A non-Archimedean interpretation of the weight zero subspaces of limit mixed Hodge structures

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To Yuri Ivanovich Manin on the occasion of his 70th birthday

Introduction

Let $\mathcal{X}$ be a proper scheme over the field F of functions meromorphic in an open neighborhood of zero in the complex plane $\mathbf{C}$. The scheme $\mathcal{X}$ gives rise to a proper morphism of complex analytic spaces $\mathcal{X}^h \to D^* = D \backslash \{0\}$, where D is an open disc with center at zero (see §3). It is well known that, after shrinking the disc D (and replacing $\mathcal{X}^h$ by its preimage), the cohomology groups $H^i(\mathcal{X}^h_t, \mathbf{Z})$ of the fiber $\mathcal{X}^h_t$ at a point $t \in D^*$ form a local system of finitely generated abelian groups, and that the corresponding action of the fundamental group $\pi_1(D^*) = \pi_1(D^*, t)$ on $H^i(\mathcal{X}^h_t, \mathbf{Z})$ is quasi-unipotent. Furthermore, the mixed Hodge structures on the above groups define a variation of mixed Hodge structures on D^*. Let $\overline{D}^* \to D^*$ be a universal covering of D^*, and $\overline{\mathcal{X}}^h = \mathcal{X}^h \times_{D^*} \overline{D}^*$. Then the cohomology group $H^i(\overline{\mathcal{X}}^h, \mathbf{Z})$ admits a mixed Hodge structure, which is the limit (in a certain sense) of the above variation of mixed Hodge structures on D^* (see [GNPP, Exp. IV, Theorem 7.4]). One of the purposes of this paper is to describe the weight zero subspace $W_0 H^i(\overline{\mathcal{X}}^h, \mathbf{Q})$ in terms of non-Archimedean analytic geometry.

Let K be the completion of the discrete valuation field F, and fix a corresponding multiplicative valuation on it. The scheme $\mathcal{X}$ gives rise to a proper K-analytic space $\mathcal{X}^{\mathrm{an}} = (\mathcal{X} \otimes_F K)^{\mathrm{an}}$ in the sense of [Ber1] and [Ber2]. Recall that, as a topological space, $\mathcal{X}^{\mathrm{an}}$ is compact and locally arcwise connected, and the topological dimension of $\mathcal{X}^{\mathrm{an}}$ is equal to the dimension of $\mathcal{X}$. If $\mathcal{X}$ is smooth, then $\mathcal{X}^{\mathrm{an}}$ is even locally contractible. Furthermore, let $\overline{\mathcal{X}}^{\mathrm{an}} = (\mathcal{X} \otimes_F \widehat{K}^{\mathrm{a}})^{\mathrm{an}}$, where $\widehat{K}^{\mathrm{a}}$ is the completion of the algebraic closure K^{a} of K, which corresponds to the universal covering $\overline{D}^* \to D^*$. Recall that the cohomology groups $H^i(\overline{\mathcal{X}}^{\mathrm{an}}, \mathbf{Z})$ of the underlying topological space of $\overline{\mathcal{X}}^{\mathrm{an}}$ are finitely generated, and there is a finite extension K'' of K in K^{a} such that they coincide with $H^i((\mathcal{X} \otimes_F K')^{\mathrm{an}}, \mathbf{Z})$ for any finite extension K' of K'' in K^{a} (see [Ber5, 10.1]).

In §3, we construct a topological space $\mathcal{X}^{\mathrm{An}}$ and a surjective continuous map $\lambda : \mathcal{X}^{\mathrm{An}} \to [0,1]$ for which there are an open embedding $\lambda^{-1}(]0,1]) \hookrightarrow \mathcal{X}^h \times]0,1]$, which is a homotopy equivalence, and a homeomorphism $\lambda^{-1}(0) \xrightarrow{\sim} \mathcal{X}^{\mathrm{an}} \times]0,r[$, where r is the radius of the disc D. We show that the induced maps $H^i(\mathcal{X}^{\mathrm{An}}, \mathbf{Z}) \to H^i(\mathcal{X}^{\mathrm{an}} \times]0,r[, \mathbf{Z}) = H^i(\mathcal{X}^{\mathrm{an}}, \mathbf{Z})$ are isomorphisms for all $i \geq 0$. In this way, we get a homomorphism $H^i(\mathcal{X}^{\mathrm{an}}, \mathbf{Z}) \to H^i(\mathcal{X}^h, \mathbf{Z})$, whose composition with the canonical map $H^i(\mathcal{X}^h, \mathbf{Z}) \to H^i(\overline{\mathcal{X}}^h, \mathbf{Z})$ gives a homomorphism $H^i(\mathcal{X}^{\mathrm{an}}, \mathbf{Z}) \to H^i(\overline{\mathcal{X}}^h, \mathbf{Z})$. The same construction applied to finite extensions of F in F^{a} gives rise to a homomorphism of $\pi_1(D^*)$-modules $H^i(\overline{\mathcal{X}}^{\mathrm{an}}, \mathbf{Z}) \to H^i(\overline{\mathcal{X}}^h, \mathbf{Z})$. Theorem 5.1 states that the latter gives rise to a functorial isomorphism $\pi_1(D^*)$-modules

$$H^i(\overline{\mathcal{X}}^{\mathrm{an}}, \mathbf{Q}) \xrightarrow{\sim} W_0 H^i(\overline{\mathcal{X}}^h, \mathbf{Q}) \ .$$

If $\mathcal{X}$ is projective and smooth, one can describe the group $W_0 H^i(\overline{\mathcal{X}}^h, \mathbf{Q})$ as follows. By the local monodromy theorem, the action of $(T^m - 1)^{i+1}$ on $H^i(\mathcal{X}_t^h, \mathbf{Z})$ is zero (for some $m \geq 1$), where T is the canonical generator of $\pi_1(D^*)$. If we fix a point of $\overline{D}^*$ over t, there is an induced isomorphism $H^i(\overline{\mathcal{X}}^h, \mathbf{Z}) \xrightarrow{\sim} H^i(\mathcal{X}_t^h, \mathbf{Z})$ which gives rise to an isomorphism between $W_0 H^i(\overline{\mathcal{X}}^h, \mathbf{Q})$ and the maximal unipotent monodromy subspace of $H^i(\mathcal{X}_t^h, \mathbf{Q})$, i.e., $(T^m - 1)^i H^i(\mathcal{X}_t^h, \mathbf{Q})$. Thus, in the case considered, there is a functorial isomorphism of $\pi_1(D^*)$-modules

$$H^i(\overline{\mathcal{X}}^{\mathrm{an}}, \mathbf{Q}) \xrightarrow{\sim} (T^m - 1)^i H^i(\mathcal{X}_t^h, \mathbf{Q}) \ .$$

The mixed Hodge theory (see [St, p. 247], [Ill, p. 29]) provides an upper bound on the dimension of the space on the right hand side, which implies the following bound on that of the left hand side

$$\dim_{\mathbf{Q}} H^i(\overline{\mathcal{X}}^{\mathrm{an}}, \mathbf{Q}) \leq \dim_F H^i(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \ .$$

The equality is achieved for a totally degenerate family of abelian varieties (see [Ber1, §6]), and for a totally degenerate family of Calabi-Yau varieties (in the strong sense). In the latter example, $\mathcal{X}^{\mathrm{an}}$ has rational cohomology of the sphere of dimension $\dim(\mathcal{X})$, and is simply connected (see Remark 4.4(ii)).

In fact $\mathcal{X}^{\mathrm{An}}$ is the underlying topological space of an analytic space over a commutative Banach ring. The idea of such an object was introduced in [Ber1, §1.5], and developed there in detail in the case when the Banach ring is a non-Archimedean field. The spaces considered in this paper are defined over the field of complex numbers $\mathbf{C}$ provided with the following Banach norm: $||a|| = \max\{|a|_\infty, |a|_0\}$ for $a \in \mathbf{C}$, where $|\ |_\infty$ is the usual Archimedean valuation, and $|\ |_0$ is the trivial valuation (i.e., $|a|_0 = 1$ for $a \neq 0$). One has $[0,1] \xrightarrow{\sim} \mathcal{M}(\mathbf{C}, \| \ \|)$. Namely, a nonzero point $\rho \in]0,1]$ corresponds to the Archimedean valuation $| \ |_\infty^\rho$, and the zero 0 corresponds to the trivial valuation $| \ |_0$. The above map λ is a canonical map $\mathcal{X}^{\mathrm{An}} \to \mathcal{M}(\mathbf{C}, \| \ \|) = [0,1]$. The preimage $\lambda^{-1}(\rho)$ of $\rho \in]0,1]$ is the restriction of the complex analytic space $\mathcal{X}^h$ to the smaller open disc $D(r^{\frac{1}{\rho}})$, and $\lambda^{-1}(0)$ is a non-Archimedean analytic space over the field $\mathbf{C}$ provided with the trivial valuation $| \ |_0$. Thus, the space $\mathcal{X}^{\mathrm{An}}$ incorporates both complex analytic and non-Archimedean analytic spaces, and the result on a non-Archimedean interpretation of the weight zero subspaces is evidence that analytic spaces over $(\mathbf{C}, \| \ \|)$ are worth studying.

In §1, we recall a construction from [Ber1, §1] that associates with an algebraic variety over a commutative Banach ring k the underlying topological space of a k-analytic space. We do not develop a theory of k-analytic spaces, but restrict ourselves with establishing basic properties necessary for this paper. In §2, we specify our study for the field $\mathbf{C}$ provided with the above Banach norm $\| \ \|$, and prove a particular case of the main result from §4. Let $\mathcal{O}_{\mathbf{C},0}$ be the local ring of functions analytic in an open neighborhood of zero in $\mathbf{C}$. In §3, we associate with an algebraic variety $\mathcal{X}$ over $\mathcal{O}_{\mathbf{C},0}$ analytic spaces of three types: a complex analytic space $\mathcal{X}^h$, a $(\mathbf{C}, \| \ \|)$-analytic space $\mathcal{X}^{\mathrm{An}}$, and a $(\mathbf{C}, | \ |_0)$-analytic space $\mathcal{X}_0^{\mathrm{An}}$. All three spaces are provided with a morphism to the corresponding open discs, and are closely interrelated. The construction gives rise to a commutative diagram of maps between topological spaces. In §4, we prove our main result (Theorem 4.1) which states that, if $\mathcal{X}$ is proper over $\mathcal{O}_{\mathbf{C},0}$, the homomorphisms between integral cohomology groups induced by certain maps from that diagram are isomorphisms. Essential ingredients of the proof are C. H. Clemens's results from [Cle] and similar results from [Ber5]. If $\mathcal{X}$ is strictly semi-stable over $\mathcal{O}_{\mathbf{C},0}$, the former provide a strong deformation retraction of $\mathcal{X}^h$ to its fiber $\mathcal{X}_s^h$ at zero, and the latter provide a similar homotopy description of the non-Archimedean space $\mathcal{X}_0^{\mathrm{An}}$. In §5, we prove Theorem 5.1 which was already formulated.

We want to emphasize that the above result is an analog of the description of the weight zero subspaces of l-adic étale cohomology groups of algebraic varieties defined over a local field in terms of cohomology groups of the associated non-Archimedean spaces (see [Ber6]). All of these results are evidence for the fact that the underlying topological space of the non-Archimedean analytic space associated with an algebraic variety somehow represents the weight zero part of the mixed motive of the variety.

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1 Topological spaces associated with algebraic varieties over a commutative Banach ring

Let k be a commutative Banach ring, i.e., a commutative ring provided with a Banach norm $\| \ \|$ and complete with respect to it. For an affine scheme $\mathcal{X} = \mathrm{Spec}(A)$ of finite type over k, let $\mathcal{X}^{\mathrm{An}}$ denote the set of all nonzero multiplicative seminorms $| \ | : A \to \mathbf{R}_+$ on the ring A whose restriction to k is bounded with respect to the norm $\| \ \|$. The set $\mathcal{X}^{\mathrm{An}}$ is provided with the weakest topology with respect to which all real valued functions of the form $| \ | \mapsto |f|$, $f \in A$, are continuous. For a point $x \in \mathcal{X}^{\mathrm{An}}$, the corresponding multiplicative seminorm $| \ |_x$ on A gives rise to a multiplicative norm on the integral domain $A/\mathrm{Ker}(| \ |_x)$ and, therefore, extends to a multiplicative norm on its field of fractions. The completion of the latter is denoted by $\mathcal{H}(x)$, and the image of an element $f \in A$ under the corresponding character $A \to \mathcal{H}(x)$ is denoted by $f(x)$. (In particular, $|f|_x = |f(x)|$ for all $f \in A$.) If $A = k \neq 0$, the space $\mathcal{X}^{\mathrm{An}}$ is the spectrum $\mathcal{M}(k)$ of k, which is a nonempty compact space, by [Ber1, 1.2.1]. If $A = k[T_1, \dots, T_n]$, the space $\mathcal{X}^{\mathrm{An}}$ is denoted by $\mathbf{A}^n$ (the n-dimensional affine space over k). Notice that the correspondence $\mathcal{X} \mapsto \mathcal{X}^{\mathrm{An}}$ is functorial in $\mathcal{X}$.

A continuous map of topological spaces $\varphi : Y \to X$ is said to be Hausdorff if, for any pair of different points $y_1, y_2 \in Y$ with $\varphi(y_1) = \varphi(y_2)$, there exist open neighborhoods $\mathcal{V}_1$ of y_1 and $\mathcal{V}_2$ of y_2 with $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$ (i.e., the image of Y in $Y \times_X Y$ is closed). Furthermore, let X be a topological space such that each point of it has a compact neighborhood. A continuous map $\varphi : Y \to X$ is said to be compact, if the preimage of a compact subset of X is a compact subset of Y (i.e., φ is proper in the usual sense, but we use the terminology of [Ber2]). Such a map is Hausdorff, it takes closed subsets of Y to closed subsets of X, and each point of Y has a compact neighborhood.

Lemma 1.1. *(i) The space $\mathcal{X}^{\mathrm{An}}$ is locally compact and countable at infinity;*
(ii) given a closed (resp. an open) immersion $\varphi : \mathcal{Y} \to \mathcal{X}$, the map $\varphi^{\mathrm{an}} : \mathcal{Y}^{\mathrm{An}} \to \mathcal{X}^{\mathrm{An}}$ induces a homeomorphism of $\mathcal{Y}^{\mathrm{An}}$ with a closed (resp. open) subset of $\mathcal{X}^{\mathrm{An}}$;
(iii) given morphisms $\varphi : \mathcal{Y} \to \mathcal{X}$ and $\mathcal{Z} \to \mathcal{X}$, the canonical map

$$(\mathcal{Y} \times_{\mathcal{X}} \mathcal{Z})^{\mathrm{An}} \to \mathcal{Y}^{\mathrm{An}} \times_{\mathcal{X}^{\mathrm{An}}} \mathcal{Z}^{\mathrm{An}}$$

is compact.

Proof. (ii) If φ is a closed immersion, the required fact is trivial. If φ is an open immersion, it suffices to consider the case of a principal open subset $\mathcal{Y} = \mathrm{Spec}(A_f)$ for an element $f \in A$. It is clear that the map φ^{An} is injective and its image is an open subset of $\mathcal{X}^{\mathrm{An}}$. A fundamental system of open sets in $\mathcal{Y}^{\mathrm{An}}$ is formed by finite intersections of sets of the form $\mathcal{U} = \{y \in \mathcal{Y}^{\mathrm{An}} \big| |\frac{g}{f^n}(y)| < r\}$ and $\mathcal{V} = \{y \in \mathcal{Y}^{\mathrm{An}} \big| |\frac{g}{f^n}(y)| > r\}$, where $g \in A$, $n \geq 0$ and $r > 0$. It suffices therefore to verify that the sets $\mathcal{U}$ and $\mathcal{V}$ are open in $\mathcal{X}^{\mathrm{An}}$. Given a point $y \in \mathcal{U}$

(resp. $\mathcal{V}$), there exist $\varepsilon, \delta > 0$ such that $\frac{|g(y)|+\delta}{|f^n(y)|-\varepsilon} < r$ (resp. $\frac{|g(y)|-\delta}{|f^n(y)|+\varepsilon} > r$). Then the set $\{z \in \mathcal{X}^{\mathrm{An}} \big| |g(z)| < |g(y)| + \delta,\ |f^n(z)| > |f^n(y)| - \varepsilon\}$ (resp. $\{z \in \mathcal{X}^{\mathrm{An}} \big| |g(z)| > |g(y)| - \delta,\ |f^n(z)| < |f^n(y)| + \varepsilon\}$) is an open neighborhood of the point y in $\mathcal{X}^{\mathrm{An}}$, which is contained in $\mathcal{U}$ (resp. $\mathcal{V}$).

(i) By (ii), it suffices to consider the case of the affine space $\mathbf{A}^n$, which is associated with the ring of polynomials $k[T] = k[T_1, \dots, T_n]$. One has $\mathbf{A}^n = \bigcup_r E(r)$, where the union is taken over tuples of positive numbers $r = (r_1, \dots, r_n)$ and $E(r)$ is the closed polydisc of radius r with center at zero $\{x \in \mathbf{A}^n \big| |T_i(x)| \le r_i$ for all $1 \le i \le n\}$. The latter is a compact space. Indeed, let $k\langle r^{-1}T\rangle = k\langle r_1^{-1}T_1, \dots, r_n^{-1}T_n\rangle$ denote the commutative Banach ring of all power series $f = \sum_\nu a_\nu T^\nu$ over k such that $||f|| = \sum_\nu ||a_\nu|| r^\nu < \infty$. By [Ber1, Theorem 1.2.1], the spectrum $\mathcal{M}(k\langle r^{-1}T\rangle)$ is a nonempty compact space, and the canonical homomorphism $k[T] \to k\langle r^{-1}T\rangle$ induces a homeomorphism $\mathcal{M}(k\langle r^{-1}T\rangle) \xrightarrow{\sim} E(r)$.

(iii) Let $\mathcal{X} = \mathrm{Spec}(A)$, $\mathcal{Y} = \mathrm{Spec}(B)$ and $\mathcal{Z} = \mathrm{Spec}(C)$. By (ii), it suffices to consider the case when $A = k[T_1, \dots, T_n]$, $B = A[U_1, \dots, U_p]$ and $C = A[V_1, \dots, V_q]$, i.e., it suffices to verify that the corresponding map $\mathbf{A}^{n+p+q} \to \mathbf{A}^{n+p} \times_{\mathbf{A}^n} \mathbf{A}^{n+q}$ is compact. This is clear since the preimage of $E(r') \times_{\mathbf{A}^n} E(r'')$ with $r' = (r_1, \dots, r_n, s_1, \dots, s_p)$ and $r'' = (r_1, \dots, r_n, t_1, \dots, t_q)$ is the polydisc $E(r)$ with $r = (r_1, \dots, r_n, s_1, \dots, s_p, t_1, \dots, t_q)$. □

Let now $\mathcal{X}$ be a scheme of finite type over k. By Lemma 1.1(ii), one can glue the spaces $\mathcal{U}^{\mathrm{An}}$ for open affine subschemes $\mathcal{U} \subset \mathcal{X}$ to get a topological space $\mathcal{X}^{\mathrm{An}}$ in which all $\mathcal{U}^{\mathrm{An}}$ are open subspaces. Here is an equivalent description of the space $\mathcal{X}^{\mathrm{An}}$. For a bounded character $\chi : k \to K$ to a valuation field K (i.e., a field complete with respect to a valuation), let $\mathcal{X}(K)^\chi$ denote the set of all K-points of $\mathcal{X}$ which induce the character χ on k. Furthermore, let $\widetilde{\mathcal{X}}^{\mathrm{An}}$ be the disjoint union of the sets $\mathcal{X}(K)^\chi$ taken over bounded characters $\chi : k \to K$ to a valuation field K. Two points $x' \in \mathcal{X}(K')^{\chi'}$ and $x'' \in \mathcal{X}(K'')^{\chi''}$ are said to be equivalent if there exist a bounded character $\chi : k \to K$, a point $x \in \mathcal{X}(K)^\chi$, and isometric embeddings $K \to K'$ and $K \to K''$ which are compatible with the characters χ' and χ'', taking x to the points x' and x'', respectively. It is really an equivalence relation, and the space $\mathcal{X}^{\mathrm{An}}$ is the set of equivalence classes in $\widetilde{\mathcal{X}}^{\mathrm{An}}$.

The correspondence $\mathcal{X} \mapsto \mathcal{X}^{\mathrm{An}}$ is functorial in $\mathcal{X}$, and the properties (ii) and (iii) of Lemma 1.1 are straightforwardly extended to arbitrary schemes of finite type over k.

Lemma 1.2. *Let $\varphi : \mathcal{Y} \to \mathcal{X}$ be a morphism of schemes of finite type over k, and let φ^{An} be the induced map $\mathcal{Y}^{\mathrm{An}} \to \mathcal{X}^{\mathrm{An}}$. Then*

(i) if φ is separated, then the map φ^{An} is Hausdorff;
(ii) if φ is projective, then the map φ^{An} is compact;
(iii) if φ is proper and either the ring k is Noetherian, or $\mathcal{Y}$ has a finite number of irreducible components, then the map φ^{An} is compact.

The assumptions in (iii) guarantee application of Chow's Lemma (see [EGAII, 5.6.1]).

Proof. We may assume that the scheme $\mathcal{X} = \mathrm{Spec}(A)$ is affine.

(i) Since φ is separated, the diagonal map $\mathcal{Y}^{\mathrm{An}} \to (\mathcal{Y} \times_{\mathcal{X}} \mathcal{Y})^{\mathrm{An}}$ has a closed image. Since the image of the latter in $\mathcal{Y}^{\mathrm{An}} \times_{\mathcal{X}^{\mathrm{An}}} \mathcal{Y}^{\mathrm{An}}$ is closed, it follows that the map φ^{An} is Hausdorff.

(ii) It suffices to consider the case when $\mathcal{Y} = \mathrm{Proj}(A[T_0, \dots, T_n])$ is the projective space over A. In this case, $\mathcal{Y} = \bigcup_{i=0}^n \mathcal{Y}_i$, where $\mathcal{Y}_i = \mathrm{Spec}(A[\frac{T_0}{T_i}, \dots, \frac{T_0}{T_i}])$. If $E_i = \{y \in \mathcal{Y}_i^{\mathrm{An}} \big| |\frac{T_0}{T_i}(y)| \le 1 \text{ for all } 0 \le j \le n\}$, then the map $E_i \to \mathcal{X}^{\mathrm{An}}$ is compact, and one has $\mathcal{Y}^{\mathrm{An}} = \bigcup_{i=0}^n E_i$. It follows that φ^{An} is a compact map.

(iii) Chow's Lemma reduces the situation to the case considered in (ii). □

2 The case of the Banach ring $(\mathbf{C}, || \, ||)$

We now consider the case when k is the field of complex numbers $\mathbf{C}$ provided with the following Banach norm: $||a|| = \max\{|a|_\infty, |a|_0\}$ for all $a \in \mathbf{C}$. Notice that there is a homeomorphism $[0,1] \xrightarrow{\sim} \mathcal{M}(\mathbf{C}, || \, ||) : \rho \mapsto p_\rho$, where the point p_0 corresponds to the trivial norm $| \, |_0$, and each point p_ρ with $\rho > 0$ corresponds to the Archimedean norm $| \, |_\infty^\rho$. Indeed, if $| \, |$ is a valuation which is different from the above ones, then it is nontrivial and not equivalent to $| \, |_\infty$. It follows that there exists a complex number $a \in \mathbf{C}$ with $|a|_\infty < 1$ and $|a| > 1$, i.e., $|a| > ||a||$, and the valuation $| \, |$ is not bounded with respect to the Banach norm $|| \, ||$.

For every scheme $\mathcal{X}$ of finite type over $\mathbf{C}$, there is a canonical surjective map $\lambda = \lambda_{\mathcal{X}} : \mathcal{X}^{\mathrm{An}} \to \mathcal{M}(\mathbf{C}, || \, ||) = [0,1]$. If $\rho \in]0,1]$, then $\mathcal{H}(p_\rho)$ is the field $\mathbf{C}$ provided with the Archimedean valuation $| \, |_\infty^\rho$. The fiber $\lambda^{-1}(1)$ is the complex analytic space $\mathcal{X}^h$ associated with $\mathcal{X}$, by complex GAGA ([Serre]). The fiber $\lambda^{-1}(0)$ is the non-Archimedean $(\mathbf{C}, | \, |_0)$-analytic space $\mathcal{X}^{\mathrm{an}}$ associated with $\mathcal{X}$, by non-Archimedean GAGA ([Ber1]).

Lemma 2.1. *There is a functorial homeomorphism* $\lambda^{-1}(]0,1]) \xrightarrow{\sim} \mathcal{X}^h \times]0,1] : x \mapsto (y, \rho)$*, which commutes with the projections onto* $]0,1]$ *and, in the case of affine* $\mathcal{X} = \mathrm{Spec}(A)$*, is defined by* $\rho = \lambda(x)$ *and* $|f(y)|_\infty = |f(x)|^{\frac{1}{\rho}}$*,* $f \in A$*.*

Proof. Assume first that $\mathcal{X} = \mathrm{Spec}(A)$ is affine. The map considered is evidently continuous. It has an inverse map $\mathcal{X}^h \times]0,1] \to \lambda^{-1}(]0,1]) : (y, \rho) \mapsto y_\rho$, defined by $|f(y_\rho)| = |f(y)|_\infty^\rho$ for $f \in A$, and, therefore, it is bijective. The inverse map is continuous since the topology on $\mathcal{X}^h \times]0,1]$ coincides with the weakest one with respect to which all functions of the form $\mathcal{X}^h \times]0,1] \to \mathbf{R}_+ : (y, \rho) \mapsto |f(y)|^\rho$ for $f \in A$ are continuous. It is trivial that the homeomorphisms are functorial in $\mathcal{X}$, and they extend to the class of all schemes of finite type over $\mathbf{C}$. □

Corollary 2.2. *If $\mathcal{X}$ is connected, then the topological space $\mathcal{X}^{\mathrm{An}}$ is also connected.*

Proof. Any $\mathbf{C}$-point of $\mathcal{X}$ defines a section $\mathcal{M}(\mathbf{C}, || \ ||) = [0,1] \to \mathcal{X}^{\mathrm{An}}$ of the canonical map $\lambda : \mathcal{X}^{\mathrm{An}} \to [0,1]$, and so the required fact follows from the corresponding facts in complex GAGA ([Serre]) and non-Archimedean GAGA ([Ber1, 3.5.3]). $\square$

Proposition 2.3. *If $\mathcal{X}$ is proper, then $H^q(\mathcal{X}^{\mathrm{An}}, \mathbf{Z}) \xrightarrow{\sim} H^q(\mathcal{X}^{\mathrm{an}}, \mathbf{Z})$ for all $q \geq 0$.*

Proof. Since the space $\mathcal{X}^{\mathrm{An}}$ is compact, it suffices to show that the cohomology groups with compact support $H^q_c(\mathcal{X}^h \times]0,1], \mathbf{Z})$ are zero for all $q \geq 0$. For this we use the Leray spectral sequence

$$E_2^{p,q} = H^p_c(]0,1], R^q\lambda_*\mathbf{Z}) \Longrightarrow H^{p+q}_c(\mathcal{X}^h \times]0,1], \mathbf{Z}) \ .$$

The sheaves $R^q\lambda_*\mathbf{Z}$ are constant and, therefore, $E_2^{p,q} = 0$ for all $p, q \geq 0$, and the required fact follows. $\square$

By Proposition 2.3, if $\mathcal{X}$ is proper, there is a homomorphism

$$H^q(\mathcal{X}^{\mathrm{an}}, \mathbf{Z}) \longrightarrow H^q(\mathcal{X}^h \times]0,1], \mathbf{Z}) = H^q(\mathcal{X}^h, \mathbf{Z}) \ .$$

Corollary 2.4. *If $\mathcal{X}$ is proper, the above homomorphism gives rise to an isomorphism*

$$H^q(\mathcal{X}^{\mathrm{an}}, \mathbf{Q}) \xrightarrow{\sim} W_0 H^q(\mathcal{X}^h, \mathbf{Q})$$

Proof. By the construction from [Del3, §6.2] and Hironaka's theorem on resolution of singularities, there exists a proper hypercovering $\mathcal{X}_\bullet \to \mathcal{X}$ such that each $\mathcal{X}_n$ is smooth. By [SGA4, Exp. V bis], it gives rise to a homomorphism of spectral sequences

$$\begin{array}{ccccc} {}'E_1^{p,q} & = & H^q(\mathcal{X}_p^{\mathrm{an}}, \mathbf{Q}) & \Longrightarrow & H^{p+q}(\mathcal{X}^{\mathrm{an}}, \mathbf{Q}) \\ & & \downarrow & & \downarrow \\ {}''E_1^{p,q} & = & H^q(\mathcal{X}_p^h, \mathbf{Q}) & \Longrightarrow & H^{p+q}(\mathcal{X}^h, \mathbf{Q}) \end{array}$$

By [Ber5, §5], the connected components of each $\mathcal{X}_p^{\mathrm{an}}$ are contractible. This implies that ${}'E_1^{p,q} = 0$ for all $q \geq 1$ and, therefore, the first spectral sequence gives rise to isomorphisms ${}'E_2^{p,0} \xrightarrow{\sim} H^p(\mathcal{X}^{\mathrm{an}}, \mathbf{Q})$. On the other hand, by [Del2, 3.2.15(ii)], the mixed Hodge structure on $H^q(\mathcal{X}_p^h, \mathbf{Q})$ has the property that $W_i = 0$ for $i < q$. Since the functor $H \mapsto W_0 H$ on the category of rational mixed Hodge structures H with $W_i H = 0$ for $i < 0$ is exact ([Del2, 2.3.5(iv)]), the latter implies that $W_0({}''E_1^{p,q}) = 0$ for all $q \geq 1$ and, therefore, the second spectral sequence gives rise to an isomorphism $W_0({}''E_2^{p,0}) \xrightarrow{\sim} W_0 H^p(\mathcal{X}^h, \mathbf{Q})$. The required fact now follows from Corollary 2.2. $\square$

Remarks 2.5. (i) It would be interesting to know if Proposition 2.3. is true for an arbitrary separated scheme $\mathcal{X}$ of finite type over F. If this is true, then the similar induced homomorphism $H^q(\mathcal{X}^{\mathrm{an}}, \mathbf{Z}) \to H^q(\mathcal{X}^h, \mathbf{Z})$ gives rise to an isomorphism analogous to that of Corollary 2.4. That such an isomorphism exists is shown in [Ber6, Theorem 1.1(c)] using the same reasoning as that used in the proof of Corollary 2.4 (see also Remark 5.3).

(ii) It is very likely that the map $\mathcal{X}^{\mathrm{an}} \to \mathcal{X}^{\mathrm{An}}$ is a homotopy equivalence at least in the case when $\mathcal{X}$ is a proper scheme over F with the property that, for every $n \geq 1$, a nonempty intersection of n irreducible components is smooth and of codimension $n-1$ (see also Remark 4.4(iii)).

3 Topological spaces associated with algebraic varieties over the ring $\mathcal{O}_{\mathbf{C},0}$

Let $\mathcal{X}$ be a scheme of finite type over the local ring $\mathcal{O}_{\mathbf{C},0}$. We are going to associate with $\mathcal{X}$ (the underlying topological spaces of) analytic spaces of three types. The first one is a classical object. This is a complex analytic space $\mathcal{X}^h$ over an open disc $D(r)$ in $\mathbf{C}$ of radius r (with center at zero). The second one is a $(\mathbf{C}, ||\ ||)$-analytic space $\mathcal{X}^{\mathrm{An}}$ over an open disc $\mathcal{D}(r)$ in $\mathbf{A}^1$. And the third one is a non-Archimedean $(\mathbf{C}, |\ |_0)$-analytic space $\mathcal{X}_0^{\mathrm{An}}$ over an open disc $D_0(r)$ in the $(\mathbf{C}, |\ |_0)$-analytic affine line $\mathbf{A}_0^1$. The first two objects are related to two representations of the ring $\mathcal{O}_{\mathbf{C},0}$ in the form of a filtered inductive limit of the same commutative rings, provided with two different commutative Banach ring structures. The third object is the analytic space associated with the base change of $\mathcal{X}$ under the homomorphism $\mathcal{O}_{\mathbf{C},0} \to \mathbf{C}[[T]] = \mathcal{O}_{\mathbf{A}_0^1,0}$, and is a particular case of an object introduced in [Ber3, §3].

For $r > 0$, let $\mathbf{C}\langle r^{-1}T\rangle$ denote the commutative Banach algebra of formal power series $f = \sum_{i=0}^{\infty} a_i T^i$ over $\mathbf{C}$ absolutely convergent at the closed disc $E(r) = \{x \in \mathbf{C} \big| |T(x)| \leq r\}$ and provided with the norm $||f|| = \sum_{i=0}^{\infty} |a_i|_\infty r^i$. The canonical homomorphism $\mathbf{C}[T] \to \mathbf{C}\langle r^{-1}T\rangle$ induces a homeomorphism $\mathcal{M}(\mathbf{C}\langle r^{-1}T\rangle) \xrightarrow{\sim} E(r)$, and one has $\mathcal{O}_{\mathbf{C},0} = \varinjlim \mathbf{C}\langle r^{-1}T\rangle$ for r tending to zero. By [EGAIV, §8], for any scheme $\mathcal{X}$ of finite type over $\mathcal{O}_{\mathbf{C},0}$, there exist $r > 0$ and a scheme $\mathcal{X}'$ of finite type over $\mathbf{C}\langle r^{-1}T\rangle$ whose base change with respect to the canonical homomorphism $\mathbf{C}\langle r^{-1}T\rangle \to \mathcal{O}_{\mathbf{C},0}$ is $\mathcal{X}$. By the construction of §1, there is an associated topological space $\mathcal{X}'^h$, and we denote by $\mathcal{X}^h$ the preimage of the open disc $D(r) = \{x \in \mathbf{C} \big| |T(x)| < r\}$ with respect to the canonical map $\mathcal{X}'^h \to E(r)$. The morphism $\mathcal{X}^h \to D(r)$ does not depend, up to a change of r, on the choice of $\mathcal{X}'$, and the construction is functorial in $\mathcal{X}$ (see Remark 3.3).

Furthermore, for $r > 0$, let $\mathbf{C}\langle\langle r^{-1}T\rangle\rangle$ denote the commutative Banach ring of formal power series $f = \sum_{i=0}^{\infty} a_i T^i$ over $\mathbf{C}$ such that $||f|| = \sum_{i=0}^{\infty} ||a_i|| r^i < \infty$. The canonical homomorphism $\mathbf{C}[T] \to \mathbf{C}\langle\langle r^{-1}T\rangle\rangle$ gives rise to a homeomorphism $\mathcal{M}(\mathbf{C}\langle\langle r^{-1}T\rangle\rangle) \xrightarrow{\sim} \mathcal{E}(r) = \{x \in \mathbf{A}^1 \big| |T(x)| \leq r\}$. If

$r \geq 1$, then $\mathbf{C}[T] \xrightarrow{\sim} \mathbf{C}\langle\langle r^{-1}T\rangle\rangle$, and, if $r < 1$, then $\mathbf{C}\langle\langle r^{-1}T\rangle\rangle \xrightarrow{\sim} \mathbf{C}\langle r^{-1}T\rangle$ (as $\mathbf{C}$-subalgebras of $\mathbf{C}[[T]]$). One has $\mathcal{O}_{\mathbf{C},0} = \varinjlim \mathbf{C}\langle\langle r^{-1}T\rangle\rangle$ for r tending to zero. By [EGAIV, §8] again, for any scheme $\mathcal{X}$ of finite type over $\mathcal{O}_{\mathbf{C},0}$, there exist $0 < r < 1$ and a scheme $\mathcal{X}'$ of finite type over $\mathbf{C}\langle\langle r^{-1}T\rangle\rangle$ whose base change with respect to the canonical homomorphism $\mathbf{C}\langle\langle r^{-1}T\rangle\rangle \to \mathcal{O}_{\mathbf{C},0}$ is $\mathcal{X}$. By the construction of §1, there is an associated topological space $\mathcal{X}'^{\mathrm{An}}$, and we denote by $\mathcal{X}^{\mathrm{An}}$ the preimage of $\mathcal{D}(r) = \{x \in \mathbf{A}^1 \big| |T(x)| < r\}$ with respect to the canonical map $\mathcal{X}'^{\mathrm{An}} \to \mathcal{E}(r)$. The map $\varphi : \mathcal{X}^{\mathrm{An}} \to \mathcal{D}(r)$ does not depend, up to a change of r, on the choice of $\mathcal{X}'$, and the construction is functorial in $\mathcal{X}$ (see again Remark 3.3).

Finally, for $r > 0$, let $\mathbf{C}\{r^{-1}T\}$ denote the commutative Banach ring of formal power series $f = \sum_{i=0}^{\infty} a_i T^i$ over $\mathbf{C}$ convergent at the closed disc $E_0(r) = \{x \in \mathbf{A}^1_0 \big| |T(x)| \leq r\}$ and provided with the norm $||f|| = \max\limits_{i\geq 0}\{|a_i|_0 r^i\}$. The canonical homomorphism $\mathbf{C}[T] \to \mathbf{C}\{r^{-1}T\}$ gives rise to a homeomorphism $\mathcal{M}(\mathbf{C}\{r^{-1}T\}) \xrightarrow{\sim} E_0(r)$. If $r \geq 1$, then $\mathbf{C}[T] \xrightarrow{\sim} \mathbf{C}\{r^{-1}T\}$, and, if $r < 1$, then $\mathbf{C}\{r^{-1}T\} \xrightarrow{\sim} \mathbf{C}[[T]]$. One has $\mathcal{O}_{\mathbf{A}^1_0,0} = \mathbf{C}[[T]] = \mathbf{C}\{r^{-1}T\}$ for every $0 < r < 1$. Thus, given a scheme $\mathcal{X}$ of finite type over $\mathcal{O}_{\mathbf{C},0}$, we set $\mathcal{X}_0 = \mathcal{X} \otimes_{\mathcal{O}_{\mathbf{C},0}} \mathbf{C}[[T]]$ and, for $0 < r < 1$, we set $\mathcal{X}'_0 = \mathcal{X}_0 \otimes_{\mathbf{C}[[T]]} \mathbf{C}\{r^{-1}T\}$. There is an associated topological space $\mathcal{X}'^{\mathrm{An}}_0$, and we denote by $\mathcal{X}^{\mathrm{An}}_0$ the preimage of $D_0(r) = \{x \in \mathbf{A}^1_0 \big| |T(x)| < r\}$ with respect to the canonical map $\mathcal{X}'^{\mathrm{An}}_0 \to E_0(r)$.

Recall that F denotes the fraction field of $\mathcal{O}_{\mathbf{C},0}$, and K denotes the completion of F with respect to a fixed valuation, which is determined by its value at T. Let ε be the latter value. The K-analytic space associated with a scheme $\mathcal{X}$ of finite type over F is denoted by $\mathcal{X}^{\mathrm{an}}$ (instead of $(\mathcal{X} \otimes_F K)^{\mathrm{an}}$).

Lemma 3.1. *Let $\mathcal{X}$ be a scheme of finite type over $\mathcal{O}_{\mathbf{C},0}$, $\mathcal{X}_\eta = \mathcal{X} \otimes_{\mathcal{O}_{\mathbf{C},0}} F$ its generic fiber, and $\mathcal{X}_s = \mathcal{X} \otimes_{\mathcal{O}_{\mathbf{C},0}} \mathbf{C}$ its closed fiber. Let $\mathcal{X}^{\mathrm{An}}$ be the associated $(\mathbf{C}, ||\ ||)$-analytic space over $\mathcal{D}(r)$ with $0 < r < 1$, and let λ, λ_η and λ_s be the canonical maps to $\mathcal{M}((\mathbf{C}, ||\ ||) = [0,1]$ from $\mathcal{X}^{\mathrm{An}}$, $\mathcal{X}^{\mathrm{An}}_\eta$ and $\mathcal{X}^{\mathrm{An}}_s$, respectively. Then*

(i) there is a homeomorphism

$$\lambda^{-1}(]0,1]) \xrightarrow{\sim} \{(x,\rho) \in \mathcal{X}^h \times]0,1] \big| |T(x)|_\infty < r^{\frac{1}{\rho}}\} : x' \mapsto (x,\rho) \ ,$$

which commutes with the projections onto $]0,1]$ and, in the case of the affine $\mathcal{X} = \mathrm{Spec}(A)$, is defined by $\rho = \lambda(x')$ and $|f(x)|_\infty = |f(x')|^{\frac{1}{\rho}}$, $f \in A$;

(ii) $\lambda^{-1}(0) \xrightarrow{\sim} \mathcal{X}^{\mathrm{An}}_0$;

(iii) there is a homeomorphism

$$\lambda_\eta^{-1}(0) \xrightarrow{\sim} \mathcal{X}^{\mathrm{an}}_\eta \times]0, r[: x' \mapsto (x,\rho) \ ,$$

which, in the case of the affine $\mathcal{X} = \mathrm{Spec}(A)$, is defined by $\rho = |T(x')|$ and $|f(x)| = |f(x')|^{\log_\rho(\varepsilon)}$, $f \in A$;

(iv) $\mathcal{X}^{\mathrm{An}} \backslash \mathcal{X}_\eta^{\mathrm{An}} = \mathcal{X}_s^{\mathrm{An}}$*, where the right hand side is the* $(\mathbf{C}, || \, ||)$*-analytic space associated with* $\mathcal{X}_s$ *in the sense of §2.*

Proof. In (i), the converse map $(x, \rho) \mapsto x_\rho$ is defined by $|f(x_\rho)| = |f(x)|_\infty^\rho$, $f \in A$, and, in (iii), the converse map $\mathcal{X}_\eta^{\mathrm{an}} \times]0, r[\stackrel{\sim}{\to} \lambda^{-1}(0) \backslash \varphi^{-1}(0) : (x, \rho) \mapsto P_{x,\rho}$ is defined by $|f(P_{x,\rho})| = |f(x)|^{\log_\varepsilon(\rho)}$, $f \in A$. The statements (ii) and (iv) are trivial. □

Corollary 3.2. *The open embedding* $\lambda^{-1}(]0, 1]) \hookrightarrow \mathcal{X}^h \times]0, 1]$ *is a homotopy equivalence.*

Proof. The formula $((x, \rho), t) \mapsto (x, \max(\rho, t))$ defines a strong deformation retraction of $\lambda^{-1}(]0, 1])$ and $\mathcal{X}^h \times]0, 1]$ to $\mathcal{X}^h \times \{1\}$. □

Remarks 3.3. (i) The spaces $\mathcal{X}^h$, $\mathcal{X}^{\mathrm{An}}$ and $\mathcal{X}_0^{\mathrm{An}}$ are in fact pro-objects (i.e., filtered projective systems of objects) of the corresponding categories of analytic spaces (see [Ber3, §2]). The functoriality of their constructions means that they give rise to functors from the category of schemes of finite type over $\mathcal{O}_{\mathbf{C},0}$ to the corresponding categories of pro-objects.

(ii) Suppose that $\mathcal{X}$ is a scheme of finite type over $\mathcal{O}_{\mathbf{C},0}$ for which the canonical morphism to $\mathrm{Spec}(\mathcal{O}_{\mathbf{C},0})$ is a composition $\mathcal{X} \stackrel{\varphi}{\to} \mathrm{Spec}(\mathcal{O}_{\mathbf{C},0}) \stackrel{\psi}{\to} \mathrm{Spec}(\mathcal{O}_{\mathbf{C},0})$, where ψ is induced by the homomorphism $\mathcal{O}_{\mathbf{C},0} \to \mathcal{O}_{\mathbf{C},0} : T \mapsto T^n$ for $n \geq 1$. Let $\mathcal{Y}$ denote the same scheme $\mathcal{X}$ but considered over $\mathrm{Spec}(\mathcal{O}_{\mathbf{C},0})$ with respect to the morphism φ. Then there is a canonical homeomorphism of topological spaces $\mathcal{Y}_0^{\mathrm{An}} \stackrel{\sim}{\to} \mathcal{X}_0^{\mathrm{An}} : y \mapsto x$ which, in the case of affine $\mathcal{X} = \mathrm{Spec}(A)$, is defined by $|f(x)| = |f(y)|^n$ for $f \in A$. It induces homeomorphisms $\mathcal{Y}_\eta^{\mathrm{An}} \times]0, r^{\frac{1}{n}}[\stackrel{\sim}{\to} \mathcal{X}_\eta^{\mathrm{An}} \times]0, r[: (y, \rho) \mapsto (x, \rho^n)$ and $\mathcal{Y}_s^{\mathrm{An}} \stackrel{\sim}{\to} \mathcal{X}_s^{\mathrm{An}} : y \mapsto x$ (see Lemma 3.1(i) and (iii)), defined by $|f(x)| = |f(y)|^n$ for $f \in A$.

4 The main result

Let $\mathcal{X}$ be a scheme of finite type over $\mathcal{O}_{\mathbf{C},0}$. By the previous subsection, for some $0 < r < 1$ there is a commutative diagram in which hook and down arrows are open embeddings, left and up arrows are closed embeddings, and all squares are cartesian:

$$\begin{tikzcd}
\mathcal{X}_\eta^h \times]0,1] \arrow[d, "*"] & \lambda_\eta^{-1}(]0,1]) \arrow[l, hook'] \arrow[r, hook, "*"] \arrow[d, "*"] & \mathcal{X}_\eta^{\mathrm{An}} \arrow[d] & \lambda_\eta^{-1}(0) \arrow[l] \arrow[r, "\sim"] \arrow[d] & \mathcal{X}_\eta^{\mathrm{an}} \times]0,r[\arrow[d] \\
\mathcal{X}^h \times]0,1] & \lambda^{-1}(]0,1]) \arrow[l, hook'] \arrow[r, hook, "*"] & \mathcal{X}^{\mathrm{An}} & \lambda^{-1}(0) \arrow[l] \arrow[r, "\sim"] & \mathcal{X}_0^{\mathrm{An}} \\
\mathcal{X}_s^h \times]0,1] \arrow[u] & \lambda_s^{-1}(]0,1]) \arrow[l, "\sim"'] \arrow[r, hook, "*"] \arrow[u] & \mathcal{X}_s^{\mathrm{An}} \arrow[u] & \lambda_s^{-1}(0) \arrow[l] \arrow[r, "\sim"] \arrow[u] & \mathcal{X}_s^{\mathrm{an}} \arrow[u]
\end{tikzcd}$$

Theorem 4.1. *Assume that* $\mathcal{X}$ *is proper (resp. proper and strictly semi-stable) over* $\mathcal{O}_{\mathbf{C},0}$*. Then for a sufficiently small* r *all of the horizontal (resp.*

vertical) arrows of the diagram, except those marked by $*$, induce an isomorphism between integral cohomology groups of the corresponding topological spaces.

The following lemma is a version of Grothendieck's Proposition 3.10.2 from [Gro]. If F is a sheaf on a topological space X, Φ is a family of supports in X, and Y is a subspace of X, then $H^q_{\Phi\cap Y}(Y,F)$ denotes the cohomology groups with coefficients in the pullback of F at Y and with supports in $\Phi \cap Y = \{A\cap Y \big| A \in \Phi\}$.

Lemma 4.2. *Let X be a paracompact locally compact topological space, $X_1 \subset X_2 \subset \dots$ an increasing sequence of closed subsets such that the union of their topological interiors in X coincides with X, and Φ a paracompactifying family of supports in X such that $A \in \Phi$ if and only if $A \cap X_i \in \Phi \cap X_i$ for all $i \geq 1$. Let F be an abelian sheaf on X, and let $q \geq 1$. Assume that for each $i \geq 1$ the image $H^{q-1}_{\Phi\cap X_{i+1}}(X_{i+1},F)$ in $H^{q-1}_{\Phi\cap X_i}(X_i,F)$ under the restriction homomorphism coincides with the image of $H^{q-1}_{\Phi\cap X_{i+2}}(X_{i+2},F)$. Then there is a canonical isomorphism*

$$H^q_\Phi(X,F) \stackrel{\sim}{\to} \varprojlim H^q_{\Phi\cap X_i}(X_i,F) \ .$$

Remark 4.3. An analog of [Gro, Proposition 3.10.2] for étale cohomology groups of non-Archimedean analytic spaces is [Ber2, Proposition 6.3.12]. In the formulation of the latter only the assumption that X is a union of all X_i's was made. This is not enough, and one has to make the stronger assumption that X is a union of the topological interiors of all X_i's (it guarantees that $F(X) \stackrel{\sim}{\to} \varprojlim F(X_i)$ for any sheaf F on X).

Proof. First of all, by the above remark and the assumptions on X_i's and Φ, for any abelian sheaf G on X one has $\Gamma_\Phi(X,G) \stackrel{\sim}{\to} \varprojlim \Gamma_{\Phi\cap X_i}(X_i,G)$. *We claim that, given an injective abelian sheaf J on X and a closed subset $Y \subset X$, the canonical map $\Gamma_\Phi(X,J) \to \Gamma_{\Phi\cap Y}(Y,J)$ is surjective.* Indeed, let g be an element from $\Gamma_{\Phi\cap Y}(Y,J)$ and let B be its support. By [God, Ch. II, Theorem 3.3.1], g is the restriction of a section g' of J over an open neighborhood $\mathcal{U}$ of B in X. Furthermore, let $A \in \Phi$ be such that $B = A \cap Y$, and let $A' \in \Phi$ be a neighborhood of A in X. Shrinking $\mathcal{U}$, we may assume that $\mathcal{U} \subset A'$. Since J is injective, the map $\Gamma(X,J) \to \Gamma(\mathcal{U} \coprod (X\backslash A'),J) = \Gamma(\mathcal{U},J) \oplus \Gamma(X\backslash A',J)$ is surjective. It follows that there exists an element $f \in \Gamma(X,J)$ whose restriction to $\mathcal{U}$ is g' and the restriction to $X\backslash A'$ is zero. Since the support of f lies in A', one has $f \in \Gamma_\Phi(X,J)$, and the claim follows.

The claim implies that the pullback of J at any closed subset $Y \subset X$ is a $(\Phi\cap Y)$-soft sheaf on Y, i.e., for any $A \in \Phi \cap Y$, the canonical map $\Gamma_{\Phi\cap Y}(Y,J) \to \Gamma(A,J)$ is surjective (see [God, Ch. II, §3.5]). Thus, given an injective resolution $0 \to F \to J^0 \to J^1 \to \dots$ of F, there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \Gamma_{\Phi}(X, J^0) & \to & \Gamma_{\Phi}(X, J^1) & \to & \Gamma_{\Phi}(X, J^2) & \to \dots \\ & & \downarrow & & \downarrow & & \downarrow & \\ 0 & \to & \Gamma_{\Phi\cap X_i}(X_i, J^0) & \to & \Gamma_{\Phi\cap X_i}(X_i, J^1) & \to & \Gamma_{\Phi\cap X_i}(X_i, J^2) & \to \dots \end{array}$$

in which the first and second rows give rise to the groups $H^q_{\Phi}(X,F)$ and $H^q_{\Phi\cap X_i}(X_i,F)$, respectively, and the vertical arrows are surjections. The injectivity of the map considered is verified by a simple diagram search in the same way as in the proof of [Ber2, Proposition 6.3.12], and verification of its surjectivity is even easier (and because of that it was omitted in loc.cit) and goes as follows.

Let $\overline{\beta}_i \in H^q_{\Phi\cap X_i}(X_i,F)$, $i \geq 1$, be a compatible system. Assume that, for $i \geq 1$, we constructed elements $\beta_j \in \Gamma_{\Phi\cap X_j}(X_j, J^q)$ each from the class of $\overline{\beta}_j$, $1 \leq j \leq i$, with $\beta_{j+1}\big|_{X_j} = \beta_j$ for $1 \leq j \leq i-1$, and let β'_{i+1} be an element from the class of $\overline{\beta}_{i+1}$. Then $\beta'_{i+1}\big|_{X_i} = \beta_i + d\gamma_i$ for some $\gamma_i \in \Gamma_{\Phi\cap X_i}(X_i, J^{q-1})$. If $\alpha \in \Gamma_{\Phi}(X, J^{q-1})$ is such that $\alpha\big|_{X_i} = \gamma_i$, then for the element $\beta_{i+1} = \beta'_{i+1} - d\alpha\big|_{X_{i+1}}$, we have $\beta_{i+1}\big|_{X_i} = \beta_i$. By the remark at the beginning of the proof, there exists an element $\beta \in \Gamma_{\Phi}(X, J^q)$ such that $\beta\big|_{X_i} = \beta_i$ for all $i \geq 1$. Then $d\beta = 0$, and the surjectivity follows. $\square$

Proof of Theorem 4.1. Step 1. First of all, the isomorphism $H^q(\mathcal{X}^{\mathrm{An}}_s, \mathbf{Z}) \xrightarrow{\sim} H^q(\lambda_s^{-1}(0), \mathbf{Z}) = H^q(\mathcal{X}^{\mathrm{an}}_s, \mathbf{Z})$ follows from Proposition 2.3. The isomorphisms

$$H^q(\mathcal{X}^h \times]0,1], \mathbf{Z}) \xrightarrow{\sim} H^q(\lambda^{-1}(]0,1]), \mathbf{Z}) \text{ and}$$

$$H^q(\mathcal{X}^h_\eta \times]0,1], \mathbf{Z}) \xrightarrow{\sim} H^q(\lambda_\eta^{-1}(]0,1]), \mathbf{Z})$$

follow from Corollary 3.2.

Step 2. To get the isomorphisms

$$H^q(\mathcal{X}^{\mathrm{An}}, \mathbf{Z}) \xrightarrow{\sim} H^q(\lambda^{-1}(0), \mathbf{Z}) \quad \text{and} \quad H^q(\mathcal{X}^{\mathrm{An}}_\eta, \mathbf{Z}) \xrightarrow{\sim} H^q(\lambda_\eta^{-1}(0), \mathbf{Z})\ ,$$

we assume that r is sufficiently small so that the groups $H^q(\mathcal{X}^h_t, \mathbf{Z})$, $t \in D^*(r)$, form a local system for all $q \geq 0$ and, therefore, $R^q\psi_{\eta *}(\mathbf{Z}_{\mathcal{X}^h_\eta})$ are locally constant quasi-unipotent sheaves of finitely generated abelian groups for all $q \geq 0$, where ψ is the canonical morphism $\mathcal{X}^h \to D(r)$. Let $\widetilde{\mathcal{X}^h}$ and $\widetilde{\mathcal{X}^h_\eta}$ denote the images of $\lambda^{-1}(]0,1])$ and $\lambda_\eta^{-1}(]0,1])$ in $\mathcal{X}^h \times]0,1]$ and $\mathcal{X}^h_\eta \times]0,1]$, respectively. (If $\mathcal{X} = \mathrm{Spec}(\mathcal{O}_{\mathbf{C},0})$, they will be denoted by $\widetilde{D(r)}$ and $\widetilde{D^*(r)}$.) It suffices to show that $H^q_{\Phi}(\widetilde{\mathcal{X}^h}, \mathbf{Z}) = 0$ and $H^q_{\Phi_\eta}(\widetilde{\mathcal{X}^h_\eta}, \mathbf{Z}) = 0$ for all $q \geq 0$, where Φ and Φ_η are families of supports in $\widetilde{\mathcal{X}^h}$ and $\widetilde{\mathcal{X}^h_\eta}$ consisting of the closed subsets which are also closed in $\mathcal{X}^{\mathrm{An}}$ and $\mathcal{X}^{\mathrm{An}}_\eta$, respectively.

Consider the following commutative diagrams in which all squares are cartesian

$$\begin{array}{ccccc} \widetilde{D(r)} & \hookrightarrow & \mathcal{D}(r) & \longleftarrow & [0,r[\\ \uparrow \widetilde{\varphi} & & \uparrow \varphi & & \uparrow \\ \widetilde{\mathcal{X}^h} & \hookrightarrow & \mathcal{X}^{\mathrm{An}} & \longleftarrow & \mathcal{X}^{\mathrm{An}}_0 \end{array} \qquad \begin{array}{ccccc} \widetilde{D^*(r)} & \hookrightarrow & \mathcal{D}^*(r) & \longleftarrow &]0,r[\\ \uparrow \widetilde{\varphi}_\eta & & \uparrow \varphi_\eta & & \uparrow \\ \widetilde{\mathcal{X}^h_\eta} & \hookrightarrow & \mathcal{X}^{\mathrm{An}}_\eta & \longleftarrow & \mathcal{X}^{\mathrm{an}}_\eta \times]0,r[\end{array}$$

Since all of the vertical maps are compact, there are spectral sequences (with initial terms $E_2^{p,q}$)

$$H^p_\Phi(\widetilde{D(r)}, R^q\widetilde{\varphi}_* \mathbf{Z}_{\widetilde{\mathcal{X}^h}}) \Longrightarrow H^{p+q}_\Phi(\widetilde{\mathcal{X}^h}, \mathbf{Z}) \text{ and}$$

$$H^p_{\Phi_\eta}(\widetilde{D^*(r)}, R^q\widetilde{\varphi}_{\eta *} \mathbf{Z}_{\widetilde{\mathcal{X}^h_\eta}}) \Longrightarrow H^{p+q}_{\Phi_\eta}(\widetilde{\mathcal{X}^h_\eta}, \mathbf{Z}) \ ,$$

where Φ and Φ_η in the $E_2^{p,q}$ terms denote the similar families of supports in $\widetilde{D(r)}$ and $\widetilde{D^*(r)}$, respectively. Thus, it suffices to verify the following fact. Let L be an abelian sheaf on $D(r)$ whose restriction to $D^*(r)$ is locally constant and quasi-unipotent, and let π denote the canonical projection $\widetilde{D(r)} \to D(r)$. *Then* $(*)$ $H^p_\Phi(\widetilde{D(r)}, \pi^*L) = 0$ *and* $(*_\eta)$ $H^p_{\Phi_\eta}(\widetilde{D^*(r)}, \pi^*L) = 0$ *for all* $p \geq 0$. Both equalities are proved in the same way using Lemma 4.2 as follows.

The equality $(*)$. The space $\mathcal{D}(r)$ is a union of the closed discs $\mathcal{E}(\rho) = \{x \in \mathcal{D}(r) \big| |T(x)| \leq \rho\}$ with $\rho < r$. Let $\widetilde{\mathcal{E}(\rho)} = \mathcal{E}(\rho) \cap \widetilde{D(r)} = \{(y,t) \big| |T(y)|_\infty \leq \rho^{\frac{1}{t}}\}$. Then $\widetilde{D(r)}$ is a union of all $\widetilde{\mathcal{E}(\rho)}$ with $\rho < r$, and, if $\rho < \rho'$, then $\mathcal{E}(\rho)$ and $\widetilde{\mathcal{E}(\rho)}$ are contained in the topological interiors of $\mathcal{E}(\rho')$ and $\widetilde{\mathcal{E}(\rho')}$ in $\mathcal{D}(r)$ and $\widetilde{D(r)}$, respectively. It follows easily that a closed subset $B \subset \widetilde{D(r)}$ is closed in $\mathcal{D}(r)$ if and only if $B \cap \widetilde{\mathcal{E}(\rho)}$ is closed in $\mathcal{E}(\rho)$ for all $\rho < r$. Since the spaces $\mathcal{E}(\rho)$ are compact, from Lemma 4.2 it follows that to prove the equality $(*)$, it suffices to show that $H^q_c(\widetilde{\mathcal{E}(\rho)}, \pi^*_\rho L) = 0$ for all $q \geq 0$, where π_ρ is the canonical projection $\widetilde{\mathcal{E}(\rho)} \to E(\rho)$.

One has $\pi^{-1}_\rho(0) \stackrel{\sim}{\to}]0,1]$ and, for $y \neq 0$, $\pi^{-1}_\rho(y) \stackrel{\sim}{\to} [t_y, 1]$, where $0 < t_y \leq 1$ is such that $|T(y)|_\infty = \rho^{\frac{1}{t_y}}$. It follows that $(R^q\pi_{\rho !}(\pi^*_\rho L))_y$ is zero, if $q \geq 1$ or $q = 0$ and $y = 0$, and coincides with L_y, if $q = 0$ and $y \neq 0$. This means that $R^q\pi_{\rho !}(\pi^*_\rho L)$ is zero for $q \geq 1$, and coincides with $j_{\rho !}(j^*_\rho L)$ for $q = 0$, where j_ρ is the canonical open embedding $E^*(\rho) \hookrightarrow E(\rho)$. The Leray spectral sequence $E_2^{p,q} = H^p_c(E(\rho), R^q\pi_{\rho !}(\pi^*_\rho L)) \Longrightarrow H^{p+q}_c(\widetilde{\mathcal{E}(\rho)}, \pi^*_\rho L)$ implies that $H^q_c(\widetilde{\mathcal{E}(\rho)}, \pi^*_\rho L) = H^q_c(E^*(\rho), L)$ for all $q \geq 0$. Thus, the equality $(*)$ is a consequence of the following simple fact: $H^q_c(E^*(\rho), L) = 0$, $q \geq 0$, *for any abelian quasi-unipotent sheaf* L *on* $E^*(\rho)$.

If L is constant, the above fact follows from the long exact sequence of cohomology with compact supports associated with the maps

$$E^*(\rho) \stackrel{j_\rho}{\hookrightarrow} E(\rho) \longleftarrow \{0\} \ .$$

It follows easily that the same is true for any unipotent abelian sheaf L. Assume now that L is quasi-unipotent. Then there exists $n \geq 1$ such that the pullback of L under the n-power map $\varphi : E^*(\rho^{\frac{1}{n}}) \to E^*(\rho) : z \mapsto z^n$ is unipotent. By the previous case, $H^q_c(E^*(\rho^{\frac{1}{n}}), \varphi^*L) = 0$ for all $q \geq 0$. The spectral sequence $E_2^{p,q} = H^p(\mathbf{Z}/n\mathbf{Z}, H^q_c(E^*(\rho^{\frac{1}{n}}), \varphi^*L)) \Longrightarrow H^{p+q}_c(E^*(\rho), L)$ implies required fact for such L.

The equality $(*_\eta)$ (see also Remark 4.4(i)). The space $\mathcal{D}^*(r)$ is a union of the closed annuli $\mathcal{A}_\rho = \{x \in \mathcal{D}(r) \big| \rho \le |T(x)| \le r - \rho\}$ with $0 < \rho < \frac{r}{2}$. Let $\widetilde{\mathcal{A}_\rho} = \mathcal{A}_\rho \cap \widetilde{D^*(r)} = \{(y,t) \big| \rho^{\frac{1}{t}} \le |T(y)|_\infty \le (r-\rho)^{\frac{1}{t}}\}$. Then $\widetilde{D^*(r)}$ is a union of $\widetilde{\mathcal{A}_\rho}$ with $0 < \rho < \frac{r}{2}$ and, for $\rho < \rho'$, $\mathcal{A}_\rho$ and $\widetilde{\mathcal{A}_\rho}$ lie in the topological interiors of $\mathcal{A}_{\rho'}$ and $\widetilde{\mathcal{A}_{\rho'}}$ in $\mathcal{D}^*(r)$ and $\widetilde{D^*(r)}$, respectively. It follows that a closed subset $B \subset \widetilde{D^*(r)}$ is closed in $\mathcal{D}^*(r)$ if and only if $B \cap \widetilde{\mathcal{A}_\rho}$ is closed in $\mathcal{A}_\rho$ for all $0 < \rho < \frac{r}{2}$. Since the spaces $\mathcal{A}_\rho$ are compact, from Lemma 4.2 it follows that to prove the equality $(*_\eta)$ it suffices to show that $H^q_c(\widetilde{\mathcal{A}_\rho}, \pi^*_\rho L) = 0$ for all $q \ge 0$, where π_ρ is the canonical projection $\widetilde{\mathcal{A}_\rho} \to E^*(r-\rho)$.

Notice that, in comparison with the previous case, the preimage of any point of $E^*(\rho)$ under the latter map is always a closed interval or a point. It follows that $R^q\pi_{\rho *}(\pi^*_\rho L)$ is zero, if $q \ge 1$, and coincides with $L\big|_{E^*(r-\rho)}$, if $q = 0$. The Leray spectral sequence of the map π_ρ implies that $H^q_c(\widetilde{\mathcal{A}_\rho}, \pi^*_\rho L) = H^q_c(E^*(r-\rho), L)$ for all $q \ge 0$, and the equality $(*_\eta)$ follows from the fact we already verified.

Step 3. It remains to show that, if $\mathcal{X}$ is proper and strictly semi-stable over $\mathcal{O}_{\mathbf{C},0}$, then the unmarked vertical arrows in the extreme left and right columns induce isomorphisms of cohomology groups. In this case, $\mathcal{X}^h_s$ is even a strong deformation retract of $\mathcal{X}^h$, by the results of C. H. Clemens (see [Cle, §6]), and *both maps* $\mathcal{X}^{\mathrm{an}}_\eta \times]0, r[\to \mathcal{X}^{\mathrm{An}}_0$ *and* $\mathcal{X}^{\mathrm{an}}_s \to \mathcal{X}^{\mathrm{An}}_0$ *are homotopy equivalences*, by results from [Ber5], we are going to explain.

Consider a more general situation. Let k be an arbitrary field (instead of $\mathbf{C}$) provided with the trivial valuation. The ring of formal power series $k[[z]]$ coincides with the ring $\mathcal{O}_{\mathbf{A}^1,0}$ of formal power series convergent in an open neighborhood of zero in the affine line $\mathbf{A}^1$ over k as well as with the ring $\mathcal{O}(D(1))$ of those power series which are convergent in the open disc $D(1)$ (of radius one with center at zero). The formal spectrum $\mathfrak{X} = \mathrm{Spf}(k[[z]])$ is a special formal scheme over $k^\circ = k$ in the sense of [Ber4], and its generic fiber $\mathfrak{X}_\eta$ coincides with $D(1)$. Notice that there is a canonical homeomorphism $[0, 1[\overset{\sim}{\to} D(1)$: $\rho \mapsto P_\rho$, where P_ρ is defined by $|z(P_\rho)| = \rho$.

Let $\mathcal{X}$ be a scheme of finite type over $k[[z]]$. For any number $0 < r < 1$, the ring $k[[z]]$ coincides with the k-affinoid algebra $k\{r^{-1}z\}$, the algebra of analytic functions on the closed disc $E(r) \subset \mathbf{A}^1$ (which is canonically homeomorphic to $[0, r]$), and so there is an associated k-analytic space $\mathcal{Y}^{\mathrm{an}}(r)$. If $r < r'$, $\mathcal{X}^{\mathrm{an}}(r)$ is identified with a closed analytic subdomain of $\mathcal{X}^{\mathrm{an}}(r')$, we set $\mathcal{X}^{\mathrm{an}} = \cup \mathcal{X}^{\mathrm{an}}(r)$. There is a canonical surjective morphism $\varphi : \mathcal{X}^{\mathrm{an}} \to D(1) \overset{\sim}{\to} [0, 1[$. The fiber $\varphi^{-1}(\rho)$ at $\rho \in [0, 1[$ is identified with the $\mathcal{H}(P_\rho)$-analytic space $\mathcal{X}^{\mathrm{an}}_\rho$ associated with the scheme $\mathcal{X} \otimes_{k[[z]]} \mathcal{H}(P_\rho)$. The formal completion $\widehat{\mathcal{X}}$ of $\mathcal{X}$ along its closed fiber $\mathcal{X}_s$ is a special formal scheme, and there is a canonical morphism of strictly k-analytic spaces $\widehat{\mathcal{X}}_\eta \to \mathcal{X}^{\mathrm{an}}$ whose composition with the above morphism φ is induced by the canonical morphism of formal schemes $\widehat{\mathcal{X}} \to \mathfrak{X}$. If $\mathcal{X}$ is separated and of finite type over $k[[z]]$, $\widehat{\mathcal{X}}_\eta$ is identified with a closed strictly analytic subdomain of $\mathcal{X}^{\mathrm{an}}$. If $\mathcal{X}$ is proper over $k[[z]]$, then $\widehat{\mathcal{X}}_\eta \stackrel{\sim}{\to} \mathcal{X}^{\mathrm{an}}$. If $\mathcal{X}$ is semi-stable over $k[[z]]$, then so is $\widehat{\mathcal{X}}$.

Assume now that $\mathfrak{Y}$ be a semi-stable formal scheme over $\mathfrak{X} = \mathrm{Spf}(k[[z]])$ (or, more generally, poly-stable in the sense of [Ber5]). For $\rho \in [0,1[$, we set $\mathfrak{Y}_\rho = \mathfrak{Y} \times_{\mathfrak{X}} \mathrm{Spf}(\mathcal{H}(P_\rho)^\circ)$. It is a semi-stable formal scheme of $\mathcal{H}(P_\rho)$, and there are canonical isomorphisms $\mathfrak{Y}_{\rho,\eta} \stackrel{\sim}{\to} \mathfrak{Y}_{\eta,\rho}$ and $\mathfrak{Y}_{\rho,s} \stackrel{\sim}{\to} \mathfrak{Y}_s$. In [Ber5, §5], we constructed a closed subset $S(\widehat{\mathfrak{Y}}_\rho)$ (the skeleton of $\mathfrak{Y}_\rho$), and a strong deformation retraction $\Phi_\rho : \mathfrak{Y}_{\rho,\eta} \times [0,1] \to \mathfrak{Y}_{\rho,\eta}$ of $\mathfrak{Y}_{\rho,\eta}$ to the skeleton $S(\widehat{\mathfrak{Y}}_\rho)$. We denote by $S(\mathfrak{Y}/\mathfrak{X})$ the union of $S(\widehat{\mathfrak{Y}}_\rho)$ over all $\rho \in [0,1[$, and by Φ the mapping $\mathfrak{Y}_\eta \times [0,1] \to \mathfrak{Y}_\eta$ that coincides with Φ_ρ at each fiber of φ. In [Ber5, §4], we also associated with the closed fiber $\mathfrak{Y}_s$ of $\mathfrak{Y}$ a simplicial set $\mathbf{C}(\mathfrak{Y}_s)$ which has a geometric realization $|\mathbf{C}(\mathfrak{Y}_s)|$. Thus, to prove the claim, it suffices to verify the following two facts:

(a) the mapping $\Phi : \mathfrak{Y}_\eta \times [0,1] \to \mathfrak{Y}_\eta$ is continuous and compact, and

(b) there is a homeomorphism $|\mathbf{C}(\mathfrak{Y}_s)| \times \mathfrak{X}_\eta \to S(\mathfrak{Y}/\mathfrak{X})$ which commutes with the canonical projections to $[0,1[$.

(a) follows from the proof of [Ber5, Theorem 7.1]. In the formulation of the latter the formal scheme $\mathfrak{X}$ was in fact assumed to be locally finitely presented over the ring of integers of the ground field (in our case $k^\circ = k$), but its proof only uses the fact that the morphism $\mathfrak{Y} \to \mathfrak{X}$ is poly-stable and works in the case when $\mathfrak{X}$ is an arbitrary special formal scheme.

(b) By the properties of the skeleton established in [Ber5, §5], the situation is easily reduced to the case when $\mathfrak{Y} = \mathrm{Spf}(B)$, where

$$B = k[[z]]\{T_0, \dots, T_m\}/(T_0 \cdot \ldots T_n - z)\ ,\ 0 \le n \le m\ .$$

If $n = 0$, then $S(\mathfrak{Y}/\mathfrak{X}) \stackrel{\sim}{\to} \mathfrak{X}_\eta \stackrel{\sim}{\to} [0,1[$, $|\mathbf{C}(\mathfrak{Y}_s)|$ is a point, and (b) follows. Assume that $n \ge 1$. Then $S(\mathfrak{Y}/\mathfrak{X})$ is identified with the set $\{(P_\rho, r_0, \dots, r_n) \in \mathfrak{X}_\eta \times [0,1]^{n+1} \big| r_0 \cdot \ldots \cdot r_n = \rho\}$, and $|\mathbf{C}(\mathfrak{Y}_s)|$ is identified with the set $\{(u_0, \dots, u_n) \in [0,1]^{n+1} \big| u_0 + \ldots + u_n = 1\}$. The required map $|\mathbf{C}(\mathfrak{Y}_s)| \times \mathfrak{X}_\eta \to S(\mathfrak{Y}/\mathfrak{X})$ takes a point $((u_0, \dots, u_n), \rho)$ to $(P_\rho, (\rho^{u_0}, \dots, \rho^{u_n}))$. □

Remarks 4.4. (i) The equality $(*_\eta)$ can be established in a different way. Namely, *we claim that there is a strong deformation retraction of* $\mathcal{D}^*(r)$ *to the subset* $\lambda_\eta^{-1}(0)$ *(identified with* $]0,r[$*)*. Indeed, let P_ρ denote the point of $\lambda_\eta^{-1}(0)$ that corresponds to $\rho \in]0,r[$ (it is a unique point from $\lambda_\eta^{-1}(0)$ with $|T(P_\rho)| = \rho$). Then the required strong deformation retraction $\Psi : \mathcal{D}^*(r) \times [0,1] \to \mathcal{D}^*(r)$ (with $\Psi(x,1) = x$ and $\Psi(x,0) \in \lambda_\eta^{-1}(0)$) is defined as follows:

(1) if $(\rho e^{i\varphi}, s) \in \widetilde{D^*(r)}$, then $\Psi((\rho e^{i\varphi}, s), t) = (\rho^{\frac{1}{t}} e^{i\varphi}, st) \in \widetilde{D^*(r)}$ for $t \in]0,1]$;
(2) $\Psi((\rho e^{i\varphi}, s), 0) = P_{\rho^s}$;
(3) if $\rho \in]0,r[$, then $\Psi(P_\rho, t) = P_\rho$ for all $t \in [0,1]$.

The claim implies that, if the sheaf L is constant, then $H^p(\mathcal{D}^*(r), \pi^* L) \stackrel{\sim}{\to} H^p(]0,r[, \pi^* L) = 0$ and, therefore, $H^p_{\Phi_\eta}(\widetilde{D^*(r)}, \pi^* L) = 0$ for all $p \ge 0$. Thus, the equality $(*_\eta)$ is true for constant L, and is easily extended to arbitrary quasi-unipotent sheaves L.

(ii) Let $\mathcal{X}$ be a connected projective scheme over F that admits a strictly semi-stable reduction over $\mathcal{O}_{\mathbf{C},0}$. *Then the fundamental group of $\mathcal{X}^{\mathrm{an}}$ is isomorphic to a quotient of the fundamental group of the fiber $\mathcal{X}^h_t$, $t \in D^*(r)$*, and, in particular, if the latter is simply connected, then so is $\mathcal{X}^{\mathrm{an}}$. Indeed, let $\mathcal{Y}$ be a projective strictly semi-stable scheme over $\mathcal{O}_{\mathbf{C},0}$ with $\mathcal{Y}_\eta = \mathcal{X}$. The canonical morphism $\mathcal{Y} \to \mathrm{Spec}(\mathcal{O}_{\mathbf{C},0})$ has a section $\mathrm{Spec}(\mathcal{O}_{\mathbf{C},0}) \to \mathcal{Y}$ (with the image in the smooth locus of that morphism), and, therefore, for some $0 < r < 1$, the canonical morphism $\mathcal{Y}^h \to D(r)$ has a section $D(r) \to \mathcal{Y}^h$. It follows the canonical surjection $\pi_1(\mathcal{X}^h) \to \pi_1(D^*)$ has a section whose image lies in the kernel of the canonical homomorphism $\pi_1(\mathcal{X}^h) \to \pi_1(\mathcal{Y}^h)$ and, therefore, the image of $\pi_1(\mathcal{X}^h_t)$ in $\pi_1(\mathcal{Y}^h)$ coincides with that of $\pi_1(\mathcal{X}^h)$. But the canonical homomorphism $\pi_1(\mathcal{X}^h) \to \pi_1(\mathcal{Y}^h)$ is surjective since the preimage of $\mathcal{X}^h$ in a universal covering of $\mathcal{Y}^h$ is connected (it is the complement of a Zariski closed subset of a connected smooth complex analytic space). Thus, $\pi_1(\mathcal{Y}^h)$ is a quotient of $\pi_1(\mathcal{X}^h_t)$. Furthermore, by the result of C. Clemens ([Cle]) used in the proof of Theorem 4.1, $\mathcal{Y}^h_s$ is a strong deformation retract of $\mathcal{Y}^h$, i.e., $\pi_1(\mathcal{Y}^h_s)$ is a quotient of $\pi_1(\mathcal{X}^h_t)$. If C is the simplicial set associated with the scheme $\mathcal{Y}_s$, there is a surjective homomorphism from $\pi_1(\mathcal{Y}^h_s)$ to the fundamental group of the geometric realization $|C|$ of C. It remains to notice that, by [Ber5, Theorem 5.2], $\mathcal{X}^{\mathrm{an}}$ is homotopy equivalent to $|C|$. (I am due to O. Gabber for the above reasoning.)

(iii) Assume that $\mathcal{X}$ is proper and strictly semi-stable over $\mathcal{O}_{\mathbf{C},0}$. It is very likely that all of the maps in the diagram from the beginning of this section, except those marked by $*$, are in fact homotopy equivalences.

(iv) It would be interesting to know whether Theorem 4.1 is true for not necessarily proper schemes.

5 An interpretation of the weight zero subspaces

Let $\mathcal{X}$ be a proper scheme over F, and let $0 < r < 1$ be small enough so that the isomorphisms $H^q(\mathcal{X}^{\mathrm{an}} \times]0, r[, \mathbf{Z}) = H^q(\mathcal{X}^{\mathrm{an}}, \mathbf{Z}) \xrightarrow{\sim} H^q(\mathcal{X}^{\mathrm{An}}, \mathbf{Z})$ from Theorem 4.1 take place. They give rise to homomorphisms $H^q(\mathcal{X}^{\mathrm{an}}, \mathbf{Z}) \to H^q(\mathcal{X}^h, \mathbf{Z})$, $q \geq 0$. Let $\overline{D}^*(r) \to D^*(r)$ be a universal covering of $D^*(r)$. The fundamental group $\pi_1(D^*) = \pi_1(D^*(r), t)$ (which does not depend on the choice of r and a point $t \in D^*(r)$) acts on $\overline{D}^*(r)$ and, therefore, it acts on $\overline{\mathcal{X}}^h = \mathcal{X}^h \times_{D^*(r)} \overline{D}^*(r)$. Furthermore, let F^{a} be the field of functions meromorphic in the preimage of an open neighborhood of zero in $\overline{D}^*(r)$, which are algebraic over F. It is an algebraic closure of F and, in particular, $\pi_1(D^*)$ acts on F^{a}. Let K^{a} be the corresponding algebraic closure of K, $\widehat{K}^{\mathrm{a}}$ the completion of K^{a}, and $\overline{\mathcal{X}}^{\mathrm{an}} = (\mathcal{X}^{\mathrm{an}} \otimes_F \widehat{K}^{\mathrm{a}})^{\mathrm{an}}$. As was mentioned in the introduction, the constructed homomorphisms of cohomology groups induce $\pi_1(D^*)$-equivariant homomorphisms $H^q(\overline{\mathcal{X}}^{\mathrm{an}}, \mathbf{Z}) \to H^q(\overline{\mathcal{X}}^h, \mathbf{Z})$, $q \geq 0$.

Theorem 5.1. *The above homomorphisms give rise to $\pi_1(D^*)$-equivariant isomorphisms*

$$H^q(\overline{\mathcal{X}}^{\mathrm{an}}, \mathbf{Q}) \xrightarrow{\sim} W_0 H^q(\overline{\mathcal{X}}^h, \mathbf{Q}),\ q \geq 0\ .$$

Proof. Consider first the case when $\mathcal{X} = \mathcal{Y}_\eta$, where $\mathcal{Y}$ is a projective strictly semi-stable scheme over $\mathcal{O}_{\mathbf{C},0}$. By Corollary 2.4, in the commutative diagram of Theorem 4.1 for such $\mathcal{Y}$ the maps from the lower row give rise to an isomorphism $H^q(\mathcal{Y}^{\mathrm{an}}, \mathbf{Q}) \xrightarrow{\sim} W_0 H^q(\mathcal{Y}_s^h, \mathbf{Q})$ and, by Steenbrink's work [St], the homomorphisms $H^q(\mathcal{Y}_s^h, \mathbf{Z}) \xrightarrow{\sim} H^q(\mathcal{Y}^h, \mathbf{Z}) \to H^q(\mathcal{X}^h, \mathbf{Z}) \to H^q(\overline{\mathcal{X}}^h, \mathbf{Z})$ give rise to an isomorphism $W_0 H^q(\mathcal{Y}_s^h, \mathbf{Q}) \xrightarrow{\sim} W_0 H^q(\overline{\mathcal{X}}^h, \mathbf{Q})$, $q \geq 0$. Since the residue field of K is algebraically closed, the canonical map $\overline{\mathcal{X}}^{\mathrm{an}} \to \mathcal{X}^{\mathrm{an}}$ is a homotopy equivalence (see [Ber5, §5]) and, in particular, $H^q(\mathcal{X}^{\mathrm{an}}, \mathbf{Z}) \xrightarrow{\sim} H^q(\overline{\mathcal{X}}^{\mathrm{an}}, \mathbf{Z})$. Thus, the required isomorphism follows from Theorem 4.1. Consider now the case when $\mathcal{X}$ is projective and smooth over F. One can find an integer $n \geq 1$ such that, if F' is the cyclic extension of F of degree n in F^{a}, then the scheme $\mathcal{X}' = \mathcal{X} \otimes_F F'$ is of the previous type over F'. The extensions $F^{\mathrm{a}} \supset F' \supset F$ correspond to morphisms $\overline{D}^*(r) \to D^*(r^{\frac{1}{n}}) \xrightarrow{z \mapsto z^n} D^*(r)$, and so there is a canonical isomorphism of complex analytic spaces $\overline{\mathcal{X}}'^h \xrightarrow{\sim} \overline{\mathcal{X}}^h$. The latter gives rise to isomorphisms $H^q(\overline{\mathcal{X}}^h, \mathbf{Z}) \xrightarrow{\sim} H^q(\overline{\mathcal{X}}'^h, \mathbf{Z})$ of cohomology groups provided with the limit mixed Hodge structures (see [GNPP, p. 126]). Similarly, one has canonical isomorphisms $H^q(\overline{\mathcal{X}}^{\mathrm{an}}, \mathbf{Z}) \xrightarrow{\sim} H^q(\overline{\mathcal{X}}'^{\mathrm{an}}, \mathbf{Z})$, and the required isomorphism follows from the previous case. Finally, if $\mathcal{X}$ is an arbitrary proper scheme over F, one gets the required isomorphism using the same reasoning as in the proof of Corollary 2.4, i.e., using a proper hypercovering $\mathcal{X}_\bullet \to \mathcal{X}$ with projective and smooth $\mathcal{X}_n$'s and the fact that the functor $H \mapsto W_0 H$ on the category of rational mixed Hodge structures H with $W_i H = 0$ for $i < 0$ is exact. □

Corollary 5.2. *In the above situation, the following is true:*

(i) $H^q(\mathcal{X}^{\mathrm{an}}, \mathbf{Q}) \xrightarrow{\sim} (W_0 H^q(\overline{\mathcal{X}}^h, \mathbf{Q}))^{T=1}$;

(ii) if $\mathcal{X}$ is projective and smooth, then

$$H^q(\mathcal{X}^{\mathrm{an}}, \mathbf{Q}) \xrightarrow{\sim} ((T^m - 1)^i H^q(\mathcal{X}_t^h, \mathbf{Q}))^{T=1}\ .$$

Here T is the canonical generator of $\pi_1(D^*)$, and m is a positive integer for which the action of $(T^m - 1)^{i+1}$ on $H^q(\mathcal{X}_t^h, \mathbf{Q})$ is zero (see the introduction).

Proof. It suffices to show that the canonical map

$$H^q(\mathcal{X}^{\mathrm{an}}, \mathbf{Q}) \to H^q(\overline{\mathcal{X}}^{\mathrm{an}}, \mathbf{Q})^{T=1}$$

is an isomorphism. For this we recall that, by [Ber5, Theorem 10.1], one has $H^q((\mathcal{X} \otimes_F K')^{\mathrm{an}}, \mathbf{Q}) \xrightarrow{\sim} H^q(\overline{\mathcal{X}}^{\mathrm{an}}, \mathbf{Q})$ for some finite Galois extension K' of K in K^{a}. Since the topological space $\mathcal{X}^{\mathrm{an}}$ is the quotient of $(\mathcal{X} \otimes_F K')^{\mathrm{an}}$ by the action of the Galois group of K' over K, the required fact follows from [Gro, Corollary 5.2.3]. □

Remark 5.3. As was mentioned at the end of the introduction, Theorem 5.1 is an analog of a similar description of the weight zero subspaces in the l-adic cohomology groups of algebraic varieties over a local field, which holds for arbitrary separated schemes of finite type (see [Ber6]). And so it is very likely that the isomorphism of Theorem 5.1 also takes place for arbitrary separated schemes of finite type over F. The latter would follow from the validity of Theorem 4.1 for that class of schemes (see Remark 4.4(iv)), and is easily extended to separated smooth schemes. (Recall that the theory of limit mixed Hodge structures on the cohomology groups $H^q(\overline{\mathcal{X}}^h, \mathbf{Q})$ for separated schemes $\mathcal{X}$ of finite type over F is developed in [EZ].)

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