Operads revisited

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To Yuri Manin, many happy returns.

This paper presents an approach to operads related to recent work in topological field theory (Costello [4]). The idea is to represent operads as symmetric monoidal functors on a symmetric monoidal category T; we recall how this works for cyclic and modular operads and dioperads in Section 1. This point of view permits the construction of all sorts of variants on the notion of an operad. As our main example, we present a simplicial variant of modular operads, related to Segal's definition of quantum field theory, as modified for topological field theory (Getzler [10]). (This definition is only correct in a cocomplete symmetric monoidal category whose tensor product preserves colimits, but this covers most cases of interest.)

An operad $\mathcal{P}$ in the category Set of sets may be presented as a symmetric monoidal category $\mathsf{t}\mathcal{P}$, called the theory associated to $\mathcal{P}$ (Boardman and Vogt [2]). The category $\mathsf{t}\mathcal{P}$ has the natural numbers as its objects; tensor product is given by addition. The morphisms of $\mathsf{t}\mathcal{P}$ are built using the operad $\mathcal{P}$:

$$\mathsf{t}\mathcal{P}(m,n) = \coprod_{f:\{1,\dots,m\}\to\{1,\dots,n\}} \prod_{i=1}^{n} \mathcal{P}(|f^{-1}(i)|).$$

The category of $\mathcal{P}$-algebras is equivalent to the category of symmetric monoidal functors from $\mathsf{t}\mathcal{P}$ to Set; this reduces the study of algebras over operads to the study of symmetric monoidal functors.

The category of contravariant functors from the opposite category T° of a small category T to the category Set of sets

$$\mathsf{T}\hat{} = [\mathsf{T}^\circ, \mathsf{Set}]$$

is called the category of presheaves of T. If T is a symmetric monoidal category, then $\mathsf{T}\hat{}$ is too (Day [5]); its tensor product is the coend

$$V * W = \int^{A,B\in\mathsf{T}} \mathsf{T}(-, A\otimes B) \times V(A) \times W(B).$$

A symmetric monoidal functor $F : \mathsf{S} \to \mathsf{T}$ between symmetric monoidal categories S and T is a functor F together with a natural equivalence

$$\Phi : \otimes \circ F \times F \Longrightarrow F \circ \otimes.$$

The functor F is lax symmetric monoidal if Φ is only a natural transformation.

If $\tau : \mathsf{S} \to \mathsf{T}$ is a symmetric monoidal functor, it is not always the case that the induced functor

$$\tau\hat{} : \mathsf{T}\hat{} \longrightarrow \mathsf{S}\hat{}$$

is a symmetric monoidal functor; in general, it is only a lax symmetric monoidal functor, with respect to the natural transformation

$$\begin{tikzcd}
\tau\hat{}(V) * \tau\hat{}(W) \arrow[r, equal] \arrow[dddd, Rightarrow, "\Phi_{V,W}"'] & \int^{A,B\in\mathsf{S}} \mathsf{S}(-, A\otimes B) \times V(\tau A) \times W(\tau B) \arrow[d, Rightarrow, "\tau \text{ is a functor}"] \\
& \int^{A,B\in\mathsf{S}} \mathsf{T}(\tau(-), \tau(A\otimes B)) \times V(\tau A) \times W(\tau B) \arrow[d, Rightarrow, "\tau \text{ is symmetric monoidal}"] \\
& \int^{A,B\in\mathsf{S}} \mathsf{T}(\tau(-), \tau(A)\otimes \tau(B)) \times V(\tau A) \times W(\tau B) \arrow[d, Rightarrow, "\text{universality of coends}"] \\
\tau\hat{}(V * W) \arrow[r, equal] & \int^{A,B\in\mathsf{T}} \mathsf{T}(\tau(-), A\otimes B) \times V(A) \times W(B)
\end{tikzcd}$$

The following definition introduces the main object of study of this paper.

Definition. A **pattern** is a symmetric monoidal functor $\tau : \mathsf{S} \to \mathsf{T}$ between small symmetric monoidal categories S and T such that $\tau\hat{} : \mathsf{T}\hat{} \to \mathsf{S}\hat{}$ is a symmetric monoidal functor (in other words, the natural transformation Φ defined above is an equivalence).

Let τ be a pattern and let $\mathcal{C}$ be a symmetric monoidal category. A **τ-preoperad** in $\mathcal{C}$ is a symmetric monoidal functor from S to $\mathcal{C}$, and a **τ-operad** in $\mathcal{C}$ is a symmetric monoidal functor from T to $\mathcal{C}$. Denote by $\mathrm{PreOp}_\tau(\mathcal{C})$ and $\mathrm{Op}_\tau(\mathcal{C})$ the categories of τ-preoperads and τ-operads respectively. If $\mathcal{C}$ is a cocomplete symmetric monoidal category, there is an adjunction

$$\tau_* : \mathrm{PreOp}_\tau(\mathcal{C}) \rightleftharpoons \mathrm{Op}_\tau(\mathcal{C}) : \tau^*.$$

In Section 2, we prove that if τ is essentially surjective, $\mathcal{C}$ is cocomplete, and the functor $A\otimes B$ on $\mathcal{C}$ preserves colimits in each variable, then the functor τ^* is monadic. We also prove that if S is a free symmetric monoidal category and $\mathcal{C}$ is locally finitely presentable, then $\mathrm{Op}_\tau(\mathcal{C})$ is locally finitely presentable.

Patterns generalize the coloured operads of Boardman and Vogt [2], which are the special case where S is the free symmetric category generated by a discrete category (called the set of colours). Operads are themselves algebras for a coloured operad, whose colours are the natural numbers (cf. Berger and Moerdijk [1]), but it is more natural to think of the colour n as having nontrivial automorphisms, namely the symmetric group $\mathbb{S}_n$.

The definition of a pattern may be applied in the setting of simplicial categories, or even more generally, enriched categories. We define and study patterns enriched over a symmetric monoidal category $\mathcal{V}$ in Section 2; we recall those parts of the formalism of enriched categories which we will need in the appendix.

In Section 3, we present examples of a simplicial pattern which arises in topological field theory. The most interesting of these is related to modular operads. Let cob be the (simplicial) groupoid of diffeomorphisms between compact connected oriented surfaces with boundary. Let $S_{g,n}$ be a compact oriented surface of genus g with n boundary circles, and let the mapping class group be the group of components of the oriented diffeomorphism group:

$$\varGamma_{g,n} = \pi_0(\mathrm{Diff}_+(S_{g,n})).$$

The simplicial groupoid cob has a skeleton

$$\bigsqcup_{g,n} \mathrm{Diff}_+(S_{g,n}),$$

and if $2g-2+n>0$, there is a homotopy equivalence $\mathrm{Diff}_+(S_{g,n}) = \varGamma_{g,n}$. In Section 3, we define a pattern whose underlying functor is the inclusion $\mathbb{S} \wr \mathsf{cob} \hookrightarrow \mathsf{Cob}$. This pattern bears a similar relation to modular operads that braided operads (Fiedorowicz [8]) bear to symmetric operads. The mapping class group $\varGamma_{0,n}$ is closely related to the ribbon braid group $B_n \wr \mathbb{Z}$. Denoting the generators of $B_n \subset B_n \wr \mathbb{Z}$ by $\{b_1, \dots, b_{n-1}\}$ and the generators of $\mathbb{Z}^n \subset B_n \wr \mathbb{Z}$ by $\{t_1, \dots, t_n\}$, Moore and Seiberg show [21, Appendix B.1] that $\varGamma_{0,n}$ is isomorphic to the quotient of $B_n \wr \mathbb{Z}$ by its subgroup $\langle (b_1 b_2 \dots b_{n-1} t_n)^n, b_1 b_2 \dots b_{n-1} t_n^2 b_{n-1} \dots b_1 \rangle$. In this sense, operads for the pattern Cob are a cyclic analogue of braided operads.

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1 Modular operads as symmetric monoidal functors

In this section, we define modular operads in terms of the symmetric monoidal category of dual graphs. Although we do not assume familiarity with the original definition (Getzler and Kapranov [11]; see also Markl et al. [18]), this section will certainly be easier to understand if this is not the reader's first brush with the subject.

We also show how modifications of this construction, in which dual graphs are replaces by forests, or by directed graphs, yielding cyclic operads and modular dioperads. Finally, we review the definition of algebras for modular operads and dioperads.

Graphs

A graph $\varGamma$ consists of the following data:

i) finite sets $V(\varGamma)$ and $F(\varGamma)$, the sets of vertices and flags of the graph;
ii) a function $p : F(\varGamma) \to V(\varGamma)$, whose fibre $p^{-1}(v)$ is the set of flags of the graph meeting at the vertex v;
iii) an involution $\sigma : F(\varGamma) \to F(\varGamma)$, whose fixed points are called the legs of $\varGamma$, and whose remaining orbits are called the edges of $\varGamma$.

We denote by $L(\varGamma)$ and $E(\varGamma)$ the sets of legs and edges of $\varGamma$, and by $n(v) = |p^{-1}(v)|$ the number of flags meeting a vertex v.

To a graph is associated a one-dimensional cell complex, with 0-cells $V(\varGamma) \sqcup L(\varGamma)$, and 1-cells $E(\varGamma) \sqcup L(\varGamma)$. The 1-cell associated to an edge $e = \{f, \sigma(f)\}$, $f \in F(G)$, is attached to the 0-cells corresponding to the vertices $p(f)$ and $p(\sigma(f))$ (which may be equal), and the 1-cell associated to a leg $f \in L(G) \subset F(G)$ is attached to the 0-cells corresponding to the vertex $p(f)$ and the leg f itself.

The edges of a graph $\varGamma$ define an equivalence relation on its vertices $V(\varGamma)$; the components of the graph are the equivalence classes with respect to this relation. Denote the set of components by $\pi_0(\varGamma)$. The Euler characteristic of a component C of a graph is $e(C) = |V(C)| - |E(C)|$. Denote by $n(C)$ the number of legs of a component C.

Dual graphs

A dual graph is a graph $\varGamma$ together with a function $g : V(\varGamma) \to \mathbb{N}$. The natural number $g(v)$ is called the genus of the vertex v.

The genus $g(C)$ of a component C of a dual graph $\varGamma$ is defined by the formula

$$g(C) = \sum_{v \in C} g(v) + 1 - e(C).$$

The genus of a component is a non-negative integer, which equals 0 if and only if C is a tree and $g(v) = 0$ for all vertices v of C.

A **stable** graph is a dual graph $\varGamma$ such that for all vertices $v \in V(\varGamma)$, the integer $2g(v) - 2 + n(v)$ is positive. In other words, if $g(v) = 0$, then $n(v)$ is at least 3, while if $g(v) = 1$, then $n(v)$ is nonzero. If $\varGamma$ is a stable graph, then $2g(C) - 2 + |L(C)| > 0$ for all components C of $\varGamma$.

The symmetric monoidal category $\mathcal{G}$

The objects of the category $\mathcal{G}$ are the dual graphs $\varGamma$ whose set of edges $E(\varGamma)$ is empty. Equivalently, an object of $\mathcal{G}$ is a pair of finite sets L and V and functions $p : L \to V$, $g : V \to \mathbb{N}$.

A morphism of $\mathcal{G}$ with source (L_1, V_1, p_1, g_1) and target (L_2, V_2, p_2, g_2) is a pair consisting of a dual graph $\varGamma$ with $F(\varGamma) = L_1$, $V(\varGamma) = V_1$, and $g(\varGamma) = g_1$, together with isomorphisms $\alpha : L_2 \to L(\varGamma)$ and $\beta : V_2 \to \pi_0(\varGamma)$ such that $p \circ \alpha = \beta \circ p_2$ and $\alpha^* g = g_2$.

The definition of the composition of two morphisms $\varGamma = \varGamma_2 \circ \varGamma_1$ in $\mathcal{G}$ is straightforward. We have dual graphs $\varGamma_1$ and $\varGamma_2$, with

$$\begin{aligned} F(\varGamma_1) &= L_1, & V(\varGamma_1) &= V_1, & g(\varGamma_1) &= g_1, \\ F(\varGamma_2) &= L_2, & V(\varGamma_2) &= V_2, & g(\varGamma_2) &= g_2, \end{aligned}$$

together with isomorphisms

$$\begin{aligned} \alpha_1 &: L_2 \longrightarrow L(\varGamma_1), & \beta_1 &: V_2 \longrightarrow \pi_0(\varGamma_1), \\ \alpha_2 &: L_3 \longrightarrow L(\varGamma_2), & \beta_2 &: V_3 \longrightarrow \pi_0(\varGamma_2). \end{aligned}$$

Since the source of $\varGamma = \varGamma_2 \circ \varGamma_1$ equals the source of $\varGamma_1$, we see that $p : F(\varGamma) \to V(\varGamma)$ is identified with $p : L_1 \to V_1$. The involution σ of L_1 is defined as follows: if $f \in F(\varGamma) = L_1$ lies in an edge of $\varGamma_1$, then $\sigma(f) = \sigma_1(f)$, while if f is a leg of $\varGamma_1$, then $\sigma(f) = \alpha_1(\sigma_2(\alpha_1^{-1}(f)))$. The isomorphisms α and β of $\varGamma$ are simply the isomorphisms α_2 and β_2 of $\varGamma_2$.

In other words, $\varGamma_2 \circ \varGamma_1$ is obtained from $\varGamma_1$ by gluing those pairs of legs of $\varGamma_1$ together which correspond to edges of $\varGamma_2$. It is clear that composition in $\mathcal{G}$ is associative.

The tensor product on objects of $\mathcal{G}$ extends to morphisms, making $\mathcal{G}$ into a symmetric monoidal category.

A morphism in $\mathcal{G}$ is invertible if the underlying stable graph has no edges, in other words, if it is simply an isomorphism between two objects of $\mathcal{G}$. Denote by $\mathcal{H}$ the groupoid consisting of all invertible morphisms of $\mathcal{G}$, and by $\tau : \mathcal{H} \hookrightarrow \mathcal{G}$ the inclusion. The groupoid $\mathcal{H}$ is the free symmetric monoidal functor generated by the groupoid h consisting of all morphisms of $\mathcal{G}$ with connected domain.

The category $\mathcal{G}$ has a small skeleton; for example, the full subcategory of $\mathcal{G}$ in which the sets of flags and vertices of the objects are subsets of the set of natural numbers. The tensor product of $\mathcal{G}$ takes us outside this category, but it is not hard to define an equivalent tensor product for this small skeleton. We will tacitly replace $\mathcal{G}$ by this skeleton, since some of our constructions will require that $\mathcal{G}$ be small.

Modular operads as symmetric monoidal functors

The following definition of modular operads may be found in Costello [4]. Let $\mathcal{C}$ be a symmetric monoidal category which is cocomplete, and such that the functor $A \otimes B$ preserves colimits in each variable. (This last condition is automatic if $\mathcal{C}$ is a closed symmetric monoidal category.) A **modular preoperad** in $\mathcal{C}$ is a symmetric monoidal functor from $\mathcal{H}$ to $\mathcal{C}$, and a **modular operad** in $\mathcal{C}$ is a symmetric monoidal functor from $\mathcal{G}$ to $\mathcal{C}$.

Let $\mathcal{A}$ be a small symmetric monoidal category, and let $\mathcal{B}$ be symmetric monoidal category. Denote by $[\mathcal{A}, \mathcal{B}]$ the category of functors and natural equivalences from $\mathcal{A}$ to $\mathcal{B}$, and by $[\![\mathcal{A}, \mathcal{B}]\!]$ the category of symmetric monoidal functors and monoidal natural equivalences.

With this notation, the categories of modular operads, respectively preoperads, in a symmetric monoidal category $\mathcal{C}$ are $\mathrm{Mod}(\mathcal{C}) = [\![\mathcal{G}, \mathcal{C}]\!]$ and $\mathrm{PreMod}(\mathcal{C}) = [\![\mathcal{H}, \mathcal{C}]\!]$.

Theorem. *i) There is an adjunction*

$$\tau_* : \mathrm{PreMod}(\mathcal{C}) \rightleftharpoons \mathrm{Mod}(\mathcal{C}) : \tau^*,$$

where τ^* is restriction along $\tau : \mathcal{H} \hookrightarrow \mathcal{G}$, and τ_* is the coend

$$\tau_* \mathcal{P} = \int^{A \in \mathcal{H}} \mathcal{G}(A, -) \times \mathcal{P}(A).$$

ii) The functor

$$\tau^* : \mathrm{Mod}(\mathcal{C}) \longrightarrow \mathrm{PreMod}(\mathcal{C})$$

is monadic. That is, there is an equivalence of categories

$$\mathrm{Mod}(\mathcal{C}) \simeq \mathrm{PreMod}(\mathcal{C})^{\mathbb{T}},$$

where $\mathbb{T}$ is the monad $\tau^* \tau_*$.

iii) The category $\mathrm{Mod}(\mathcal{C})$ is locally finitely presentable if $\mathcal{C}$ is locally finitely presentable.

In the original definition of modular operads (Getzler and Kapranov [11]), there was an additional stability condition, which may be phrased in the following terms. Denote by $\mathcal{G}_+$ the subcategory of $\mathcal{G}$ consisting of stable graphs, and let $\mathcal{H}_+ = \mathcal{H} \cap \mathcal{G}_+$. Then a stable modular preoperad in $\mathcal{C}$ is a symmetric monoidal functor from $\mathcal{H}_+$ to $\mathcal{C}$, and a stable modular operad in $\mathcal{C}$ is a symmetric monoidal functor from $\mathcal{G}_+$ to $\mathcal{C}$. These categories of stable modular preoperads and operads are equivalent to the categories of stable $\mathbb{S}$-modules and modular operads of loc. cit.

Algebras for modular operads

To any object M of a closed symmetric monoidal category $\mathcal{C}$ is associated the monoid $\mathrm{End}(M) = [M, M]$: an A-module is a morphism of monoids $\rho : A \to$

End(M). The analogue of this construction for modular operads is called a $\mathcal{P}$-algebra.

A bilinear form with domain M in a symmetric monoidal category $\mathcal{C}$ is a morphism $t : M \otimes M \to \mathbb{1}$ such that $t \circ \sigma = t$. Associated to a bilinear form (M, t) is a modular operad $\mathsf{End}(M, t)$, defined on a connected object Γ of $\mathcal{G}$ to be

$$\mathsf{End}(M,t)(\Gamma) = M^{\otimes L(\Gamma)}.$$

If $\mathcal{P}$ is a modular operad, a $\mathcal{P}$-algebra M is an object M of $\mathcal{C}$, a bilinear form $t : M \otimes M \to \mathbb{1}$, and a morphism of modular operads $\rho : \mathcal{P} \to \mathsf{End}(M, t)$.

This modular operad has an underlying stable modular operad $\mathsf{End}_+(M, t)$, defined by restriction to the stable graphs in $\mathcal{G}$.

Cyclic operads

A variant of the above definition of modular operads is obtained by taking the subcategory of forests $\mathcal{G}_0$ in the category $\mathcal{G}$: a forest is a graph each component of which is simply connected. Symmetric monoidal functors on $\mathcal{G}_0$ are cyclic operads.

Denote the cyclic operad underlying $\mathsf{End}(M, t)$ by $\mathsf{End}_0(M, t)$: thus

$$\mathsf{End}_0(M,t)(\Gamma) = M^{\otimes L(\Gamma)}.$$

If $\mathcal{P}$ is a cyclic operad, a $\mathcal{P}$-algebra M is bilinear form t with domain M and a morphism of cyclic operads $\rho : \mathcal{P} \to \mathsf{End}_0(M, t)$.

There is a functor $\mathcal{P} \mapsto \mathcal{P}_0$, which associates to a modular operad its underlying cyclic operad: this is the restriction functor from $[\![\mathcal{G}, \mathcal{C}]\!]$ to $[\![\mathcal{G}_0, \mathcal{C}]\!]$.

There is also a stable variant of cyclic operads, in which $\mathcal{G}_0$ is replaced by its stable subcategory $\mathcal{G}_{0+}$, defined by restricting to forests in which each vertex meets at least three flags.

Dioperads and modular dioperads

Another variant of the definition of modular operads is obtained by replacing the graphs Γ in the definition of modular operads by digraphs (directed graphs):

A **digraph** Γ is a graph together with a partition

$$F(\Gamma) = F_+(\Gamma) \sqcup F_-(\Gamma)$$

of the flags into outgoing and incoming flags, such that each edge has one outgoing and one incoming flag. Each edge of a digraph has an orientation, running towards the outgoing flag. The set of legs of a digraph are partitioned into the outgoing and incoming legs: $L_\pm(\Gamma) = L(\Gamma) \cap F_\pm(\Gamma)$. Denote the number of outgoing and incoming legs by $n_\pm(\Gamma)$.

A dual digraph is a digraph together with a function $g : V(\Gamma) \to \mathbb{N}$. Imitating the construction of the symmetric monoidal category of dual graphs $\mathcal{G}$, we may construct a symmetric monoidal category of dual digraphs $\mathcal{D}$. A **modular dioperad** is a symmetric monoidal functor on $\mathcal{D}$. (These are the wheeled props studied in a recent preprint of Merkulov [20].)

If $M_\pm$ are objects of $\mathcal{C}$ and $t : M_+ \otimes M_- \to \mathbb{1}$ is a pairing, we may construct a modular dioperad $\mathsf{End}^{\rightharpoonup}(M_\pm, t)$, defined on a connected object Γ of $\mathcal{D}$ to be

$$\mathsf{End}^{\rightharpoonup}(M)(\Gamma) = M_+^{\otimes L_+(\Gamma)} \otimes M_-^{\otimes L_-(\Gamma)}. \tag{1}$$

If $\mathcal{P}$ is a modular dioperad, a $\mathcal{P}$-algebra is a pairing $t : M_+ \otimes M_- \to \mathbb{1}$ and a morphism of modular dioperads $\rho : \mathcal{P} \to \mathsf{End}^{\rightharpoonup}(M_\pm, t)$.

A directed forest is a directed graph each component of which is simply connected; let $\mathcal{D}_0$ be the subcategory of $\mathcal{D}$ of consisting of directed forests. A **dioperad** is a symmetric monoidal functor on $\mathcal{D}_0$ (Gan [9]).

If M is an object of $\mathcal{C}$, we may construct a dioperad $\mathsf{End}_0^{\rightharpoonup}(M)$, defined on a connected object Γ of $\mathcal{D}_0$ to be

$$\mathsf{End}_0^{\rightharpoonup}(M)(\Gamma) = \mathrm{Hom}\big(M^{\otimes L_+(\Gamma)}, M^{\otimes L_-(\Gamma)}\big).$$

When $M = M_-$ is a rigid object with dual $M^\vee = M_+$, this is a special case of (1). If $\mathcal{P}$ is a dioperad, a $\mathcal{P}$-algebra M is an object M of $\mathcal{C}$ and a morphism of dioperads $\rho : \mathcal{P} \to \mathsf{End}_0^{\rightharpoonup}(M)$.

props

MacLane's notion [17] of a prop also fits into the above framework. There is a subcategory $\mathcal{D}_{\mathsf{P}}$ of $\mathcal{D}$, consisting of all dual digraphs Γ such that each vertex has genus 0, and Γ has no directed circuits. A prop is a symmetric monoidal functor on $\mathcal{D}_{\mathsf{P}}$. (This follows from the description of free props in Enriquez and Etingof [7].) Note that $\mathcal{D}_0$ is a subcategory of $\mathcal{D}_{\mathsf{P}}$: thus, every prop has an underlying dioperad. Note also that the dioperad $\mathsf{End}_0^{\rightharpoonup}(M)$ is in fact a prop; if $\mathcal{P}$ is a prop, we may define a $\mathcal{P}$-algebra M to be an object M of $\mathcal{C}$ and a morphism of props $\rho : \mathcal{P} \to \mathsf{End}_0^{\rightharpoonup}(M)$.

2 Patterns

Patterns abstract the approach to modular operads sketched in Section 1. This section develops the theory of patterns enriched over a complete, cocomplete, closed symmetric monoidal category $\mathcal{V}$. In fact, we are mainly interested in the cases where $\mathcal{V}$ is the category Set of sets or the category sSet of simplicial sets. We refer to the appendices for a review of the needed enriched category theory.

Symmetric monoidal $\mathcal{V}$-categories

Let $\mathcal{A}$ be a $\mathcal{V}$-category. Denote by $\mathbb{S}_n$ the symmetric group on n letters. The wreath product $\mathbb{S}_n \wr \mathcal{A}$ is the $\mathcal{V}$-category

$$\mathbb{S}_n \wr \mathcal{A} = \mathbb{S}_n \times \mathcal{A}^n.$$

If $\alpha, \beta \in \mathbb{S}_n$, the composition of morphisms $(\alpha, \varphi_1, \dots, \varphi_n)$ and $(\beta, \psi_1, \dots, \psi_n)$ is

$$(\beta \circ \alpha, \psi_{\alpha_1} \circ \varphi_1, \dots, \psi_{\alpha_n} \circ \varphi_n).$$

Define the wreath product $\mathbb{S} \wr \mathcal{A}$ to be

$$\mathbb{S} \wr \mathcal{A} = \coprod_{n=0}^{\infty} \mathbb{S}_n \wr \mathcal{A}.$$

In fact, $\mathbb{S} \wr (-)$ is a 2-functor from the 2-category $\mathcal{V}\text{-}\mathsf{Cat}$ to itself. This 2-functor underlies a 2-monad $\mathbb{S} \wr (-)$ on $\mathcal{V}\text{-}\mathsf{Cat}$: the composition

$$m : \mathbb{S} \wr \mathbb{S} \wr (-) \longrightarrow \mathbb{S} \wr (-)$$

is induced by the natural inclusions

$$(\mathbb{S}_{n_1} \times \mathcal{A}_1^n) \times \cdots \times (\mathbb{S}_{n_k} \times \mathcal{A}_k^n) \hookrightarrow \mathbb{S}_{n_1 + \cdots + n_k} \times \mathcal{A}^{n_1 + \cdots + n_k},$$

and the unit $\eta : 1_{\mathcal{V}\text{-}\mathsf{Cat}} \to \mathbb{S} \wr (-)$ is induced by the natural inclusion

$$\mathcal{A} \cong \mathbb{S}_1 \times \mathcal{A} \hookrightarrow \mathbb{S} \wr \mathcal{A}.$$

Definition 2.1. A **symmetric monoidal** $\mathcal{V}$-category $\mathcal{C}$ is a pseudo $\mathbb{S} \wr (-)$-algebra in $\mathcal{V}\text{-}\mathsf{Cat}$.

If $\mathcal{C}$ is a symmetric monoidal category, we denote the object obtained by acting on the object $(A_1, \dots, A_n)$ of $\mathbb{S}_n \wr \mathcal{C}$ by $A_1 \otimes \cdots \otimes A_n$. When $n = 0$, we obtain an object $\mathbb{1}$ of $\mathcal{C}$, called the identity. When $n = 1$, we obtain a $\mathcal{V}$-endofunctor of $\mathcal{C}$, which is equivalent by the natural $\mathcal{V}$-equivalence ι in the definition of a pseudo $\mathbb{S}_n \wr \mathcal{C}$-algebra to the identity $\mathcal{V}$-functor. When $n = 2$, we obtain a $\mathcal{V}$-functor $(A, B) \mapsto A \otimes B$ from $\mathcal{C} \times \mathcal{C}$ to $\mathcal{C}$, called the tensor product. Up to $\mathcal{V}$-equivalence, all of the higher tensor products are obtained by iterating the tensor product $A \otimes B$: if $n > 2$, there is a natural $\mathcal{V}$-equivalence between the functors $A_1 \otimes \cdots \otimes A_n$ and $(A_1 \otimes \cdots \otimes A_{n-1}) \otimes A_n$.

$\mathcal{V}$-patterns

If $\mathcal{A}$ is a small symmetric monoidal $\mathcal{V}$-category, the $\mathcal{V}$-category of presheaves $\mathcal{A}\hat{}$ on $\mathcal{A}$ is a symmetric monoidal $\mathcal{V}$-category: the convolution of presheaves $V_1, \dots, V_n \in \mathcal{A}\hat{}$ is the $\mathcal{V}$-coend

$$V_1 * \cdots * V_n = \int^{A_1,\dots,A_n \in \mathcal{A}} V_1(A_1) \otimes \cdots \otimes V_n(A_n) \otimes y(A_1 \otimes \cdots \otimes A_n).$$

The Yoneda functor $y : \mathcal{A} \to \mathcal{A}\hat{}$ is a symmetric monoidal $\mathcal{V}$-functor. (See Day [5] and Im and Kelly [13].)

If $\tau : \mathsf{S} \to \mathsf{T}$ is a symmetric monoidal $\mathcal{V}$-functor between small symmetric monoidal $\mathcal{V}$-categories, the pull-back functor $F\hat{}$ is a lax symmetric monoidal $\mathcal{V}$-functor; we saw this in the unenriched case in the introduction, and the proof in the enriched case is similar.

Definition 2.2. A **$\mathcal{V}$-pattern** is a symmetric monoidal $\mathcal{V}$-functor $\tau : \mathsf{S} \to \mathsf{T}$ between small symmetric monoidal $\mathcal{V}$-categories S and T such that

$$\tau\hat{} : \mathsf{T}\hat{} \longrightarrow \mathsf{S}\hat{}$$

is a symmetric monoidal $\mathcal{V}$-functor.

Let $\mathcal{C}$ be a symmetric monoidal $\mathcal{V}$-category which is cocomplete, and such that the $\mathcal{V}$-functors $A \otimes B$ preserves colimits in each variable. The $\mathcal{V}$-categories of τ-preoperads and τ-operads in $\mathcal{C}$ are respectively the $\mathcal{V}$-categories $\mathrm{PreOp}_\tau(\mathcal{C}) = [\![\mathsf{S}, \mathcal{C}]\!]$ and $\mathrm{Op}_\tau(\mathcal{C}) = [\![\mathsf{T}, \mathcal{C}]\!]$ of symmetric monoidal $\mathcal{V}$-functors from S and T to $\mathcal{C}$.

The monadicity theorem

We now construct a $\mathcal{V}$-functor τ_*, which generalizes the functor taking a preoperad to the free operad that it generates.

Proposition 2.3. *Let τ be a $\mathcal{V}$-pattern. The $\mathcal{V}$-adjunction*

$$\tau_* : [\mathsf{S}, \mathcal{C}] \rightleftharpoons [\mathsf{T}, \mathcal{C}] : \tau^*$$

induces a $\mathcal{V}$-adjunction between the categories of τ-preoperads and τ-operads

$$\tau_* : \mathrm{PreOp}_\tau(\mathcal{C}) \rightleftharpoons \mathrm{Op}_\tau(\mathcal{C}) : \tau^*.$$

Proof. Let G be a symmetric monoidal $\mathcal{V}$-functor from S to $\mathcal{C}$. The left Kan $\mathcal{V}$-extension $\tau_* G$ is the $\mathcal{V}$-coend

$$\tau_* G(B) = \int^{A \in \mathsf{S}} \mathsf{T}(\tau A, B) \otimes G(A).$$

For each n, there is a natural $\mathcal{V}$-equivalence

$$
\begin{aligned}
\tau_*G(B_1)\otimes\cdots\otimes\tau_*G(B_n) &= \textstyle\bigotimes_{k=1}^n \int^{A_k\in\mathsf{S}}\mathsf{T}(\tau A_k,B_k)\otimes G(A_k)\\
&\cong \textstyle\int^{A_1,\dots,A_n\in\mathsf{S}} \bigotimes_k \mathsf{T}(\tau A_k,B_k)\otimes\bigotimes_{k=1}^n G(A_k)\\
&\qquad\text{since } \otimes \text{ preserves } \mathcal{V}\text{-coends}\\
&\cong \textstyle\int^{A_1,\dots,A_n\in\mathsf{S}} \bigotimes_k \mathsf{T}(\tau A_k,B_k)\otimes G\big(\bigotimes_{k=1}^n A_k\big)\\
&\qquad\text{since } G \text{ is symmetric monoidal}\\
&\cong \textstyle\int^{A_1,\dots,A_n\in\mathsf{S}} \bigotimes_k \mathsf{T}(\tau A_k,B_k)\otimes\int^{A\in\mathsf{S}}\mathsf{S}\big(A,\bigotimes_{k=1}^n A_k\big)\otimes G(A)\\
&\qquad\text{by the Yoneda lemma}\\
&\cong \textstyle\int^{A\in\mathsf{S}}\int^{A_1,\dots,A_n\in\mathsf{S}} \bigotimes_{k=1}^n \mathsf{T}(\tau A_k,B_k)\otimes\mathsf{S}\big(A,\bigotimes_{k=1}^n A_k\big)\otimes G(A)\\
&\qquad\text{by Fubini's theorem for } \mathcal{V}\text{-coends}\\
&\cong \textstyle\int^{A\in\mathsf{S}}\mathsf{T}\big(\tau A,\bigotimes_{k=1}^n B_k\big)\otimes G(A)\\
&\qquad\text{since } \tau\hat{\ } \text{ is symmetric monoidal}\\
&= \tau_*G\big(\bigotimes_{k=1}^n B_k\big).
\end{aligned}
$$

This natural $\mathcal{V}$-equivalence makes τ_*G into a symmetric monoidal $\mathcal{V}$-functor.

The unit and counit of the $\mathcal{V}$-adjunction between τ_* and τ^* on $\mathrm{PreOp}_\tau(\mathcal{C})$ and $\mathrm{Op}_\tau(\mathcal{C})$ are now induced by the unit and counit of the $\mathcal{V}$-adjunction between τ_* and τ^* on $[\mathsf{S},\mathcal{C}]$ and $[\mathsf{T},\mathcal{C}]$. □

For the unenriched version of the following result on reflexive $\mathcal{V}$-coequalizers, see, for example, Johnstone [14], Corollary 1.2.12; the proof in the enriched case is identical.

Proposition 2.4. *Let $\mathcal{C}$ be a symmetric monoidal $\mathcal{V}$-category with reflexive $\mathcal{V}$-coequalizers. If the tensor product $A\otimes -$ preserves reflexive $\mathcal{V}$-coequalizers, then so does the functor $\mathbb{S}\wr(-)$.*

Corollary 2.5. *The $\mathcal{V}$-categories $\mathrm{PreOp}_\tau(\mathcal{C})$ and $\mathrm{Op}_\tau(\mathcal{C})$ have reflexive coequalizers.*

Proof. The $\mathcal{V}$-coequalizer $\mathcal{R}$ in $[\mathsf{S},\mathcal{C}]$ of a reflexive parallel pair $\mathcal{P}\overset{f}{\underset{g}{\rightrightarrows}}\mathcal{Q}$ in $[\![\mathsf{S},\mathcal{C}]\!]$ is computed pointwise: for each $X\in\mathrm{Ob}(\mathsf{S})$,

$$\mathcal{P}(X)\overset{f(X)}{\underset{g(X)}{\rightrightarrows}}\mathcal{Q}(X)\longrightarrow\mathcal{R}(X)$$

is a reflexive $\mathcal{V}$-coequalizer in $\mathcal{C}$. By Proposition 2.4, $\mathcal{R}$ is a symmetric monoidal $\mathcal{V}$-functor; thus, $\mathcal{R}$ is the $\mathcal{V}$-coequalizer of the reflexive pair $\mathcal{P}\overset{f}{\underset{g}{\rightrightarrows}}\mathcal{Q}$ in $[\![\mathsf{S},\mathcal{C}]\!]$. The same argument works for $[\![\mathsf{T},\mathcal{C}]\!]$. □

Proposition 2.6. *If τ is an essentially surjective $\mathcal{V}$-pattern, the $\mathcal{V}$-functor*

$$\tau^* : \mathrm{Op}_\tau(\mathcal{C}) \longrightarrow \mathrm{PreOp}_\tau(\mathcal{C})$$

creates reflexive $\mathcal{V}$-coequalizers.

Proof. The proof of Corollary 2.5 shows that the horizontal $\mathcal{V}$-functors in the diagram

$$\begin{tikzcd} \llbracket \mathsf{T}, \mathcal{C} \rrbracket \arrow[r] \arrow[d, dashed, "\tau^*"'] & {[\mathsf{T}, \mathcal{C}]} \arrow[d, "\tau^*"] \\ \llbracket \mathsf{S}, \mathcal{C} \rrbracket \arrow[r] & {[\mathsf{S}, \mathcal{C}]} \end{tikzcd}$$

create, and hence preserve, reflexive $\mathcal{V}$-coequalizers. The $\mathcal{V}$-functor

$$\tau^* : [\mathsf{T}, \mathcal{C}] \longrightarrow [\mathsf{S}, \mathcal{C}]$$

creates all $\mathcal{V}$-colimits, since $\mathcal{V}$-colimits are computed pointwise and τ is essentially surjective. It follows that the $\mathcal{V}$-functor $\tau^* : \llbracket \mathsf{T}, \mathcal{C} \rrbracket \to \llbracket \mathsf{S}, \mathcal{C} \rrbracket$ creates reflexive $\mathcal{V}$-coequalizers. □

Recall that a $\mathcal{V}$-functor $R : \mathcal{A} \to \mathcal{B}$, with left adjoint $L : \mathcal{B} \to \mathcal{A}$, is $\mathcal{V}$-monadic if there is an equivalence of $\mathcal{V}$-categories $\mathcal{A} \simeq \mathcal{B}^{\mathbb{T}}$, where $\mathbb{T}$ is the $\mathcal{V}$-monad associated to the $\mathcal{V}$-adjunction

$$L : \mathcal{A} \rightleftharpoons \mathcal{B} : R.$$

The following is a variant of Theorem II.2.1 of Dubuc [6]; reflexive $\mathcal{V}$-coequalizers are substituted for contractible $\mathcal{V}$-coequalizers, but otherwise, the proof is the same.

Proposition 2.7. *A $\mathcal{V}$-functor $R : \mathcal{A} \to \mathcal{B}$, with left adjoint $L : \mathcal{B} \to \mathcal{A}$, is $\mathcal{V}$-monadic if $\mathcal{B}$ has, and R creates, reflexive $\mathcal{V}$-coequalizers.*

Corollary 2.8. *If τ is an essentially surjective $\mathcal{V}$-pattern, then the $\mathcal{V}$-functor*

$$\tau^* : \mathrm{Op}_\tau(\mathcal{C}) \longrightarrow \mathrm{PreOp}_\tau(\mathcal{C})$$

is $\mathcal{V}$-monadic.

In practice, the $\mathcal{V}$-patterns of interest all have the following property.

Definition 2.9. A $\mathcal{V}$-pattern $\tau : \mathsf{S} \to \mathsf{T}$ is **regular** if it is essentially surjective and S is equivalent to a free symmetric monoidal $\mathcal{V}$-category.

Denote by s a $\mathcal{V}$-category such that S is equivalent to the free symmetric monoidal $\mathcal{V}$-category $\mathbb{S} \wr \mathsf{s}$. The $\mathcal{V}$-category s may be thought of as a generalized set of colours; the theory associated to a coloured operad is a regular pattern with discrete s.

Theorem 2.10. *If τ is a regular pattern and $\mathcal{C}$ is locally finitely presentable, then $\mathrm{Op}_\tau(\mathcal{C})$ is locally finitely presentable.*

Proof. The $\mathcal{V}$-category $\mathrm{PreOp}_\tau(\mathcal{C})$ is equivalent to the $\mathcal{V}$-category $[\mathsf{s}, \mathcal{C}]$, and hence is locally finitely presentable. By Lemma 3.8, $\mathrm{Op}_\tau(\mathcal{C})$ is cocomplete. The functor τ_* takes finitely presentable objects of $\mathcal{C}$ to finitely presentable objects of $\mathrm{Op}_\tau(\mathcal{C})$, and hence takes finitely presentable strong generators of $\mathrm{PreOp}_\tau(\mathcal{C})$ to finitely presentable strong generators of $\mathrm{Op}_\tau(\mathcal{C})$. □

3 The simplicial patterns Cob and $\mathsf{Cob}^{\rightharpoonup}$

In this section, we construct simplicial patterns $\mathsf{Cob} = \mathsf{Cob}[d]$, associated with gluing of oriented d-dimensional manifolds along components of their boundary.

Definition 3.1. An object S of Cob is a compact d-dimensional manifold, with boundary ∂S and orientation o.

In the above definition, we permit the manifold S to be disconnected. In particular, it may be empty. (Note that an empty manifold of dimension d has a unique orientation.) However, the above definition should be refined in order to produce a set of objects of Cob: one way to do this is to add to the data defining an object of Cob an embedding into a Euclidean space $\mathbb{R}^N$, together with a collared neigbourhood of the boundary ∂S. We call this a decorated object.

We now define a simplicial set $\mathsf{Cob}(S,T)$ of morphisms between objects S and T of Cob.

Definition 3.2. A hypersurface γ in an object S of Cob consists of a closed $(d-1)$-dimensional manifold M together with an embedding $\gamma : M \hookrightarrow S$.

Given a hypersurface γ in S, let $S[\gamma]$ be the manifold obtained by cutting S along the image of γ. The orientation o if S induces an orientation $o[\gamma]$ of $S[\gamma]$.

A hypersurface in a decorated object is an open embedding of the manifold $M \times [-1,1]$ into the complement in S of the collared neighbourhoods of the boundary. This induces a collaring on the boundary of $S[\gamma]$. To embed $S[\gamma]$ into $\mathbb{R}^{N+1}$, we take the product of the embedding of S into $\mathbb{R}^N$ and the function $\chi\circ\pi_2\circ\gamma^{-1}$, where $\pi_2\circ\gamma^{-1}$ is the function on S equal to the coordinate $t \in [-1,1]$ on the image of γ and undefined elsewhere, and $\chi(t) = \mathrm{sgn}(t)\varphi(t)$. Here, $\varphi \in C^\infty_c(-1,1)$ is a non-negative smooth function of compact support equal to 1 in a neighbourhood of $0 \in (-1,1)$.

A k-simplex in the simplicial set $\mathsf{Cob}(S,T)$ of morphisms from S to T consists of the following data:

i) a closed $(d-1)$-manifold M;

ii) an isotopy of hypersurfaces, that is a commutative diagram

$$\begin{tikzcd} \Delta^k \times M \arrow[rr, hook, "\gamma"] \arrow[dr] & & \Delta^k \times T \arrow[dl] \\ & \Delta^k & \end{tikzcd}$$

in which γ is an embedding;

iii) a fibred diffeomorphism

$$\begin{tikzcd} \Delta^k \times S \arrow[rr, "\varphi"] \arrow[dr] & & \Delta^k \times T[\gamma] \arrow[dl] \\ & \Delta^k & \end{tikzcd}$$

compatible with the orientations on its domain and target.

It is straightforward to extend this definition to decorated objects: the only difference is that γ is a fibred open embedding of $\Delta^k \times M \times [-1,1]$ into $\Delta^k \times T$.

We now make Cob into a symmetric monoidal simplicial category. The composition of k-simplices $(M,\gamma,\varphi) \in \mathsf{Cob}(S,T)_k$ and $(N,\delta,\psi) \in \mathsf{Cob}(T,U)_k$ is the k-simplex consisting of the embedding

$$\begin{tikzcd} \Delta^k \times (M \sqcup N) \arrow[rr, hook, "(\psi\circ\gamma)\sqcup\delta"] \arrow[dr] & & \Delta^k \times U \arrow[dl] \\ & \Delta^k & \end{tikzcd}$$

and the fibred diffeomorphism

$$\begin{tikzcd} \Delta^k \times S \arrow[rr, "\psi\circ\varphi"] \arrow[dr] & & \Delta^k \times U[(\psi\circ\gamma)\sqcup\delta] \arrow[dl] \\ & \Delta^k & \end{tikzcd}$$

In the special case that the diffeomorphisms φ and ψ are the identity, this composition is obtained by taking the union of the disjoint hypersurfaces $\Delta^k \times M$ and $\Delta^k \times N$ in $\Delta^k \times U$. It is clear that composition is associative, and compatible with the face and degeneracy maps between simplices.

The identity 0-simplex 1_S in $\mathsf{Cob}(S,S)$ is associated to the empty hypersurface in S and the identity diffeomorphism of S.

The tensor product of Cob is simple to describe: it is disjoint union. When S_i are decorated objects of Cob, $1 \le i \le k$, embedded in $\mathbb{R}^{N_i}$, we embed $S_1 \otimes \ldots \otimes S_k$ in $\mathbb{R}^{\max(N_i)+1}$ by composing the embedding of S_i with the inclusion $\mathbb{R}^{N_i} \hookrightarrow \mathbb{R}^{\max(N_i)+1}$ defined by

$$(t_1,\ldots,t_{N_i}) \mapsto (t_1,\ldots,t_{N_i},0,\ldots,0,i).$$

Just as modular operads have a directed version, modular dioperads, the simplicial pattern $\mathsf{Cob}[d]$ has a directed analogue.

Definition 3.3. An object S of $\mathsf{Cob}^{\rightharpoonup}$ is a compact d-dimensional manifold, with orientations o and ∂o of S and its boundary ∂S.

The boundary ∂S of an object S of $\mathsf{Cob}^{\rightharpoonup}$ is partitioned into incoming and outgoing parts $\partial S = \partial_- S \sqcup \partial_+ S$, according to whether the boundary is positive or negatively oriented by ∂o with respect to the orientation o of S.

Definition 3.4. A hypersurface γ in an object S of $\mathsf{Cob}^{\rightharpoonup}$ consists of a closed $(d-1)$-dimensional oriented manifold M together with an embedding $\gamma : M \hookrightarrow S$.

Given a hypersurface γ in S, let $S[\gamma]$ be the manifold obtained by cutting S along the image of γ. The orientation o if S induces an orientation $o[\gamma]$ of $S[\gamma]$, while the orientations of ∂S and M induce an orientation of its boundary $\partial S[\gamma]$.

A k-simplex in the simplicial set $\mathsf{Cob}^{\rightharpoonup}(S,T)$ of morphisms from S to T consists of the following data:

i) a closed oriented $(d-1)$-manifold M;
ii) an isotopy of hypersurfaces, that is a commutative diagram

$$\begin{tikzcd} \Delta^k \times M \arrow[rr, hook, "\gamma"] \arrow[dr] & & \Delta^k \times T \arrow[dl] \\ & \Delta^k & \end{tikzcd}$$

in which γ is an embedding;
iii) a fibred diffeomorphism

$$\begin{tikzcd} \Delta^k \times S \arrow[rr, "\varphi"] \arrow[dr] & & \Delta^k \times T[\gamma] \arrow[dl] \\ & \Delta^k & \end{tikzcd}$$

compatible with the orientations on its domain and target.

We may make $\mathsf{Cob}^{\rightharpoonup}$ into a symmetric monoidal simplicial category in the same way as for Cob. The definition of decorated objects and decorated morphisms is also easily extended to this setting.

Let cob be the simplicial groupoid of connected oriented surfaces and their oriented diffeomorphisms, with skeleton

$$\bigsqcup_{g,n} \mathrm{Diff}_+(S_{g,n}).$$

The embedding $\mathsf{cob} \hookrightarrow \mathsf{Cob}$ extends to an essentially surjective $\mathcal{V}$-functor $\mathbb{S} \wr \mathsf{cob} \to \mathsf{Cob}$. Similarly, if $\mathsf{cob}^{\rightharpoonup}$ is the simplicial groupoid of connected oriented surfaces with oriented boundary and their oriented diffeomorphisms, the $\mathcal{V}$-functor $\mathbb{S} \wr \mathsf{cob}^{\rightharpoonup} \to \mathsf{Cob}^{\rightharpoonup}$ is essentially surjective.

We can now state the main theorem of this section.

Theorem 3.5. *The simplicial functors* $\mathsf{cob} \hookrightarrow \mathsf{Cob}$ *and* $\mathsf{cob}^{\rightharpoonup} \hookrightarrow \mathsf{Cob}^{\rightharpoonup}$ *induce regular simplicial patterns* $\mathbb{S} \wr \mathsf{cob} \to \mathsf{Cob}$ *and* $\mathbb{S} \wr \mathsf{cob}^{\rightharpoonup} \to \mathsf{Cob}^{\rightharpoonup}$

A d-dimensional modular operad, respectively dioperad, is an operad for the simplicial pattern $\mathbb{S} \wr \mathsf{cob} \to \mathsf{Cob}$, respectively $\mathbb{S} \wr \mathsf{cob}^{\rightharpoonup} \to \mathsf{Cob}^{\rightharpoonup}$.

One-dimensional modular operads

When $d = 1$, the category cob has a skeleton with two objects, the interval I and the circle S. The simplicial group $\mathsf{cob}(I, I) \cong \mathrm{Diff}_+[0, 1]$ is contractible and the simplicial group $\mathsf{cob}(S, S)$ is homotopy equivalent to $\mathrm{SO}(2)$.

The simplicial set $\mathsf{Cob}(I^{\otimes k}, I)$ is empty for $k = 0$ and contractible for each $k > 0$, and the simplicial set $\mathsf{Cob}(I^{\otimes k}, S)$ is empty for $k = 0$ and homotopy equivalent to $\mathrm{SO}(2)$ for each $k > 0$. Thus, a 1-dimensional modular operad in a symmetric monoidal category $\mathcal{C}$ consists of a homotopy associative algebra A, and a homotopy trace from A to a homotopy $\mathrm{SO}(2)$-module M. If $\mathcal{C}$ is discrete, a 1-dimensional modular operad is simply a non-unital associative algebra A in $\mathcal{C}$ together with an object M in $\mathcal{C}$ and a trace $\mathrm{tr} : A \to M$.

One-dimensional modular dioperads

When $d = 1$, the category $\mathsf{cob}^{\rightharpoonup}$ has a skeleton with five objects, the intervals I_-^-, I_-^+, I_+^- and I_+^+, representing the 1-manifold $[0, 1]$ with the four different orientations of its boundary, and the circle S. The simplicial groups $\mathsf{cob}^{\rightharpoonup}(I_a, I_a)$ are contractible, and the simplicial group $\mathsf{cob}^{\rightharpoonup}(S, S)$ is homotopy equivalent to $\mathrm{SO}(2)$.

A 1-dimensional modular dioperad $\mathcal{P}$ in a discrete symmetric monoidal category $\mathcal{C}$ consists of associative algebras $A = \mathcal{P}(I_-^+)$ and $B = \mathcal{P}(I_+^-)$ in $\mathcal{C}$, a (A, B)-bimodule $Q = \mathcal{P}(I_-^-)$, a (B, A)-bimodule $R = \mathcal{P}(I_+^+)$, and an object $M = \mathcal{P}(S)$, together with morphisms

$$\alpha : Q \otimes_B R \longrightarrow A, \quad \beta : R \otimes_A Q \longrightarrow B, \quad \mathrm{tr}_A : A \longrightarrow M, \quad \mathrm{tr}_B : B \longrightarrow M.$$

Denote the left and right actions of A and B on Q and R by $\lambda_Q : A \otimes Q \to Q$, $\rho_Q : Q \otimes B \to Q$, $\lambda_R : B \otimes R \to R$ and $\rho_R : R \otimes A \to R$. The above data must in addition satisfy the following conditions:

- α and β are morphisms of (A, A)-bimodules and (B, B)-bimodules respectively;
- tr_A and tr_B are traces;
- $\lambda_Q \circ (\alpha \otimes Q) = \rho_Q \circ (R \otimes \beta) : Q \otimes_B R \otimes_A Q \to Q$ and $\lambda_R \circ (\beta \otimes R) = \rho_R \circ (Q \otimes \alpha) : R \otimes_A Q \otimes_B R \to R$;
- $\mathrm{tr}_A \circ \alpha : Q \otimes_B R \to M$ and $\mathrm{tr}_B \circ \beta : R \otimes_A Q \to M$ are equal on the isomorphic objects $(Q \otimes_B R) \otimes_{A^\circ \otimes A} \mathbb{1}$ and $(R \otimes_A Q) \otimes_{B^\circ \otimes B} \mathbb{1}$.

That is, a 1-dimensional modular dioperad is the same thing as a **Morita context** (Morita [22]) $(A, B, P, Q, \alpha, \beta)$ with trace $(\mathrm{tr}_A, \mathrm{tr}_B)$.

Two-dimensional topological field theories

Let $\mathcal{G}$ be the discrete pattern introduced in Section 1 whose operads are modular operads. There is a natural morphism of patterns

$$\alpha : \mathsf{Cob}[2] \longrightarrow \mathcal{G};$$

thus, application of α^* to a modular operad gives rise to a 2-dimensional modular operad. But modular operads and what we call 2-dimensional modular operads are quite different: a modular preoperad $\mathcal{P}$ consists of a sequence of $\mathbb{S}_n$-modules $\mathcal{P}(\!(g,n)\!)$, while a 2-dimensional modular preoperad is a cob-module.

An example of a 2-dimensional modular operad is the terminal one, for which $\mathcal{P}(S)$ is the unit $\mathbb{1}$ for each surface S. A more interesting one comes from conformal field theory: the underlying 2-dimensional modular operad associates to an oriented surface with boundary S the moduli space $\mathcal{N}(S)$ of conformal structures on S. (More accurately, $\mathcal{N}(S)$ is the simplicial set whose n-simplices are the n-parameter smooth families of conformal structures parametrized by the n-simplex.) The space $\mathcal{N}(S)$ is contractible for all S, since it is the space of smooth sections of a fibre bundle over S with contractible fibres. To define the structure of a 2-dimensional modular operad on $\mathcal{N}$ amounts to showing that conformal structures may be glued along circles: this is done by choosing a Riemannian metric in the conformal class which is flat in a neighbourhood of the boundary, such that the boundary is geodesic, and each of its components has length 1.

A bilinear form $t : M \otimes M \to \mathbb{1}$ on a cochain complex M is **non-degenerate** if it induces a quasi-isomorphism between M and $\operatorname{Hom}(M, \mathbb{1})$.

Definition 3.6. Let $\mathcal{P}$ be a 2-dimensional modular operad. A $\mathcal{P}$-algebra is a morphism of 2-dimensional modular operads

$$\rho : \mathcal{P} \longrightarrow \alpha^* \,\mathsf{End}(M,t),$$

where M is a cochain complex with non-degenerate bilinear form $t : M \otimes M \to \mathbb{1}$.

A topological conformal field theory is a $C_*(\mathcal{N})$-algebra.

In the theory of infinite loop spaces, one defines an E_∞-algebra as an algebra for an operad $\mathcal{E}$ such that $\mathcal{E}(n)$ is contractible for all n. Similarly, as shown by Fiedorowicz [8], an E_2-algebra is an algebra for a braided operad $\mathcal{E}$ such that $\mathcal{E}(n)$ is contractible for all n. Motivated by this, we make the following definition.

Definition 3.7. A 2-dimensional topological field theory is a pair consisting of a 2-dimensional modular operad $\mathcal{E}$ in the category of cochain complexes such that $\mathcal{E}(S)$ is quasi-isomorphic to $\mathbb{1}$ for all surfaces S, and an $\mathcal{E}$-algebra (M,t).

In particular, a topological conformal field theory is a 2-dimensional topological field theory.

Appendix. Enriched categories

In this appendix, we recall some results of enriched category theory. Let $\mathcal{V}$ be a closed symmetric monoidal category; that is, $\mathcal{V}$ is a symmetric monoidal category such that the tensor product functor $-\otimes Y$ from $\mathcal{V}$ to itself has a right adjoint for all objects Y, denoted $[Y,-]$: in other words,

$$[X\otimes Y,Z]\cong[X,[Y,Z]].$$

Throughout this paper, we assume that $\mathcal{V}$ is complete and cocomplete. Denote by $A\mapsto A_0$ the continuous functor $A_0=\mathcal{V}(\mathbb{1},A)$ from $\mathcal{V}$ to Set, where $\mathbb{1}$ is the unit of $\mathcal{V}$.

Let $\mathcal{V}$-Cat be the 2-category whose objects are $\mathcal{V}$-categories, whose 1-morphisms are $\mathcal{V}$-functors, and whose 2-morphisms are $\mathcal{V}$-natural transformations. Since $\mathcal{V}$ is closed, $\mathcal{V}$ is itself a $\mathcal{V}$-category.

Applying the functor $(-)_0:\mathcal{V}\to\mathsf{Set}$ to a $\mathcal{V}$-category $\mathcal{C}$, we obtain its underlying category $\mathcal{C}_0$; in this way, we obtain a 2-functor

$$(-)_0:\mathcal{V}\text{-}\mathsf{Cat}\longrightarrow\mathsf{Cat}.$$

Given a small $\mathcal{V}$-category $\mathcal{A}$, and a $\mathcal{V}$-category $\mathcal{B}$, there is a $\mathcal{V}$-category $[\mathcal{A},\mathcal{B}]$, whose objects are the $\mathcal{V}$-functors from $\mathcal{A}$ to $\mathcal{B}$, and such that

$$[\mathcal{A},\mathcal{B}](F,G)=\int_{A\in\mathcal{A}}\mathcal{B}(FA,GA).$$

There is an equivalence of categories $\mathcal{V}\text{-}\mathsf{Cat}(\mathcal{A},\mathcal{B})\simeq[\mathcal{A},\mathcal{B}]_0$.

The Yoneda embedding for $\mathcal{V}$-categories

The opposite of a $\mathcal{V}$-category $\mathcal{C}$ is the $\mathcal{V}$-category $\mathcal{C}^\circ$ with

$$\mathcal{C}^\circ(A,B)=\mathcal{C}(B,A).$$

If $\mathcal{A}$ is a small $\mathcal{V}$-category, denote by $\mathcal{A}\hat{}$ the $\mathcal{V}$-category of presheaves

$$\mathcal{A}\hat{}=[\mathcal{A}^\circ,\mathcal{V}].$$

By the $\mathcal{V}$-Yoneda lemma, there is a full, faithful $\mathcal{V}$-functor $y:\mathcal{A}\to\mathcal{A}\hat{}$, with

$$y(A)=\mathcal{A}(-,A),\quad A\in\mathrm{Ob}(\mathcal{A}).$$

If $F:\mathcal{A}\to\mathcal{B}$ is a $\mathcal{V}$-functor, denote by $F\hat{}:\mathcal{B}\hat{}\to\mathcal{A}\hat{}$ the $\mathcal{V}$-functor induced by F.

Cocomplete $\mathcal{V}$-categories

A $\mathcal{V}$-category $\mathcal{C}$ is **tensored** if there is a $\mathcal{V}$-functor $\otimes : \mathcal{V} \otimes \mathcal{C} \to \mathcal{C}$ together with a $\mathcal{V}$-natural equivalence of functors $\mathcal{C}(X \otimes A, B) \cong [X, \mathcal{C}(A, B)]$ from $\mathcal{V}^\circ \otimes \mathcal{C}^\circ \otimes \mathcal{C}$ to $\mathcal{V}$. For example, the $\mathcal{V}$-category $\mathcal{V}$ is itself tensored.

Let $\mathcal{A}$ be a small $\mathcal{V}$-category, let F be a $\mathcal{V}$-functor from $\mathcal{A}^\circ$ to $\mathcal{V}$, and let G be a $\mathcal{V}$-functor from $\mathcal{A}$ to a $\mathcal{V}$-category $\mathcal{C}$. If B is an object of $\mathcal{C}$, denote by $\langle G, B\rangle : \mathcal{C}^\circ \to \mathcal{V}$ the presheaf such that $\langle G, C\rangle(A) = [GA, C]$. The **weighted colimit** of G, with weight F, is an object $\int^{A\in\mathcal{A}} FA \otimes GA$ of $\mathcal{C}$ such that there is a natural isomorphism

$$\mathcal{C}\hat{}(F, \langle G, -\rangle) \cong \mathcal{C}\big(\textstyle\int^{A\in\mathcal{A}} FA \otimes GA, -\big).$$

If $\mathcal{C}$ is tensored, $\int^{A\in\mathcal{A}} F(A) \otimes G(A)$ is the coequalizer of the diagram

$$\coprod_{A_0, A_1 \in \mathrm{Ob}(\mathcal{A})} \mathcal{A}(A_0, A_1) \otimes FA_1 \otimes GA_0 \rightrightarrows \coprod_{A \in \mathrm{Ob}(\mathcal{A})} FA \otimes GA,$$

where the two morphisms are induced by the action on F and coaction on G respectively.

A $\mathcal{V}$-category $\mathcal{C}$ is **cocomplete** if it has all weighted colimits, or equivalently, if it satisfies the following conditions:

i) the category $\mathcal{C}_0$ is cocomplete;
ii) for each object A of $\mathcal{C}$, the functor $\mathcal{C}(-, A) : \mathcal{C}_0 \to \mathcal{V}$ transforms colimits into limits;
iii) $\mathcal{C}$ is tensored.

In particular, $\mathcal{V}$-categories $\mathcal{A}\hat{}$ of presheaves are cocomplete.

Let $\mathcal{C}$ be a cocomplete $\mathcal{V}$-category. A $\mathcal{V}$-functor $F : \mathcal{A} \to \mathcal{B}$ between small $\mathcal{V}$-categories gives rise to a $\mathcal{V}$-adjunction

$$F_* : [\mathcal{A}, \mathcal{C}] \rightleftharpoons [\mathcal{B}, \mathcal{C}] : F^*,$$

that is, an adjunction in the 2-category $\mathcal{V}$-Cat. The functor F_* is called the (pointwise) left $\mathcal{V}$-Kan extension of along F; it is the $\mathcal{V}$-coend

$$F_*G(-) = \int^{A\in\mathcal{A}} \mathcal{A}(FA, -) \otimes GA.$$

Cocomplete categories of $\mathcal{V}$-algebras

A $\mathcal{V}$-monad $\mathbb{T}$ on a $\mathcal{V}$-category $\mathcal{C}$ is a monad in the full sub-2-category of $\mathcal{V}$-Cat with unique object $\mathcal{C}$. If $\mathcal{C}$ is small, this is the same thing as a monoid in the monoidal category $\mathcal{V}$-$\mathsf{Cat}(\mathcal{C}, \mathcal{C})$.

The following is the enriched version of a result of Linton [16], and is proved in exactly the same way.

Lemma 3.8. *Let $\mathbb{T}$ be a $\mathcal{V}$-monad on a cocomplete $\mathcal{V}$-category $\mathcal{C}$ such that the $\mathcal{V}$-category of algebras $\mathcal{C}^{\mathbb{T}}$ has reflexive $\mathcal{V}$-coequalizers. Then the $\mathcal{V}$-category of $\mathbb{T}$-algebras $\mathcal{C}^{\mathbb{T}}$ is cocomplete.*

Proof. We must show that $\mathcal{C}^{\mathbb{T}}$ has all weighted colimits. Let $\mathcal{A}$ be a small $\mathcal{V}$-category, let F be a weight, and let $G : \mathcal{A} \to \mathcal{C}^{\mathbb{T}}$ be a diagram of $\mathbb{T}$-algebras. Then the weighted colimit $\int^{A\in\mathcal{A}} F(A) \otimes G(A)$ is a reflexive coequalizer

$$\mathbb{T}\big(\textstyle\int^{A\in\mathcal{A}} FA \otimes \mathbb{T}RGA\big) \rightleftarrows \mathbb{T}\big(\textstyle\int^{A\in\mathcal{A}} FA \otimes RGA\big). \qquad \square$$

Locally finitely presentable $\mathcal{V}$-categories

In studying algebraic theories using categories, locally finite presentable categories plays a basic role. When the closed symmetric monoidal category $\mathcal{V}$ is locally finitely presentable, with finitely presentable unit $\mathbb{1}$, these have a generalization to enriched category theory over $\mathcal{V}$, due to Kelly [15]. (There is a more general theory of locally presentable categories, where the cardinal $\aleph_0$ is replaced by an arbitrary regular cardinal; this extension is straightforward, but we do not present it here in order to simplify exposition.)

Examples of locally finitely presentable closed symmetric monoidal categories include the categories of sets, groupoids, categories, simplicial sets, and abelian groups — the finitely presentable objects are respectively finite sets, groupoids and categories, simplicial sets with a finite number of nondegenerate simplices, and finitely presentable abelian groups. A less obvious example is the category of symmetric spectra of Hovey et al. [12].

An object A in a $\mathcal{V}$-category $\mathcal{C}$ is **finitely presentable** if the functor

$$\mathcal{C}(A,-) : \mathcal{C} \longrightarrow \mathcal{V}$$

preserves filtered colimits. A **strong generator** in a $\mathcal{V}$-category $\mathcal{C}$ is a set $\{G_i \in \mathrm{Ob}(\mathcal{C})\}_{i\in I}$ such that the functor

$$A \mapsto \bigotimes_{i\in I} \mathcal{C}(G_i, A) : \mathcal{C} \mapsto \mathcal{V}^{\otimes I}$$

reflects isomorphisms.

Definition 3.9. A **locally finitely presentable** $\mathcal{V}$-category $\mathcal{C}$ is a cocomplete $\mathcal{V}$-category with a finitely presentable strong generator (i.e. the objects making up the strong generator are finitely presentable).

Pseudo algebras

A 2-monad $\mathbb{T}$ on a 2-category $\mathbb{C}$ is by definition a Cat-monad on $\mathbb{C}$. Denote the composition of the 2-monad $\mathbb{T}$ by $m : \mathbb{T}\mathbb{T} \to \mathbb{T}$, and the unit by $\eta : 1 \to \mathbb{T}$.

Associated to a 2-monad $\mathbb{T}$ is the 2-category $\mathbb{C}^{\mathbb{T}}$ of pseudo $\mathbb{T}$-algebras (Bunge [3] and Street [23]; see also Marmolejo [19]). A pseudo $\mathbb{T}$-algebra is an object $\mathcal{A}$ of $\mathbb{C}$, together with a morphism $a : \mathbb{T}\mathcal{A} \to \mathcal{A}$, the composition, and invertible 2-morphisms

$$\begin{tikzcd}
 & \mathbb{T}\mathcal{A} \arrow[dr, "m_{\mathcal{A}}"] \arrow[dd, phantom, "\Downarrow\theta"] & \\
\mathbb{T}\mathbb{T}\mathcal{A} \arrow[ur, "\mathbb{T}a"] \arrow[dr, "m_{\mathcal{A}}"'] & & \mathcal{A} \\
 & \mathbb{T}\mathcal{A} \arrow[ur, "a"'] &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
 & \mathbb{T}\mathcal{A} \arrow[dr, "a"] \arrow[d, phantom, "\Downarrow\iota"] & \\
\mathcal{A} \arrow[ur, "\eta_{\mathcal{A}}"] \arrow[rr, equal] & {} & \mathcal{A}
\end{tikzcd}$$

such that

$$\begin{tikzcd}
 & \mathbb{T}\mathbb{T}\mathcal{A} \arrow[rr, "\mathbb{T}a"] \arrow[dr, "\mathbb{T}m_{\mathcal{A}}"] & & \mathbb{T}\mathcal{A} \arrow[dr, "a"] \arrow[d, phantom, "\Downarrow\theta"] & \\
\mathbb{T}\mathbb{T}\mathbb{T}\mathcal{A} \arrow[ur, "\mathbb{T}\mathbb{T}a"] \arrow[dr, "m_{\mathbb{T}\mathcal{A}}"'] & & \mathbb{T}\mathcal{A} \arrow[rr, "a"] & {} \arrow[d, phantom, "\Downarrow\theta"] & \mathcal{A} \\
 & \mathbb{T}\mathbb{T}\mathcal{A} \arrow[ur, "\mathbb{T}a"] \arrow[rr, "m_{\mathcal{A}}"'] & & \mathbb{T}\mathcal{A} \arrow[ur, "a"'] &
\end{tikzcd}
\quad \text{equals} \quad
\begin{tikzcd}
 & \mathbb{T}\mathbb{T}\mathcal{A} \arrow[rr, "\mathbb{T}a"] \arrow[d, phantom, "\Downarrow\mathbb{T}\theta"] & & \mathbb{T}\mathcal{A} \arrow[dr, "a"] & \\
\mathbb{T}\mathbb{T}\mathbb{T}\mathcal{A} \arrow[ur, "\mathbb{T}\mathbb{T}a"] \arrow[r, "\mathbb{T}m_{\mathcal{A}}"'] \arrow[dr, "m_{\mathbb{T}\mathcal{A}}"'] & {} & \mathbb{T}\mathbb{T}\mathcal{A} \arrow[ur, "m_{\mathcal{A}}"] \arrow[dr, "\mathbb{T}a"'] & \Downarrow\theta & \mathcal{A} \\
 & \mathbb{T}\mathbb{T}\mathcal{A} \arrow[rr, "m_{\mathcal{A}}"'] & & \mathbb{T}\mathcal{A} \arrow[ur, "a"'] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & \mathbb{T}\mathbb{T}\mathcal{A} \arrow[dr, "\mathbb{T}a"] \arrow[d, phantom, "\Downarrow\mathbb{T}\iota"] & & \\
\mathbb{T}\mathcal{A} \arrow[ur, "\mathbb{T}\eta_{\mathcal{A}}"] \arrow[rr, equal] \arrow[dr, "\mathbb{T}\eta_{\mathcal{A}}"'] & {} & \mathbb{T}\mathcal{A} \arrow[r, "a"] & \mathcal{A} \\
 & \mathbb{T}\mathbb{T}\mathcal{A} \arrow[ur, "m_{\mathcal{A}}"'] & &
\end{tikzcd}$$

equals

$$\begin{tikzcd}
 & & \mathbb{T}\mathcal{A} \arrow[dr, "m_{\mathcal{A}}"] \arrow[dd, phantom, "\Downarrow\theta"] & \\
\mathbb{T}\mathcal{A} \arrow[r, "\mathbb{T}\eta_{\mathcal{A}}"] & \mathbb{T}\mathbb{T}\mathcal{A} \arrow[ur, "\mathbb{T}a"] \arrow[dr, "m_{\mathcal{A}}"'] & & \mathcal{A} \\
 & & \mathbb{T}\mathcal{A} \arrow[ur, "a"'] &
\end{tikzcd}$$

The lax morphisms of $\mathbb{C}^{\mathbb{T}}$ are pairs (f, φ) consisting of a morphism $f : \mathcal{A} \to \mathcal{B}$ and a 2-morphism

$$\begin{tikzcd}
 & \mathbb{T}\mathcal{B} \arrow[dr, "b"] \arrow[dd, phantom, "\Downarrow\varphi"] & \\
\mathbb{T}\mathcal{A} \arrow[ur, "\mathbb{T}f"] \arrow[dr, "a"'] & & \mathcal{B} \\
 & \mathcal{A} \arrow[ur, "f"'] &
\end{tikzcd}$$

such that

$$
\begin{tikzcd}
& \mathbb{T}\mathbb{T}\mathcal{B} \arrow[rr, "\mathbb{T}b"] \arrow[dr, "m_{\mathcal{B}}"'] & & \mathbb{T}\mathcal{B} \arrow[dr, "b"] \arrow[dl, phantom, "\Downarrow\theta"] & \\
\mathbb{T}\mathbb{T}\mathcal{A} \arrow[ur, "\mathbb{T}\mathbb{T}f"] \arrow[dr, "m_{\mathcal{A}}"'] & & \mathbb{T}\mathcal{B} \arrow[rr, "b"] \arrow[dr, phantom, "\Downarrow\varphi"] & & \mathcal{B} \\
& \mathbb{T}\mathcal{A} \arrow[rr, "a"'] \arrow[ur, "\mathbb{T}f"] & & \mathcal{A} \arrow[ur, "f"'] &
\end{tikzcd}
$$

equals

$$
\begin{tikzcd}
& \mathbb{T}\mathbb{T}\mathcal{B} \arrow[rr, "\mathbb{T}b"] \arrow[dr, phantom, "\Downarrow\mathbb{T}\varphi"] & & \mathbb{T}\mathcal{B} \arrow[dr, "b"] & \\
\mathbb{T}\mathbb{T}\mathcal{A} \arrow[ur, "\mathbb{T}f"] \arrow[r, "\mathbb{T}a"] \arrow[dr, "m_{\mathcal{A}}"'] & {} & \mathbb{T}\mathcal{A} \arrow[ur, "\mathbb{T}f"] \arrow[dr, "a"] \arrow[rr, phantom, "\Downarrow\theta"] \arrow[dl, phantom, "\Downarrow\theta"] & & \mathcal{B} \\
& \mathbb{T}\mathcal{A} \arrow[rr, "a"'] & & \mathcal{A} \arrow[ur, "f"'] &
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
& \mathbb{T}\mathcal{A} \arrow[dr, "\mathbb{T}f"] & & \\
\mathcal{A} \arrow[ur, "\eta_{\mathcal{A}}"] \arrow[dr, "f"'] & & \mathbb{T}\mathcal{B} \arrow[dr, "b"] \arrow[d, phantom, "\Downarrow\iota"] & \\
& \mathcal{B} \arrow[ur, "\eta_{\mathcal{B}}"] \arrow[rr, equal] & {} & \mathcal{B}
\end{tikzcd}
$$

equals

$$
\begin{tikzcd}
& & \mathbb{T}\mathcal{B} \arrow[dr, "b"] \arrow[dd, phantom, "\Downarrow\varphi"] & \\
& \mathbb{T}\mathcal{A} \arrow[ur, "\mathbb{T}f"] \arrow[dr, "a"] \arrow[d, phantom, "\Downarrow\iota"] & & \mathcal{B} \\
\mathcal{A} \arrow[ur, "\eta_{\mathcal{A}}"] \arrow[rr, equal] & {} & \mathcal{A} \arrow[ur, "f"'] &
\end{tikzcd}
$$

A lax morphism (f, φ) is a morphism if the 2-morphism φ is invertible.

The 2-morphisms $\gamma : (f, \varphi) \to (\tilde{f}, \tilde{\varphi})$ of $\mathbb{C}^{\mathbb{T}}$ are 2-morphisms $\gamma : f \to \tilde{f}$ such that

$$
\begin{tikzcd}
& \mathbb{T}\mathcal{B} \arrow[dr, "b"] \arrow[dd, phantom, "\Downarrow\varphi"] & \\
\mathbb{T}\mathcal{A} \arrow[ur, "\mathbb{T}f"] \arrow[dr, "a"'] & & \mathcal{B} \\
& \mathcal{A} \arrow[ur, bend left, "f"] \arrow[ur, bend right, "\tilde{f}"'] \arrow[ur, phantom, "\Downarrow\gamma"] &
\end{tikzcd}
\quad \text{equals} \quad
\begin{tikzcd}
& \mathbb{T}\mathcal{B} \arrow[dr, "a"] \arrow[dd, phantom, "\Downarrow\tilde{\varphi}"] & \\
\mathbb{T}\mathcal{A} \arrow[ur, bend left, "\mathbb{T}f"] \arrow[ur, bend right, "\mathbb{T}\tilde{f}"'] \arrow[ur, phantom, "\Downarrow\mathbb{T}\gamma"] \arrow[dr, "a"'] & & \mathcal{B} \\
& \mathcal{A} \arrow[ur, "\tilde{f}"'] &
\end{tikzcd}
\tag{2}
$$

$\mathcal{V}$-categories of pseudo algebras

If $\mathcal{A}$ and $\mathcal{B}$ are pseudo $\mathbb{T}$-algebras, and the underlying category of $\mathcal{A}$ is small, we saw that there is a $\mathcal{V}$-category $[\mathcal{A}, \mathcal{B}]$ whose objects are $\mathcal{V}$-functors $f : \mathcal{A} \to \mathcal{B}$ between the underlying $\mathcal{V}$-categories. Using $[\mathcal{A}, \mathcal{B}]$, we now define a $\mathcal{V}$-category $[\![\mathcal{A}, \mathcal{B}]\!]$ whose objects are morphisms $f : \mathcal{A} \to \mathcal{B}$ of pseudo $\mathbb{T}$-algebras. If f_0

and f_1 are morphisms of pseudo $\mathbb{T}$-algebras, $[\![\mathcal{A},\mathcal{B}]\!](f_0,f_1)$ is defined as the equalizer

$$[\![\mathcal{A},\mathcal{B}]\!](f_0,f_1) \xrightarrow{\psi_{f_0f_1}} [\mathcal{A},\mathcal{B}](f_0,f_1) \underset{(-\circ a)\varphi_0}{\overset{\varphi_1(b\circ\mathbb{T}-)}{\rightrightarrows}} [\mathbb{T}\mathcal{A},\mathcal{B}](b\circ\mathbb{T}f_0, f_1\circ a)$$

Of course, this is the internal version of (2): the 2-morphisms $\gamma : f_0 \to f_1$ of $\mathbb{C}^{\mathbb{T}}$ are the elements of the set $|[\![\mathcal{A},\mathcal{B}]\!](f_0,f_1)|$.

The composition morphism

$$[\![\mathcal{A},\mathcal{B}]\!](f_0,f_1)\otimes[\![\mathcal{A},\mathcal{B}]\!](f_1,f_2) \xrightarrow{m^{[\![\mathcal{A},\mathcal{B}]\!]}_{f_0f_1f_2}} [\![\mathcal{A},\mathcal{B}]\!](f_0,f_2)$$

is the universal arrow for the coequalizer $[\![\mathcal{A},\mathcal{B}]\!](f_0,f_2)$, whose existence is guaranteed by the commutativity of the diagram

$$\begin{aligned}&[\![\mathcal{A},\mathcal{B}]\!](f_0,f_1)\otimes[\![\mathcal{A},\mathcal{B}]\!](f_1,f_2) \xrightarrow{\psi_{f_0f_1}\otimes\psi_{f_1f_2}} [\mathcal{A},\mathcal{B}](f_0,f_1)\otimes[\mathcal{A},\mathcal{B}](f_1,f_2)\\ &\qquad\xrightarrow{m^{[\mathcal{A},\mathcal{B}]}_{f_0f_1f_1}} [\mathcal{A},\mathcal{B}](f_0,f_2) \underset{(-\circ a)\varphi_0}{\overset{\varphi_2(b\circ\mathbb{T}-)}{\rightrightarrows}} [\mathbb{T}\mathcal{A},\mathcal{B}](b\circ\mathbb{T}f_0, f_2\circ c)\end{aligned}$$

Indeed, we have

$$\begin{aligned}&\varphi_2(b\circ\mathbb{T}-)\cdot m^{[\mathcal{A},\mathcal{B}]}_{f_0f_1f_1}\cdot(\psi_{f_0f_1}\otimes\psi_{f_1f_2})\\ &\qquad= m^{[\mathbb{T}\mathcal{A},\mathcal{B}]}_{b\circ\mathbb{T}f_0,b\circ\mathbb{T}f_1,f_2\circ a}\cdot((b\circ\mathbb{T}-)\cdot\psi_{f_0f_1}\otimes\varphi_2(b\circ\mathbb{T}-)\cdot\psi_{f_1f_2})\\ &\qquad= m^{[\mathbb{T}\mathcal{A},\mathcal{B}]}_{b\circ\mathbb{T}f_0,b\circ\mathbb{T}f_1,f_2\circ a}\cdot((b\circ\mathbb{T}-)\cdot\psi_{f_0f_1}\otimes(-\circ a)\varphi_1\cdot\psi_{f_1f_2})\\ &\qquad= m^{[\mathbb{T}\mathcal{A},\mathcal{B}]}_{b\circ\mathbb{T}f_0,f_1\circ a,f_2\circ a}\cdot(\varphi_1(b\circ\mathbb{T}-)\cdot\psi_{f_0f_1}\otimes(-\circ a)\cdot\psi_{f_1f_2})\\ &\qquad= m^{[\mathbb{T}\mathcal{A},\mathcal{B}]}_{b\circ\mathbb{T}f_0,f_1\circ a,f_2\circ a}\cdot((-\circ a)\varphi_0\cdot\psi_{f_0f_1}\otimes(-\circ a)\cdot\psi_{f_1f_2})\\ &\qquad= (-\circ a)\varphi_0\cdot m^{[\mathcal{A},\mathcal{B}]}_{f_0f_1f_1}\cdot(\psi_{f_0f_1}\otimes\psi_{f_1f_2}).\end{aligned}$$

Associativity of this composition is proved by a calculation along the same lines for the iterated composition map

$$[\![\mathcal{A},\mathcal{B}]\!](f_0,f_1)\otimes[\![\mathcal{A},\mathcal{B}]\!](f_1,f_2)\otimes[\![\mathcal{A},\mathcal{B}]\!](f_2,f_3)\longrightarrow[\![\mathcal{A},\mathcal{B}]\!](f_0,f_3).$$

The unit $v_f : \mathbb{1} \to [\![\mathcal{A},\mathcal{B}]\!](f,f)$ of the $\mathcal{V}$-category $[\![\mathcal{A},\mathcal{B}]\!]$ is the universal arrow for the coequalizer $[\![\mathcal{A},\mathcal{B}]\!](f,f)$, whose existence is guaranteed by the commutativity of the diagram

$$\mathbb{1} \xrightarrow{u_f} [\mathcal{A},\mathcal{B}](f,f) \underset{(-\circ a)\varphi}{\overset{\varphi(b\circ\mathbb{T}-)}{\rightrightarrows}} [\mathbb{T}\mathcal{A},\mathcal{B}](b\circ\mathbb{T}f, f\circ a)$$

Indeed, both $\varphi(b\circ\mathbb{T}-)\cdot u_f$ and $(-\circ a)\varphi\cdot u_f$ are equal to

$$\varphi \in \mathcal{V}(\mathbb{1}, [\mathbb{T}\mathcal{A}, \mathcal{B}](b \circ \mathbb{T}f, f \circ a)).$$

In a remark at the end of Section 3 of [13], Im and Kelly discount the possible existence of the $\mathcal{V}$-categories of pseudo algebras. The construction presented here appears to get around their objections.

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