De Jong-Oort Purity for p-Divisible Groups

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Dedicated to Professor Yuri Manin

1 Introduction

De Jong-Oort purity states that for a family of p-divisible groups $X \to S$ over a noetherian scheme S the geometric fibres have all the same Newton polygon if this is true outside a set of codimension bigger than 2. A more general result was first proved in [JO] and an alternative proof is given in [V1]. We present here a short proof which is based on the fact that a formal p-divisible group may be defined by a display ([Z1], [Me2]). There are two other ingredients of the proof which are known for a long time. One is the boundedness principal for crystals over an algebraically closed field ([O], [V1], [V2]) and the other is the existence of a slope filtration for a p-divisible group over a non-perfect field ([Z2]). The last fact was already mentioned in a letter of Grothendieck to Barsotti [G]. The boundedness property is also an important ingredient in the proof given by Vasiu in [V1].

We discuss in detail some elementary consequences of the display structure. The other two ingredients can be found in the literature above. Therefore we discuss them only briefly. I thank B.Messing for pointing out the correct formulation of Proposition 3 below.

2 Frobenius Modules

We fix a prime number p. Let R be a commutative ring, such that p is nilpotent in R. The ring of Witt vectors with respect to p is denoted by $W(R)$. We write $I_R = VW(R)$ for the Witt vectors whose first component is 0. The Witt polynomials are denoted by $\mathbf{w}_n : W(R) \to R$. The truncated Witt vectors of length n are denoted by $W_n(R)$. If $pR = 0$ the Frobenius endomorphism F of the ring $W(R)$ induces an endomorphism $F : W_n(R) \to W_n(R)$.

Definition 1. *A Frobenius module over R is a pair (M, F), where M is a projective finitely generated $W(R)$-module of some fixed rank h and $F : M \to M$ is a Frobenius-linear homomorphism, such that $\det F = p^d \epsilon$ locally for the Zariski topology on R, where $\epsilon : \det M \to \det M$ is a Frobenius-linear isomorphism and $d \geq 0$ is some integer. We call h the height of the Frobenius module and d the dimension.*

This definition implies that the factorization $\det F = p^d \epsilon$ exists even globally, but we will never use this. Since the kernel of $\mathbf{w}_0 : W(R) \to R$ is in the radical of $W(R)$, there is always a covering $\operatorname{Spec} R = \bigcup_i \operatorname{Spec} R_{f_i}$, such that $W(R_{f_i}) \otimes_{W(R)} M$ is a free $W(R_{f_i})$-module for each i. Therefore we will often consider the case where M is a free $W(R)$-module. If we choose a basis of M we may view $\det F$ as an element of $W(R)$. Then (M, F) is a Frobenius module iff $\det F = p^d \eta$ for some unit $\eta \in W(R)$. In a question which is local on $\operatorname{Spec} R$ we will consider $\det F$ as an element of $W(R)$ without futher notice.

In this article a display over R is a 3n-display in the sense of [Z1]. The displays of [Z1] are called nilpotent displays. If $\mathcal{P} = (P, Q, F, F_1)$ is a display over R then (P, F) is a Frobenius module over R.

Let X be a p-divisible over R and assume that p is nilpotent in R. If we evaluate the Grothendieck-Messing crystal of X at $W(R)$ we obtain a finitely generated locally free $W(R)$-module M_X, which is endowed with a Frobenius linear map $F : M_X \to M_X$. If X is the formal p-divisible group associated to a nilpotent display $\mathcal{P}$ then $(M_X, F) = (P, F)$ is a Frobenius module. The pair (M_Y, F) is also a Frobenius module, if Y is an extension of an étale p-divisible group by X.

If we assume moreover that R is a complete local noetherian ring (M_X, F) is a Frobenius module for an arbitrary p-divisible group X over R. Indeed if the special fibre of X has no étale part then (M_X, F) comes from a display and is therefore a Frobenius module. Since X is an extension of an étale p-divisible group by a p-divisible group with no étale part in the special fibre, we see that (M_X, F) is a Frobenius module in general.

By these remarks any (M_X, F) appearing in this work are Frobenius modules.

We add that Lau [L] in a forthcoming paper will associate a display to any p-divisible group over a ring R, where p is nilpotent. Thereby he obtains a functor from p-divisible groups to Frobenius modules. If we could use this functor it would be more satisfying then the remark above.

The following lemma is mainly a motivation for the definitions we are going to make:

Lemma 2. *Let $\mathcal{P}$ and $\mathcal{P}'$ be displays over a ring R of the same height and dimension. Let $\alpha : \mathcal{P} \to \mathcal{P}'$ be a homomorphism.*

Locally on $\operatorname{Spec} R$ the element $\det \alpha \in W(R)$ satisfies an equation:

$$^F\!\det \alpha = \varepsilon \cdot \det \alpha,$$

where $\varepsilon \in W(R)^*$ is a unit.

Proof. We choose normal decompositions

$$\begin{array}{l} P = L \oplus T, \quad Q = L \oplus I_R T \\ P' = L' \oplus T', Q' = L' \oplus I_R T'. \end{array}$$

Without loss of generality we may assume that L, L', T, T' are free $W(R)$-modules. We choose identifications

$$L \simeq W(R)^l \simeq L', \quad T \simeq W(R)^t \simeq T'.$$

Then operators F_1 and F_1' are given by invertible block-matrices with coefficient in $W(R)$:

$$F_1 \begin{pmatrix} \underline{x} \\ {}^V\underline{y} \end{pmatrix} = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \begin{pmatrix} {}^F\underline{x} \\ \underline{y} \end{pmatrix}$$

$$F_1' \begin{pmatrix} \underline{x} \\ {}^V\underline{y} \end{pmatrix} = \begin{pmatrix} X' & Y' \\ Z' & W' \end{pmatrix} \begin{pmatrix} {}^F\underline{x} \\ \underline{y} \end{pmatrix}$$

The block-matrices are invertible by the definition of a display. We also represent α by a block matrix

$$\alpha \begin{pmatrix} \underline{x} \\ {}^V\underline{y} \end{pmatrix} = \begin{pmatrix} A & B \\ {}^VC & D \end{pmatrix} \begin{pmatrix} \underline{x} \\ {}^V\underline{y} \end{pmatrix}$$

Since α commutes with the operators F_1 and F_1' we find

$$\begin{pmatrix} X' & Y' \\ Z' & W' \end{pmatrix} \begin{pmatrix} {}^FA & p\,{}^FB \\ C & {}^FD \end{pmatrix} = \begin{pmatrix} A & B \\ {}^VC & D \end{pmatrix} \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \tag{1}$$

We see that

$${}^F\begin{pmatrix} A & B \\ {}^VC & D \end{pmatrix} = \begin{pmatrix} {}^FA & {}^FB \\ pC & {}^FD \end{pmatrix}$$

has the same determinant as

$$\begin{pmatrix} {}^FA & p\,{}^FB \\ C & {}^FD \end{pmatrix}$$

But then taking determinants in (1) gives the result. □

Proposition 3. *Let R be a noetherian ring such that* $\operatorname{Spec} R$ *is connected. We assume that $pR = 0$. Let $\alpha : \mathcal{P} \to \mathcal{P}'$ be a homomorphism of displays of the same height h and the same dimension d.*

If $\det \alpha \neq 0$ then there is a nonnegative integer u, such that locally on $\operatorname{Spec} R$ *the following equation holds:*

$$\det \alpha = p^u \varepsilon, \quad \textit{where} \quad \varepsilon \in W(R)^*,\ u \in \mathbb{Z}_{\geq 0}.$$

Proof. If the number u exists locally it is clearly a locally constant function. Therefore the question is local. We may replace $\operatorname{Spec} R$ by a small affine connected neighbourhood.

We set $\eta = \det \alpha$. By the last proposition we find:

$$^{F}\eta = \zeta \cdot \eta \text{ for some } \zeta \in W(R)^{*}. \tag{2}$$

We write $\eta = {}^{V^t}\xi$, such that $\mathbf{w}_0(\xi) \neq 0$. We claim that (2) implies:

$$^{F}\xi = {}^{F^t}\zeta \cdot \xi. \tag{3}$$

To verify this we may assume that $t > 0$. We obtain:

$$^{FV^t}\xi = \zeta\, {}^{V^t}\xi = {}^{V^t}({}^{F^t}\zeta\xi)$$

Since $pR = 0$ the operators F and V acting on $W(R)$ commute. Therefore we deduce (3)

Let $\mathbf{w}_0(\xi) = x$ and $\mathbf{w}_0({}^{F^t}\zeta) = e \in R^*$. We apply $\mathbf{w}_0$ to the equation (3) and obtain:

$$x^p = ex. \tag{4}$$

Since the product

$$x(x^{p-1} - e) = 0$$

has relatively prime factors, it follows that

$$\begin{aligned} D(x) \cup D(x^{p-1} - e) &= \operatorname{Spec} R \\ D(x) \cap D(x^{p-1} - e) &= \quad \emptyset. \end{aligned}$$

Hence by connectedness either $D(x) = \operatorname{Spec} R$ or $D(x) = \emptyset$. In the first case x is nilpotent. But then we find $x = 0$, by iterating the equation (4). This is a contradiction to our choices. Therefore $D(x) = \operatorname{Spec} R$ and x is a unit. Then ξ is a unit too. We find

$$^{F^t}\eta = {}^{F^tV^t}\xi = p^t\xi.$$

But by (2) ${}^{F^t}\eta$ may be expressed as the product of η by a unit. This proves the result. □

Definition 4. *A homomorphism as in the proposition is called an isogeny of displays.*

Let R be a ring such that $pR = 0$. Assume that the ideal of nilpotent elements of R is nilpotent. Let $\alpha : \mathcal{P} \to \mathcal{P}'$ be a homomorphism of nilpotent displays of the same height and dimension. By the functor from the category of nilpotent displays to the category of formal p-divisible groups ([Z1] 3.1) we obtain from α a morphism $\phi : X \to X'$ of p-divisible groups. It follows from loc.cit. Prop. 66 and Prop. 99 that α is an isogeny, iff ϕ is an isogeny of p-divisible groups.

Since $pR = 0$ the Frobenius endomorphism on $W(R)$ induces a Frobenius endomorphism on the truncated Witt vectors $F : W_n(R) \to W_n(R)$. Therefore we may consider truncated Frobenius modules. We are going to prove a version of Proposition 3 for truncated Frobenius modules.

Definition 5. *Let R be a ring such that $pR = 0$. A truncated Frobenius module of level n, dimension d, and height h over R is a finitely generated projective $W_n(R)$-module M of rank h equipped with a Frobenius linear operator $F : M \to M$, such that locally on* $\operatorname{Spec} R$ *the determinant has the form*

$$\det F = p^d \varepsilon$$

where $\varepsilon : \det M \to \det M$ is a Frobenius linear isomorphism.

A Frobenius module M over R induces a truncated Frobenius module, if we tensor it by $W_n(R)$.

Definition 6. *Let M and N be truncated Frobenius modules of level n and of the same dimension d and height h. A morphism of Frobenius modules $\alpha : M \to N$ is called an isogeny if there is a natural number $u < n$ such that the determinant of α has locally on* $\operatorname{Spec} R$ *the form:*

$$^{F^d}\det \alpha = p^u \varepsilon, \quad \varepsilon \in W_n(R)^*.$$

The number u is called the height of the isogeny.

Proposition 7. *Let M and N be truncated Frobenius modules of level n and of the same dimension d and height h over a ring R such that* $\operatorname{Spec} R$ *is connected and $pR = 0$.*

Let $u \geq 0$ be an integer, such that $n > u + d$. Let $\alpha : M \to N$ be a homomorphism of Frobenius modules such that

$$^{F^d}\det \alpha \notin V^{u+1} W_{n-u-1}(R). \tag{5}$$

Then α becomes an isogeny if we truncate it to level $n - d$:

$$\alpha[n-d] : M[n-d] \to N[n-d].$$

Proof. We may assume that M and N are free $W_n(R)$-modules. We choose isomorphisms

$$\det M \simeq W_n(R) \simeq \det N$$

and view $\theta := \det \alpha$ as an element of $W(R)$. Then we obtain a commutative diagram

$$\begin{array}{ccc} \det M & \xrightarrow{\theta} & \det N \\ {\scriptstyle p^d \tau_M F}\downarrow & & \downarrow{\scriptstyle p^d \tau_N F} \\ \det M & \xrightarrow{\theta} & \det N, \end{array}$$

where $\tau_M, \tau_N \in W_n(R)^*$ are units. We obtain

$$p^d \tau_N \ {}^F\theta = \theta p^d \tau_M. \tag{6}$$

Using $p^d = V^d F^d$ in $W_n(R)$, we can divide (6) by V^d. We then obtain an equality in $W_{n-d}(R)$:

$$ {}^{F^{d+1}}\theta[n-d] = {}^{F^d}\theta[n-d]\rho. \tag{7} $$

Here $\theta[n-d]$ denotes the image of θ by the natural restriction $W_n(R) \to W_{n-d}(R)$ and $\rho \in W_{n-d}(R)^*$ is a unit.

On the other hand we may write by assumption:

$$ {}^{F^d}\theta = {}^{V^{u_1}}\sigma, \tag{8} $$

where $u_1 \leq u$, and $\mathbf{w}_0(\sigma) = s_0 \neq 0$. Clearly we may assume $u = u_1$. Since $n - d > u$ we obtain from equation (7)

$$ s_0^p = s_0 e $$

for some unit $e \in R^*$. As in the proof of Proposition 3 (see: (4)) we conclude that s_0 is a unit. Then σ is a unit too. From (8) we obtain

$$ {}^{F^{d+u}}\theta = p^u \sigma. $$

We truncate this equation to $W_{n-d}(R)$ and use (7) to obtain

$$ {}^{F^d}\theta[n-d] = p^u \varepsilon $$

for some unit $\varepsilon \in W_{n-d}(R)^*$. □

Let $n > u$ be natural numbers. It is clear that a morphism of displays $\alpha : \mathcal{P} \to \mathcal{P}'$ is an isogeny of height u, iff the map of the truncated Frobenius modules $\alpha[n] : (P[n], F) \to (P'[n], F)$ is an isogeny of height u.

3 Proof of Purity

For the proof of the purity theorem of de Jong and Oort for p-divisible groups we need to recall a few facts on completely slope divisible p-divisible groups (abbreviated: c.s.d. groups) from [Z2] and [OZ] Definition 1.2. We will use truncated Frobenius modules of p-divisible groups over any scheme U. These are locally free $W_n(\mathcal{O}_U)$-modules.

Lemma 8. *Let Y be a c.s.d. group over a normal noetherian scheme U over $\bar{\mathbb{F}}_p$. Let n be a natural number. Then there is a finite morphism $U' \to U$, such that the truncated Frobenius module $M_Y[n]$ of Y over U' is obtained by base change from a truncated Frobenius module over $\bar{\mathbb{F}}_p$, i.e. we can find a Frobenius module N over $\bar{\mathbb{F}}_p$ such that there is an isomorphism of Frobenius modules*

$$ W_n(\mathcal{O}_{U'}) \otimes_{W_n(\mathcal{O}_U)} M_Y[n] \simeq W_n(\mathcal{O}_{U'}) \otimes_{W(\bar{\mathbb{F}}_p)} N \tag{9} $$

Proof. This is an immediate consequence of [OZ] Proposition 1.3, since it says that this is true if we take for U' the perfect hull of the universal pro-étale cover of U. Another proof is obtained by substituting in the proof of loc.cit. Frobenius modules. □

Proposition 9. *Let T be a regular connected 1-dimensional scheme over $\mathbb{F}_p$. Then any p-divisible group X with constant Newton polygon over T is isogenous to a c.s.d. group.*

Proof. This follows from the main result of [OZ] Thm. 2.1. for any normal noetherian scheme T. But under under the assumptions made the proof is much easier (compare [Z2] proof of Thm. 7). Indeed let $K = K(T)$ be the function field of T. Then we find by over K an isogeny to a c.s.d. group:

$$X_K \to \overset{\circ}{Y} \tag{10}$$

Let $\overset{\circ}{G}$ be the finite group scheme which is the kernel of (10) and let $G \subset X$ be its scheme theoretic closure. We set $Y = X/G$. Using the fact that X has constant Newton polygon one proves that Y is c.s.d. □

The third ingredient is the boundedness principle, which seems to have been known for a long time [M].

Proposition 10. *Let k be an algebraic closed of characteristic p field. Let h be a natural number. Then there is a constant $c \in \mathbb{N}$ with the following property:*

Let M_1 and M_2 be Frobenius modules of height $\leq h$ over k. Let $n \in \mathbb{N}$ be arbitrary and let $\bar{\alpha} : M_1/p^n M_1 \to M_2/p^n M_2$ be a morphism of truncated Frobenius modules which lifts to a morphism of truncated Frobenius modules $M_1/p^{n+c}M_1 \to M_2/p^{n+c}M_2$. Then $\bar{\alpha}$ lifts to a morphism of Frobenius modules $\alpha : M_1 \to M_2$.

A weaker version of this is contained in [O], where the existence of the constant c is only asserted for given modules M_1 and M_2. But one can show that for given modules N_1 resp. N_2 in the isogeny class of M_1 resp. M_2, there are always isogenies $N_1 \to M_1$ resp. $N_1 \to M_1$ whose degrees are bounded by a constant only depending on h. This is another well-known boundedness principle. As an alternative to this proof the reader may use the much stronger results discussed in the introduction of [V2].

Theorem 11. *(de Jong-Oort) Let R be a noetherian local ring of Krull dimension ≥ 2 with $p \cdot R = 0$. Let $U = \operatorname{Spec} R \setminus \{\mathfrak{m}\}$, the complement of the closed point. A p-divisible group X over SpecR, which has constant Newton polygon over U has constant Newton polygon over* $\operatorname{Spec} R$.

Proof. It is not difficult to reduce to the case where R is complete, normal of Krull dimension 2 with algebraically closed residue class field $k = R/\mathfrak{m}$ ([JO]).

Then U is a 1-dimensional regular scheme. We find by Proposition 9 a c.s.d. group Y over U and an isogeny

$$\alpha : Y \to X_{|U}, \tag{11}$$

Let d be the dimension of X let u be the height of α and let c be the number from Proposition 10. We choose a natural number $n > c+u+d$.. After a finite extension of R we may assume by Lemma 8 that the truncated Frobenius module of Y is constant

$$M_Y[n] \simeq W_n(\mathcal{O}_U) \otimes_{W(\bar{\mathbb{F}}_p)} N \tag{12}$$

where N is a Frobenius module over $\bar{\mathbb{F}}_p$. In particular the Newton polygons of N and Y must be the same by the boundedness principle applied to the field $\bar{K}$, where K is the field of fractions of R.

Combining (11) and (12) we find an isogeny of height u of truncated Frobenius modules

$$W_n(\mathcal{O}_U) \otimes_{W(\bar{\mathbb{F}}_p)} N \to W_n(\mathcal{O}_U) \otimes_R M_X[n]. \tag{13}$$

By the normality of R we find $\Gamma(U, W_n(\mathcal{O}_U)) = W_n(R)$. Taking the global section of (13) over U we obtain a morphism of truncated Frobenius modules

$$W_n(R) \otimes_{W(\bar{\mathbb{F}}_p)} N \to M_X[n]. \tag{14}$$

We know that (14) is an isogeny over K of height u. Therefore Proposition 3 is applicable to the morphism (14). We obtain therefore an isogeny of height u of truncated Frobenius modules over R:

$$W_{n-d}(R) \otimes_{W(\bar{\mathbb{F}}_p)} N \to M_X[n-d],$$

It is clear that the base change of an isogeny of truncated Frobenius modules is again an isogeny. Making the base change $R \to k$ we obtain an isogeny:

$$W_{n-d}(k) \otimes_{W(\bar{\mathbb{F}}_p)} N \to W_{n-d}(k) \otimes_{W(R)} M_X[n-d] = M_{X_k}[n-d].$$

The boundedness principle shows that X_k and N have the same Newton polygon. □

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