Self-correspondences of K3 surfaces via moduli of sheaves

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To Yuri Ivanovich Manin for his 70th birthday

Summary. Let X be an algebraic K3 surface with Picard lattice $N(X)$ and $M_X(v)$ the moduli space of sheaves on X with given primitive isotropic Mukai vector $v = (r, H, s)$. In [14] and [3], we described all the divisors in moduli of polarized K3 surfaces (X, H) (that is, all pairs $H \in N(X)$ with $\operatorname{rank} N(X) = 2$) for which $M_X(v) \cong X$. These provide certain Mukai self-correspondences of X.

Applying these results, we show that there exists a Mukai vector v and a codimension 2 subspace in moduli of (X, H) (that is, a pair $H \in N(X)$ with $\operatorname{rank} N(X) = 3$) for which $M_X(v) \cong X$, but such that this subspace does not extend to a divisor in moduli having the same property. There are many similar examples.

Aiming to generalize the results of [14] and [3], we discuss the general problem of describing all subspaces of moduli of K3 surfaces with this property, and the Mukai self-correspondences defined by these and their composites, in an attempt to outline a possible general theory.

1 Introduction

We consider algebraic K3 surfaces X over $\mathbb{C}$; recall that a nonsingular projective algebraic (or compact Kähler) surface X is a K3 surface if its canonical class K_X is zero and its irregularity $q = \dim \Omega^1[X] = 0$. We write $N(X)$ for the Picard lattice of X, $\rho(X) = \operatorname{rank} N(X)$ for its rank, and $T(X)$ for the transcendental lattice.

Consider a primitive isotropic Mukai vector on X

$$v = (r, l, s), \quad \text{with } r \in \mathbb{N},\ s \in \mathbb{Z} \text{ and } l \in N(X) \text{ such that } l^2 = 2rs, \tag{1}$$

and denote by $Y = M_X(v) = M_X(r, l, s)$ the K3 surface obtained as the minimal resolution of singularities of the moduli space of sheaves on X with Mukai vector v. For details, see Mukai [4]–[7] and Yoshioka [19]. Under these assumptions, by results of Mukai [5], the quasi-universal sheaf on $X \times Y$ and its Chern class defines a 2-dimensional algebraic cycle on $X \times Y$ and a correspondence between X and Y with nice geometric properties. For more details, see Section 5.

If $Y \cong X$, this provides an important 2-dimensional algebraic cycle on $X \times X$, and a correspondence from X to itself; the question of when $Y \cong X$ is thus very interesting. The answer when $\rho(X) = 1$, probably already known to specialists, is given in Section 2.

Tensoring by any $D \in N(X)$ gives a natural isomorphism:

$$T_D\colon M_X(r, l, s) \cong M_X(r, l + rD, s + \tfrac{1}{2}rD^2 + D \cdot l)$$
$$\text{defined by } \mathcal{E} \mapsto \mathcal{E} \otimes \mathcal{O}(D).$$

For $r, s > 0$, we have an isomorphism called *reflection*

$$\delta\colon M_X(r, l, s) \cong M_X(s, l, r);$$

see for example [5] and [16], [17], [20]. For integers $d_1, d_2 > 0$ with $(d_1, d_2) = (d_1, s) = (r, d_2) = 1$, we have an isomorphism

$$\nu(d_1, d_2)\colon M_X(r, l, s) \cong M_X(d_1^2 r, d_1 d_2 l, d_2^2 s)$$

and its inverse $\nu(d_1, d_2)^{-1}$; see [5], [6], [14], [3].

In Theorem 2.1 and Corollary 2.2, we show that if $\rho(X) = 1$ and X is general, then for two primitive isotropic Mukai vectors v_1 and v_2, the moduli spaces $M_X(v_1)$ and $M_X(v_2)$ are isomorphic if and only if there exists an isomorphism between them obtained by composing the above three natural isomorphisms. They give *universal isomorphisms* between moduli of sheaves on X.

For $l \in N(X)$ with $\pm l^2 > 0$, it is known that we have a *Tyurin isomorphism* (see for example Tyurin [17])

$$\mathrm{Tyu} = \mathrm{Tyu}(\pm l) : \ M_X(\pm l^2/2, l, \pm 1) \cong X\,. \tag{2}$$

Corollary 2.6 shows that if $\rho(X) = 1$, then $M_X(r, H, s)$ and X are isomorphic if and only if there exists an isomorphism between them which is a composite of the above three universal isomorphisms between moduli of sheaves, and a Tyurin isomorphism (see also Remark 3.4). Compare [14] for a similar result.

We showed in [14] and interpreted geometrically in [3] (together with Carlo Madonna) that analogous results hold if $\rho(X)=2$ and X is general with its Picard lattice, i.e., the automorphism group of the transcendental periods is trivial: $\mathrm{Aut}\big(T(X), H^{2,0}(X)\big)=\pm 1$ (See also [1], [2], [13] about important particular cases of these results.) We review these results in Section 3; see Theorems 3.1, 3.2 and 3.3 for precise statements. These results show that in this case (i.e., when $\rho(X)=2$ and X is general with its Picard lattice), $M_X(r,H,s)\cong X$ if and only if there exists an isomorphism between $M_X(r,H,s)$ and X which is a composite of the universal isomorphisms T_D, δ and $\nu(d_1,d_2)$ between moduli of sheaves on X and a Tyurin isomorphism between moduli spaces of sheaves on X and X itself. The above results for $\rho(X)=1$ clarify the appearance of the natural isomorphisms T_D, δ, $\nu(d_1,d_2)$, Tyu in these results for Picard number 2.

The importance of the results for $\rho(X)=2$ and general X is that they describe all the divisorial conditions on moduli of algebraic polarized K3 surfaces (X,H) that imply $M_X(r,H,s)\cong X$. More exactly, the results for $\rho(X)=2$ describe all abstract polarized Picard lattices $H\in N$ with $\mathrm{rank}\, N=2$ such that $H\in N\subset N(X)$ and $N\subset N(X)$ is primitive implies $M_X(r,H,s)\cong X$. Recall that such X have codimension 1 in the 19-dimensional moduli of polarized K3 surfaces. Applying these results, we give in Theorems 3.6 and 3.8 a necessary condition on a Mukai vector (r,H,s) and a polarized K3 surface X in order for the isomorphism $M_X(r,H,s)\cong X$ to follow from a divisorial condition on the moduli of X. In Example 3.7, we give an exact numerical example when this necessary condition is not satisfied. Thus for the K3 surfaces X in this example, the isomorphism $M_X(r,H,s)\cong X$ is not a consequence of any divisorial condition on moduli of polarized K3 surfaces. In other words, $M_X(r,H,s)\cong X$, but this isomorphism cannot be deduced from any divisorial condition on K3 surfaces X' implying $M_{X'}(r,H,s)\cong X'$.

Applying these results, in Section 4, Theorem 4.1, we give an exact example of a type of primitive isotropic Mukai vector (r,H,s) and a pair $H\in N$ of an (abstract) polarized K3 Picard lattice with $\mathrm{rank}\, N=3$ such that for any polarized K3 surface (X,H) with $H\in N\subset N(X)$ and primitive $N\subset N(X)$ one has $M_X(r,H,s)\cong X$, but this isomorphism does not follow from any divisorial condition (i.e., from Picard number 2) on the moduli of polarized K3 surfaces. Thus these polarized K3 surfaces have codimension 2 in moduli, and they cannot be extended to a divisor in moduli of polarized K3 surfaces preserving the isomorphism $M_X(r,H,s)\cong X$. *This is the main result of this paper.* Section 4 gives many similar examples for Picard number $\rho(X)\geq 3$.

These results give important corollaries for higher Picard number $\rho(X)\geq 3$ of the above results for Picard number 1 and 2; they also show that the case $\rho(X)\geq 3$ is very nontrivial. These are the main subjects of this paper. Another important aim is to formulate some general concepts, and predict the general structure of possible results for higher Picard number $\rho(X)\geq 3$.

At the end of Section 4, for a type (r,H,s) of primitive isotropic Mukai vector, we introduce a notion of *critical polarized K3 Picard lattice* $H\in N$

(critical for the problem of K3 self-correspondences). Roughly speaking, it means that $M_X(r, H, s) \cong X$ for any polarized K3 surface X with $H \in N \subset N(X)$ where $N \subset N(X)$ is primitive, but the same does not hold for any primitive strict sublattice $H \in N_1 \subset N$. Thus the corresponding moduli space of K3 surfaces has dimension $20 - \operatorname{rank} N$, and is not a specialization of higher dimensional moduli spaces of K3 surfaces.

The classification of critical polarized K3 Picard lattices is the main problem of self-correspondences of a K3 surface via moduli of sheaves. Our results for $\rho = 1$ and $\rho = 2$ can be interpreted as a classification of all critical polarized K3 Picard lattices of rank one and two. The example of Theorem 4.1 mentioned above gives an example of a rank 3 critical polarized K3 Picard lattice N. In Theorem 4.10 we prove that a critical polarized K3 Picard lattice N has $\operatorname{rank} N \leq 12$. In Problem 4.11, we raise the problem of the exact bound for the rank of a critical polarized K3 Picard lattice for a fixed type of primitive isotropic Mukai vector. This problem is now solved only for very special types: we know all primitive isotropic Mukai vectors when the exact bound is one.

In Section 5, we interpret the above results in terms of isometric actions of correspondences and their composites on $H^2(X, \mathbb{Q})$. For example, the Tyurin isomorphisms of (2) give reflections in elements $l \in N(X)$, and generate the full automorphism group $O(N(X) \otimes \mathbb{Q})$. Every isotropic primitive Mukai vector (r, H, s) on X with $M_X(r, H, s) \cong X$ then generates some class of isometries in $O(N(X) \otimes \mathbb{Q})$. See Section 5 for exact statements. Thus the main problem of self-correspondences of X via moduli of sheaves is to find all these generators and the relations between them. In this connection, we state problems (1–4) at the end of Section 5; these show that, in principle, the general results for any $\rho(X)$ should look similar to the now known results for $\rho(X) = 1, 2$.

Our general idea should be clear: for a K3 surface X that is general for its Picard lattice, the very complicated structure of self-correspondences of X via moduli of sheaves is hidden inside the abstract lattice $N(X)$; we try to recover this structure. This should lead to some nontrivial constructions involving the abstract Picard lattice $N(X)$, and should relate it more closely to the geometry of the K3 surface.

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1.1 Preliminary notation for lattices

We use the notation and terminology of [10] for lattices, and their discriminant groups and forms. A *lattice* L is a nondegenerate integral symmetric bilinear form. That is, L is a free $\mathbb{Z}$-module of a finite rank with a symmetric pairing $x \cdot y \in \mathbb{Z}$ for $x, y \in L$, assumed to be nondegenerate. We write $x^2 = x \cdot x$. The *signature* of L is the signature of the corresponding real form $L \otimes \mathbb{R}$.

The lattice L is called *even* if x^2 is even for any $x \in L$. Otherwise, L is called *odd*. The *determinant* of L is defined to be $\det L = \det(e_i \cdot e_j)$ where $\{e_i\}$ is some basis of L. The lattice L is *unimodular* if $\det L = \pm 1$. The *dual lattice* of L is $L^* = \mathrm{Hom}(L, \mathbb{Z}) \subset L \otimes \mathbb{Q}$. The *discriminant group* of L is $A_L = L^*/L$; it has order $|\det L|$, and is equipped with a *discriminant bilinear form* $b_L \colon A_L \times A_L \to \mathbb{Q}/\mathbb{Z}$ and, if L is even, with a *discriminant quadratic form* $q_L \colon A_L \to \mathbb{Q}/2\mathbb{Z}$. To define these, we extend the form on L to a form on the dual lattice L^* with values in $\mathbb{Q}$.

An embedding $M \subset L$ of lattices is called *primitive* if L/M has no torsion. Similarly, a non-zero element $x \in L$ is called *primitive* if $\mathbb{Z}x \subset L$ is a primitive sublattice.

2 Isomorphisms between $M_X(v)$ and X for a general K3 surface X and a primitive isotropic Mukai vectors v

We consider algebraic K3 surfaces X over $\mathbb{C}$. Further, $N(X)$ denotes the Picard lattice of X, and $T(X)$ its transcendental lattice. We consider primitive isotropic Mukai vectors (1) on X. We denote by $Y = M_X(v) = M_X(r, l, s)$ the K3 surface obtained as the minimal resolution of singularities of the moduli space of sheaves on X with Mukai vector v. Compare Mukai [4]–[7] and Yoshioka [19].

In this section, we say that an algebraic K3 surface is *general* if its Picard number $\rho(X) = \operatorname{rank} N(X) = 1$ and the automorphism group of the transcendental periods of X is trivial over $\mathbb{Q}$: $\mathrm{Aut}\big(T(X) \otimes \mathbb{Q},\ H^{2,0}(X)\big) = \pm 1$.

We now consider the following question: for a general algebraic K3 surface X and two primitive isotropic Mukai vectors $v_1 = (r_1, l_1, s_1)$ and $v_2 = (r_2, l_2, s_2)$, when are the moduli spaces $M_X(v_1)$ and $M_X(v_2)$ isomorphic?

We have the following three *universal isomorphisms* between moduli spaces of sheaves over a K3 surface. (Here universal means that they are valid for all algebraic K3 surfaces.)

Let $D \in N(X)$. Then one has the natural isomorphism given by the tensor product

$$T_D \colon M_X(r, l, s) \cong M_X(r, l + rD, s + r(D^2/2) + D \cdot l), \quad \mathcal{E} \mapsto \mathcal{E} \otimes \mathcal{O}(D).$$

Moreover, here the Mukai vectors

$$v = (r, l, s) \quad \text{and} \quad T_D(v) = (r, l + rD, s + r(D^2/2) + D \cdot l)$$

have the same general common divisor and the same square under the Mukai pairing. In particular, one is primitive and isotropic if and only if the other is.

Taking $D = kH$ for H a hyperplane section and $k > 0$, using the isomorphisms T_D, we can always replace $M_X(r, l, s)$ by an isomorphic $M_X(r, l', s')$

where l' is ample, so that $l'^2 > 0$. Thus in our problem, we can also assume that $v = (r, l, s)$ where $r > 0$ and l is ample. Then $l^2 = 2rs > 0$ and $r, s > 0$.

For $r,\ s > 0$, one has an isomorphism called *reflection*

$$\delta \colon M_X(r, l, s) \cong M_X(s, l, r).$$

See for example [5] and [16], [17], [20]. Thus using reflection, we can also assume that $0 < r \le s$.

For integers $d_1 > 0$, $d_2 > 0$ such that $(d_1, d_2) = (d_1, s) = (r, d_2) = 1$, one has an isomorphism

$$\nu(d_1, d_2) \colon M_X(r, l, s) \cong M_X(d_1^2 r, d_1 d_2 l, d_2^2 s)$$

and its inverse $\nu(d_1, d_2)^{-1}$; see [5], [6], [14], [3]. Using the isomorphisms $\nu(d_1, d_2)$, $\nu(d_1, d_2)^{-1}$ and reflection δ, we can always assume that the primitive isotropic Mukai vector $v = (r, l, s)$ satisfies:

$$0 < r \le s, \quad l^2 = 2rs, \quad \text{and} \quad l \in N(X) \text{ is primitive and ample.} \tag{3}$$

We call such a primitive isotropic Mukai vector a *reduced primitive isotropic Mukai vector* (for $\rho(X) = 1$).

We have the following result.

Theorem 2.1. *Let X be a general algebraic K3 surface, i.e., $N(X) = \mathbb{Z}H$ where H is a primitive polarization of X and* $\operatorname{Aut}\bigl(T(X) \otimes \mathbb{Q}, H^{2,0}(X)\bigr) = \pm 1$. *Let $v = (r, H, s)$ and $v' = (r', H, s')$ be two reduced primitive isotropic Mukai vectors on X (see (3)), i.e., $0 < r \le s$ and $0 < r' \le s'$.*

Then $M_X(v) \cong M_X(v')$ if and only if $v = v'$, i.e., $r' = r$, $s' = s$.

It follows that the above universal isomorphisms T_D, δ and $\nu(d_1, d_2)$ are sufficient to find all the isomorphic moduli spaces of sheaves with primitive isotropic Mukai vectors for a general K3 surface.

Corollary 2.2. *Let X be a general algebraic K3 surface and v, v' primitive isotropic Mukai vectors on X. Then $M_X(v) \cong M_X(v')$ if and only if there exists an isomorphism between $M_X(v)$ and $M_X(v')$ which is a composite of the universal isomorphisms T_D, δ and $\nu(d_1, d_2)$.*

Proof. The following considerations are similar to the more general and difficult calculations of [14], Section 2.3. We have

$$N(X) = \mathbb{Z}H = \bigl\{x \in H^2(X, \mathbb{Z}) \bigm| x \cdot H^{2,0}(X) = 0\bigr\},$$

and the transcendental lattice of X is

$$T(X) = N(X)^{\perp}_{H^2(X,\mathbb{Z})}.$$

The lattices $N(X)$ and $T(X)$ are orthogonal complements to one another in the unimodular lattice $H^2(X, \mathbb{Z})$, and $N(X) \oplus T(X) \subset H^2(X, \mathbb{Z})$ is a sublattice of finite index; here and in what follows $\oplus$ denotes the orthogonal sum. Since $H^2(X,\mathbb{Z})$ is unimodular and $N(X)=\mathbb{Z}H$ a primitive sublattice, there exists $u\in H^2(X,\mathbb{Z})$ such that $u\cdot H=1$.

We denote the dual lattices by $N(X)^*=\mathbb{Z}\cdot\frac{1}{2rs}H\subset N(X)\otimes\mathbb{Q}$ and $T(X)^*\subset T(X)\otimes\mathbb{Q}$. Then $H^2(X,\mathbb{Z})\subset N(X)^*\oplus T(X)^*$, and

$$u=\tfrac{1}{2rs}H+t^*(H)\quad\text{with}\quad t^*(H)\in T(X)^*.$$

The element

$$t^*(H)\mod T(X)\in T(X)^*/T(X)\cong\mathbb{Z}/2rs\mathbb{Z}$$

is canonically defined by the primitive element $H\in H^2(X,\mathbb{Z})$. Obviously,

$$H^2(X,\mathbb{Z})=\big[N(X),T(X),u=\tfrac{1}{2rs}H+t^*(H)\big]$$

where $[\,\cdot\,]$ means "generated by". The element $t^*(H)\bmod T(X)$ distinguishes between the different polarized K3 surfaces with Picard number one and the same transcendental periods; more precisely, for another polarized K3 surface (X',H') with transcendental periods $\big(T(X'),H^{2,0}(X')\big)$, the periods of X and X' are isomorphic (and then $X\cong X'$ by the Global Torelli Theorem [15]) if and only if there exists an isomorphism of transcendental lattices $\phi\colon T(X)\cong T(X')$ such that $(\phi\otimes\mathbb{C})(H^{2,0}(X))=H^{2,0}(X')$ and

$$(\phi\otimes\mathbb{Q})(t^*(H))\mod T(X)=t^*(H')\mod T(X').$$

Thus the calculation of the periods of X in terms its transcendental periods is contained in the following statement.

Proposition 2.3. *Let (X,H) be a polarized K3 surface with a primitive polarization H such that $H^2=2rs$. Assume that $N(X)=\mathbb{Z}H$ (i.e., $\rho(X)=1$). Then*

$$H^2(X,\mathbb{Z})=\big[N(X)=\mathbb{Z}H,T(X),\tfrac{1}{2rs}H+t^*(H)\big]$$

where $t^*(H)\in T(X)^*$. The element $t^*(H)\bmod T(X)$ is uniquely defined.

Moreover, $H^{2,0}(X)\subset T(X)\otimes\mathbb{C}$. (More generally, for $\rho(X)\geq 1$, one should replace $T(X)$ by $H^{\perp}_{H^2(X,\mathbb{Z})}$.)

Let $Y=M_X(r,H,s)$. We calculate the periods of Y. The Mukai lattice of X is defined by

$$\widetilde{H}(X,\mathbb{Z})=H^0(X,\mathbb{Z})+H^2(X,\mathbb{Z})+H^4(X,\mathbb{Z})=U\oplus H^2(X,\mathbb{Z})$$

where $+$ is direct sum, and $\oplus$ orthogonal direct sum of lattices. Here $H^2(X,\mathbb{Z})$ is the cohomology lattice of X with its intersection pairing and $U=\mathbb{Z}e_1+\mathbb{Z}e_2$ is the hyperbolic plane, where canonically $\mathbb{Z}e_1=H^0(X,\mathbb{Z})$ and $\mathbb{Z}e_2=H^4(X,\mathbb{Z})$ with the Mukai pairing $e_1^2=e_2^2=0$ and $e_1\cdot e_2=-1$.

We have

$$v = re_1 + se_2 + H. \tag{4}$$

By Mukai [5], we have

$$H^2(Y,\mathbb{Z}) = v^\perp/\mathbb{Z}v, \tag{5}$$

and $H^{2,0}(Y) = H^{2,0}(X)$ by the canonical projection. This determines the periods of the K3 surface Y and its isomorphism class (by Global Torelli Theorem [15]). We calculate the periods of Y as in Proposition 2.3.

Any element f of $\widetilde{H}(X,\mathbb{Z})$ can be written in a unique way as

$$f = xe_1 + ye_2 + \alpha\tfrac{1}{2rs}H + \beta t^*, \quad \text{with } x,y,\alpha \in \mathbb{Z} \text{ and } t^* \in T(X)^*.$$

We have $f\cdot v = -sx - ry + \alpha$, so $f \in v^\perp$ if and only if $-sx - ry + \alpha = 0$, and then

$$f = xe_1 + ye_2 + (sx+ry)\tfrac{1}{2rs}H + \beta t^*.$$

By Proposition 2.3, $f \in \widetilde{H}(X,\mathbb{Z})$ if and only if $t^* = (sx+ry)t^*(H) \bmod T(X)$. Since $T(X) \subset v^\perp$, we can write

$$f = xe_1 + ye_2 + (sx+ry)\left(\tfrac{1}{2rs}H + t^*(H)\right) \mod T(X) \quad \text{with } x,y \in \mathbb{Z}.$$

Set

$$c = (r,s), \quad a = r/c, \quad b = s/c.$$

Then $(a,b) = 1$. We have $h = -ae_1 + be_2 \in v^\perp$ and $h^2 = 2ab = 2rs/c^2$. Moreover, $h \perp T(X)$ and then $h \perp H^{2,0}(X)$. Thus

$$h \mod \mathbb{Z}v = -ae_1 + be_2 \mod \mathbb{Z}v \tag{6}$$

gives an element of the Picard lattice $N(Y)$. We have

$$e_1 = \frac{v - ch - H}{2r}, \quad e_2 = \frac{v + ch - H}{2s}.$$

It follows that

$$f = \frac{sx+ry}{2rs}v + \frac{c(-sx+ry)}{2rs}h + (sx+ry)t^*(H) \mod T(X), \tag{7}$$

for some $x,y \in \mathbb{Z}$. Here $f \bmod \mathbb{Z}v$ gives all the elements of $H^2(Y,\mathbb{Z})$, and $H^{2,0}(Y) = H^{2,0}(X) \subset T(X)\otimes\mathbb{C}$.

It follows that $f \bmod \mathbb{Z}v \in T(Y)$ (where $\mathbb{Z}v$ gives the kernel of $v^\perp$ and $H^2(Y,\mathbb{Z}) = v^\perp/\mathbb{Z}v$) if and only if $-sx+ry = 0$. Equivalently $-bx+ay = 0$, or (since $(a,b)=1$) $x = az$, $y = bz$ where $z \in \mathbb{Z}$, and then

$$(sx+ry)t^*(H) = z(sa+rb)t^*(H) = z\,2abc\,t^*(H) \quad \text{for some } z \in \mathbb{Z}.$$

It follows that

$$T(Y) = [T(X), 2abc\,t^*(H)]. \tag{8}$$

Since $t^*(H) \bmod T(X)$ has order $2rs = 2abc^2$ in $T(X)^*/T(X) \cong \mathbb{Z}/2rs\mathbb{Z}$, it follows that $[T(Y)\colon T(X)] = c$ (this is a result of Mukai, [5]).

By (7) and (8), we have $f \perp H^{2,0}(Y) = H^{2,0}(X)$, that is $f \bmod \mathbb{Z}v \in N(Y)$, if and only if

$$f = \frac{sx+ry}{2rs}v + \frac{c(-sx+ry)}{2rs}h$$

where $sx+ry \equiv 0 \bmod 2abc$. Thus $acx+bcy \equiv 0 \bmod 2abc$ and $ax+by \equiv 0 \bmod 2ab$. Since $(a,b)=1$, it follows that $x = b\widetilde{x}$, $y = a\widetilde{y}$ where $\widetilde{x},\ \widetilde{y} \in \mathbb{Z}$, and $\widetilde{x}+\widetilde{y} \equiv 0 \bmod 2$. Thus $\widetilde{y} = -\widetilde{x}+2k$ where $k \in \mathbb{Z}$. It follows that

$$f = \frac{k}{c}v + (-\widetilde{x}+k)h, \quad \text{for some } \widetilde{x},\ k \in \mathbb{Z}.$$

Thus $h \bmod \mathbb{Z}v$ generates the Picard lattice $N(Y)$, and we can consider $h \bmod \mathbb{Z}v$ as the polarization of Y (or $-h \bmod \mathbb{Z}v$, which makes no difference from the point of view of periods and isomorphism class of Y).

Let us calculate $t^*(h) \in T(Y)^*$. Then in (7) we should take an element f with $c(-sx+ry)/(2rs) = 1/(2ab)$. Thus $-sx+ry = c$ or $-bx+ay = 1$. Then

$$t^*(h) = (sx+ry)t^*(H) \mod T(Y).$$

By (8), $T(Y)^* = [T(X), ct^*(H)]$ and $T(Y)^*/T(Y) \cong \mathbb{Z}/2ab\mathbb{Z}$.

Thus $t^*(h) = (bx+ay)\big(ct^*(H) \bmod [T(X), 2ab\,(ct^*(H))]\big)$ is defined by $m \equiv bx+ay \bmod 2ab$. Since $-bx+ay = 1$, we have $m \equiv 2ay-1 \equiv -1 \bmod 2a$ and $m \equiv 2bx+1 \equiv 1 \bmod 2b$. This defines $m \bmod 2ab$ uniquely. We call such $m \bmod 2ab$ a Mukai element (compare with [6]). Thus $m(a,b) \bmod 2ab$ is called *Mukai element* if

$$m(a,b) \equiv -1 \mod 2a \quad \text{and} \quad m(a,b) \equiv 1 \mod 2b. \tag{9}$$

Thus $t^*(h) = m(a,b)\,ct^*(H) \bmod [T(X), 2abc\,t^*(H)]$.

Thus we have finally completed the calculation of the periods of Y in terms of those of X (see Proposition 2.3).

Proposition 2.4. *Let (X,H) be a polarized K3 surface with a primitive polarization H such that $H^2 = 2rs$ with $r,s > 0$. Assume that $N(X) = \mathbb{Z}H$ (i.e., $\rho(X)=1$). Let $Y = M_X(r,H,s)$ and set $c = (r,s)$ and $a = r/c$, $b = s/c$.*

Then $N(Y) = \mathbb{Z}h$ where $h^2 = 2ab$,

$$T(Y) = [T(X), 2abc\,t^*(H)], \quad T(Y)^* = [T(X), ct^*(H)]$$

and $t^*(h) \bmod T(Y) = m(a,b)ct^*(H) \bmod T(Y)$ where $m(a,b) \bmod 2ab$ is the Mukai element: $m(a,b) \equiv -1 \bmod 2a$, $m(a,b) \equiv 1 \bmod 2b$. Thus

$$\begin{aligned} H^2(Y,\mathbb{Z}) &= \big[N(Y), T(Y), \tfrac{1}{2ab}h + t^*(h)\big] \\ &= \big[\mathbb{Z}h,\ [T(X),\ 2abc\,t^*(H)],\ \tfrac{1}{2ab}h + m(a,b)ct^*(H)\big]. \end{aligned}$$

(More generally, when $\rho(X) \geq 1$, one should replace $T(X)$ by $H^{\perp}_{H^2(X,\mathbb{Z})}$ and $T(Y)$ by $h^{\perp}_{H^2(Y,\mathbb{Z})}$.)

Now let us prove Theorem 2.1. We need to recover r and s from the periods of Y. By Proposition 2.4, we have $N(Y)=\mathbb{Z}h$ where $h^2=2ab$. Thus we recover ab. Since $c^2=2rs/2ab$, we recover c.

We have $(T(X)\otimes\mathbb{Q},H^{2,0}(X))\cong(T(Y)\otimes\mathbb{Q},H^{2,0}(Y))$. Since X is general, there exists only one such isomorphism up to multiplication by ±1. It follows that (up to multiplication by ±1) there exists only one embedding $T(X)\subset T(Y)$ of lattices which identifies $H^{2,0}(X)$ and $H^{2,0}(Y)$. By Proposition 2.4, then $t^*(h) \bmod T(Y)=\widetilde{m}(a,b)ct^*(H) \bmod T(Y)$, where $\widetilde{m}(a,b)\equiv\pm m(a,b) \bmod 2ab$ and $m(a,b)$ is the Mukai element. Assume $p^\alpha\mid ab$ and $p^{\alpha+1}$ does not divide ab where p is prime and $\alpha>0$. Then $\widetilde{m}(a,b)\equiv\pm1 \bmod 2p^\alpha$. Clearly, only one sign ±1 is possible here; we denote by a the product of all the p^α having $\widetilde{m}(a,b)\equiv-1 \bmod 2p^\alpha$, and by b the product of all the other p^α having $\widetilde{m}(a,b)\equiv1 \bmod 2p^\alpha$. If $a>b$, we must exchange a and b. Thus we recover a and b and the reduced primitive Mukai vector $(r,H,s)=(ac,H,bc)$ such that periods of $M_X(r,H,s)$ are isomorphic to the periods of Y.

This completes the proof. □

Remark 2.5. Propositions 2.3 and 2.4 and their proofs remain valid for any algebraic K3 surface X and a primitive element $H\in N(X)$ with $H^2=2rs\neq0$, provided we replace $T(X)$ by the orthogonal complement $H^\perp_{H^2(X,\mathbb{Z})}$.

As an example of an application of Theorem 2.1, let us consider the case when $M_X(r,l,s)\cong X$. It is known (see for example [17]) that for $l\in N(X)$ and $\pm l^2>0$, one has the Tyurin isomorphism

$$\mathrm{Tyu}=\mathrm{Tyu}(\pm l):\ M_X(\pm l^2/2,l,\pm1)\cong X\,. \tag{10}$$

The existence of such an isomorphism follows at once from the Global Torelli Theorem for K3 surfaces [15] using Propositions 2.3, 2.4 and Remark 2.5.

Thus for a general K3 surface X and a primitive isotropic Mukai vector $v=(r,H,1)$ where $r=H^2/2$, we have $M_X(r,H,1)\cong X$. By Theorem 2.1, we then obtain the following result, where we also use the well-known fact that $\mathrm{Aut}\big(T(X),H^{2,0}(X)\big)=\pm1$ if $\rho(X)=1$ (see (33) below); it is sufficient to consider the automorphism group over $\mathbb{Z}$ for this result.

Corollary 2.6. *Let X be an algebraic K3 surface with $\rho(X)=1$, i.e., $N(X)=\mathbb{Z}H$ where H is a primitive polarization of X. Let $v=(r,H,s)$ be a reduced primitive isotropic Mukai vector on X (see* (3)*), i.e., $0<r\le s$.*

Then $M_X(v)\cong X$ if and only if $v=(1,H,H^2/2)$, i.e. $r=1$, $s=H^2/2$.

3 Isomorphisms between $M_X(v)$ and X for X a general K3 surface with $\rho(X)=2$

We now consider general K3 surfaces X with $\rho(X)=\mathrm{rank}\,N(X)=2$; here a *K3 surface X* is called *general with its Picard lattice* if the transcendental periods have trivial automorphism group, $\mathrm{Aut}\big(T(X),H^{2,0}(X)\big)=\pm1$.

For $\rho(X) \geq 2$, we do not know when $M_X(v_1) \cong M_X(v_2)$ for primitive isotropic Mukai vectors v_1 and v_2 on X. But we still have the universal isomorphisms T_D, $D \in N(X)$, the reflection δ, the isomorphism $\nu(d_1, d_2)$ and the Tyurin isomorphism Tyu considered in Section 2. They are *universal isomorphisms,* i.e., they are defined for all K3 surfaces.

We start by reviewing the results of [14] and [3], where we found all the primitive isotropic Mukai vectors v with $M_X(v) \cong X$ for general K3 surfaces X with $\rho(X) = 2$. In particular, we know when $M_X(v_1) \cong M_X(v_2)$ in the case when both moduli spaces are isomorphic to X. The result is that $M_X(v) \cong X$ if and only if there exists such an isomorphism which is a composite of the universal isomorphisms δ, T_D and $\nu(d_1, d_2)$ between moduli of sheaves over X and the Tyurin isomorphism Tyu between moduli of sheaves over X and X itself. More exactly, the results are as follows.

Using the universal isomorphisms T_D, we can assume that the primitive isotropic Mukai vector is

$$v = (r, H, s), \quad \text{with } r > 0,\ s > 0 \text{ and } H^2 = 2rs.$$

(We can even assume that H is ample.) We are interested in the case when $Y = M_X(r, H, s) \cong X$.

We set $c = (r, s)$ and $a = r/c$, $b = s/c$. Then $(a, b) = 1$. Suppose that H is divisible by $d \in \mathbb{N}$ where $\widetilde{H} = H/d$ is primitive in $N(X)$. The primitivity of $v = (r, H, s)$ means that $(r, d, s) = (c, d) = 1$. Since $\widetilde{H}^2 = 2abc^2/d^2$ is even, we have $d^2 \mid abc^2$. Since $(a, b) = (c, d) = 1$, it follows that $d = d_a d_b$ where $d_a = (d, a)$, $d_b = (d, b)$, and we can introduce integers

$$a_1 = \frac{a}{d_a^2} \quad \text{and} \quad b_1 = \frac{b}{d_b^2},$$

obtaining $\widetilde{H}^2 = 2a_1 b_1 c^2$. Define $\gamma = \gamma(\widetilde{H})$ by $\widetilde{H} \cdot N(X) = \gamma\mathbb{Z}$, in other words, $H \cdot N(X) = \gamma d\mathbb{Z}$. Clearly, $\gamma \mid \widetilde{H}^2 = 2a_1 b_1 c^2$. We write

$$n(v) = (r, s, d\gamma) = (r, s, \gamma). \tag{11}$$

By Mukai [5], we have $T(X) \subset T(Y)$, and

$$n(v) = [T(Y) \colon T(X)], \tag{12}$$

where $T(X)$ and $T(Y)$ are the transcendental lattices of X and Y. Thus

$$Y \cong X \implies n(v) = (r, s, d\gamma) = (c, d\gamma) = (c, \gamma) = 1. \tag{13}$$

Assuming that $Y \cong X$ and then $n(v) = 1$, we have $\gamma \mid 2a_1 b_1$, and we can introduce

$$\gamma_a = (\gamma, a_1), \quad \gamma_b = (\gamma, b_1) \quad \text{and} \quad \gamma_2 = \frac{\gamma}{\gamma_a \gamma_b}. \tag{14}$$

Clearly, $\gamma_2 \mid 2$.

In [14], Theorem 4.4, we obtained the following general theorem (see important particular cases of it in [1], [2] and [13]). In the theorem, we use the notation c, a, b, d, d_a, d_b, a_1, b_1 introduced above. The same notation γ, γ_a, γ_b and γ_2 as above is used when we replace $N(X)$ by a 2-dimensional primitive sublattice $N \subset N(X)$, e.g., $\widetilde{H} \cdot N = \gamma\mathbb{Z}$ with $\gamma > 0$. We write $\det N = -\gamma\delta$ and $\mathbb{Z}f(\widetilde{H})$ for the orthogonal complement to $\widetilde{H}$ in N.

Theorem 3.1. *Let X be a K3 surface and H a polarization of X such that $H^2 = 2rs$ where $r, s \in \mathbb{N}$. Assume that the Mukai vector (r, H, s) is primitive. Let $Y = M_X(r, H, s)$ be the K3 surface which is the moduli of sheaves over X with isotropic Mukai vector $v = (r, H, s)$. Let $\widetilde{H} = H/d$ for $d \in \mathbb{N}$ be the corresponding primitive polarization.*

We have $Y \cong X$ if there exists $\widetilde{h}_1 \in N(X)$ such that $\widetilde{H}$ and $\widetilde{h}_1$ belong to a 2-dimensional primitive sublattice $N \subset N(X)$ such that $\widetilde{H} \cdot N = \gamma\mathbb{Z}$, $\gamma > 0$, $(c, d\gamma) = 1$, and the element $\widetilde{h}_1$ belongs to the a-series or the b-series described below:

$\widetilde{h}_1$ belongs to the a-series if

$$\widetilde{h}_1^2 = \pm 2b_1c, \quad \widetilde{H} \cdot \widetilde{h}_1 \equiv 0 \mod \gamma(b_1/\gamma_b)c, \quad f(\widetilde{H}) \cdot \widetilde{h}_1 \equiv 0 \mod \delta b_1 c \tag{15}$$

(where $\gamma_b = (\gamma, b_1)$);

$\widetilde{h}_1$ belongs to the b-series if

$$\widetilde{h}_1^2 = \pm 2a_1c, \quad \widetilde{H} \cdot \widetilde{h}_1 \equiv 0 \mod \gamma(a_1/\gamma_a)c, \quad f(\widetilde{H}) \cdot \widetilde{h}_1 \equiv 0 \mod \delta a_1 c \tag{16}$$

(where $\gamma_a = (\gamma, a_1)$).

These conditions are necessary to have $Y \cong X$ if $\rho(X) \leq 2$ and X is a general K3 surface with its Picard lattice.

In [3], we interpreted geometrically Theorem 3.1 as follows.

Theorem 3.2. *Let X be a K3 surface and H a polarization of X such that $H^2 = 2rs$ where $r, s \in \mathbb{N}$. Assume that the Mukai vector (r, H, s) is primitive. Let $Y = M_X(r, H, s)$ be the K3 surface which is the moduli of sheaves over X with isotropic Mukai vector $v = (r, H, s)$. Let $\widetilde{H} = H/d$ with $d \in \mathbb{N}$ be the corresponding primitive polarization.*

Assume that there exists $\widetilde{h}_1 \in N(X)$ such that $\widetilde{H}$ and $\widetilde{h}_1$ belong to a 2-dimensional primitive sublattice $N \subset N(X)$ such that $\widetilde{H} \cdot N = \gamma\mathbb{Z}$, $\gamma > 0$, $(c, d\gamma) = 1$, and the element $\widetilde{h}_1$ belongs to the a-series or to the b-series described in (15) *and* (16) *above.*

If $\widetilde{h}_1$ belongs to the a-series, then

$$\widetilde{h}_1 = d_2\widetilde{H} + b_1cD \quad \textit{for some} \quad d_2 \in \mathbb{N},\ D \in N, \tag{17}$$

which defines an isomorphism

$$\mathrm{Tyu}(\pm\widetilde{h}_1)\cdot T_D\cdot \nu(1,d_2)\cdot\delta\cdot\nu(d_a,d_b)^{-1}\colon Y=M_X(r,H,s)\cong X. \tag{18}$$

If $\widetilde{h}_1$ belongs to the b-series, then

$$\widetilde{h}_1=d_2\widetilde{H}+a_1cD \quad \textit{for some} \quad d_2\in\mathbb{N},\ D\in N, \tag{19}$$

which defines an isomorphism

$$\mathrm{Tyu}(\pm\widetilde{h}_1)\cdot T_D\cdot \nu(1,d_2)\cdot\nu(d_a,d_b)^{-1}\colon Y=M_X(r,H,s)\cong X. \tag{20}$$

Since the conditions of Theorems 3.1, 3.2 are necessary for general K3 surfaces with $\rho(X)\le 2$, we obtain the following result.

Theorem 3.3. *Let X be a K3 surface with a polarization H such that $H^2=2rs$, $r,s\ge 1$ and the Mukai vector (r,H,s) is primitive. Let $Y=M_X(r,H,s)$ be the moduli space of sheaves over X with isotropic Mukai vector (r,H,s). Assume that $\rho(X)\le 2$ and X is general with its Picard lattice. Let $\widetilde{H}=H/d$, $d\in\mathbb{N}$, be the corresponding primitive polarization.*

Then $Y=M_X(r,H,s)$ is isomorphic to X if and only if there exists $d_2\in\mathbb{N}$ and $D\in N=N(X)$ such that either

$$\widetilde{h}_1=d_2\widetilde{H}+b_1cD \quad \textit{has} \quad \widetilde{h}_1^2=\pm 2b_1c, \tag{21}$$

defining an isomorphism

$$\mathrm{Tyu}(\pm\widetilde{h}_1)\cdot T_D\cdot \nu(1,d_2)\cdot\delta\cdot\nu(d_a,d_b)^{-1}\colon Y=M_X(r,H,s)\cong X, \tag{22}$$

or

$$\widetilde{h}_1=d_2\widetilde{H}+a_1cD \quad \textit{has} \quad \widetilde{h}_1^2=\pm 2a_1c, \tag{23}$$

defining an isomorphism

$$\mathrm{Tyu}(\pm\widetilde{h}_1)\cdot T_D\cdot \nu(1,d_2)\cdot\nu(d_a,d_b)^{-1}\colon Y=M_X(r,H,s)\cong X. \tag{24}$$

Theorem 2.1 clarifies the appearance of the isomorphisms T_D, δ, $\nu(d_1,d_2)$ and Tyu in these results for Picard number 2. These are universal and exist for all K3 surfaces; moreover, they are all isomorphisms which are necessary to obtain all isomorphisms from moduli spaces $M_X(v)$ to X for isotropic Mukai vectors v on a general K3 surface X (i.e. with $\rho(X)=1$). Thus the appearance of the isomorphisms T_D, δ, $\nu(d_1,d_2)$ and Tyu is very natural in the above results.

Remark 3.4. For Picard number $\rho(X)=1$, Theorems 3.1, 3.2 and 3.3 are formally equivalent to Corollary 2.6. In fact, for $\rho(X)=1$ we have $\gamma=2a_1b_1c^2$. Thus $(\gamma,c)=1$ implies that $c=1$. Then $\gamma=2a_1b_1$ and $\gamma_2=2$, $\gamma_a=a_1$, $\gamma_b=b_1$. The conditions of Theorem 3.1 can only be satisfied for $\widetilde{h}_1=\widetilde{H}$, which implies that $a_1=1$ for the a-series and $b_1=1$ for the b-series (we can formally put $f(\widetilde{H})=0$).

Thus for $\rho(X)=1$ we have $Y\cong X$ if and only if $c=1$ and either $a_1=1$ or $b_1=1$. This is equivalent to Corollary 2.6.

Under the conditions of Theorem 3.1, assume that for a primitive rank 2 sublattice $N \subset N(X)$ an element $\widetilde{h}_1 \in N$ with $\widetilde{h}_1^2 = \pm 2b_1c$ belongs to the a-series. This is equivalent to the condition (17) of Theorem 3.2. Replacing $\widetilde{h}_1$ by $-\widetilde{h}_1$ if necessary, we see that (17) is equivalent to

$$\widetilde{h}_1 = d_2\widetilde{H} + b_1c\widetilde{D}, \quad d_2 \in \mathbb{Z},\ \widetilde{D} \in N. \tag{25}$$

Since $\widetilde{H}$ is primitive, the lattice N has a basis $\widetilde{H}$, $D \in N$, i.e., $N = [\widetilde{H}, D]$. Since $\widetilde{H} \cdot N = \gamma\mathbb{Z}$ where $(\gamma, c) = 1$, the matrix of N in this basis is

$$\begin{pmatrix} \widetilde{H}^2 & \widetilde{H} \cdot D \\ \widetilde{H} \cdot D & D^2 \end{pmatrix} = \begin{pmatrix} 2a_1b_1c^2 & \gamma k \\ \gamma k & 2t \end{pmatrix} \tag{26}$$

where $k, t \in \mathbb{Z}$ and $\gamma \mid 2a_1b_1$, $(\gamma, c) = 1$ and $(2a_1b_1c^2/\gamma, k) = 1$.

The condition of a-series (25) is then equivalent to the existence of $\widetilde{h}_1 \in [\widetilde{H}, b_1cN] = [\widetilde{H}, b_1cD]$ with $\widetilde{h}_1^2 = \pm 2b_1c$. Thus the lattice $N_1 = [\widetilde{H}, b_1cD]$ with the matrix

$$\begin{pmatrix} 2a_1b_1c^2 & b_1c\gamma k \\ b_1c\gamma k & b_1^2c^2 2t \end{pmatrix} \tag{27}$$

must have $\widetilde{h}_1$ with $\widetilde{h}_1^2 = \pm 2b_1c$. Writing $\widetilde{h}_1$ as $\widetilde{h}_1 = x\widetilde{H} + yb_1cD$, we obtain that the quadratic equation $a_1cx^2 + \gamma kxy + b_1cty^2 = \pm 1$ must have an integral solution. Similarly, for b-series we obtain the equation $b_1cx^2 + \gamma kxy + a_1cty^2 = \pm 1$. Thus we finally obtain a very elementary reformulation of the above results.

Lemma 3.5. *For the matrix* (26) *of the lattice N in Theorems 3.1, 3.2 and 3.3, the conditions of a-series are equivalent to existence of an integral solution of the equation*

$$a_1cx^2 + \gamma kxy + b_1cty^2 = \pm 1, \tag{28}$$

and for b-series of the equation

$$b_1cx^2 + \gamma kxy + a_1cty^2 = \pm 1. \tag{29}$$

This calculation has a very important corollary. Assume that $p \mid \gamma_b = (\gamma, b_1)$ for a prime p. Then (28) gives a congruence $a_1cx^2 \equiv \pm 1 \mod p$. Thus $\pm a_1c$ is a quadratic residue $\mod p$. Similarly, for the equation (29), we obtain that $\pm b_1c$ is a quadratic residue $\mod p$ for a prime $p \mid \gamma_a = (\gamma, a_1)$.

We thus obtain an important necessary condition for $Y = M_X(v) \cong X$ when $\rho(X) = 2$.

Theorem 3.6. *Let X be a K3 surface with a polarization H such that $H^2 = 2rs$, $r, s \geq 1$ and the Mukai vector (r, H, s) is primitive. Let $Y = M_X(r, H, s)$ be the moduli of sheaves over X with isotropic Mukai vector (r, H, s). Assume that $\rho(X) \leq 2$ and X is general with its Picard lattice. Let $\widetilde{H} = H/d$, $d \in \mathbb{N}$, be the corresponding primitive polarization, $\widetilde{H} \cdot N(X) = \gamma\mathbb{Z}$ and $(\gamma, c) = 1$.*

Then $Y = M_X(r,H,s) \cong X$ implies that for one of $\pm$ either

$$\forall\ p \mid \gamma_b \implies \left(\frac{\pm a_1 c}{p}\right) = 1 \tag{30}$$

or

$$\forall p \mid \gamma_a \implies \left(\frac{\pm b_1 c}{p}\right) = 1. \tag{31}$$

Here p means any prime, and $\left(\frac{x}{2}\right) = 1$ means that $x \equiv 1 \bmod 8$.

Thus if for either choice of ± 1,

$$\exists p \mid \gamma_b \text{ such that } \left(\frac{\pm a_1 c}{p}\right) = -1 \quad \text{and} \quad \exists p \mid \gamma_a \text{ such that } \left(\frac{\pm b_1 c}{p}\right) = -1 \tag{32}$$

then $Y = M_X(r,H,s)$ is not isomorphic to X for X a K3 surface with $\rho(X) \le 2$ which is general with its Picard lattice.

Example 3.7. Assume that $a_1 = 5$, $b_1 = 13$, $c = 1$ and $\gamma = 5 \cdot 13$ (or $\gamma = 2 \cdot 5 \cdot 13$). Then (32) obviously holds. Thus for

$$v = (5, H, 13), \quad H^2 = 2 \cdot 5 \cdot 13, \text{ and } \gamma = 5 \cdot 13 \text{ or } 2 \cdot 5 \cdot 13$$

(then H is always primitive), for any general K3 surface X with $\rho(X) = 2$ and any $H \in N(X)$ with $H^2 = 2 \cdot 5 \cdot 13$ and $H \cdot N(X) = \gamma\mathbb{Z}$, the moduli space $Y = M_X(v)$ is not isomorphic to X.

There are many such Picard lattices given by (26).

In [13], we showed that any primitive isotropic Mukai vector $v = (r,H,s)$ with $H^2 = 2rs$ and $\gamma = 1$ is realized by a general K3 surface with Picard number 2 and $Y = M_X(v) \cong X$. Theorem 3.6 may possibly give all the necessary conditions for a similar result to hold for any γ; we hope to return to this problem later.

The importance of these results for general K3 surfaces X with $\rho(X) = 2$ is that they describe *all divisorial conditions on moduli of polarized K3 surfaces that imply* $Y = M_X(r,H,s) \cong X$. Let us consider the corresponding simple general arguments.

It is well known (see [9] and [11] where, it seems, it was first observed) that $\operatorname{Aut}\big(T(X), H^{2,0}(X)\big) \cong C_m$ is a finite cyclic group of order $m > 1$, and its representation in $T(X) \otimes \mathbb{Q}$ is the sum of irreducible representations of dimension $\phi(m)$ (where ϕ is the Euler function). $H^{2,0}(X)$ is a line in one of the eigenspaces of C_m. In particular, $\phi(m) \mid \operatorname{rank} T(X)$ and if $m > 2$ the dimension of moduli of these X is equal to

$$\dim \operatorname{Mod}(X) = \operatorname{rank} T(X)/\phi(m) - 1. \tag{33}$$

If $m = 2$, then $\dim \operatorname{Mod}(X) = \operatorname{rank} T(X) - 2$.

Consider polarized K3 surfaces (X, H) with $H^2 = 2rs$ and a primitive Mukai vector (r, H, s) with $r, s > 0$. Assume $Y = M_X(r, H, s) \cong X$.

If $\rho(X) = 1$, then $\operatorname{rank} T(X) = 21$ and $\phi(m) \mid 21$. Since 21 is odd, it follows that $m = 2$. Thus $\operatorname{Aut}\big(T(X), H^{2,0}(X)\big) = \pm 1$, and then $c = 1$ and either $a_1 = 1$ or $b_1 = 1$ by Corollary 2.6 (or Remark 3.4). By the specialization principle (see [14], Lemma 2.1.1), then $Y \cong M_X(r, H, s)$ for all K3 surfaces X and a Mukai vector with these invariants:

$$c = 1 \quad \text{and} \quad \text{either } a_1 = 1 \text{ or } b_1 = 1. \tag{34}$$

Now assume that (r, H, s) does not satisfy (34), but $Y = M_X(r, H, s) \cong X$; then $\rho(X) \neq 1$ by Corollary 2.6. Hence $\rho(X) \geq 2$ and

$$\dim \operatorname{Mod}(X) \leq 20 - \rho(X) \leq 18.$$

Thus a divisorial condition on moduli or polarized K3 surfaces (X, H) to have $Y = M_X(r, H, s) \cong X$ means that $\rho(X) = 2$ for a general K3 surface satisfying this condition. All these conditions are described by the isomorphism classes of $H \in N(X)$ where $\operatorname{rank} N(X) = 2$ and $H \in N(X)$ satisfies Theorems 3.1, 3.2 or 3.3 (which in this case are all equivalent). If $H \in N \subset N(X)$ is a primitive sublattice of rank two and $H \in N$ satisfies the equivalent Theorems 3.1 and 3.2, then $Y = M_X(r, H, s) \cong X$ by the specialization principle. This means that X belongs to the closure of the divisor defined by the moduli of polarized K3 surfaces (X', H) with Picard lattice $N(X') = N$ of rank two. Thus $Y' = M_X(r, H, s) \cong X'$ because X' satisfies the divisorial condition $H \in N$ where $H \in N \subset N(X')$.

By Theorem 3.6 we obtain the following result.

Theorem 3.8. *For $r, s \geq 1$ let*

$$v = (r, H, s),\ d, \qquad H^2 = 2rs,\ (c, d) = 1,\ d^2 | ab$$

be a type of primitive isotropic Mukai vector on K3, and $\gamma \mid 2a_1b_1$ and $(\gamma, c) = 1$.

Then if (32) *holds, there does not exist any divisorial condition on moduli of polarized K3 surfaces (X, H) that implies $Y = M_X(r, H, s) \cong X$ and $H \cdot N(X) = \gamma\mathbb{Z}$. Thus these K3 surfaces have codimension ≥ 2 in the* 19*-dimensional moduli space of polarized K3 surfaces (X, H).*

For example, this holds for $r = 5$, $s = 13$ (then H is primitive and $d = 1$), and $\gamma = 5 \cdot 13$ (or $\gamma = 2 \cdot 5 \cdot 13$).

In the next section, we will show that the numerical example of Theorem 3.8 can be satisfied by K3 surfaces X with $\rho(X) = 3$ and $Y = M_X(r, H, s) \cong X$. Thus these K3 surfaces define a 17-dimensional submanifold in the moduli of polarized K3 surfaces that does not extend to a divisor in moduli preserving the condition $Y = M_X(r, H, s) \cong X$.

4 Isomorphisms between $M_X(v)$ and X for a general K3 surface X with $\rho(X) \geq 3$

Here we show that it is interesting and nontrivial to generalize the results of the previous section to $\rho(X) \geq 3$.

Let $K = [e_1, e_2, (e_1+e_2)/2]$ be a negative definite 2-dimensional lattice with $e_1^2 = -6$, $e_2^2 = -34$ and $e_1 \cdot e_2 = 0$. Then $((e_1+e_2)/2)^2 = (-6-34)/4 = -10$ is even, and the lattice K is even. Since $6x^2 + 34y^2 = 8$ has no integral solutions, it follows that K has no elements $\delta \in K$ with $\delta^2 = -2$. Consider the lattice

$$S = \mathbb{Z}H \oplus K$$

which is the orthogonal sum of $\mathbb{Z}H$ with $H^2 = 2 \cdot 5 \cdot 13$ and the lattice K. By standard results about K3 surfaces, there exists a polarized K3 surface (X, H) with the Picard lattice S and the polarization $H \in S$. (E.g., see [15] and [9].) We then have $H \cdot S = 2 \cdot 5 \cdot 13\,\mathbb{Z}$. Thus $\gamma = 2 \cdot 5 \cdot 13$.

Let $Y = M_X(5, H, 13)$. We have the following result, perhaps the main result of the paper.

Theorem 4.1. *For any polarized K3 surface (X, H) with $N(X) = S$, where S is the hyperbolic lattice of rank 3 defined above, one has $Y = M_X(5, H, 13) \cong X$ which gives a 17-dimensional moduli space M_S of polarized K3 surfaces (X, H) with $Y = M_X(5, H, 13) \cong X$.*

On the other hand, M_S is not contained in any 18-dimensional moduli space M_N of polarized K3 surfaces (X', H) where $H \in N(X') = N \subset S$ is a primitive sublattice of rank $N = 2$ *and $M_{X'}(5, H, 13) \cong X'$. Thus M_S is not defined by any divisorial condition on moduli of polarized K3 surfaces (X, H) implying $M_X(5, H, 13) \cong X$. (That is, M_S is not a specialization of a divisor with this condition.)*

Proof. For this case, $c = (5, 13) = 1$ and $(\gamma, c) = 1$. By Mukai's results (5) and (12), the transcendental periods $(T(X), H^{2,0}(X))$ and $(T(Y), H^{2,0}(Y))$ are then isomorphic. The discriminant group $A_S = S^*/S$ of the lattice $S = T(X)^\perp$ is a cyclic group $\mathbb{Z}/(2 \cdot 5 \cdot 13 \cdot 3 \cdot 17)$. Thus the minimal number $l(A_S)$ of generators of A_S is one. Thus $l(A_S) \leq \operatorname{rank} S - 2$. By [10], Theorem 1.14.4, a primitive embedding of $T(X)$ into the cohomology lattice of K3 (which is an even unimodular lattice of signature $(3, 19)$) is then unique, up to isomorphisms. It follows that the isomorphism between transcendental periods of X and Y can be extended to an isomorphism of periods of X and Y. By the Global Torelli Theorem for K3 surfaces [15], the K3 surfaces X and Y are isomorphic. (These considerations are now standard.)

Let $H \in N \subset S$ be a primitive sublattice with $\operatorname{rank} N = 2$. Since $H \cdot S = H \cdot H\mathbb{Z} = 2 \cdot 5 \cdot 13\mathbb{Z}$, it follows that $H \cdot N = 2 \cdot 5 \cdot 13\mathbb{Z}$, and the invariant $\gamma = 2 \cdot 5 \cdot 13$ is the same for any sublattice $N \subset S$ containing H. By Theorem 3.8, $M_{X'}(r, H, s)$ is not isomorphic to X' for any general K3 surface (X', H) with $N(X') = N$.

This completes the proof. $\square$

Similar arguments can be used to prove the following general statement for $\rho(X) \geq 12$. This shows that there are many cases when $Y = M_X(r,H,s) \cong X$ which do not follow from divisorial conditions on moduli. Its first statement is well known (see for example [1], Proposition 2.2.1).

Theorem 4.2. *Let (X,H) be a polarized K3 surface with $\rho(X) \geq 12$, and for $r,s \geq 1$, let (r,H,s) be a primitive isotropic Mukai vector on X, i.e., $H^2 = 2rs$ and $(c,d) = 1$. Assume that $H \cdot N(X) = \gamma\mathbb{Z}$.*

Then $Y = M_X(r,H,s) \cong X$ if $(\gamma, c) = 1$ (Mukai's necessary condition).

On the other hand, if (32) *holds, the isomorphism $Y = M_X(r,H,s) \cong X$ does not follow from any divisorial condition on moduli of polarized K3 surfaces. That is, for any primitive 2-dimensional sublattice $H \in N \subset N(X)$, there exists a polarized K3 surface (X',H) with $N(X') = N$ such that $Y' = M_{X'}(r,H,s)$ is not isomorphic to X'.*

Proof. Since $\rho(X) \geq 12$,

$$\operatorname{rank} T(X) \leq 22 - 12 = 10 \quad \text{and} \quad l(A_{T(X)}) \leq \operatorname{rank} T(X) = 10.$$

Since $N(X)$ and $T(X)$ are orthogonal complements to one another in the unimodular lattice $H^2(X,\mathbb{Z})$, it follows that $A_{N(X)} \cong A_{T(X)}$ and $l(A_{N(X)}) \leq 10 \leq \operatorname{rank} N(X) - 2$. By [10], Theorem 1.14.4, a primitive embedding of $T(X)$ into the cohomology lattice of K3 is then unique up to isomorphisms. As in the proof of Theorem 4.1, it follows that $Y \cong X$.

We prove the second statement. Since $H \cdot N(X) = \gamma\mathbb{Z}$ and $H \in N \subset N(X)$, it follows that $H \cdot N = \gamma(N)\mathbb{Z}$ where $\gamma \mid \gamma(N)$. Let X' be a general K3 surface with $N(X') = N$. If $(c, \gamma(N)) > 1$, then $Y' = M_{X'}(r,H,s)$ is not isomorphic to X' because $[T(Y') \colon T(X')] = (c, \gamma(N)) > 1$ by Mukai's result (12). Assume $(c, \gamma(N)) = 1$. Obviously, (32) for γ implies (32) for $\gamma(N)$. By Theorem 3.6, $Y' = M_{X'}(r,H,s)$ is not isomorphic to X'.

This completes the proof. □

Theorems 4.1 and 4.2 can be unified in the following statement, which is the most general known: when does $Y = M_X(r,H,s) \cong X$ hold for any primitive isotropic Mukai vector on X satisfying Mukai's necessary condition. We remind that two lattices have the same genus if they are isomorphic over $\mathbb{R}$ and rings $\mathbb{Z}_p$ of p-adic integers for all prime p.

Theorem 4.3. *Let X be a K3 surface. Assume that the Picard lattice $N(X)$ is unique in its genus, and the natural homomorphism*

$$O(N(X)) \to O(q_{N(X)})$$

is surjective, where $q_{N(X)}$ is the discriminant quadratic form of $N(X)$. Equivalently, any isomorphism of the transcendental periods of X and another K3 surface extends to an isomorphism of the periods of X and the other K3 surface.

Then for any primitive isotropic Mukai vector $v = (r, H, s)$ *on* X *such that* $(c, \gamma) = 1$ *(Mukai's necessary condition), one has* $Y = M_X(r, H, s) \cong X$*.*

On the other hand, if X *is general with its Picard lattice and* (32) *holds, then the isomorphism* $Y = M_X(r, H, s) \cong X$ *does not follow from any divisorial condition on moduli of polarized K3 surfaces* (X, H)*. That is, for any primitive* 2*-dimensional sublattice* $H \in N \subset N(X)$*, there exists a polarized K3 surface* (X', H) *with* $N(X') = N$ *such that* $Y' = M_{X'}(r, H, s)$ *is not isomorphic to* X'*.*

These results and those of Section 3 suggest the following general notions. Let $r \in \mathbb{N}$ and $s \in \mathbb{Z}$. We formally put $H^2 = 2rs$ and introduce $c = (r, s)$ and $a = r/c$, $b = s/c$. Let $d \in \mathbb{N}$, $(d, c) = 1$ and $d^2 \mid ab$. We call

$$(r, H, s), \quad H^2 = 2rs, \quad d, \tag{35}$$

a *type of primitive isotropic Mukai vector for a K3.* Clearly, *a Mukai vector of type* (35) *on a K3 surface* X is just an element $H \in N(X)$ such that $H^2 = 2rs$ and $\widetilde{H} = H/d$ is primitive. As above, we introduce $d_a = (d, a)$, $d_b = (d, b)$ and put $a_1 = a/d_a^2$, $b_1 = b/d_b^2$. Then $\widetilde{H}^2 = 2a_1 b_1 c^2$.

Let N be a lattice that embeds primitively into the Picard lattice of some algebraic K3 surface (equivalently, there exists a Kähler K3 surface with this Picard lattice). It is equivalent to say that N is either negative definite, or semi-negative definite with 1-dimensional kernel, or hyperbolic (i.e., N has signature $(1, \rho - 1)$), and has a primitive embedding into an even unimodular lattice of signature $(3, 19)$. Moreover, we say that N is an *abstract K3 Picard lattice* (or just a K3 Picard lattice). Let $H \in N$; we say that $H \in N$ is a *polarized (abstract) K3 Picard lattice* (despite the fact that H^2 can be nonpositive). We consider such pairs up to natural isomorphism. Another polarized K3 Picard lattice $H' \in N'$ is *isomorphic* to $H \in N$ if there exists an isomorphism $f\colon N \cong N'$ of lattices with $f(H) = H'$.

Definition 4.4. Fix a type (35) of primitive isotropic Mukai vector of K3. A polarized K3 Picard lattice $H \in N$ is *critical for self-correspondences of a K3 surface via moduli of sheaves for the type* (35) *of Mukai vector* if $H^2 = 2rs$ and $\widetilde{H} = H/d \in N$ is primitive and $H \in N$ satisfies the following two conditions:

(a) for any K3 surface X such that $H \in N \subset N(X)$ is a primitive sublattice, one has $Y = M_X(r, H, s) \cong X$.

(b) the above condition (a) does not hold if one replaces $H \in N$ by $H \in N_1$ for any primitive sublattice $H \in N_1 \subset N$ of N of strictly smaller rank $\operatorname{rank} N_1 < \operatorname{rank} N$.

In what follows we abbreviate this, saying that $H \in N$ is a *critical polarized K3 Picard lattice for the type* (35).

On the one hand, [14], Theorem 2.3.3 gave a criterion for a polarized K3 Picard lattice $H \in N$, for a general (and then any) K3 surface with

$H \in N = N(X)$ to have $Y = M_X(r,H,s) \cong X$. On the other hand, by the specialization principle (Lemma 2.1.1 in [14]), if this criterion is satisfied, then $Y = M_X(r,H,s) \cong X$ for any K3 surface X such that $H \in N \subset N(X)$ is a primitive sublattice. Thus for the problem of describing in terms of Picard lattices, of all K3 surfaces X such that $Y = M_X(r,H,s) \cong X$, the main problem is as follows.

Problem 4.5. For a given type of primitive isotropic Mukai vector (35) for a K3, describe all *critical polarized K3 Picard lattices* $H \in N$ (for the problem of self-correspondences of a K3 surface via moduli of sheaves).

Now we have the following examples of solution of this problem.

By (10), or Corollary 2.6, or Remark 3.4, we have classified the critical polarized K3 Picard lattices of rank one.

Example 4.6. For the type (r,H,s), $H^2 = 2rs$, d where $c = 1$ and either $a_1 = 1$ or $b_1 = \pm 1$, we obtain that $N = \mathbb{Z}\widetilde{H}$ where $\widetilde{H}^2 = 2a_1b_1$ gives all critical polarized K3 Picard lattices $H = d\widetilde{H} \in N$ of rank one.

Example 4.7. For the type of Mukai vector which is different from Example 4.6, classification of the critical polarized K3 Picard lattices of rank 2 is given by equivalent Theorems 3.1, 3.2 or 3.3.

Example 4.8. For the Mukai vector of the type $(5,H,13)$ with $H^2 = 2\cdot 5\cdot 13$ and $d = 1$, the polarized Picard lattice $H \in S$ of Theorem 4.1 is critical of rank rank S=3, by Theorem 4.1. Obviously, there are plenty of similar examples. It would be very interesting and nontrivial to find all critical polarized K3 Picard lattices $H \in S$ of rank 3.

Example 4.9. By Theorem 4.2, we should expect that there exist critical polarized K3 Picard lattices of the rank more than 3. On the other hand, the same Theorem 4.2 gives that the rank of a critical polarized K3 Picard lattice is ≤ 12.

Theorem 4.10. *For any type (r,H,s), $H^2 = 2rs$ and d of a primitive isotropic Mukai vector of K3, the rank of a critical polarized K3 Picard lattice $H \in N$ is at most 12: we have* $\operatorname{rank} N \le 12$.

Proof. Let $H \in N$ be a critical polarized K3 Picard lattice of this type and $\operatorname{rank} N \ge 13$. Let us take any primitive sublattice $H \in N' \subset N$ of the $\operatorname{rank} N' = 12$ such that $\widetilde{H}\cdot N' = \widetilde{H}\cdot N$. Obviously, it does exist. Let X be an algebraic K3 surface such that $H \in N' \subset N(X)$. Then $\operatorname{rank} N(X) \ge 12$ and $Y = M_X(r,H,s) \cong X$ by Theorem 4.2.

Then the condition (b) of Definition 4.4 is not satisfied, and we get a contradiction. Thus $\operatorname{rank} N \le 12$.

This completes the proof. $\square$

It would be very interesting to give an exact estimate for the rank of critical polarized K3 Picard lattices.

Problem 4.11. For a given type (35) of primitive isotropic Mukai vector of K3, give the exact estimate of the rank $\operatorname{rank} N$ of a critical polarized K3 Picard lattices $H \in N$ of this type (for the problem of self-correspondences of K3 surfaces).

Now we don't know the answer to this problem for any type (35) different from Example 4.6.

5 Composing self-correspondences of a K3 surface via moduli of sheaves and the general classification problem

We want to interpret the above results in terms of the action of correspondences on the 2-dimensional cohomology lattice of a K3 surface. Moreover, we attempt to formulate the general problem of classification of self-correspondences of a K3 surface via moduli of sheaves.

Let $v = (r, H, s)$ be a primitive isotropic Mukai vector on a K3 surface X and $Y = M_X(r, H, s)$. Write π_X, π_Y for the projections of $X \times Y$ to X and Y. By Mukai [5], Theorem 1.5, the algebraic cycle

$$Z_{\mathcal{E}} = (\pi_X^* \sqrt{\mathrm{td}_X}) \cdot \mathrm{ch}(\mathcal{E}) \cdot (\pi_Y^* \sqrt{\mathrm{td}_Y})/\sigma(\mathcal{E}) \tag{36}$$

arising from the quasi-universal sheaf $\mathcal{E}$ on $X \times Y$ defines an isomorphism of the full cohomology groups

$$f_{Z_{\mathcal{E}}} \colon H^*(X, \mathbb{Q}) \to H^*(Y, \mathbb{Q}), \quad t \mapsto \pi_{Y*}(Z_{\mathcal{E}} \cdot \pi_X^* t) \tag{37}$$

with their Hodge structures (see [5], Theorem 1.5 for details). Moreover, according to Mukai, it defines an isomorphism of lattices (an isometry)

$$f_{Z_{\mathcal{E}}} \colon v^{\perp} \to H^4(Y, \mathbb{Z}) \oplus H^2(Y, \mathbb{Z})$$

where $f_{Z_{\mathcal{E}}}(v) = w \in H^4(Y, \mathbb{Z})$ is the fundamental cocycle, and the orthogonal complement $v^{\perp}$ is taken in the Mukai lattice $\widetilde{H}(X, \mathbb{Z})$. This gives the relation (5) already used in Section 2.

In particular, composing $f_{Z_{\mathcal{E}}}$ with the projection $\pi \colon H^4(Y, \mathbb{Z}) \oplus H^2(Y, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$ gives an embedding of lattices

$$\pi \cdot f_{Z_{\mathcal{E}}} \colon H^{\perp}_{H^2(X,\mathbb{Z})} \to H^2(Y, \mathbb{Z})$$

that extends to an isometry

$$\widetilde{f}_{Z_{\mathcal{E}}} \colon H^2(X, \mathbb{Q}) \to H^2(Y, \mathbb{Q}) \tag{38}$$

of quadratic forms over $\mathbb{Q}$ by Witt's Theorem. If $H^2 = 0$, this extension is unique.

If $H^2 \neq 0$, there are two such extensions, differing by ± 1 on $\mathbb{Z}H$. We agree to take

$$\widetilde{f}_{Z\varepsilon}(\widetilde{H}) = ch, \tag{39}$$

where h is defined in (6), and we use Proposition 2.4 to relate the periods of X and Y.

The Hodge isometry (38) can be viewed as a minor modification of Mukai's algebraic cycle (36) to give an isometry in H^2. It is also clearly defined by some algebraic cycle, because it only changes the Mukai isomorphism (37) in the algebraic part.

By Proposition 2.4, the isomorphism $\widetilde{f}_{Z\varepsilon}$ is given by embeddings

$$\begin{gathered}\widetilde{H}^{\perp} \subset h^{\perp} = \left[\widetilde{H}^{\perp}, 2a_1b_1ct^*(\widetilde{H})\right], \quad \mathbb{Z}\widetilde{H} \subset \mathbb{Z}h, \quad \widetilde{H} = ch \\ \text{and} \quad H^{2,0}(X) = H^{2,0}(Y).\end{gathered} \tag{40}$$

This identifies the quadratic forms $H^2(X,\mathbb{Q}) = H^2(Y,\mathbb{Q})$ over $\mathbb{Q}$, and the lattices $H^2(X,\mathbb{Z})$, $H^2(Y,\mathbb{Z})$ as two sublattices of this. Let

$$\begin{aligned}O(H^2(X,\mathbb{Q}))_0 &= \left\{f \in O(H^2(X,\mathbb{Q})) \mid f|T(X) = \pm 1\right\} \\ &\cong O(N(X)\otimes\mathbb{Q}) \times \{\pm 1_{T(X)}\},\end{aligned}$$

and

$$O(H^2(X,\mathbb{Z}))_0 = O(H^2(X,\mathbb{Z})) \cap O(H^2(X,\mathbb{Q}))_0.$$

By the Global Torelli Theorem for K3 surfaces of [15], we obtain at once:

Proposition 5.1. *If a K3 surface X is general with its Picard lattice, then $Y = M_X(r,H,s) \cong X$ if and only if there exists an automorphism $\phi(r,H,s) \in O(H^2(X,\mathbb{Q}))_0$ such that $\phi(H^2(X,\mathbb{Z})) = H^2(Y,\mathbb{Z})$.*

If $Y \cong X$ we can give the definition.

Definition 5.2. If $Y = M_X(r,H,s) \cong X$ and X is general with its Picard lattice, then the isomorphism of Proposition 5.1

$$\phi(r,H,s) \mod O(H^2(X,\mathbb{Z}))_0 \in O(H^2(X,\mathbb{Q}))_0/O(H^2(X,\mathbb{Z}))_0$$

is called the *action* on $H^2(X,\mathbb{Q})$ of the self-correspondence of a general K3 surface X (general with its Picard lattice) via moduli of sheaves $Y = M_X(r,H,s)$ with primitive isotropic Mukai vector $v = (r,H,s)$.

By the Global Torelli Theorem for K3 surfaces [15], the group $O(H^2(X,\mathbb{Z}))_0$ mod ± 1 can be considered as generated by correspondences defined by graphs of automorphisms of X and by the reflections in elements $\delta \in N(X)$ with $\delta^2 = -2$ given by $s_\delta\colon x \mapsto x + (x\cdot\delta)\delta$ for $x \in H^2(X,\mathbb{Z})$. By the Riemann–Roch theorem for K3 surfaces, $\pm\delta$ contains an effective curve E. If $\Delta \subset X\times X$

is the diagonal, the effective 2-dimensional algebraic cycle $\Delta + E \times E \subset X \times X$ acts as the reflection s_δ in $H^2(X, \mathbb{Z})$ (I learnt this from Mukai [8]). Thus considering actions of correspondences modulo $O(H^2(X, \mathbb{Z}))$ mod ± 1 is very natural.

Consider the Tyurin isomorphism (10) defined by the Mukai vector $v = (\pm H^2/2, H, \pm 1)$, where $H \in N(X)$ has $\pm H^2 > 0$. Then $M_X(\pm H^2/2, H, \pm 1) \cong M_X(\pm \widetilde{H}^2/2, \widetilde{H}, \pm 1)$ where $\widetilde{H} = H/d$ is primitive.

Then $c = 1$, $a_1 = \pm \widetilde{H}^2/2$ and $b_1 = \pm 1$, $m(a_1, b_1) \equiv -1 \bmod 2a_1 b_1$, $h = \widetilde{H}$, and we have

$$H^2(X, \mathbb{Z}) = \left[\mathbb{Z}\widetilde{H}, \widetilde{H}^\perp, \widetilde{H} + t^*(\widetilde{H})\right] \quad \text{and} \quad H^2(Y, \mathbb{Z}) = \left[\mathbb{Z}\widetilde{H}, \widetilde{H}^\perp, \widetilde{H} - t^*(\widetilde{H})\right].$$

Then the reflection $s_{\widetilde{H}}$ in $\widetilde{H}$

$$s_{\widetilde{H}}(x) = x - \frac{2(x \cdot \widetilde{H})\widetilde{H}}{\widetilde{H}^2} \quad \text{for } x \in H^2(X, \mathbb{Q})$$

belongs to $O(H^2(X, \mathbb{Q}))_0$, and $s_{\widetilde{H}}(H^2(X, \mathbb{Z})) = H^2(Y, \mathbb{Z})$. Moreover, the reflections s_H and $s_{\widetilde{H}}$ coincide.

This gives the following result.

Proposition 5.3. *For a K3 surface X and $H \in N(X)$ with $\pm H^2 > 0$, the Tyurin isomorphism*

$$M_X(\pm H^2, H, \pm 1) \cong X$$

defines a self-correspondence of X with the action

$$s_H \bmod O(H^2(X, \mathbb{Z})_0),$$

where s_H is the reflection in H.

By classical and well-known results, their composites generate the full group $O(H^2(X, \mathbb{Q}))_0$ mod ± 1.

5.1 General problem of classifying self-correspondences of a K3 surface via moduli of sheaves

We need some notation. For a primitive sublattice $N \subset N(X)$, we introduce

$$O(N \otimes \mathbb{Q})_0 = \left\{ f \in O(H^2(X, \mathbb{Q})) \;\middle|\; f|N^\perp_{H^2(X,\mathbb{Z})} = \pm 1 \right\}$$

and

$$O(N)_0 = O(H^2(X, \mathbb{Z})) \cap O(N \otimes \mathbb{Q})_0.$$

We denote by $[\,\cdot\,]_{pr} \subset L$ the primitive sublattice of L generated by $\cdot$.

Let X be a K3 surface which is general with its Picard lattice $N(X)$. From our current point of view, the problem of classifying self-correspondences of X via moduli of sheaves consists of the following problems:

(1) Find all primitive isotropic Mukai vectors (r, H, s) on X such that $Y = M_X(r, H, s) \cong X$.
(2) For a primitive isotropic Mukai vector (r, H, s) as in (1), find all primitive critical polarized Picard sublattices $H \in N(r, H, s) \subset N(X)$.

For either of these problems, the action $\phi(r, H, s)$ of Definition 5.2 can be taken to be in $O(N(r, H, s) \otimes \mathbb{Q})_0$. We denote it by $\phi_{N(r,H,s)}$, and it looks like a reflection with respect to $N(r, H, s)$. For two critical polarized Picard sublattices $H \in N(r, H, s)$ and $H' \in N'(r, H, s)$ as in (2), the automorphisms $\phi_{N(r,H,s)}$ and $\phi_{N'(r,H,s)}$ differ by an automorphism in $O(H^2(X, \mathbb{Z}))_0$.

(3) *The structures (1) and (2) are important for the following reason:* given any two primitive isotropic Mukai vectors (r, H, s) and (r', H', s') as in (1) and two critical polarized Picard sublattices $H \in N(r, H, s)$ and $H' \in N(r', H', s')$ for them as in (2), the isomorphism

$$\phi(r', H', s') \circ \phi(r, H, s)^{-1} \colon M_X(r, H, s) \to M_X(r', H', s')$$

comes from K3 surfaces with the Picard sublattice

$$[N(r, H, s) + N(r', H', s')]_{pr} \subset N(X),$$

and it can be viewed as a natural isomorphism between these moduli .
(4) All these generators $\phi_{N(r,H,s)} \bmod O(N(r, H, s))_0$ can be considered as natural generators for self-correspondences of X via moduli of sheaves, together with automorphisms of X and reflections s_δ, $\delta \in N(X)$ and $\delta^2 = -2$. They and their relations are the natural subject to study.

Problems (1)—(4) are solved for $\rho(X) = 1$ and 2 in Sections 2 and 3. The results of Section 4 show that these problems are very nontrivial for $\rho(X) \geq 3$.

As an example, take a general K3 surface X with the rank 3 Picard lattice $N(X) = S$ of Theorem 4.1 (or any other Picard lattice of rank 3 satisfying Theorem 4.3). Let $v = (r, H, s)$ be a primitive isotropic Mukai vector on X. Then $Y = M_X(r, H, s) \cong X$ if and only if $(\gamma, c) = 1$ where $\widetilde{H} \cdot S = \gamma\mathbb{Z}$. Moreover, we have three cases:

(a) If $c = 1$ and either $a_1 = 1$ or $b_1 = \pm 1$ (Tyurin's case), then the critical sublattice is $N(v) = \mathbb{Z}\widetilde{H}$, it has rank one and is unique. The corresponding $\phi_{N(v)} = s_H \bmod O(H^2(X, \mathbb{Z}))_0$.
(b) If $v = (r, H, s)$ is different from (a), but the critical sublattice $N(v)$ has rank two (the divisorial case), then all critical sublattices $N(v)$ are primitively generated by $\widetilde{H}$ and $\widetilde{h}_1 \in [\widetilde{H}, a_1 c N(X)]$ with $\widetilde{h}_1^2 = \pm 2a_1 c$ or $\widetilde{h}_1 \in [\widetilde{H}, b_1 c N(X)]$ with $\widetilde{h}_1^2 = \pm 2b_1 c$ (see the theorems of Section 3). All these $N(v)$ give automorphisms $\phi_{N(v)}$ that differ by elements of $O(H^2(X, \mathbb{Z}))_0$.
(c) If $v = (r, H, s)$ is different from (a) and (b), then the critical sublattice $N(v) = N(X)$ has rank 3. These cases really happen by Theorem 4.1. We get $\phi_{N(v)} \bmod O(H^2(X, \mathbb{Z}))_0$.

Any two v_1, v_2 satisfying one of these conditions (a–c), together with any two critical sublattices $N(v_1)$, $N(v_2)$ for them, generate natural isomorphisms $\phi_{N(v_2)} \circ \phi^{-1}_{N(v_1)}$ between the corresponding moduli spaces of sheaves over X (all of which are isomorphic to X), which are specializations of isomorphisms from the Picard sublattice $[N(v_1) + N(v_2)]_{pr} \subset N(X)$.

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