g) \in \mathbb{Z}$ iff the multiplicity of $(1,1)$ in ξ is even. Here δ stands for "discrepancy". We will see that $\delta \geq 0$. We will see that $\delta = 0$ and $\delta > 0$ are possible.

6.3. Theorem.

$$\mathrm{sdim}(\xi) = \mathrm{minsd}(\xi), \ \textit{and}\, \mathrm{maxsd}(\xi) = \mathrm{cdp}(\xi) + \mathrm{idu}(\xi) = \frac{1}{2}(g-1)g + f(\xi) - \delta(\xi).$$

6.4. Corollary/Examples. *Suppose* $\xi = \sum(m_i, n_i)$ *with* $gcd(m_i, n_i) = 1$ *for all* i. *Then:*

$$\delta(\xi) = 0 \quad \Longleftrightarrow \quad \min(m_i, n_i) = 1, \quad \forall i.$$

We see that $0 \leq \delta(\xi) \leq \lceil g/2 \rceil - 2$. *We see that*

$$\mathrm{maxsd}(\xi) = \frac{1}{2}(g-1)g/2 + f \quad \Longleftrightarrow \quad \delta(\xi) = 0.$$

We see that $\delta(\xi) > 0$ for example in the following cases:
$g = 5$ and $\delta((3,2) + (2,3)) = 1$, $\quad g = 8$ and $\delta((4,3) + (1,1) + (3,4)) = 2$,
more generally, $g = 2k+1$, and $\delta((k+1,k) + (k,k+1)) = k-1$,
$g = 2k+2$, and $\delta((k+1,k) + (1,1) + (k,k+1)) = k-1$.

Knowing this theorem one can construct many examples of pairs of symmetric Newton polygons $\zeta \prec \xi$ such that

$$\mathcal{W}^0_\zeta(\mathcal{A}_g) \quad \not\subset \quad \left(\mathcal{W}^0_\xi(\mathcal{A}_g)\right)^{\mathrm{Zar}}.$$

6.5. Proof of 6.3. Let T be an irreducible component of $\mathcal{W}^0_\xi(\mathcal{A}_g) \otimes k$. Let $\eta \in T$ be the generic point. There exist a finite extension $[L_1 : k(\eta)] < \infty$ and (B,μ) over L_1 such that $[(B,\mu)] = \eta$. There exist a finite extension $[L : L_1] < \infty$ and an isogeny $\varphi : (B_L, \mu_L) \to (C, \lambda)$, where (C,λ) is a principally polarized abelian variety over L. Let T' be the normalization of T in $k(\eta) \subset L$. Let $N = \mathrm{Ker}(\varphi)$. By flat extension there exists a dense open subscheme $T^0 \subset T'$, and a flat extension $\mathcal{N} \subset \mathcal{B}^0 \to T^0$ of $(N \subset B_L)/L$. Hence we arrive at a morphism $(\mathcal{B}^0, \mu) \to (\mathcal{C}, \lambda)$, with $\mathcal{C} := \mathcal{B}^0/\mathcal{N}$, of polarized abelian schemes over T^0. This gives the moduli morphism $\psi : T^0 \to \mathcal{W}^0_\xi(\mathcal{A}_{g,1}) \otimes k$.

We study $\mathrm{Isog_g}$ as in [9], VII.4. The morphism $\psi : T^0 \to \mathcal{W}^0_\xi(\mathcal{A}_{g,1}) \otimes k$ extends to an isogeny correspondence. This is proper in its both projections by [9], VII.4.3. As T is an irreducible component of $\mathcal{W}^0_\xi(\mathcal{A}_g) \otimes k$ this implies that the image of ψ is dense in a component T'' of $\mathcal{W}^0_\xi(\mathcal{A}_{g,1}) \otimes k$. Hence $\dim(T) \geq \dim(T'')$. By 7.11 we have $\dim(T'') = \mathrm{sdim}(\xi)$. This proves the first claim of the theorem.

Let $(A_0, \mu_0) \in \mathcal{W}^0_\xi(\mathcal{A}_g) \otimes k$, and define $(X_0, \mu_0) = (A_0, \mu_0)[p^\infty]$. We obtain

$$\mathrm{Def}(A_0, \mu_0) \quad = \quad \mathrm{Def}(X_0, \mu_0) \quad \subset \quad \mathrm{Def}(X_0),$$

the first equality by the Serre-Tate theorem, and the inclusion is a closed immersion. Moreover $\mathcal{I}_{(A_0,\mu_0)}(D(A_0, \mu_0)) \subset \mathcal{I}_{X_0}(D(X_0))$. This shows that

$$\mathrm{maxsd}(\xi) \leq \mathrm{cdp}(\xi) + \mathrm{idu}(\xi).$$

We show that in certain cases, for certain degrees of polarization, equality holds.

We choose A_0 such that $X_0 = A_0[p^\infty]$ is minimal. Let J be an irreducible component of $\mathcal{I}_{X_0}(D(X_0))$. Let $\varphi : (Y_0 \times J) \to X$ be the universal family over J defining this isogeny leaf. Let $q = p^n$ be the degree of φ. Define $r = p^{2gn}$. We are going to prove that in $\mathcal{I}_{X_0}\left(\mathcal{W}^0_\xi((\mathcal{A}_{g,r})_k)\right)$ there exists a component I with $I = J$. Hence inside $\mathcal{W}^0_\xi((\mathcal{A}_{g,r})_k)$ there is a component of dimension $\mathrm{cdp}(\xi) + \mathrm{idu}(\xi)$. Choose $[(A_0, \mu_0)] \in \mathcal{W}^0_\xi((\mathcal{A}_{g,r})_k)$ such that $\mathrm{Ker}(\mu_0) = A_0[p^n]$; as X_0 is minimal, this is possible by [32], 3.7.
Claim. In this case

$$\mathcal{I}_{(A_0,\mu_0)}(D(A_0, \mu_0)) \supset I = J \subset \mathcal{I}_{X_0}(D(X_0)).$$

Let τ be the quasi-polarization on Y_0 obtained by pulling back μ_0 via $Y_0 \to X_0$. Note that the kernel of φ is totally isotropic under the form given by $\tau = \varphi^*(\mu)$. Hence the conditions imposed by the polarization do not give any restrictions and we have proved the claim. This finishes the proof of $\mathrm{maxsd}(\xi) = \mathrm{cdp}(\xi) + \mathrm{idu}(\xi)$.

By 4.5 and 7.16 and by 5.4 we see that $\mathrm{maxsd}(\xi) = \mathrm{cdp}(\xi) + \mathrm{idu}(\xi)$ is the cardinality of set of (integral points) in the following regions:

$$\triangle(\xi) \cup \{(x,y) \mid (x,y) \precneqq \xi^*, g < x,\ y < g\} \cup \{(x,y) \mid (x,y) \in \xi^*, g < x,\ y < g\}.$$

Note that

$$\{(x,y) \mid (x,y) \precneqq \xi^*,\quad g < x,\quad y < g\} \quad \cong \quad \{(x,y) \mid (x,y) \succneqq \xi, x < g,\ y > 0\},$$

and

$$\{(x,y) \mid (x,y) \in \xi^*,\ \ g < x,\ \ y < g\} \quad \cong \quad \{(x,y) \mid (x,y) \in \xi,\ \ f(\xi) < x < g\}.$$

Hence

$$\mathrm{cdp}(\xi) + \mathrm{idu}(\xi) \quad = \quad \frac{1}{2}(g-1)g + f - \delta(\xi).$$

$\square$6.3

Remark. Let $q = p^n$ be a as above. Actually we can already construct inside $\mathcal{W}^0_\xi(\mathcal{A}_{g,q}) \otimes k$ a component of dimension equal to $\mathrm{maxsd}(\xi)$; in this way the relevant part of the proof above can be given.

6.6. Explanation. We see the curious fact that on a Newton polygon stratum *the dimension of a central leaf is independent of the degree of the polarization* (which supports the "feeling" that these leaves look like moduli spaces in characteristic zero), while the dimension of an isogeny leaf in general *depends on the degree of the polarization. As we know, Hecke correspondences are finite-to-finite above central leaves, and may blow up and down subsets of isogeny leaves.*

7 Some results used in the proofs

7.1. A basic theorem tells us that the isogeny class of a p-divisible group over an algebraically closed field $k \supset \mathbb{F}_p$ is "the same" as its Newton polygon, see below. Let X be a simple p-divisible group of dimension m and height h over k. In that case we define $\mathcal{N}(X)$ as the isoclinic polygon (all slopes are equal) of slope equal to m/h with multiplicity h. Such a simple p-divisible group exists, see the construction of $G_{m,n}$, [21], page 50, see 1.2; in the *covariant* theory of Dieudonné module this group can be given (over any perfect field) by the module generated by one element e over the Dieudonné ring, with relation $(\mathcal{V}^n - \mathcal{F}^m)e$. Any p-divisible group X over an algebraically field closed k is isogenous with a product

$$X \quad \sim_k \quad \Pi_i \ (G_{m_i,n_i}),$$

where $m_i \geq 0$, $n_i \geq 0$ and $\gcd(m_i, n_i) = 1$ for every i. In this case the Newton polygon $\mathcal{N}(X)$ of X is defined by all slopes $m_i/(m_i + n_i)$ with multiplicity $h_i := m_i + n_i$.

7.2. Theorem (Dieudonné and Manin), see [21], "Classification theorem " on page 35.

$$\{X\}/\sim_k \quad \xrightarrow{\sim} \quad \{\text{Newton polygon}\}, \qquad X \mapsto \mathcal{N}(X).$$

This means: for every p-divisible group X over a field we define its Newton polygon $\mathcal{N}(X)$; over en algebraically closed field, every Newton polygon comes from a p-divisible group and

$$X \quad \sim_k \quad Y \quad \Longleftrightarrow \quad \mathcal{N}(X) = \mathcal{N}(Y).$$

7.3. Minimal p-divisible groups. In [36] and [37] we study the following question:

Starting from a p-divisible group X we obtain a BT_1 group scheme:

$$[p] : \{X \mid \text{a } p\text{-divisible group}\}/\cong_k \quad \longrightarrow \{G \mid \text{a } \mathrm{BT}_1\}/\cong_k; \quad X \mapsto G := X[p].$$

This map is known to be surjective. Does $G = X[p]$ determine the isomorphism class of X? This seems a strange question, and in general the answer is "NO". It is the main theorem of [36] that the fiber of this map over G up to $\cong_k$ is precisely one p-divisible group X if G is minimal:

7.4. Theorem. *If $G = G(\zeta)$ is minimal over k, and X and Y are p-divisible groups with $X[p] \cong G \cong Y[p]$, then $X \cong Y$; hence $X \cong H(\zeta) \cong Y$.*

For the notation $H(\zeta)$ see 1.5.

However things are different if G is not minimal: it is one of the main results of [37] that for a non-minimal BT_1 group scheme G there are infinitely many isomorphism classes X with $X[p] \cong G$.

Note the following important corollaries.

7.5. *Suppose X is a p-divisible group and $G = X[p]$; let $D = D(X)$. Study the inclusion $\mathcal{C}_X(D) \subset \mathcal{S}_G(D)$. Then:*

$$X \quad \text{is minimal} \quad \Rightarrow \quad \mathcal{C}_X(D) = \mathcal{S}_G(D).$$

7.6. Corollary. *Let (A_0, μ) be a polarized abelian variety. If $A_0[p]$ is minimal, then every irreducible component of $\mathcal{C}_{(A_0,\mu)[p^\infty]}(\mathcal{A}_g)$ is an irreducible component of $\mathcal{S}_{A[p]}(\mathcal{A}_g)$.*

7.7. Remark. *Let (X, λ') be a quasi-polarized p-divisible group over k, with $\mathcal{N}(X) = \xi$. There exists an isogeny between (X, ξ') and $(H(\xi), \lambda)$, where λ is a principal quasi-polarization.*

See [32], 3.7. □

7.8. Newton polygon strata: a theorem by Grothendieck - Katz. A theorem by Grothendieck and Katz, see [17], Th. 2.3.1 on page 143 says that for any $\mathcal{X} \to S$ and for any Newton polygon ζ

$$\mathcal{W}_\zeta(S) \subset S \text{ is a } \textit{closed} \text{ set.}$$

Hence

$$\mathcal{W}^0_\zeta(S) \subset S \text{ is a } \textit{locally closed} \text{ set.}$$

Notation. We do not know a natural way of defining a scheme structure on these sets. These set can be considerd as schemes by introducing the *reduced scheme structure* on these sets.

Sometimes we will write $W_\xi = \mathcal{W}_\xi(\mathcal{A}_{g,1})$ and $W^0_\xi = \mathcal{W}^0_\xi(\mathcal{A}_{g,1})$ for a symmetric Newton polygon ξ and the moduli space of *principally* polarized abelian varieties.

7.9. Remark. For $\xi = \sigma$, the *supersingular* Newton polygon, the locus W_σ has many components (for $p \gg 0$), see [19], 4.9. However in [4] we find: for a *non-supersingular* Newton polygon the locus $W_\xi = \mathcal{W}_\xi(\mathcal{A}_{g,1})$ is geometrically irreducible.

7.10. Theorem (the dimension of Newton polygon strata in the unpolarized case), see [29], Th. 3.2 and [31], Th. 2.10. *Let X_0 be a p-divisible group over a field K; let $\zeta \succ \mathcal{N}(X_0)$. Then:*

$$\dim(\mathcal{W}_\zeta(D(X_0))) \quad = \quad \dim(\zeta).$$

See 4.2 for the definition of $\dim(\xi)$. □

7.11. (the dimension of Newton polygon strata in the principally polarized case), see [29], Th. 3.4 and [31], Th. 4.1. *Let ξ be a symmetric Newton polygon. Then:*

$$\dim\left(\mathcal{W}_\xi(\mathcal{A}_{g,1} \otimes \mathbb{F}_p)\right) = \mathrm{sdim}(\xi).$$

See 5.1 for the definition of $\mathrm{sdim}(\xi)$. See Section 6 for what happens for *non-principally* polarized abelian varieties and Newton polygon strata in their moduli spaces.

7.12. see [32], Th. 2.3.

$$\mathcal{C}_X(S) \subset \mathcal{W}^0_{\mathcal{N}(X)}(S)$$

is a closed set.

A proof can be given using 7.13, 7.14 and 7.15. □

7.13. Definition. *Let S be a scheme, and let $X \to S$ be a p-divisible group. We say that X/S is* geometrically fiberwise constant, *abbreviated* gfc *if there exist a field K, a p-divisible group X_0 over K, a morphism $S \to \mathrm{Spec}(K)$, and for every $s \in S$ an algebraically closed field $k \supset \kappa(s) \supset K$ containing the residue class field of s and an isomorphism $X_0 \otimes k \cong_k X_s \otimes k$.*

The analogous terminology will be used for quasi-polarized p-divisible groups and for (polarized) abelian schemes.

See [32], 1.1.

7.14. Theorem (T. Zink & FO). *Let S be an integral, normal Noetherian scheme. Let $\mathcal{X} \to S$ be a p-divisible group with constant Newton polygon. Then there exist a p-divisible $\mathcal{Y} \to S$ and an S-isogeny $\varphi : \mathcal{Y} \to \mathcal{X}$ such that $\mathcal{Y}/S$ is* gfc.

See [44], [38], 2.1, and [32], 1.8. □

7.15. *Let S be a scheme which is integral, and such that the normalization $S' \to S$ gives a noetherian scheme. Let $\mathcal{X} \to S$ be a p-divisible group; let $n \in \mathbb{Z}_{\geq 0}$. Suppose that $\mathcal{X} \to S$ is* gfc*. Then there exists a finite surjective morphism $T_n = T \to S$, such that $\mathcal{X}[p^n] \times_S T$ is constant over T.*

See [32], 1.3. □

Note that we gave a "point-wise" definition of $\mathcal{C}_X(S)$; we can consider $\mathcal{C}_X(S) \subset S$ as a closed set, or as a subscheme with induced reduced structure; however is this last definition "invariant under base change"? It would be much better to have a "functorial definition" and a nature-given scheme structure on $\mathcal{C}_X(S)$.

Note that the proof of Theorem 7.12 is quite involved. One of the ingredients is the notion of "completely slope divisible p-divisible groups" introduced by T. Zink, and theorems on p-divisible groups over a normal base, see [44] and [38].

Considering the situation in the moduli space with enough level structure in order to obtain a fine moduli scheme, we see that $C(x) = \mathcal{C}_{(A,\lambda)[p^\infty]}(\mathcal{A}_{g,*,n} \otimes \mathbb{F}_p)$ is regular (as a stack, or regular as a scheme in case sufficiently high level structure is taken into account).

We write $\mathcal{C}_x$ for the irreducible component of $\mathcal{C}_{(A,\lambda)[p^\infty]}\mathcal{A}$ passing through $[(A,\lambda)] = x$.

Remark/Theorem (Chai & FO). In fact, for $\mathcal{N}(A) \neq \sigma$, i.e. A is not supersingular, it is known that $\mathcal{C}_{(A,\lambda)[p^\infty]}(\mathcal{A})$ is *geometrically irreducible in every irreducible component of* $\mathcal{A}_g$; see [4].

Central and isogeny leaves in a deformation space. We give additional results on deformation spaces of p-divisible groups analogously to the results in the polarized case in [32]. We choose a p-divisible group over a perfect field K. We write $D = D(X)$.

7.16. Proposition. *The central leaf $\mathcal{C}_X(D) \subset D$ is closed. There exists an isogeny leaf (a maximal H_α-subscheme as in [32], §4), $\mathcal{I}_X(D) = I(X) \subset D$. The intersection $\mathcal{C}_X(D) \cap I(X) \subset D$ equals the closed point $[X] = 0 \in D$. There is a natural, finite epimorphism $\mathcal{C}_X(D) \times I(X) \to D$. Hence*

$$\mathrm{cdu}(\zeta) + \mathrm{idu}_X(\zeta) = \dim(D).$$

Here $\mathrm{idu}_X(\zeta)$ is the dimension of $I(X) \subset D$.
The proof of this proposition follows as in [32], §4, (5.1), (5.3). □

7.17. Corollary. *Let ζ be a Newton polygon. There exists a number* $\mathrm{idu}(\zeta)$ *such that for every X with $\mathcal{N}(X) = \zeta$ the isogeny leaf in $D = D(X)$ has pure dimension equal to* $\mathrm{idu}(\zeta)$.

This follows because $\dim(\mathcal{C}_X(D))$ and $\dim(D)$ only depend on ζ, see 4.5 and 7.21. □

7.18. Theorem. Isogeny correspondences, unpolarized case. *Let $\psi : X \to Y$ be an isogeny between p-divisible groups. Then the isogeny correspondence contains an integral scheme T with two finite surjective morphisms*

$$\mathcal{C}_X(D(X)) \quad \twoheadleftarrow \quad T \quad \twoheadrightarrow \quad \mathcal{C}_Y(D(Y))$$

such that T contains a point corresponding with ψ.

7.19. *The dimension of $\mathcal{C}_X(D(X))$ only depends on the isogeny class of X.* $\square$

7.20. Isogeny correspondences, polarized case. Let $\psi : A \to B$ be an isogeny, and let λ respectively μ be a polarization on A, respective on B, and suppose there exists an integer $n \in \mathbb{Z}_{>0}$ such that $\psi^*(\mu) = n{\cdot}\lambda$. Then there exist finite surjective morphisms

$$\mathcal{C}_{(A,\lambda)[p^\infty]}(\mathcal{A}_g \otimes \mathbb{F}_p) \quad \twoheadleftarrow \quad T \quad \twoheadrightarrow \quad \mathcal{C}_{(B,\mu)[p^\infty]}(\mathcal{A}_g \otimes \mathbb{F}_p).$$

See [32], 3.16.

7.21. The dimension of $\mathcal{C}_{(X,\lambda)}(\mathcal{A}_g \otimes \mathbb{F}_p)$ only depends on the isogeny class of (X, λ). $\square$

Remark / Notation. In fact, this dimension only depends on the isogeny class of X. We write

$$c(\xi) := \dim\left(\mathcal{C}_{(X,\lambda)}(\mathcal{A}_g \otimes \mathbb{F}_p)\right), \quad X = A[p^\infty], \quad \xi := \mathcal{N}(X);$$

this is well-defined: all irreducible components have the same dimension.

A proof of all previous results on isogeny correspondences, and the independence of the dimension of the leaf in an isogeny class can be given using 7.13, 7.14 and 7.15; see [32], 2.7 and 3.13.

7.22. Remark. Isogeny correspondences in characteristic p in general blow up and down in a rather wild pattern. The dimension of Newton polygon strata and of EO-strata in general depends very much on the degree of the polarization in consideration. However for p-rank strata the dimension in the whole of $\mathcal{A}_g \otimes \mathbb{F}_p$ solely depends on the p-rank, see [27], Theorem 4.1. It seems a miracle that the dimension on central leaves does not depend on the degree of a polarization. See 6.6.

In [32] we also find the definition of *isogeny leaves*, and we see that any irreducible component of $\mathcal{W}_\xi(\mathcal{A}_g \otimes \mathbb{F}_p)$ up to a finite morphism is isomorphic with the product of a central leaf and an isogeny leaf, see [32], 5.3. Note that all central leaves with the same Newton polygon have the same dimension, see 7.19 and 7.21. However for Newton polygon strata, and hence also for isogeny leaves, the dimension in general depends very much on the degree of the polarization; for more information see Section 6.

7.23. Cayley - Hamilton. See [29]. We like to compute the dimension of a Newton polygon stratum. In 7.10 and 7.11 we have seen "easy" formulas to compute these dimensions (in the unpolarized, or in the principally polarized case). However, up to now there seems to be no really easy proof that these are indeed the correct formulas. In [19] the dimension of the supersingular locus $W_\sigma \subset \mathcal{A}_{g,1} \otimes \mathbb{F}_p$ is computed: every irreducible component of W_σ has dimension equal to $[g^2/4]$. Using purity, see [15], and if one would know beforehand that Newton polygon strata in $\mathcal{A}_{g,1} \otimes \mathbb{F}_p$ are nested as predicted by the Newton polygon graph, we have a proof of 7.11. However *proofs work the other way around.*

7.24. For a group scheme G over a perfect field K we write $a(G) := \dim_K(\mathrm{Hom}(\alpha_p, G))$. For a local-local p-divisible group X the fact $a(X) = 1$ implies that this Dieudonné module $\mathbb{D}(X)$ is generated by one element over the Dieudonné ring. It turns out that Newton polygon strata around a points where $a = 1$ are smooth (in the local deformation space in the unpolarized case, and in the principally polarized case). In this case the local dimension of the deformation space is computed in [29]. More precisely:

7.25. (CH - unpolarized). *Let X_0 be a p-divisible group over a perfect field K. Suppose $a(X_0) = 1$. Let $\gamma = \mathcal{N}(X_0)$, and let $\gamma \prec \beta$. Let ρ be the ordinary Newton polygon of the same dimension and height as X_0. Define R_β by:*

$$\mathcal{W}_\beta(\mathrm{Def}(X_0)) = \mathrm{Spf}(R_\beta);$$

then

$$R_\beta \quad \cong \quad \frac{K[[z_{x,y} \mid (x,y) \in \Diamond(\rho)]]}{(z_{x,y} \mid (x,y) \notin \Diamond(\beta))} \quad \cong \quad K[[z_{x,y} \mid (x,y) \in \Diamond(\beta)]].$$

7.26. (CH - polarized). *Let (A_0, λ) be a principally polarized abelian variety over a perfect field K. Suppose $a(A_0) = 1$. Let $\zeta = \mathcal{N}(A_0)$, and let $\zeta \prec \xi$. Let ρ be the ordinary Newton polygon. Define R_ξ by:*

$$\mathcal{W}_\xi(\mathrm{Def}(A_0, \lambda_0)) = \mathrm{Spf}(R_\xi);$$

then

$$R_\xi \quad \cong \quad \frac{K[[z_{x,y} \mid (x,y) \in \triangle(\rho)]]}{(z_{x,y} \mid (x,y) \notin \triangle(\beta))} \quad \cong \quad K[[z_{x,y} \mid (x,y) \in \triangle(\beta)]].$$

A theorem by Torsten Wedhorn on the dimension of EO*-strata*, see [42]. Let X be a p-divisible group, and $G := X[p]$. Consider the EO-stratum $S_G(D(X))$.

7.27. (Wedhorn).

$$\dim(S_G(D(X))) = \dim(\mathrm{Def}(X)) - \dim\left(\underline{\mathrm{Aut}}(G)\right).$$

See [42], 6.10. $\square$

Let(X, λ) be a principally quasi-polarized p-divisible group over a field of characteristic $\mathrm{char}(k) = p > 2$. Write $(G, \lambda) = (X, \lambda)[p]$.

7.28. Theorem (Wedhorn) $\boxed{p > 2}$**.**

$$\dim(S_{(G,\lambda)}(D(X, \lambda))) = \dim(\mathrm{Def}(X, \lambda)) - \dim\left(\underline{\mathrm{Aut}}((G, \lambda))\right).$$

See [42], 2.8 and 6.10. $\square$

7.29. In [43] we find a theorem which shows that the previous result also holds in case the characteristic of the base field equals 2.

8 Some questions and some remarks

8.1. In general the number of lattice points in a region need not be equal to its volume. For example in the case $\rho = g(1,0) + g(0,1)$ and $\triangle(\rho)$. Same remark in general for $\Diamond(\beta)$ and for $\triangle(\xi; \xi^*)$. However:

Remark (I thank Cathy O'Neil for this observation). The number $\#(\Diamond(\zeta; \zeta^*)) = \mathrm{cdu}(\zeta)$ as defined and computed in Section 2 is equal to the volume of the region between ζ^* and ζ for every ζ.

8.2. Remark. Using the result 7.8 by Grothendieck and Katz we have defined open and closed Newton polygon strata. Suppose we have symmetric Newton polygons $\zeta \prec \xi$. Then by the definitions we see

$$\mathcal{W}^0_\zeta(\mathcal{A}_g) \quad \subset \quad \mathcal{W}_\xi(\mathcal{A}_g) \quad \supset \quad \overline{\mathcal{W}^0_\xi(\mathcal{A}_g)}.$$

In general the last inclusion is not an equality. For example for $\zeta = \sigma$, the supersingular Newton polygon, and for $\xi = (2,1) + (1,2)$ we can see that $\mathcal{W}^0_\sigma(\mathcal{A}_{3,p^3}) = \mathcal{W}_\sigma(\mathcal{A}_{3,p^3})$ is not contained in the closure of $\mathcal{W}^0_\xi(\mathcal{A}_{3,p^3})$. Using the results of Section 6 we see many more of such examples do exist.

However for every symmetric ξ in the *principally polarized* case we do have:

$$\mathcal{W}_\xi(\mathcal{A}_g) \quad = \quad \overline{\mathcal{W}^0_\xi(\mathcal{A}_g)}.$$

We consider the central leaf $\mathcal{C}_Y(D(X_0) \subset D(X_0)$. Can we describe this locus in the coordinates $z_{x,y}$ given as in 7.25? I.e. does the inclusion $\triangle(\beta; \beta^*) \subset \triangle(\beta)$ induce the inclusion $\mathcal{C}_Y(D) \subset \mathcal{W}_\beta(D)$?

8.3. Question. *Under the identification given in 7.25 it might be that the formal completion C of the central leaf $\mathcal{C}_Y(D(X_0))$ can be described by:*

$$C \overset{?}{=} \mathrm{Spf}(K[[z_{x,y} \mid (x,y) \in \Diamond(\beta;\beta^*)]]) \subset \mathcal{W}_\beta(D) = \mathrm{Spf}(K[[z_{x,y} \mid (x,y) \in \Diamond(\beta)]]).$$

8.4. Question. *Under the identification given in 7.26 is it true that the formal completion C of the central leaf $\mathcal{C}_{(B,\mu)}(D(A_0,\lambda))$ can be described by:*

$$C \overset{?}{=} \mathrm{Spf}(K[[z_{x,y} \mid (x,y) \in \triangle(\xi,\xi^*)]]) \subset \mathcal{W}_\xi(D) = \mathrm{Spf}(K[[z_{x,y} \mid (x,y) \in \triangle(\xi)]]).$$

8.5. It seems desirable to have an explicit formula for the elementary sequence of a principally quasi-polarized minimal p-divisible group. If there are only two slopes this is easy. For every explicitly given Newton polygon this can be computed. In case there are more than two slopes, I do not know a general formula. However Harashita has proven, see [12], that for symmetric Newton polygons $\zeta \prec \xi$ and their minimal p divisible groups we have $\mathrm{ES}(H(\zeta)) \subset \mathrm{ES}(H(\xi))$.

8.6. *Give a simple criterion, in terms of φ and ξ, which decides when an elementary sequence φ appears on an open Newton polygon stratum, i.e. when $S_\varphi \cap W^0_\xi \neq \emptyset$.*

8.7. Conjecture. Let $\psi_\xi = \mathrm{ES}(H(\xi))$. I expect:

$$S_\varphi \cap W^0_\xi \neq \emptyset \quad \Rightarrow \quad \psi_\xi \subset \varphi.$$

The notation $\psi_\xi \subset \varphi$ stands for $S_{\psi_\xi} \subset \overline{S_\varphi}$, see [30], 14.3. Note that the opposite implication is false in general. Hence a proof of this conjecture does not fully answer the previous question.

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