Lie algebra theory without algebra

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Dedicated to Professor Yu I. Manin, on his 70th. birthday.

1 Introduction

This is an entirely expository piece: the main results discussed are very well-known and the approach we take is not really new, although the presentation may be somewhat different to what is in the literature. The author's main motivation for writing this piece comes from a feeling that the ideas deserve to be more widely known.

Let $\mathbf{g}$ be a Lie algebra over $\mathbf{R}$ or $\mathbf{C}$. A vector subspace $I \subset \mathbf{g}$ is an *ideal* if $[I, \mathbf{g}] \subset I$. The Lie algebra is called *simple* if it is not abelian and contains no proper ideals. A famous result of Cartan asserts that any simple complex Lie algebra has a compact real form (that is to say, the complex Lie algebra is the complexification of the Lie algebra of a compact group). This result underpins the theory of real Lie algebras, their maximal compact subgroups and the classification of symmetric spaces. In the standard approach, Cartan's result emerges after a good deal of theory: the Theorems of Engel and Lie, Cartan's criterion involving the nondegeneracy of the Killing form, root systems etc. On the other hand if one assumes this result known–by some means–then one can immediately read off much of the standard structure theory of complex Lie groups and their representations. Everything is reduced to the compact case (Weyl's "unitarian trick"), and one can proceed directly to develop the detailed theory of root systems etc.

In [4], Cartan wrote

J'ai trouvé effectivement une telle forme pour chacun des types de groupes simples. M. H. Weyl a démontré ensuite l'existence de cette forme par une raisonnement général s'appliquant à tous les cas à fois. On peut se demander si les calculs qui l'ont conduit à ce résultat ne pourraient pas encore se simplifier, ou plutôt si l'on ne pourrait pas, par une raissonnement a priori, démontrer *ce théorème; une telle démonstration permettrait de simplifier notablement l'exposition de la theorie des groupes simples. Je ne suis a cet égard arrivé à aucun résultat; j'indique simplement l'idée qui m'a guidé dans mes recherches infructueuses.*

The direct approach that Cartan outlined (in which he assumed known the nondegeneracy of the Killing form) was developed by Helgason (see page 196 in [5]), and a complete proof was accomplished by Richardson in [15]. In this article we revisit these ideas and present an almost entirely geometric proof of the result. This is essentially along the same lines as Richardson's, so it might be asked what we can add to the story. One point is that, guided by modern developments in Geometric Invariant Theory and its relations with differential geometry, we can nowadays fit this into a much more general context and hence present the proofs in a (perhaps) simpler way. Another is that we are able to remove more of the algebraic theory; in particular, the nondegeneracy of the Killing form. We show that the results can be deduced from a general principle in Riemannian geometry (Theorem 4). The arguments apply directly to real Lie groups, and in our exposition we will work mainly in that setting. In the real case the crucial concept is the following. Suppose V is a Euclidean vector space. Then there is a transposition map $A \mapsto A^T$ on the Lie algebra End V. We say a subalgebra $\mathbf{g} \subset \text{End } V$ is *symmetric* with respect to the Euclidean structure if it is preserved by the transposition map.

Theorem 1. *Let* $\mathbf{g}$ *be a simple real Lie algebra. Then there is a Euclidean vector space* V*, a Lie algebra embedding* $\mathbf{g} \subset \text{End}(V)$*, and a Lie group* $G \subset SL(V)$ *with Lie algebra* $\mathbf{g}$*, such that* $\mathbf{g}$ *is symmetric with respect to the Euclidean structure. Moreover, any compact subgroup of* G *is conjugate in* G *to a subgroup of* $G \cap SO(V)$*.*

We explain in (5.1) below how to deduce the existence of the compact real form, in the complex case. Theorem 1 also leads immediately to the standard results about real Lie algebras and symmetric spaces, as we will discuss further in (5.1).

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2 More general setting

Consider any representation

$$\rho : SL(V) \to SL(W),$$

where V, W are finite-dimensional real vector spaces. Let w be a nonzero vector in W and let G_w be the identity component of the stabiliser of w in $SL(V)$. Then we have

Theorem 2. *If V is an ireducible representation of G_w then there is a Euclidean metric on V such that the Lie algebra of G_w is symmetric with respect to the Euclidean structure, and any compact subgroup of G_w is conjugate in G_w to a subgroup of $G_w \cap SO(V)$.*

Now we will show that Theorem 2 implies Theorem 1. Given a simple real Lie algebra $\mathbf{g}$, consider the action of $SL(\mathbf{g})$ on the vector space W of skew symmetric bilinear maps from $\mathbf{g} \times \mathbf{g}$ to $\mathbf{g}$. The Lie bracket of $\mathbf{g}$ is a point w in W. The group G_w is the identity component of the group of Lie algebra automorphisms of $\mathbf{g}$, and the Lie algebra of G_w is the algebra $\mathrm{Der}(\mathbf{g})$ of derivations of $\mathbf{g}$, that is, linear maps $\delta : \mathbf{g} \to \mathbf{g}$ with

$$\delta[x, y] = [\delta x, y] + [x, \delta y].$$

The adjoint action gives a Lie algebra homomorphism

$$\mathrm{ad} : \mathbf{g} \to \mathrm{Der}(\mathbf{g}).$$

The kernel of ad is an ideal in $\mathbf{g}$. This is not the whole of $\mathbf{g}$ (since $\mathbf{g}$ is not abelian) so it must be the zero ideal (since $\mathbf{g}$ is simple). Hence ad is injective. If U is a vector subspace of $\mathbf{g}$ preserved by G_w then any derivation δ must map U to U. In particular ad_ξ maps U to U for any ξ in $\mathbf{g}$, so $[\mathbf{g}, U] \subset U$ and U is an ideal. Since $\mathbf{g}$ is simple we see that there can be no proper subspace preserved by G_w and the restriction of the representation is irreducible. By Theorem 2 there is a Euclidean metric on $\mathbf{g}$ such that $\mathrm{Der}(\mathbf{g})$ is preserved by transposition. Now we want to see that in fact $\mathrm{Der}(\mathbf{g}) = \mathbf{g}$. For $\alpha \in \mathrm{Der}(\mathbf{g})$ and $\xi \in \mathbf{g}$ we have

$$[ad_\xi, \alpha] = ad_{\alpha(\xi)},$$

so $\mathbf{g}$ is an ideal in $\mathrm{Der}(\mathbf{g})$. Consider the bilinear form

$$B(\alpha_1, \alpha_2) = \mathrm{Tr}(\alpha_1 \alpha_2)$$

on $\mathrm{Der}(\mathbf{g})$. This is nondegenerate, since $\mathrm{Der}(\mathbf{g})$ is preserved by transposition and $B(\alpha, \alpha^T) = |\alpha|^2$. We have

$$B([\alpha, \beta], \gamma) + B(\beta, [\alpha, \gamma]) = 0$$

for all $\alpha, \beta, \gamma \in \mathrm{Der}(\mathbf{g})$. Thus the subspace

$$\mathbf{g}^{\mathrm{perp}} = \{\alpha \in \mathrm{Der}\mathbf{g} : B(\alpha, ad_\xi) = 0 \ \text{ for all } \xi \in \mathbf{g}\}$$

is another ideal in $\mathrm{Der}(\mathbf{g})$. On the other hand the map $\alpha \mapsto -\alpha^T$ is an automorphism of $\mathrm{Der}(\mathbf{g})$, so $\mathbf{g}^T$ is also an ideal in $\mathrm{Der}(\mathbf{g})$. Suppose that $\mathbf{g} \cap \mathbf{g}^T \neq 0$. Then we can find a non-zero element α of $\mathbf{g} \cap \mathbf{g}^T$ with $\alpha^T = \pm\alpha$ and then $B(\alpha, \alpha) = \pm|\alpha|^2 \neq 0$, so the restriction of B to $\mathbf{g}$ is not identically zero. This means that $I = \mathbf{g} \cap \mathbf{g}^{\mathrm{perp}}$ is not the whole of $\mathbf{g}$, but I is an ideal in $\mathbf{g}$ so, since $\mathbf{g}$ is simple, we must have $I = 0$.

We conclude from the above that if $\mathbf{g}$ were a proper ideal in $\mathrm{Der}(\mathbf{g})$ there would be another proper ideal J in $\mathrm{Der}(\mathbf{g})$ such that $J \cap \mathbf{g} = 0$. (We take J to be either $\mathbf{g}^T$ or $\mathbf{g}^{\mathrm{perp}}$.) But then for $\alpha \in J$ we have $[\alpha, \mathbf{g}] = 0$, but this means that α acts trivially on $\mathbf{g}$, which gives a contradiction.

Finally, the statement about compact subgroups in Theorem 1 follows immediately from that in Theorem 2.

(The argument corresponding to the above in the complex case (see (5.1)) is more transparent.)

3 Lengths of vectors

We will now begin the proof of Theorem 2. The idea is to find a metric by minimising the associated norm of the vector w. In the Lie algebra situation, which we are primarily concerned with here, this is in essence the approach suggested by Cartan and carried through by Richardson. In the general situation considered in Theorem 2 the ideas have been studied and applied extensively over the last quarter century or so, following the work of Kempf-Ness [9], Ness [13] and Kirwan [7]. Most of the literature is cast in the setting of complex representations. The real case has been studied by Richardson and Slodowy [14] and Marian [12] and works in just the same way.

Recall that we have a representation ρ of $SL(V)$ in $SL(W)$, where V and W are real vector spaces, a fixed vector w in W and we define G_w to be the identity component of the stabiliser of w in $SL(V)$. Suppose we also have some compact subgroup (which could be trivial) $K_0 \subset G_w$. We fix any Euclidean metric $| \, |_1$ on V which is preserved by K_0. Now it is standard that we can choose a Euclidean metric $| \, |_W$ on W which is invariant under the restriction of ρ to $SO(V)$. We want to choose this metric $| \, |_W$ with the further property that the derivative $d\rho$ intertwines transposition in $\mathrm{End}V$ (defined by $| \, |_1$) and transposition in $\mathrm{End}W$ (defined by $| \, |_W$); that is to say

$$d\rho(\xi^T) = (d\rho(\xi))^T \, .$$

To see that this is possible we can argue as follows. We complexify the representation to get $\rho_{\mathbf{C}} : SL(V \otimes \mathbf{C}) \to SL(W \otimes C)$. Then the compact group generated by the action of $SU(V \otimes \mathbf{C})$ and complex conjugation acts on $W \otimes \mathbf{C}$ and we can choose a Hermitian metric on $W \otimes \mathbf{C}$ whose norm function is invariant under this group. Invariance under complex conjugation means that this Hermitian metric is induced from a Euclidean metric on W. Then the fact that $\rho_{\mathbf{C}}$ maps $SU(V \otimes \mathbf{C})$ to $SU(W \otimes \mathbf{C})$ implies that $d\rho$ has the property desired. (The author is grateful to Professors He and Zhang for pointing out the need for this argument. In our main application, to Theorem 1, the standard metric on W already has the desired property.)

Now define a function $\tilde{F}$ on $SL(V)$ by

$$\tilde{F}(g) = |g(w)|^2_W.$$

For $u \in SO(V)$ and $\gamma \in G_w$ we have

$$\tilde{F}(ug\gamma) = |ug\gamma(w)|^2_W = |ug(w)|^2_W = \tilde{F}(g)$$

So $\tilde{F}$ induces a function F on the quotient space $\mathcal{H} = SL(V)/SO(V)$, invariant under the natural action of $G_w \subset SL(V)$. We can think about this in another, equivalent, way. We identify $\mathcal{H}$ with the Euclidean metrics on V of a fixed determinant. Since $\rho : SL(V) \to SL(W)$ maps $SO(V)$ to $SO(W)$ it induces a map from $SL(V)/SO(V)$ to $SL(W)/SO(W)$ and so a metric on V with the same determinant as $| \ |_1$ induces a metric on W. Then F is given by the square of the induced norm of the fixed vector w. Explicitly, the identification of $SL(V)/SO(V)$ with metrics is given by $[g] \mapsto | \ |_g$ where

$$|v|^2_g = |gv|^2_1 = \langle v, g^T gv\rangle_1.$$

This function F has two crucial, and well-known, properties, which we state in the following Lemmas

Lemma 3. *Suppose F has a critical point at $H \in \mathcal{H}$. Then the Lie algebra of the stabiliser G_w is symmetric with respect to the Euclidean structure H on V.*

To prove this, there is no loss in supposing that H is the original metric $| \ |_1$. (For we can replace w by gw for any $g \in SL(V)$.) The fact that ρ maps $SO(V)$ to $SO(W)$ implies that its derivative takes transposition in End$V$ (defined by $| \ |_1$) to transposition in End$W$ defined by $| \ |_W$. The condition for $\tilde{V}$ to be stationary is that

$$\langle d\rho(\xi)w, w\rangle_W = 0$$

for all ξ in the Lie algebra of $SL(V)$. In particular consider elements of the form $\xi = [\eta, \eta^T]$ and write $A = d\rho(\eta)$. Then we have

$$0 = \langle d\rho[\eta, \eta^T]w, w\rangle_W = \langle [A, A^T]w, w\rangle_W = |A^T w|^2_W - |Aw|^2_W.$$

By definition η lies in the Lie algebra of G_w if and only if $Aw = 0$. By the identity above, this occurs if and only if $A^T w = 0$, which is just when η^T lies in the Lie algebra of G_w.

For the second property of the function we need to recall the standard notion of geodesics in $\mathcal{H}$. We can identify $\mathcal{H}$ with the positive definite symmetric elements of $SL(V)$, with the quotient map $SL(V) \to \mathcal{H}$ given by $g \mapsto g^T g$. Then the geodesics in $\mathcal{H}$ are paths of the form

$$\gamma(t) = g^T \exp(St) g, \tag{1}$$

where g and S are fixed, with $g \in SL(V)$ and S a trace-free endomorphism which is symmetric with respect to $| \ |_1$. Another way of expressing this is that a geodesics through any point $H \in \mathcal{H}$ is the orbit of H under a 1-parameter subgroup $e(t)$ in $SL(V)$ where $e(t) = \exp(\sigma t)$ with σ a symmetric endomorphism with respect to the metric H.

Lemma 4. *1. For any geodesic γ the function $F \circ \gamma$ is convex i.e.*

$$\frac{d^2}{dt^2} F(\gamma(t)) \geq 0.$$

2. If F achieves its minimum in $\mathcal{H}$ then G_w acts transitively on the set of minima.

To prove the first part, note that, replacing w by gw, we can reduce to considering a geodesic through the base point $[1] \in \mathcal{H}$, so of the form $\exp(St)$ where S is symmetric with respect to $|\ |_1$. Now the derivative $d\rho$ maps the symmetric endomorphism S to a symmetric endomorphism $A \in \mathrm{End}(W)$. We can choose an orthonormal basis in W so that A is diagonal, with eigenvalues λ_i say. Then if w has coordinates w_i in this basis we have

$$F(\exp(St)) = \tilde{F}(\exp(St/2)) = \sum |\exp(\lambda_i t/2) w_i|_W^2 = \sum |w_i|^2 \exp(\lambda_i t),$$

and this is obviously a convex function of t.

To prove the second part note that, in the above, the function $F(\exp(St)$ is either strictly convex or constant, and the latter only occurs when $\lambda_i = 0$ for each index i such that $w_i \neq 0$, which is the same as saying that $\exp(St)w = w$ for all t, or that the 1-parameter subgroup $\exp(St)$ lies in G_w. More generally if we write a geodesic through a point H as the orbit of H under a 1-parameter subgroup $e(t)$ in $SL(V)$ then the function is constant if and only if the 1-parameter subgroup lies in G_w. Suppose that H_1, H_2 are two points in $\mathcal{H}$ where F is minimal. Then F must be constant on the geodesic between H_1, H_2. Thus H_2 lies in the orbit of H_1 under a 1-parameter subgroup in G_w. So G_w acts transitively on the set of minima.

We now turn back to the proof of Theorem 2. Suppose that the convex function F on $\mathcal{H}$ achieves a minimum at $H_1 \in \mathcal{H}$. Then by Lemma 3 the Lie algebra of G_w is symmetric with respect to the Euclidean structure H_1 on V. It only remains to see that the compact subgroup K_0 of G_w is conjugate to a subgroup of the orthogonal group for this Euclidean structure. For each $H \in \mathcal{H}$ we have a corresponding special orthogonal group $SO(H, V) \subset SL(V)$. For $g \in SL(V)$ the groups $SO(H, V), SO(g(H), V)$ are conjugate by g in $SL(V)$. Recall that we chose the metric $|\ |_1$ to be K_0 invariant. This means that K_0 fixes the base point $[1]$ in $\mathcal{H}$. Suppose we can find a point H_0 in $\mathcal{H}$ which minimises F and which is also K_0-invariant. Then K_0 is contained in $SO(H_0, V)$. But by the second part of Lemma 4 there is a $\gamma \in G_w$ such that $\gamma(H_0) = H_1$. Thus conjugation by γ takes $SO(H_0, V)$ to $SO(H_1, V)$ and takes K_0 to a subgroup on $SO(H_1, V)$, as required.

To sum up, Theorem 2 will be proved if we can establish the following result.

Theorem 5. *Let F be a convex function on $\mathcal{H}$, invariant under a group $G_w \subset SL(V)$. Let K_0 be a compact subgroup of G_w and let $[1] \in \mathcal{H}$ be fixed by K_0. Then if V is an ireducible representation of G_w there is a point $H_0 \in \mathcal{H}$ where F achieves its minimum and which is fixed by K_0.*

(Notice that the hypothesis here that there is a point $[1] \in \mathcal{H}$ fixed by K_0 is actually redundant, since any compact subgroup of $SL(V)$ fixes some metric.)

4 Riemannian geometry argument

In this section we will see that Theorem 5 is a particular case of a more general result in Riemannian geometry. Let M be a complete Riemannian manifold, so for each point $p \in M$ we have a surjective exponential map

$$\exp_p : TM_p \to M.$$

We suppose M has the following property

Property (*)
For each point p in M the exponential map $\exp_p$ is distance-increasing:

$$d(\exp_p(\xi), \exp_p(\eta)) \geq |\xi - \eta|.$$

Readers with some background in Riemannian geometry will know that it is equivalent to say that M is simply connected with nonpositive sectional curvature, but we do not need to assume knowledge of these matters. The crucial background we need to know is

Fact
There is a metric on $\mathcal{H} = SL(V)/SO(V)$ for which the action of $SL(V)$ is isometric, with the geodesics described in (1) above and having Property (*).

This Riemannian metric on $\mathcal{H}$ can be given by the formula

$$\|\delta H\|_H^2 = \mathrm{Tr}\left(\delta H H^{-1}\right)^2.$$

The distance-increasing property can be deduced from the fact that $\mathcal{H}$ has non-positive curvature and standard comparison results for Jacobi fields. For completeness, we give a self-contained proof of the Fact in the Appendix.

The piece of theory we need to recall in order to state our Theorem is the notion of the "sphere at infinity" associated to a manifold M with Property

(*). This will be familiar in the prototype cases of Euclidean space and hyperbolic space. In general, for $x \in M$ write S_x for the unit sphere in the tangent space TM_x and define

$$\Theta_x : M \setminus \{x\} \to S_x$$

by

$$\Theta_x(z) = \frac{\exp_x^{-1}(z)}{|\exp_x^{-1}(z)|}.$$

If y is another point in M and R is greater than the distance $d = d(x,y)$ we define

$$F_{R,x,y} : S_y \to S_x$$

by

$$F_{R,x,y}(\nu) = \Theta_x \exp_y(R\nu).$$

Lemma 6. *For fixed x, y, ν the norm of the derivative of $F_{R,x,y,\nu}$ with respect to R is bounded by*

$$|\frac{\partial}{\partial R} F_{R,x,y}(\nu)| \leq \frac{d}{R(R-d)}.$$

Let γ be the geodesic $\gamma(t) = \exp_y(t\nu)$, let w be the point $\gamma(R)$ and let σ be the geodesic from x to w. The distance-increasing property of $\exp_x$ implies that the norm of the derivative appearing in the statement is bounded by $d(x,w)^{-1}$ times the component of $\gamma'(R)$ orthogonal to the tangent vector of σ at w. Thus

$$|\frac{\partial}{\partial R} F_{R,x,y}(\nu)| \leq \frac{\sin\phi}{d(x,w)},$$

where ϕ is the angle between the geodesics γ, σ at w. By the triangle inequality $d(x,w) \geq R - d$. In a Euclidean triangle with side lengths d, R the angle opposite to the side of length d is at most $\sin^{-1}(d/R)$. It follows from the distance-increasing property of $\exp_z$ that $\sin\phi \leq d/R$. Thus

$$\frac{\sin\phi}{d(x,w)} \leq \frac{d}{R(R-d)},$$

as required.

Since the integral of the function $1/R(R-d)$, with respect to R, from $R = 2d$ (say) to $R = \infty$, is finite, it follows from the Lemma that $F_{R,x,y}$ converges uniformly as $R \to \infty$ to a continuous map $F_{x,y} : S_y \to S_x$, and obviously $F_{x,x}$ is the identity. Let z be another point in M and ν be a unit tangent vector at z. Then we have an identity, which follows immediately from the definitions,

$$F_{R,x,z}(\nu) = F_{R',x,y} \circ F_{R,y,z}(\nu),$$

where $R' = d(y, \exp_z(R\nu))$. Since, by the triangle inequality again,

$$R' \geq R - d(y,z),$$

we can take the limit as $R \to \infty$ to obtain

$$F_{x,z} = F_{x,y} \circ F_{y,z} : S_z \to S_x.$$

In particular, $F_{y,x}$ is inverse to $F_{x,y}$ so the maps $F_{x,y}$ give a compatible family of homeomorphisms between spheres in the tangent spaces. We define the *sphere at infinity* $S_\infty(M)$ to be the quotient of the unit sphere bundle of M by these homeomorphisms, with the topology induced by the identification with S_{x_0} for any fixed base point x_0.

Now suppose that a topological group Γ acts by isometries on M. Then Γ acts on $S_\infty(M)$, as a set. Explicitly, if we fix a base point x_0 and identify the sphere at infinity with S_{x_0},, the action of a group element $g \in \Gamma$ is given by

$$g(\nu) = \lim_{R\to\infty} \Theta_{x_0} g(\exp_{x_0} R\nu).$$

Write the action as

$$A : \Gamma \times S_{x_0} \to S_{x_0}.$$

Given a compact set $P \subset \Gamma$ we can define

$$A_R : P \times S_{x_0} \to S_{x_0},$$

for sufficiently large R, by

$$A_R(g, \nu) = \Theta_{x_0} g(\exp_{x_0} R\nu).$$

Since $g(\exp_{x_0} R\nu) = \exp_{g(x_0)}(Rg_*\nu)$ the maps A_R converge uniformly as $R \to \infty$ to the restriction of A to $P \times S_{x_0}$. It follows that the action A is continuous. With these preliminaries in place we can state our main technical result.

Theorem 7. *Suppose that the Riemannian manifold M has Property (*). Suppose that Γ acts by isometries on M and F is a convex Γ-invariant function on M. Then either there is a fixed point for the action of Γ on $S_\infty(M)$ or the function F attains its minimum in M. Moreover, in the second case, if K_0 is a subgroup of Γ which fixes a point $x \in M$, then there is a point $x' \in M$ where F attains its minimum in M and with x' fixed by K_0.*

Return now to our example $\mathcal{H}$. The tangent space at the identity matrix [1] is the set of trace-free symmetric matrices. We define a *weighted flag* $(\mathcal{F}, \underline{\mu})$ to be a strictly increasing sequence of vector subspaces

$$0 = F_0 \subset F_1 \subset F_2 \ldots \subset F_r = V$$

with associated weights $\mu_1 > \mu_2 \ldots > \mu_r$, subject to the conditions

$$\sum n_i\mu_i = 0, \sum n_i\mu_i^2 = 1,$$

where $n_i = \dim F_i/F_{i-1}$. If S is a trace-free symmetric endomorphism with $\mathrm{Tr}\, S^2 = 1$ then we associate a weighted flag to S as follows. We take μ_i to be the eigenvalues of S, with eigenspaces E_i, and form a flag with

$$F_1 = E_1\ ,\ F_2 = E_1 \oplus E_2, \dots .$$

It is clear then that the unit sphere $S_{[1]}$ in the tangent space of $\mathcal{H}$ at [1] can be identified with the set of all weighted flags. Now there is an obvious action of $SL(V)$ on the set of weighted flags and we have:

Lemma 8. *The action of $SL(V)$ on the sphere at infinity in $\mathcal{H}$ coincides with the obvious action under the identifications above.*

This is clearly true for the subgroup $SO(V)$. We use the fact that given any weighted flag $(\mathcal{F}, \underline{\mu})$ and $g \in SL(V)$ we can write $g = uh$ where $u \in SO(V)$ and h preserves $\mathcal{F}$. (This is a consequence of the obvious fact that $SO(V)$ acts transitively on the set of flags of a given type.) Thus it suffices to show that such h fix the point S in the unit sphere corresponding to $(\mathcal{F}, \underline{\mu})$ in the differential-geometric action. By the $SO(V)$ invariance of the set-up we can choose a basis so that $\mathcal{F}$ is the standard flag

$$0 \subset \mathbf{R}^{n_1} \subset \mathbf{R}^{n_1} \oplus \mathbf{R}^{n_2} \dots \subset \mathbf{R}^n.$$

Then S is the diagonal matrix with diagonal entries $\mu_1, \dots, \mu_r$, repeated according to the multiplicities $n_1, \dots, n_r$. The matrix h is upper triangular in blocks with respect to the flag. Now consider, for a large real parameter R the matrix

$$M_R = \exp(-\frac{RS}{2}) h \exp(\frac{RS}{2}).$$

Consider a block h_{ij} of h. The corresponding block of M_R is

$$(M_R)_{ij} = e^{R(\mu_i - \mu_j)/2} h_{ij}.$$

Since h is upper-triangular in blocks and the μ_i are increasing, we see that M_R has a limit as R tends to infinity, given by the diagonal blocks in h. Since these diagonal blocks are invertible the limit of $M(R)$ is invertible, hence

$$\delta_R = \mathrm{Tr}\,(\log(M_R M_R^*))^2$$

is a bounded function of R. But $\delta_R^{1/2}$ is the distance in $\mathcal{H}_n$ between $\exp(RS)$ and $h \exp(RS) h^T$. It follows from the comparison argument, as before, that the angle between $\Theta_{[1]}(h \exp(RS) h^T)$ and S tends to zero as $R \to \infty$, hence h fixes S in the differential geometric action.

Now Theorem 3 is an immediate consequence of Theorem 7 and Lemma 8, since if G_w fixes a point on the sphere at infinity in $\mathcal{H}$ it fixes a flag, hence some non-trivial subspace of V, and V is reducible as a representation of G_w.

Remarks 9. • The advantage of this approach is that Theorem 4 seems quite accessible to geometric intuition. For example it is obviously true in the case when M is hyperbolic space, taking the ball model, and we suppose that F extends continuously to the boundary of the ball. For then F attains its minimum on the closed ball and if there are no minimising points in the interior the minimiser on the boundary must be unique (since there is a geodesic asymptotic to any two given points in the boundary).

- The author has not found Theorem 7 in the literature, but it does not seem likely that it is new. There are very similar results in [1] for example. The author has been told by Martin Bridson that a more general result of this nature holds, in the context of proper CAT(0) spaces. The proof of this more general result follows in an obvious way from Lemma 8.26 of [3] (see also Corollary 8.20 in that reference).
- The hypothesis on the existence of a fixed point x for K_0 in the statement of Theorem 7 is redundant, since any compact group acting on a manifold with Property (*) has a fixed point, by a theorem of Cartan (see the remarks at the end of Section 3 above, and at the end of (5.2) below). However we do not need to use this.

We now prove Theorem 7. We begin by disposing of the statement involving the compact group K_0. Suppose that F attains its minimum somewhere in M. Then, by convexity, the minimum set is a totally geodesic submanifold $\Sigma \subset M$. The action of K_0 preserves Σ, since F is Γ-invariant and K_0 is contained in Γ. Let x' be a point in Σ which minimises the distance to the K_0 fixed-point x. Then if x'' is any other point in Σ the geodesic segment from x' to x'' lies in Σ and is orthogonal to the geodesic from x to x' at x'. By the distance-increasing property of the exponential map at x' it follows that the distance from x to x'' is strictly greater than the distance from x to x'. Thus the distance-minimising point x' is unique, hence fixed by K_0.

To prove the main statement in Theorem 7 we use the following Lemma.

Lemma 10. *Suppose that M has Property (*) and N is any set of isometries of M. If there is a sequence x_i in M with $d(x_0, x_i) \to \infty$ and for each $g \in N$ there is a C_g with $d(x_i, gx_i) \leq C_g$ for all i, then there is a point in $S_\infty(M)$ fixed by N.*

Set $R_i = d(x_0, x_i)$ and $\nu_i = \Theta_{x_0}(x_i) \in S_{x_0}$. By the compactness of this sphere we may suppose, after perhaps taking a subsequence, that the ν_i converge as i tends to infinity to some $\nu \in S_{x_0}$. Then for each $g \in N$ we have, from the definitions,

$$A_{R_i}(g, \nu_i) = \Theta_{x_0}(gx_i).$$

Fixing g, let ϕ_i be the angle between the unit tangent vectors $\nu_i = \Theta_{x_0}(x_i)$ and $\Theta_{x_0}(gx_i)$. The distance increasing property implies, as in Lemma 6, that

$$\sin\phi_i \leq C_g/R_i.$$

The angle ϕ_i can be regarded as the distance dist(,) between the points ν_i and $A_{R_i}(g,\nu_i)$ in the sphere S_{x_0}. In other words we have

$$\text{dist }((\nu_i, A_{R_i}(g,\nu_i)) \leq \sin^{-1}(\frac{C_g}{R_i}).$$

Now take the limit as $i \to \infty$: we see that dist $(\nu, A(g,\nu)) = 0$, which is to say that ν is fixed by g.

To prove Theorem 7, consider the gradient vector field grad F of the function F, and the associated flow

$$\frac{dx}{dt} = -\text{grad } F_x$$

on M. By the standard theory, given any initial point there is a solution $x(t)$ defined for some time interval $(-T,T)$.

Lemma 11. *If $x(t)$ and $y(t)$ are two solutions of the gradient flow equation, for $t \in (-T,T)$, then $d(x(t),y(t))$ is a non-increasing function of t.*

If $x(t)$ and $y(t)$ coincide for some t then they must do so for all t, by uniqueness to the solution of the flow equation, and in that case the result is certainly true. If $x(t)$ and $y(t)$ are always different then the function $D(t) = d(x(t),y(t))$ is smooth: we compute the derivative at some fixed t_0. Let $\gamma(s)$ be the geodesic from $x(t_0) = \gamma(0)$ to $y(t_0) = \gamma(D)$. Clearly

$$D'(t_0) = \langle \text{grad}F_{x(t)}, \gamma'(0)\rangle - \langle \text{grad}F_{y(t)}, \gamma'(D)\rangle.$$

But $\langle \text{grad}F_{\gamma(s)}, \gamma'(s)\rangle$ is the derivative of the function $F\circ\gamma(s)$, which is non-decreasing in s by the convexity hypothesis, so $D'(t_0) \geq 0$, as required.

A first consequence of this Lemma—applied to $y(t) = x(t+\delta)$ and taking the limit as $\delta \to 0$— is that the velocity $|\frac{dx}{dt}|$of a gradient path is decreasing. Thus for finite positive time $x(t)$ stays in an *a priori* determined compact subset of M (since this manifold is complete). It follows that the flow is actually defined for all positive time, for any initial condition. Consider an arbitrary initial point x_0 and let $x(t)$ be this gradient path, for $t \geq 0$. If there is a sequence $t_i \to \infty$ such that $x(t_i)$ is bounded then, taking a subsequence, we can suppose that $x(t_i)$ converges and it follows in a standard way that the limit is a minimum of F. If there is no such minimum then we can take a sequence such that $x_i = x(t_i)$ tends to infinity. Suppose that g is in Γ, so the action of g on M preserves F and the metric. Then $y(t) = g(x(t))$ is a gradient path with initial value $g(x_0)$ and $d(x_i, gx_i) \leq C_g = d(x_0, gx_0)$. Then, by Lemma 10, there is a fixed point for the action of Γ on $S_\infty(M)$.

There is an alternative argument which is perhaps more elementary, although takes more space to write down in detail. With a fixed base point x_0 choose c with $\inf_M F < c < F(x_0)$ and let Σ_c be the hypersurface $F^{-1}(c)$. Let $z_c \in \Sigma_c$ be a point which minimises the distance to x_0, so that x_0 lies on a geodesic γ from z_c normal to Σ_c. The convexity of F implies that the second fundamental form of Σ_c at z_c is positive with respect to the normal given by the geodesic from z_c to x_0. A standard comparison argument in "Fermi coordinates" shows that the exponential map on the normal bundle of Σ_c is distance increasing on the side towards x_0. In particular, let w be another point in Σ_c and $y = \exp(R\xi)$ where ξ is the unit normal to Σ_c at w pointing in the direction of increasing F and $R = d(x_0, z_c)$. Then we have $d(z_c, w) \leq d(x_0, y)$. Now suppose g is in Γ. Then g preserves Σ_c and if we take $w = g(z_c)$ above we have $y = g(x_0)$. So we conclude from this comparison argument that $d(z_c, g(z_c)) \leq d(x_0, gx_0)$. Now take a sequence c_i decreasing to $\inf F$ (which could be finite or infinite). We get a sequence $x_i = z_{c_i}$ of points in M. If (x_i) contains a bounded subsequence then we readily deduce that there is a minimum of F. If x_i tends to infinity we get a sequence to which we can apply Lemma 5, since $d(x_i, gx_i) \leq C_g = d(x_0, gx_0)$.

5 Discussion

5.1 Consequences of Theorem 1

- We start with a simple Lie algebra $\mathbf{g}$ and use Theorem 1 to obtain an embedding $\mathbf{g} \subset \mathrm{End}(V)$, for a Euclidean space V, with $\mathbf{g}$ preserved by the transposition map. We also have a corresponding Lie group $G \subset SL(V)$. We write K for the identity component of $G \cap SO(V)$. It follows immediately from Theorem 1 that K is a maximal compact connected subgroup of G, and any maximal compact connected subgroup is conjugate to K.
- The involution $\alpha \mapsto -\alpha^T$ on $\mathrm{End}(V)$ induces a Cartan involution of $\mathbf{g}$ so we have an eigenspace decomposition

$$\mathbf{g} = \mathbf{k} \oplus \mathbf{p}$$

  with $\mathbf{k} = \mathrm{Lie}(K)$ and

$$[\mathbf{k}, \mathbf{k}] \subset \mathbf{k}\ ,\ \ [\mathbf{k}, \mathbf{p}] \subset \mathbf{p}\ ,\ \ [\mathbf{p}, \mathbf{p}] \subset \mathbf{k}. \tag{2}$$

  Notice that $\mathbf{k}$ is non-trivial, for otherwise $\mathbf{g}$ would be abelian.
- Consider the bilinear form $B(\alpha, \beta) = \mathrm{Tr}(\alpha\beta)$ on $\mathbf{g}$. Clearly this is positive definite on $\mathbf{p}$, negative definite on $\mathbf{k}$ and the two spaces are B-orthogonal. Thus B is nondegenerate. The Killing form $\hat{B}$ of $\mathbf{g}$ is negative-definite on $\mathbf{k}$ (since the restriction of the adjoint action to K preserves some metric and $\mathbf{k}$ is not an ideal). So the Killing form is not identically zero and must be a positive multiple of B (otherwise the relative eigenspaces would be

proper ideals). In fact we do not really need this step, since in our proof of Theorem 1 the vector space W is $\mathbf{g}$ itself, and B is trivially equal to the Killing form.

- Either $\mathbf{p}$ is trivial, in which case G is itself compact, or there is a nontrivial Riemannian symmetric space of negative type $M^-_{\mathbf{g}} = G/K$ associated to $\mathbf{g}$. This can be described rather explicitly. Let us now fix on the specific representation $\mathbf{g} \subset \mathrm{End}(\mathbf{g})$ used in the proof of Theorem 1. Say a Euclidean metric on $\mathbf{g}$ is "optimal" if the adjoint embedding is symmetric with respect to the metric, as in Theorem 1. (It is easy to see that the optimal metrics are exactly those which minimise the norm of the bracket, among all metrics of a given determinant.) Then $M^-_{\mathbf{g}}$ can be identified with the set of optimal metrics, a totally geodesic submanifold of $\mathcal{H} = SL(\mathbf{g})/SO(\mathbf{g})$.
- So far we have worked exclusively in the real setting. We will now see how to derive the existence of compact real forms of a simple complex Lie algebra.

Lemma 12. *If $\mathbf{g}$ is a simple complex Lie algebra then it is also simple when regarded as a real Lie algebra.*

To see this, suppose that $A \subset \mathbf{g}$ is a proper real ideal: a real vector subspace with $[A, \mathbf{g}] \subset A$. By complex linearity of the bracket $A \cap iA$ is a complex ideal, so we must have $A \cap iA = 0$. But then since $i\mathbf{g} = \mathbf{g}$ we have $[A, \mathbf{g}] = [A, i\mathbf{g}] = i[A, \mathbf{g}] \subset iA$, so $[A, \mathbf{g}] = 0$. But $A + iA$ is another complex ideal, so we must have $\mathbf{g} = iA \oplus A$ and $\mathbf{g}$ is Abelian.
Next we have

Lemma 13. *Let $\mathbf{g}$ be a simple complex Lie algebra and let $\mathbf{g} = \mathrm{Lie}(G) \subset \mathrm{End}V$ be an embedding provided by Theorem 1, regarding $\mathbf{g}$ as a real Lie algebra. Then $\mathbf{g}$ is the complexification of the Lie algebra of the compact group $K = G \cap SO(V)$.*

The inclusions (2) imply that

$$I = (\mathbf{p} \cap i\mathbf{k}) + (\mathbf{k} \cap i\mathbf{p}),$$

is a complex ideal in $\mathbf{g}$, so either $I = \mathbf{g}$ or $I = 0$. In the first case we have $i\mathbf{k} = \mathbf{p}$ and $\mathbf{g}$ is the complexification of $\mathbf{k}$, as required. So we have to rule out the second case. If this were to hold we have $\mathbf{k} \cap i\mathbf{p} = 0$ so $\mathbf{g} = \mathbf{k} \oplus (i\mathbf{p})$. Then $[i\mathbf{p}, i\mathbf{p}] \subset \mathbf{k}$ so the map σ on $\mathbf{g}$ given by multiplication by 1 on $\mathbf{k}$ and by -1 on $i\mathbf{p}$ is another involution of $\mathbf{g}$, regarded as a real Lie algebra. Now let $\hat{B}_{\mathbf{C}}$ be the Killing form regarded as a complex Lie algebra. So $\hat{B} = 2\mathrm{Re}\hat{B}_{\mathbf{C}}$. The fact that σ is an involution of $\mathbf{g}$ means that $\hat{B}(\mathbf{k}, i\mathbf{p}) = 0$. But we know that $\mathbf{p}$ is the orthogonal complement of $\mathbf{k}$ with respect to B and $\hat{B}$, so we must have $i\mathbf{p} = \mathbf{p}$. But $\hat{B}$ is positive definite on $\mathbf{p}$ while $\hat{B}(i\alpha, i\alpha) = 2\mathrm{Re}\hat{B}_{\mathbf{C}}(i\alpha, i\alpha) = -\hat{B}(\alpha, \alpha)$ so $\mathbf{p} \cap i\mathbf{p} = 0$. This means that $\mathbf{p} = 0$ and $\mathbf{g} = \mathbf{k}$ which is clearly impossible (by the same argument with the Killing form).

- The argument above probably obscures the picture. If one is interested in the complex situation it is much clearer to redo the whole proof in this setting, working with Hermitian metrics on complex representation spaces. The proof goes through essentially word-for-word, using the fact that the standard metric on $SL(n, \mathbf{C})/SU(n)$ has Property (*).Then one can deduce the real case from the complex case rather than the other way around, as we have done above.
- Returning to the case of a simple real Lie algebra $\mathbf{g}$, which is not the Lie algebra of a compact group, we can also give an explicit description of the symmetric space $M^+_{\mathbf{g}}$ of positive type dual to $M^-_{\mathbf{g}}$. Fix an optimal metric on $\mathbf{g}$ and extend it to a Hermitian metric H on $\mathbf{g} \otimes \mathbf{C}$. Then $M^+_{\mathbf{g}}$ is the set of real forms $\mathbf{g}' \subset \mathbf{g} \otimes \mathbf{C}$ which are conjugate by G^c to $\mathbf{g}$ and such that the restriction of Re H to $\mathbf{g}'$ is an optimal metric on $\mathbf{g}'$. This is a totally geodesic submanifold of $SU(\mathbf{g} \otimes \mathbf{C})/SO(\mathbf{g})$.

5.2 Comparison with other approaches

The approach we have used, minimising the norm of the Lie bracket, is essentially the same as that suggested by Cartan, and carried through by Richardson, with the difference that we do not assume known that the Killing form is nondegenerate so we operate with a special linear group rather than an orthogonal group. The crucial problem is to show that the minimum is attained when the Lie algebra is simple. This can be attacked by considering points in the closure of the relevant orbit for the action on the projectivized space. Richardson gives two different arguments. One uses the fact that a semisimple Lie algebra is rigid with respect to small deformations; the other uses the fact that a semisimple Lie algebra is its own algebra of derivations, so the orbits in the variety of semisimple Lie algebras all have the same dimension.

There is a general procedure for testing when an orbit contains a minimal vector, using Hilbert's 1-parameter subgroup criterion for stability in the sense of Geometric Invariant Theory [10]. In the Lie algebra situation this gives a criterion involving the nonexistence of filtrations of a certain kind, but the author does not know an easy argument to show that simple Lie algebras do not have such filtrations. However it is also a general fact that, in the unstable case, there is a preferred maximally destabilising 1-parameter subgroup. This theory was developed by Kempf [8], and Hesselink [6], Bogomolov [2] and Rousseau [16] in the algebraic setting, and—in connection with the moment map and the length function—by Kirwan [7] and Ness [13]. The argument we give in Section 4 is essentially a translation of this theory into a differential geometric setting. Lauret [11] has applied this general circle of ideas (Geometric Invariant Theory/Moment maps/minimal vectors) to more sophisticated questions in Lie algebra theory—going beyond the case of simple algebras.

One advantage of this method, in the real case, is that the uniqueness of maximal compact subgroups up to conjugacy emerges as part of the package. In the usual approach ([5], Theorem 13.5) this is deduced from a separate argument: Cartan's fixed point theorem for spaces of negative curvature. We avoid this, although the techniques we apply in Section 4 are very similar in spirit.

6 Appendix

We give a simple proof of the well-known fact stated in Section 4: that the manifold $\mathcal{H}$ has Property (*). We identify $\mathcal{H}$ with $n \times n$ positive definite symmetric matrices of determinant 1. It suffices to prove the statement for the exponential map at the identity matrix. Recall that the metric on $\mathcal{H}$ is given by $|\delta H|_H^2 = \mathrm{Tr}\left((\delta H)H^{-1}\right)^2$. For fixed symmetric matrices S, α and a small real parameter h define

$$H(h) = (\exp(S + h\alpha) - \exp(S)) \exp(-S),$$

and

$$v = \frac{dH}{dh}|_{h=0}.$$

we need to show that, for any S and α, we have

$$\mathrm{Tr}\ v^2 \geq \mathrm{Tr}\ \alpha^2.$$

To see this we introduce another real parameter t and set

$$H(t, h) = (\exp(t(S + h\alpha)) - \exp(tS)) \exp(-tS).$$

Then one readily computes

$$\frac{\partial H}{\partial t} = [S, H] + h\alpha \exp(t(S + h\alpha)) \exp(-tS).$$

Now differentiate with respect to h and evaluate at $h = 0$ to get a matrix valued function $V(t)$. Then we have

$$\frac{dV}{dt} = \frac{\partial^2 H}{\partial h \partial t}|_{h=0} = [S, V] + \alpha.$$

Clearly $v = V(1)$ and $V(0) = 0$, so our result follows from the following

Lemma 14. *Let S, α be real, symmetric $n \times n$ matrices and let $V(t)$ be the matrix valued function which is the solution of the ODE*

$$\frac{dV}{dt} = [S, V] + \alpha$$

with $V(0) = 0$. Then

$$\mathrm{Tr}\ V(t)^2 \geq t^2 \mathrm{Tr}\ \alpha^2$$

for all t.

To see this, consider first a scalar equation

$$\frac{df^+}{dt} = \lambda f^+ + a,$$

with λ, a constants and with the initial condition $f^+(0) = 0$. The solution is

$$f^+(t) = \left(\frac{e^{\lambda t} - 1}{\lambda}\right) a,$$

where we understand the expression in brackets is to be interpreted as t in the case when $\lambda = 0$. Let $f^-(t)$ satisfy the similar equation

$$\frac{df^-}{dt} = -\lambda f^- + a,$$

with $f^-(0) = 0$. Then

$$f^+(t)f^-(t) = t^2 a^2 Q(t),$$

where

$$Q(t) = \frac{(e^{\lambda t} - 1)(1 - e^{-\lambda t})}{\lambda^2 t^2} = \frac{2(\cosh(\lambda t) - 1)}{\lambda^2 t^2}.$$

It is elementary that $Q(t) \geq 1$, so $f^+(t)f^-(t) \geq t^2 a^2$.

Now consider the operator ad_S acting on $n \times n$ matrices. We can suppose S is diagonal with eigenvalues λ_i. Then a basis of eigenvectors for ad_S is given by the standard elementary matrices E_{ij} and

$$\mathrm{ad}_S(E_{ij}) = \lambda_{ij} E_{ij},$$

where $\lambda_{ij} = \lambda_i - \lambda_j$. Thus the matrix equation reduces to a collection of scalar equations for the components $V_{ij}(t)$. Since $\lambda_{ji} = -\lambda_{ij}$ and $\alpha_{ij} = \alpha_{ji}$, each pair V_{ij}, V_{ji} satisfies the conditions considered for f^+, f^- above and we have

$$V_{ij}(t)V_{ji}(t) \geq \alpha_{ij}^2 t^2.$$

(This is also true, with equality, when $i = j$). Now summing over i, j gives the result.

This proof is not very different from the usual discussion of the Jacobi equation in a symmetric space. It is also much the same as the proof of Helgason's formula for the derivative of the exponential map ([5], Theorem 1.7).

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