Potential automorphy of odd-dimensional symmetric powers of elliptic curves, and applications

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To Yuri Ivanovich Manin

Introduction

The present article was motivated by a question raised independently by Barry Mazur and Nick Katz. The articles [CHT,HST,T] contain a proof of the Sato-Tate conjecture for an elliptic curve E over a totally real field whose j-invariant $j(E)$ is not an algebraic integer. The Sato-Tate conjecture for E is an assertion about the equidistribution of Frobenius angles of E, or equivalently about the number of points $|E(\mathbb{F}_p)|$ on E modulo p as p varies. The precise statement of the conjecture, which is supposed to hold for any elliptic curve without complex multiplication, is recalled in §5. Now suppose E and E' are two elliptic curves without complex multiplication, and suppose E and E' are not isogenous. The question posed by Mazur and Katz is roughly the following: are the distributions of the Frobenius angles of E and E', or equivalently of the numbers $p+1-|E(\mathbb{F}_p)|$ and $p+1-|E'(\mathbb{F}_p)|$, independent?

The Sato-Tate conjecture, in the cases considered in [CHT,HST,T], is a consequence of facts proved there about L-functions of symmetric powers of the Galois representation on the Tate module $T_\ell(E)$ of E, following a strategy elaborated by Serre in [S]. These facts in turn follow from one of the main theorems of [CHT,HST,T], namely that, if n is *even*, the $(n-1)$st symmetric power of $T_\ell(E)$ is potentially automorphic, in that it is associated to a cuspidal automorphic representation of $GL(n)$ over some totally real Galois extension of the original base field. The restriction to even n is inherent in the approach to potential modularity developed in [HST], which applies only to even-dimensional representations. The necessary properties of all symmetric power L-functions follow from this result for even-dimensional symmetric powers, together with basic facts about Rankin-Selberg L-functions proved by Jacquet-Shalika-Piatetski-Shapiro and Shahidi. In a similar way, the Mazur-Katz question can be resolved affirmatively if we can prove potential automorphy for *all* symmetric power L-functions of E and E' over the same field.

The main purpose of the present article is to explain how to prove potential automorphy for odd-dimensional symmetric power L-functions, thus providing a response to the question of Mazur and Katz. The principal innovation is a tensor product trick that converts an odd-dimensional representation to an even-dimensional representation. Briefly, in §§2 and 3 one tensors with a two-dimensional representation. One has to choose a two-dimensional representation of the right kind, which is not difficult. The challenge is then to recover the odd-dimensional symmetric power unencumbered by the extraneous two-dimensional factor; this is the subject of §4. I say "explain how to prove" rather than "prove" because the proofs of the main results of this article make use of stronger modularity theorems than those proved in [CHT] and [T], and are thus conditional. I explain in §1 how I expect these modularity results to result from a strengthening of known theorems associating compatible families of ℓ-adic Galois representations to certain kinds of automorphic representations. These stronger theorems, stated as Expected Theorems 1.2 and 1.4, are the subject of work in progress by participants in the Paris automorphic forms seminar, and described in [H]. This work has progressed to a point where it seems legitimate to admit these Expected Theorems. Nevertheless, the present article should be viewed as a promissory note which will not be negotiable until the project outlined in [H] has been completed.

One can of course generalize the question and ask whether the distributions of the Frobenius angles of n pairwise non-isogenous elliptic curves without complex multiplication are independent. For $n \geq 3$ this seems completely inaccessible by current techniques in automorphic forms.

The article concludes with some speculations regarding additional applications of the tensor product trick.

I met Yuri Ivanovich Manin briefly near the beginning of my career. Later, as a National Academy of Sciences exchange fellow I had the remarkable good fortune of spending a year as his guest in Moscow, and as a (mostly passive but deeply appreciative) participant in his seminar at Moscow State University during what may well have been its final year, and saw first-hand what the Moscow mathematical community owed to his insight and personality. The influence of Yuri Ivanovich on my own work is pervasive, and the present work is no exception: though it is not apparent in what follows, the article [HST], on which all the results presented here are based, can be read as an extended meditation on the Gauss-Manin connection as applied to a particular family of Calabi-Yau varieties. It is an honor to dedicate this article to Yuri Ivanovich Manin.

I thank Barry Mazur and Nick Katz for raising the question that led to this paper. Apart from the tensor product trick, practically all the ideas in this paper are contained in [CHT], [HST], and [T]; I thank my coauthors – Laurent Clozel, Nick Shepherd-Barron, and Richard Taylor – for their collaboration over many years. I thank Richard Taylor specifically for his help with the proof of the crucial Lemma 4.2. Finally, I thank the referee for a careful reading, and for helping me to clarify a number of important points.

1. Reciprocity for n-dimensional Galois representations

All finite-dimensional representations of Galois groups are assumed to be continuous. When E is a number field, contained in a fixed algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$, we let Γ_E denote $Gal(\overline{\mathbb{Q}}/E)$. Let ρ be a (finite-dimensional) ℓ-adic representation of Γ_E. Say ρ is *pure* of weight w if for all but finitely many primes v of E the restriction ρ_v of ρ to the decomposition group Γ_v is unramified and if the eigenvalues of $\rho_v(Frob_v)$ are all algebraic numbers whose absolute values equal $q_v^{\frac{w}{2}}$; here q_v is the order of the residue field k_v at v and $Frob_v$ is geometric Frobenius. If ρ is pure of weight w, the *normalized* L-function of ρ is

$$L^{norm}(s,\rho) = L(s+\frac{w}{2},\rho);$$

here we assume we have a way to define the local Euler factors at primes dividing ℓ (for example, ρ belongs to a compatible system of λ-adic representations). Then $L^{norm}(s,\rho)$ converges absolutely for $Re(s) > 1$.

Let F be a CM field, $F^+ \subset F$ its maximal totally real subfield, so that $[F : F^+] \leq 2$. Let $c \in Gal(F/F^+)$ be complex conjugation; by transport of structure it acts on automorphic representations of $GL(n,F)$. The following theorem is the basis for many of the recent results on reciprocity for Galois representations of dimension > 2. For the purposes of the following theorem, a unitary Harish-Chandra module σ for $GL(n,\mathbb{C})$ will be called "cohomological" if $\sigma \otimes ||\det||^{\frac{n-1}{2}}||$ is cohomological in the usual sense, i.e. if there is a finite-dimensional irreducible representation W of $GL(n,\mathbb{C})$ such that

$$H^\bullet(Lie(GL(n,\mathbb{C})), U(n); \sigma \otimes ||\det||^{\frac{n-1}{2}}|| \otimes W) \neq 0.$$

The half-integral twist is required by the unitarity.

Theorem 1.1 ([C2], [Ko], [HT], [TY]). *In what follows, Π denotes a cuspidal automorphic representation of $GL(n,F)$, and $\{\rho_{\bullet,\lambda}\}$ denotes a compatible family of n-dimensional λ-adic representations of Γ_F.*

There is an arrow $\Pi \mapsto \{\rho_{\Pi,\lambda}\}$, where λ runs through non-archimedean completions of a certain number field $E(\Pi)$, under the following hypotheses:

$$\left\{\begin{array}{l} \textit{(1) The factor } \Pi_\infty \\ \textit{ is cohomological} \\ \textit{(2) } \Pi\circ c \cong \Pi^\vee \\ \textit{(3) } \exists v_0, \Pi_{v_0} \\ \quad \textit{discrete series} \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \textit{(a) } \rho = \rho_{\Pi,\lambda} \textit{ geometric,} \\ \textit{ HT regular} \\ \textit{(b) } \rho\otimes\rho\circ c \ \to \mathbb{Q}_\ell(1-n) \\ \textit{(c) local condition} \\ \quad \textit{at } v_0 \end{array}\right\}$$

This correspondence has the following properties:

(i) For any finite place v prime to the residue characteristic ℓ of λ,

$$[\rho_{\Pi,\lambda}\mid_{WD_v}]^{Frob-ss} = \mathcal{L}(\Pi_v \otimes |\bullet|_v^{\frac{1-n}{2}}).$$

Here WD_v is the local Weil-Deligne group at v, $\mathcal{L}$ is the **normalized** *local Langlands correspondence, and $Frob-ss$ denotes Frobenius semisimplification;*

(ii) The representation $\rho_{\Pi,\lambda}\mid_{G_v}$ is potentially semistable, in Fontaine's sense, for any v dividing ℓ, and the Hodge-Tate weights at v are explicitly determined by the infinitesimal character of the Harish-Chandra module Π_∞.

The local Langlands correspondence is given the unitary normalization. This means that the correspondence identifies $L(s,\Pi)$ and $L^{norm}(s,\rho_{\Pi,\lambda})$, so that the functional equations always exchange values at s and $1-s$.

The term "geometric" is used in the sense of Fontaine-Mazur: each $\rho_{\Pi,\lambda}$ is unramified outside a finite set of places of F, in addition to the potential semistability mentioned in the statement of the theorem. The condition "HT regular" (Hodge-Tate) means that the Hodge-Tate weights at v have multiplicity at most one.

For the local condition (c), we can take the condition that the representation of the decomposition group at v_0 is indecomposable as long as v_0 is prime to the residue characteristic of λ, or equivalently that this representation of the decomposition group at v_0 corresponds to a discrete series representation of $GL(n,F_{v_0})$. The conditions on both sides of the diagram match: (1) $\leftrightarrow$ (a), (2) $\leftrightarrow$ (b), (3) $\leftrightarrow$ (c).

When Π is a base change of a representation Π^+ of $GL(n,F^+)$, condition (2) just means that Π is self-dual.

In what follows, we will admit the following extension of Theorem 1.1:

Expected Theorem 1.2. *The assertions of Theorem 1.1 remain true provided Π satisfies conditions (1) and (2); then $\rho_{\Pi,\lambda}$ satisfies conditions (a) and (b), as well as (i) and (ii).*

Here "Expected Theorem" means something more than conjecture. The claim of Expected Theorem 1.2 is a very special case of the general Langlands conjectures, in the version for Galois representations developed in Clozel's article [C1]. This specific case is the subject of work in progress on the part of participants in the Paris automorphic forms seminar and others, and an outline of the various steps in the proof can be found in [H]. There are quite a lot of intermediate steps, the most difficult of which involve analysis of the stable trace formula, twisted or not, but I would "expect" that they will all have been verified, and the theorem completely proved, by 2010 at the latest.

I single out one of the intermediate steps. By a *unitary group* over F^+ I will mean the group of automorphisms of a vector space V/F preserving a nondegenerate hermitian form. The group is denoted $U(V)$, the hermitian form being understood. We will consider a hermitian vector space V of dimension n with the following properties:

(1.3.1) For every real place σ of F^+, the local group $U(V_\sigma)$ is compact (the hermitian form is totally definite);
(1.3.2) For every finite place v of F^+, the local group $U(V_v)$ is quasi-split and split over an unramified extension.

We write $G_0 = U(V)$. Such a unitary group always exists when n is odd, provided F/F^+ is everywhere unramified. When n is even, there is a sign obstruction that can be removed by replacing F^+ by a totally real quadratic extension. Such restrictions are harmless for applications (see [H]).

We let $K = \prod_v K_v \subset G_0(\mathbf{A}^f)$ be an open compact level subgroup. Hypothesis (1.3.2) guarantees that $G_0(F_v^+)$ contains a hyperspecial maximal compact subgroup for all finite v. If v is split then any maximal compact subgroup is hyperspecial and conjugate to $GL(n, \mathcal{O}_v)$. We assume

Hypothesis 1.3.3. *K_v is hyperspecial maximal compact for all v that remain inert in F.*

For any ring R, let

$$M_K(G_0, R) = C(G_0(F^+)\backslash G_0(\mathbf{A})/G_0(\mathbb{R}) \cdot K, R),$$

where for any topological space X, $C(X, R)$ means the R-module of continuous functions from X to R, the latter endowed with the discrete topology. The Hecke algebra $\mathcal{H}_K(R)$ of double cosets of K in $G_0(\mathbf{A}^f)$ with coefficients in R acts on $M_K(G_0, R)$. It contains a subring $\mathcal{H}_K^{hyp}(R)$ generated by the double cosets of K_v in $G_0(F_v^+)$ where v runs over primes that split in F at which K_v is hyperspecial maximal compact. We denote by $\mathbb{T}_K(R)$ the image of $\mathcal{H}_K^{hyp}(R)$ in $End_R(M_K(G_0, R))$. The algebra $\mathbb{T}_K(R)$ is reduced if R is a semisimple algebra flat over $\mathbb{Z}$ (cf. [CHT], Corollary 2.3.3).

We can also consider $M(G_0, R)$, the direct limit of $M_K(G_0, R)$ over all K, including those not satisfying (1.3.3). This is a representation of $G_0(\mathbf{A}^f)$ and decomposes as a sum of irreducible representations when R is an algebraically closed field of characteristic zero. Let $\pi \subset M(G_0, \mathbb{C})$ be an irreducible summand. Write $\pi = \pi_\infty \otimes \pi_f$, $\pi_f = \otimes'_v \pi_v$, the restricted tensor product over finite primes v of F^+ of representations of $G_0(F_v^+)$. With our hypotheses π_∞ is the trivial representation of $G_{0,\mathbb{R}} = \prod_\sigma U(V_\sigma)$, where σ is as in (1.3.1). Suppose $\pi^K \neq \{0\}$ for some K satisfying (1.3.3). Then for every finite v one can define the local base change $\Pi_v = BC_{F_v/F_v^+}\pi_v$, a representation of $G_0(F_v^+) \xrightarrow{\sim} \prod_{w|v} GL(n, F_w)$. If v is inert, then by (1.3.3) we know that π_v is an unramified representation, and so is Π_v. If not, then π_v is a representation of $GL(n, F_v^+)$ and $\Pi_v \xrightarrow{\sim} \pi_v \otimes \pi_v^\vee$, with the appropriate normalization. Thus

Lemma 1.3.4. *The local factor π_v is uniquely determined by Π_v.*

Expected Theorem 1.4. *There is a cohomological representation Π_∞ of $GL(n, F_\infty) = GL(n, F \otimes_{\mathbb{Q}} \mathbb{R})$ such that the formal base change*

$$\Pi = BC_{F/F^+}\pi = \Pi_\infty \otimes \bigotimes_v{}' \Pi_v$$

is an automorphic representation of $GL(n, F)$. Moreover, there is a partition $n = a_1 + a_2 + \cdots + a_r$ and an automorphic representation $\bigotimes_j \Pi_j$ of the group $\prod_j GL(a_j, \mathbf{A}_F)$ such that each Π_j is in the discrete automorphic spectrum of $GL(a_j, F)$ and Π, as a representation of $GL(n, \mathbf{A}_F)$, is parabolically induced from the inflation of $\bigotimes_j \Pi_j$ to the standard parabolic subgroup $P(\mathbf{A}) \subset GL(n, \mathbf{A}_F)$ associated to the partition. Moreover, each Π_j satisfies conditions (1) and (2) of Theorem 1.1, where "cohomological" is understood as in loc. cit..

We say π is *F/F^+-cuspidal* if Π is cuspidal, in which case it follows from the classification of generic cohomological representations that Π_∞ is necessarily tempered and is uniquely determined by the condition that π_∞ is trivial. This representation is denoted $\Pi_{\infty,0}$, or $\Pi_{\infty,0}(n, F)$ when this is necessary.

Fix a prime ℓ and let $\mathcal{O}$ be the ring of integers in a finite extension of $\mathbb{Q}_\ell$. The ring $\mathbb{T}_K(\mathcal{O})$ is semilocal and $\mathbb{T}_K(\mathcal{O} \otimes \bar{\mathbb{Q}}_\ell)$ is a product of fields. Let $\mathfrak{m} \subset \mathbb{T}_K(\mathcal{O})$ be a maximal ideal and let $I \subset \mathbb{T}_K(\mathcal{O} \otimes \bar{\mathbb{Q}}_\ell)$ be any prime ideal whose intersection with $\mathbb{T}_K(\mathcal{O})$ is contained in $\mathfrak{m}$. Then I determines an irreducible $G_0(\mathbf{A}^f)$-summand π_0 of $M(G_0, \bar{\mathbb{Q}}_\ell)$, or equivalently of $M(G_0, \mathbb{C})$, if one identifies the algebraic closures of $\mathbb{Q}$ in $\mathbb{C}$ and in $\bar{\mathbb{Q}}_\ell$, as in [HT], p. 20. More precisely, I only determines $\pi_{0,v}$ locally at v for which K_v is hyperspecial, but this includes all inert primes. Let S be the set at which K_v is not hyperspecial. By Expected Theorem 1.4, I determines a collection of cohomological automorphic representations Π of $GL(n, F)$ which are isomorphic outside the finite set S. By strong multiplicity one, Π is in fact unique; we denote it Π_I. Then π is unique by Lemma 1.3.4.

Combining Expected Theorems 1.4 and 1.2, we thus obtain an n-dimensional representation $\rho_{\Pi,\ell} = \rho_{I,\ell}$ of Γ_F. We say I is *Eisenstein* at ℓ if the reduction mod ℓ, denoted $\bar{\rho}_{\Pi,\ell}$, is not absolutely irreducible. It follows in the usual way from property (i) of the correspondence between $\rho_{\Pi,\ell}$ and Π that if I is Eisenstein at ℓ then every prime ideal of $\mathbb{T}_K(\mathcal{O} \otimes \bar{\mathbb{Q}}_\ell)$ lying above $\mathfrak{m}$ is Eisenstein. In that case we say $\mathfrak{m}$ is Eisenstein at ℓ.

Lemma 1.5. *Admit Expected Theorems 1.4 and 1.2. Fix a prime ℓ, and suppose $\mathfrak{m} \subset \mathbb{T}_K(\mathcal{O})$ is not an Eisenstein ideal at ℓ. Then any prime ideal $I \subset \mathbb{T}_K(\mathcal{O} \otimes \bar{\mathbb{Q}}_\ell)$ lying above $\mathfrak{m}$ has the property that Π_I is cuspidal (in that case we say $\mathfrak{m}$ and I are $F/F^+ -$ cuspidal).*

Sketch of proof. This follows from the classification of automorphic representations of $GL(n)$ and from properties of base change. Suppose I corresponds to $\pi \subset M(G_0, \mathbb{C})$. If the base change Π of π belongs to the discrete spectrum of $GL(n, F)$ – that is, if the partition in Expected Theorem 1.4 is a singleton – then it is either cuspidal or in the non-tempered discrete spectrum. In the latter case, the Moeglin-Waldspurger classification implies that n factors as ab, with $a > 1, b > 1$, and Π is the Speh representation attached to a cuspidal automorphic representation Π_1 of $GL(b, F)$. Clozel has checked in [C3] that Π_1 satisfies properties (1) and (2) of Expected Theorem 1.2 for $GL(b)$, hence is associated to a b-dimensional ℓ-adic representation ρ_1 of Γ_F. It follows from condition (i) of Expected Theorem 1.2 that the semisimple representation $\rho_{\Pi,\ell}$ decomposes as a sum of a constituents, each of which is an abelian twist of ρ_1.

Suppose Π does not belong to the discrete spectrum of $GL(n, F)$. Then π is endoscopic, hence is associated to a partition $n = \sum a_j$, with each $a_j > 0$, and an automorphic representation $\otimes_j \Pi_{a_j}$ in the discrete spectrum of $\prod_j GL(a_j, F)$, such that each Π_{a_j}, satisfies properties (1) and (2) of Expected Theorem 1.2. Then $\rho_{\Pi,\ell}$ decomposes as a sum of $r > 1$ pieces of dimensions a_j. □

I recall the main modularity lifting theorem of [T], whose proof builds on and completes the main results of [CHT].

1.6. Modularity lifting theorem. *Let $\ell > n$ be a prime unramified in F^+ (resp. and such that every divisor of ℓ in F^+ splits in F) and let*

$$r : \ \Gamma_{F^+} \to GL(n, \bar{\mathbb{Q}}_\ell) \ \textit{(resp. } r : \ \Gamma_F \to GL(n, \bar{\mathbb{Q}}_\ell)\textit{)}$$

be a continuous irreducible representation satisfying the following properties:

(a) r ramifies at only finitely many primes, is crystalline at all primes dividing ℓ, and is Hodge-Tate regular;
(b) $r \simeq r^\vee(1-n) \cdot \chi$ (resp. $r^c \simeq r^\vee(1-n)$) where $(1-n)$ is the Tate twist and χ is a character whose value is constant on all complex conjugations (resp. c denotes complex conjugation);
(c) At some finite place v not dividing ℓ, r_v corresponds to a square-integrable representation of $GL(n, F_v^+)$ under the local Langlands correspondence, and satisfies the final "minimality" hypotheses of [CHT, 4.3.4 (5)] or [T, 5.2 (5)]

In addition, we assume that $\bar{r}$

(d) has "big" image in the sense of Definition 3.1 below;
(e) is absolutely irreducible;
(f) is of the form $\bar{\rho}_{\Pi,\ell}$ for some cuspidal automorphic representation Π of $GL(n, F^+)$ satisfying conditions (1)-(3) of Theorem 1.1.

Then r is of the form $\rho_{\Pi',\ell}$ for some cuspidal automorphic representation Π' of $GL(n, F^+)$ satisfying conditions (i)-(iii) of Theorem 1.1.

I have not written out the last part of hypothesis (c) in detail, because it will be dropped in the remainder of the article. More precisely, if we admit Expected Theorems 1.2 and 1.4, then we obtain the

1.7. Expected modularity lifting theorem. *Let ℓ and r be as in 1.6, but we no longer assume condition (c), and in (f) we drop condition (3). Then r is of the form $\rho_{\Pi',\ell}$ for some cuspidal automorphic representation Π' of $GL(n, F^+)$ satisfying properties (i)-(ii) of Expected Theorem 1.2.*

I Expect the arguments of [CHT] and [T] will apply without change to yield this Expected Theorem. The various rings of deformations of $\bar{r}$ are defined exactly as in [CHT] and [T], and the target Hecke algebra is the localization of $\mathbb{T}_K(\mathcal{O})$ at the maximal ideal $\mathfrak{m}$ associated to $\bar{r}$. Hypothesis (e) guarantees that $\mathfrak{m}$ is not Eisenstein at ℓ, and the remaining arguments should go through unchanged. However, there are many steps to the proof of Theorem 1.6, and it is necessary to check carefully that, as I believe, Expected Theorems 1.2 and 1.4 eliminate all dependence on the local condition (c).

In the remainder of the paper, I draw consequences from 1.7.

2. Potential modularity of a Galois representation

Let F and F^+ be as in §1. The article [HST] develops a method for proving that certain n-dimensional ℓ-adic representations ρ of Γ_{F^+} (resp. Γ_F) that look like they arise from automorphic representations via the correspondence of Theorem 1.1, are *potentially* automorphic in the following sense: there exists a totally real Galois extension F'/F^+ such that $\rho \mid_{\Gamma_{F'}}$ (resp. $\rho \mid_{\Gamma_{F\cdot F'}}$) does correspond to a cuspidal automorphic representation of $GL(n, F')$ (resp. $GL(n, F \cdot F')$). The relevant result is Theorem 3.1 of [HST], which is in turn based on Theorem 1.6. Although the latter theorem is valid for representations of arbitrary dimension n, Theorem 3.1 only applies to an even-dimensional representation of Γ_{F^+} endowed with an alternating form that is preserved by Γ_{F^+} up to a multiplier.

If we admit the expected results of §1, then Theorem 3.1 of [HST] admits the following simplification. The constant $C(n_i)$ is a positive number introduced in [HST], Corollary 1.11; its precise definition is irrelevant to the applications. For a finite prime w, G_w denotes a decomposition group, $I_w \subset G_w$ the inertia group.

Theorem 2.1 [HST]. *Assume the Expected Theorems of §1. Let $F^+/F^{0,+}$ be a Galois extension of totally real fields and let $n_1, \ldots, n_r$ be even positive integers. Suppose that $\ell > max\{C(n_i), n_i\}$ is a prime which is unramified in F and which splits in $\mathbb{Q}(\zeta_{n_i+1})$, $i = 1, \ldots, r$. Let $\mathcal{L}$ be a finite set of primes of F^+ not containing primes above ℓ and let M be a finite extension of F.*

Suppose that for $i = 1, \ldots, r$

$$r_i : \Gamma_{F^+} \to GSp(n_i, \mathbb{Z}_\ell)$$

is a continuous representation with the following properties.

- *r_i has multiplier $\omega_\ell^{1-n_i}$, where ω_ℓ is the ℓ-adic cyclotomic character.*

- *r_i ramifies at only finitely many primes.*
- *The image $\bar{r}_i(\Gamma_{F^+(\zeta_\ell)})$ is big, in the sense of Definition 3.1 below, where ζ_ℓ is a primitive ℓ-th root of 1.*
- *The fixed field F_i of $\ker ad(\bar{r}_i) \subset \Gamma_{F^+}$ does not contain $F(\zeta_\ell)$.*
- *r_i is unramified at all primes in $\mathcal{L}$.*
- *If $w \mid \ell$ is a prime of F then $r_i \mid_{G_w}$ is crystalline with Hodge-Tate weights $0, 1, \ldots, n_i - 1$, with the conventions of [HST]. Moreover,*

$$\bar{r}_i \mid_{I_w} \simeq \bigoplus_{j=0}^{n_i-1} \omega_\ell^{-j}.$$

Then there is a totally real field $F'^{,+}/F^+$, Galois over F_0^+ and linearly disjoint from the compositum of the F_i with M over F^+, with the property that each $r_{i,F'^{,+}} = r_i \mid_{\Gamma_{F'^{,+}}}$ corresponds to an automorphic representation Π_i of $GL(n_i, F'^{,+})$. If $F'/F'^{,+}$ is a CM quadratic extension, then the base change $\Pi_{i,F'}$ has archimedean constituent isomorphic to $\Pi_{\infty,0}(n_i, F')$ (cf. the remarks after 1.4). Finally, all primes of $\mathcal{L}$ and all primes of F dividing ℓ are unramified in F'.

Apart from a few slight changes in notation, this theorem is practically identical to Theorem 3.1 of [HST]. There is no field M in [HST] but the proof yields an $F'^{,+}$ linearly disjoint over F^+ from any fixed extension. Only condition (7) of Theorem 3.1 of [HST], corresponding to condition (3) of Theorem 1.1, has been eliminated. The proof is identical but simpler: references to Theorem 1.6 are replaced by references to Expected Theorem 1.7, and all arguments involving the primes q and q' in [HST] are no longer necessary.

Let E be an elliptic curve over F^+, and let $\rho_{E,\ell} : \Gamma_{F^+} \to GL(2, \mathbb{Q}_\ell)$ denote the representation on $H^1(E_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$, i.e. the dual of the ℓ-adic Tate module. For $n \geq 1$ let

$$\rho_{E,\ell}^n = Sym^{n-1}\rho_{E,\ell} : \Gamma_{F^+} \to GL(n, \mathbb{Q}_\ell).$$

We will always assume E has no complex multiplication. Then $\rho_{E,\ell}^n$ is irreducible by a theorem of Serre, for all n, and for almost all $\ell > n$, $Im(\bar{\rho})$ contains the image of $SL(2, \mathbb{F}_\ell)$ under the symmetric power representation, and hence is absolutely irreducible. When $F^+ = \mathbb{Q}$ it was proved in the series of papers initiated by Wiles and Taylor-Wiles and completed by Breuil, Conrad, Diamond, and Taylor that $L(s, \rho_{E,\ell})$ is automorphic, which in this case means is attached to a classical new form of weight 2. The prototype for Theorem 2.1 is the theorem proved by Taylor in [T02], which shows that $L(s, \rho_{E,\ell})$ is potentially automorphic for any F^+.

In [HST] and [T], Theorem 2.1 is notably applied to show that $L(s, \rho_{E,\ell}^n)$ is potentially automorphic for any *even* n, provided E has non-integral j-invariant. One can hardly hope to apply Theorem 2.1 as such when n is odd, given that symplectic groups are only attached to even integers. Moreover, when n is odd $\rho_{E,\ell}^n$ has an orthogonal polarization rather than a symplectic polarization. These two related flaws – the oddness of n and the orthogonality of ρ^n – can be cured simultaneously by tensoring $\rho^n_{E,\ell}$ by a two-dimensional representation $\tau : \Gamma_{F^+} \rightarrow GL(2,\mathbb{Z}_\ell)$. Such a representation is necessarily symplectic, with multiplier $\det\tau$. We suppose τ has determinant ω_ℓ^{-n}. In order to preserve the hypotheses of Theorem 2.1, specifically 2.1 (6), we need to assume

Hypothesis 2.2. *If $w \mid \ell$ is a prime of F then $\tau \mid_{G_w}$ is crystalline with Hodge-Tate weights $0, n$, with the conventions of [HST]. Moreover,*

$$\bar{\tau} \mid_{I_w} \simeq 1 \oplus \omega_\ell^{-n}.$$

For example, let f be a classical new form of weight $n+1$ for $\Gamma_0(N)$, for some integer N. Associated to f is a number field $\mathbb{Q}(f)$, generated by the Fourier coefficients of f, and a compatible system of 2-dimensional λ-adic representations $\tau_{f,\lambda}$ of $\Gamma_\mathbb{Q}$ as λ varies over the primes of $\mathbb{Q}(f)$. We choose a prime ℓ that splits completely in $\mathbb{Q}(f)$ and such that $\ell \nmid N$. Fix λ dividing ℓ, and write $\tau = \tau_{f,\lambda}$. Then τ takes values in $GL(2,\mathbb{Z}_\ell)$; since f has trivial nebentypus, the determinant of τ is indeed ω_ℓ^{-n}. The hypothesis regarding $\bar{\tau} \mid_{I_w}$ is in general a serious restriction, but we will find explicit examples.

(2.3)

We say the residual representation $\bar{\tau}$ is "big enough" if its image contains a non-commutative subgroup of the normalizer $N(T)$ of a maximal torus $T \subset SL(2,\mathbb{F}_\ell)$, and more specifically that $Im(\bar{\tau})$ contains an element h of T with distinct eigenvalues that acts trivially on the cyclotomic field $\mathbb{Q}(\zeta_\ell)$, and an element w of order 2 that does not commute with h.

Here are the representations we will use. Let L be an imaginary quadratic field not contained in F, and let η_L be the corresponding quadratic Dirichlet character, viewed as an idèle class character of $\mathbb{Q}$. Let χ be a Hecke character of (the idèles of) L whose restriction to the idèles of $\mathbb{Q}$ is the product $|\bullet|_{\mathbf{A}}^{-n}\cdot\eta_L$, where $|\bullet|_{\mathbf{A}}$ is the idèle norm. Choose an isomorphism $L_\infty = L\otimes_\mathbb{Q}\mathbb{R}\overset{\sim}{\longrightarrow}\mathbb{C}$, let z be the corresponding coordinate function on L_∞, and assume $\chi_\infty(z) = z^{-n}$. Then χ is an algebraic Hecke character and is associated to a compatible system of ℓ-adic characters $\chi_\lambda : \Gamma_L \rightarrow \mathbb{Q}(\chi)^+_\lambda$, where $\mathbb{Q}(\chi)$ is the field of coefficients of χ, a finite extension of $\mathbb{Q}$, and λ runs through the places of $\mathbb{Q}(\chi)$. Choose a prime $\ell > 2n+1$ that splits in L and in $\mathbb{Q}(\chi)$; then for any λ dividing ℓ we can view χ_λ as a $\mathbb{Q}_\ell^\times$-valued character of Γ_L; we write $\chi_\ell = \chi_\lambda$.

For such an ℓ, we define the monomial induced representation

$$\tau_\ell = Ind_{\Gamma_L}^{\Gamma_\mathbb{Q}}\chi_\ell : \Gamma_\mathbb{Q} \rightarrow GL(2,\mathbb{Q}_\ell).$$

Lemma 2.4. *Under the above hypotheses, $\bar{\tau}_\ell$ is "big enough" in the sense of (2.3).*

Proof. Let v, v' be the two primes of L dividing ℓ. Let $k(v), k(v')$ denote the corresponding residue fields; then the product of the respective Teichmüller characters defines an inclusion

$$k(v)^{\times} \times k(v')^{\times} \hookrightarrow \mathcal{O}_v^{\times} \times \mathcal{O}_{v'}^{\times} \subset \mathbf{A}_L^{\times}.$$

Let b and b' denote generators of the images of the cyclic groups $k(v)^{\times}$ and $k(v')^{\times}$ in $\mathcal{O}_v^{\times}$ and $\mathcal{O}_{v'}^{\times}$, respectively, and let $s(b), s(b')$ denote their images in $Gal(L^{ab}/L)$ under the reciprocity map. Our hypotheses imply that $s(b)$ acts on $\bar{\tau}$ with eigenvalues $b^n, 1$, and likewise for $s(b')$. Moreover, the trivial eigenspace for $s(b)$ is the non-trivial eigenspace for $s(b')$, and vice versa. We identify b and b' with roots of unity in $\mathbb{Q}_{\ell}{}^{\times}$. The element $h = \bar{\tau}(s(b) \cdot s(b')^{-1})$ then belongs to $SL(2, \mathbb{F}_{\ell})$ and has eigenvalues $b^{\pm n}$. Since $\ell - 1 > 2n$, these eigenvalues are distinct; in particular, h does not commute with the image w of complex conjugation in $Im(\bar{\tau})$. Since the action of $\mathcal{O}_v^{\times} \times \mathcal{O}_{v'}^{\times}$ on $\mathbb{Q}(\zeta_{\ell})$ factors through the norm to $\mathbb{Z}_{\ell}^{\times}$, we see that h acts trivially on $\mathbb{Q}(\zeta_{\ell})$. This completes the proof of Lemma 2.4. □

Corollary 2.5. *Let $\tau = \tau_{f,\lambda}$ be as in the preceding paragraph and satisfy Hypothesis 2.2. We continue to admit the Expected Theorems of §1. We use the same notation for $\tau \mid_{\Gamma_{F^+}}$. Let E be an elliptic curve over F. Let n be an odd positive integer, and suppose there is a prime $\ell > 2n+1$, unramified in F^+, such that*

(i) E has good ordinary reduction at all primes w dividing ℓ
(ii) ℓ does not divide the conductor N of f, and

$$\bar{\tau} \mid_{I_w} \simeq 1 \oplus \omega_{\ell}^{-n}.$$

(iii) ℓ splits in $\mathbb{Q}(\zeta_{2i+1})$, $i = 1, \ldots, n-1$; in particular, $\ell \equiv 1 \pmod{n}$ (take $i = \frac{n-1}{2}$).
(iv) $\frac{\ell-1}{n} > 2$.

Suppose $\bar{\tau}$ is "big enough" in the sense of (2.3). Let M be an arbitrary extension of F^+. For every integer $i \leq n$, let $r_i = \rho_{E,\ell}^{2i}$; let $r_{\tau} = \rho_{E,\ell}^{n} \otimes \tau$. Then there is a totally real Galois extension $F'^{,+}/F^+$, linearly disjoint from M over F^+, with the property that for $i = 1, \ldots, n$, $r_{i,F'^{,+}} = r_i \mid_{\Gamma_F'^{,+}}$ corresponds to an automorphic representation Π_i of $GL(2i, F'^{,+})$, and such that r_{τ} corresponds to an automorphic representation Π_{τ} of $GL(2n, F'^{,+})$. If $F'/F'^{,+}$ is a CM quadratic extension, then the base change $\Pi_{i,F'}$ (resp. $\Pi_{\tau,F'}$) has archimedean constituent isomorphic to $\Pi_{\infty,0}(2i, F')$ (resp. $\Pi_{\infty,0}(2n, F')$).

Proof. Under our hypotheses on ℓ and the image of $\bar{\tau}$, $\bar{r}_{\tau}$ is absolutely irreducible. We first prove the corollary under the hypothesis that for each w dividing ℓ,

$$\bar{\rho}_{E,\ell} \mid_{I_w} \simeq 1 \oplus \omega_{\ell}^{-1}. \tag{2.5.1}$$

Conditions (1), (2), and (6) of Theorem 2.1 are clearly satisfied. Since ℓ is unramified in F^+, F^+ and $\mathbb{Q}(\zeta_\ell)$ are linearly disjoint over $\mathbb{Q}$. By condition (1), the intersection of $Im(\bar{r}_i)$ with the center of $GSp(n_i, \mathbb{F}_\ell)$ maps onto the subgroup of $Gal(\mathbb{Q}(\zeta_\ell)/\mathbb{Q})$ generated by $2(1-n_i)$-th powers, which implies condition (4) for all r_i, and for r_τ as well.

Condition (5) is irrelevant. It remains to verify condition (3). For $\bar{r}_i$, $i = 1, \ldots, n$, this is Lemma 3.2 of [HST]; the case of $\bar{r}_\tau$ is Lemma 3.2.

This completes the proof under hypothesis (2.5.1). We reduce to this case as in the proof of Theorem 3.3 of [HST], replacing ℓ by a second prime $\ell' > 2n+1$ also split in L and in $\mathbb{Q}(\chi)$, $\bar{\tau}_\ell$ by $\bar{\tau}_{\ell'}$, and E by a curve E' (unfortunately denoted E in [*loc. cit.*]) such that $\rho_{E',\ell} \xrightarrow{\sim} \rho_{E,\ell}$ but $\rho_{E',\ell'}$ satisfies hypothesis (2.5.1) at ℓ'. $\square$

Remark 2.6. Recall that the results of this section are all conditional on the Expected Theorems of §1. In particular, we are not assuming that E has potentially multiplicative reduction at some place. If we do assume $j(E)$ is not integral, then the automorphic representations Π_i are constructed unconditionally in [CHT, HST, T]. However, the local condition on $j(E)$ does not suffice to impose a strong enough local condition on r_τ (corresponding to a discrete series representation on the automorphic side).

3. A lemma about certain residual representations

Definition 3.1. *Let $V/\bar{\mathbb{F}}_\ell$ be a finite dimensional vector space. Let $ad^0(V) \subset ad(V) = Hom(V, V)$ be the subspace of trace 0 endomorphisms. A subgroup $\Delta \subset GL(V)$ is big if the following hold:*

(*a*) $H^i(\Delta, ad^0 V) = (0)$ *for* $i = 0, 1$.

(*b*) *For all irreducible $\bar{\mathbb{F}}_\ell[\Delta]$-submodules $W \subset Hom(V, V)$ we can find $h \in \Delta$ and $\alpha \in \bar{\mathbb{F}}_\ell$ with the following properties. The α-generalized eigenspace $V_{h,\alpha}$ of h on V is one-dimensional. Let*

$$\pi_{h,\alpha} : V \to V_{h,\alpha};\ i_{h,\alpha} : V_{h,\alpha} \hookrightarrow V$$

denote, respectively, the h-equivariant projection and the h-equivariant inclusions of the indicated spaces. Then $\pi_{h,\alpha} \circ W \circ i_{h,\alpha} \neq (0)$.

Remark. It is not the case that if Δ contains a subgroup Δ' which is big in the above sense, then Δ itself is necessarily big (bigger than big is not necessarily big). This is because condition (a) is not preserved under passage to a bigger group. To check condition (b), on the other hand, it clearly suffices to show that it holds for some subgroup $\Delta' \subset \Delta$.

Lemma 3.2. *Let F^+, τ, $\rho^n = \rho^n_{E,\ell}$, and r_τ be as in the statement of Corollary 2.5, with τ as in the proof of Lemma 2.4. Suppose $\ell > 4n-1$. Then $\bar{r}_\tau(\Gamma_{F^+(\zeta_\ell)})$ is big in the sense of Definition 3.1.*

Proof. We begin by establishing notation. Write $\bar{\rho}^n = \bar{\rho}^n_{E,\ell}$. We define ad^0 as in Definition 3.1, and write

$$ad\ \bar{\rho}^n = ad^0\ \bar{\rho}^n \oplus 1, ad\ \bar{\tau} = ad^0\ \bar{\tau} \oplus 1,$$

where 1 denotes the trivial representation. Then

$$ad^0\ \bar{r}_\tau = ad^0\ \bar{\rho}^n \otimes ad^0\ \bar{\tau} \oplus ad^0\ \bar{\rho}^n \oplus ad^0\ \bar{\tau}. \tag{3.2.1}$$

Let $\Delta = \bar{r}_\tau(\Gamma_{F^+(\zeta_\ell)})$, and define

$$\tilde{\Delta} = \bar{\rho}^n(\Gamma_{F^+(\zeta_\ell)}) \times \bar{\tau}((\Gamma_{F^+(\zeta_\ell)}).$$

The tensor product defines an exact sequence

$$1 \to C \to \tilde{\Delta} \to \Delta \to 1$$

where the kernel C maps injectively to the center of $GL(2, \mathbb{F}_\ell)$, viewed as the group of linear transformations of the space of $\bar{\tau}$. In particular, the order of C is prime to ℓ. (In fact, as the referee pointed out, under our hypotheses one checks easily that C is trivial, but this makes no difference in the sequel.) Finally, let Δ_ρ denote the image of $\bar{\rho}^n(\Gamma_{F^+(\zeta_\ell)}) \subset \tilde{\Delta}$ in Δ. This is a normal subgroup of Δ isomorphic to the simple finite group $PSL(2, \mathbb{F}_\ell)$, since n is odd and $\ell > 2n+1$. Moreover, $\Delta_\tau := \Delta/\Delta_\rho$ is of order prime to ℓ, since the image of $\bar{\tau}$ is contained in the normalizer of a maximal torus. It follows from the inflation-restriction sequence that

$$H^1(\Delta, W) \xrightarrow{\sim} H^0(\Delta_\tau, H^1(\Delta_\rho, W)) \tag{3.2.2}$$

for any summand W of (3.2.1).

Proof of (a): We first note that Δ acts irreducibly on $\bar{r}_\tau$, hence $H^0(\Delta, ad^0\ \bar{r}_\tau) = (0)$. We apply (3.2.2) to show that $H^1(\Delta, W) = 0$ for each summand W of (3.2.1). Indeed, it suffices to show that $H^1(PSL(2, \mathbb{F}_\ell), W) = 0$ for each W. But as a representation of $PSL(2, \mathbb{F}_\ell)$, each W is a direct sum of copies of i-dimensional symmetric powers Sym^{i-1} of the standard representation $PSL(2, \mathbb{F}_\ell)$, where i runs through (odd) integers at most equal to $2n-1$. Since $\ell > 2n+1$, it is well known that $H^1(PSL(2, \mathbb{F}_\ell), Sym^{i-1}) = 0$ for $i \leq 2n-1$.

Proof of (b): Let b denote a generator of the cyclic group $k(v)^\times \simeq \mu_{\ell-1}$, as in the proof of Lemma 2.4. As remarked above, if $\Delta' \subset \Delta$ is a subgroup that satisfies 3.1(b) for a given summand W of (3.2.1), then Δ also satistifes this property for the given W. We may thus assume $Im(\bar{\tau})$ is contained in the normalizer of a maximal torus T in $SL(2, \mathbb{F}_\ell)$ and contains an element $t_0 \in T$ with the two distinct eigenvalues b^n, b^{-n} on $\bar{\tau}$, with corresponding eigenvectors v_1 and v_2, as well as the element $w \notin T$ with eigenvalues 1 and -1. With appropriate normalizations, we can assume the corresponding eigenvectors are $v_1 + v_2$ and $v_1 - v_2$, respectively. Write $k = \mathbb{F}_\ell$. We can write

$$ad\ \bar{\tau} = k_+ \oplus k_- \oplus U,$$

where $U = Ind_{\Gamma_L \cdot F^+}^{\Gamma_{F^+}} \bar{\chi}_\ell / \bar{\chi}_\ell^c$ and k_+ and k_- are representations of Γ_{F^+} that factor through $Gal(L \cdot F^+/F^+)$, with the non-trivial element acting by the indicated sign. We thus have

$$Hom(\bar{r}_\tau, \bar{r}_\tau) = ad\ \bar{\rho}^n \otimes [k_+ \oplus k_- \oplus U]. \tag{3.2.3}$$

For $i, j \in \{1, 2\}$), let $p_{i,j} \in End(\bar{\tau})$ be the endomorphism that takes v_i to v_j and vanishes on v_k if $k \neq i$. Then k_+ (resp. k_-) is spanned by $p_{1,1} + p_{2,2}$ (resp. $p_{1,1} - p_{2,2}$, whereas U is spanned by $p_{1,2}$ and $p_{2,1}$.

Let $t \in Im(\bar{\rho}^n)$ be the image of an element of a split maximal torus of $SL(2, \mathbb{F}_\ell)$, with n distinct eigenvalues under $\bar{\rho}^n$, as in the proof of Lemma 3.2 of [HST]. More precisely, we can take t to be the diagonal element $diag(b, b^{-1})$, so that $\bar{\rho}^n(t)$ has eigenvalues $b^{n-1}, b^{n-3}, \ldots, b^{1-n}$. The formula (3.2.3) expresses $Hom(\bar{r}_\tau, \bar{r}_\tau)$ as a sum of four copies of $ad\ \bar{\rho}^n$, as representation of t. Let h_0, resp h_w denote the image in Δ of $(t, t_0) \in \tilde{\Delta}$, resp. the image of (t, w). Since $\ell > 4n - 1$, $b^i \neq 1$ for any $i < 4n - 1$, hence no ratio of eigenvalues of $\bar{\rho}^n(t)$ equals a ratio of eigenvalues of $\bar{\tau}(t_0)$, nor of $\tau(\bar{w})$. It follows that all the generalized eigenspaces of h_0 and h_w in $\bar{r}_\tau$ are of dimension 1. It was shown in the proof of Lemma 3.2 of [HST] that $ad\ \bar{\rho}^n$ satisfies 3.1(b), with Δ replaced by $Im(\bar{\rho}^n)$ and with t playing the role of h; here we use the hypothesis that $\ell > 2n + 1$. It thus follows that, if W is an irreducible summand of $ad\ \bar{\rho}^n \otimes [k_+ \oplus k_-]$, then 3.1(b) is satisfied for this W with $h = h_0$. On the other hand, if W is an irreducible summand of $ad\ \bar{\rho}^n \otimes U$, then 3.1(b) is satisfied for this W with $h = h_w$. Indeed, it suffices to observe that the element $(p_{1,2} + p_{2,1}) \in U$ takes the eigenvector $v_1 + v_2$ to itself. □

4. Removing τ

We fix an odd number n as above. The hypotheses of the earlier sections remain in force; in particular, we admit the Expected Theorems of §1.

Corollary 4.1. *Let $F^+(\bar{\rho}_{E,\ell})$ denote the splitting field of $\bar{\rho}_{E,\ell}$ and $M = L \cdot F^+(\bar{\rho}_{E,\ell})$. Then there is a totally real Galois extension $F^{\prime,+}/F^+$, linearly disjoint from M over F^+, with the property that for $i = 0, \ldots, n$, $r_{i,F^{\prime,+}} = r_i \mid_{\Gamma_F^{\prime,+}}$ corresponds to a cuspidal automorphic representation Π_i of $GL(2i, F^{\prime,+})$, and such that $r_{\tau,F^{\prime,+}} = r_\tau \mid_{\Gamma_F^{\prime,+}}$ corresponds to a cuspidal automorphic representation Π_τ of $GL(2n, F^{\prime,+})$.*

Let E be a number field, and let ρ be an ℓ-adic representation of Γ_E. We assume ρ to be pure of some weight w, as in §1; thus we can define $L^{norm}(s, \rho)$. We say $L(s, \rho)$ (or $L^{norm}(s, \rho)$ is *invertible* if it extends to a meromorphic function on $\mathbb{C}$ and if $L^{norm}(s, \rho)$ has no zeroes for $Res \geq 1$ and no poles for $Res \geq 1$ except for a possible pole at $s = 1$.

Let $L' = L \cdot F'^{,+}$, and let $c \in Gal(L'/F'^{,+})$ denote complex conjugation The proof of the following lemma was devised with a great deal of help from Richard Taylor.

Lemma 4.2. *The representation Π_τ of $GL(2n, F'^{,+})$ is isomorphic to the automorphic induction from L' of some cuspidal automorphic representation $\Pi_1(\tau)$ of $GL(n, L')$. After possibly replacing $\Pi_1(\tau)$ by its Galois conjugate $\Pi_1(\tau)^c$, the tensor product $\Pi_1(\tau) \otimes \chi$ is isomorphic to its c-conjugate, hence descends to a cuspidal automorphic representation π of $GL(n, F'^{,+})$.*

Proof. Let $\eta_{L'}$ be the quadratic character of $F'^{,+}$ associated to the extension L'. By construction, $\tau_\ell \mid_{\Gamma_{F',+}} \otimes \eta_{L'} \stackrel{\sim}{\longrightarrow} \tau_\ell \mid_{\Gamma_{F',+}}$, hence

$$\Pi_\tau \otimes \eta_{L'} \stackrel{\sim}{\longrightarrow} \Pi_\tau. \tag{4.2.1}$$

It follows from [AC, Chapter 3, Theorem 4.2 (b)] that there exists a cuspidal automorphic representation $\Pi_1(\tau)$ of $GL(n, L')$ such that Π_τ is isomorphic to the automorphic induction of $\Pi_1(\tau)$ from L' to $F'^{,+}$. This means in particular that

$$\Pi_1(\tau)^c \not\simeq \Pi_1(\tau),$$

and

$$L(s, \rho_{\tau,L'}) = L(s, \Pi_{\tau,L'}) = L(s, \Pi_1(\tau))L(s, \Pi_1(\tau)^c) \tag{4.2.2}$$

It follows from Corollary 2.5 that $L(s, \rho^m_{E,F})$, and more generally $L(s, \rho^m_{E,F} \otimes \xi_\ell)$, is entire for all even $m \leq 2n$, $F = F'^{,+}$ or $F = L'$, when ξ_ℓ is the ℓ-adic Galois avatar of an algebraic Hecke character ξ of $\mathbf{A}_F^\times$. The proof of Theorem 4.2 of [HST] shows that $L(s, \rho^m_{E,F} \otimes \xi_\ell)$ is invertible for *all* $m \leq 2n$ and for all algebraic Hecke characters ξ, for $F = F'^{,+}$ or $F = L'$. Of course $L(s, \rho_{\tau,F',+})$ is also entire and $L(s, \rho_{\tau,L'})$ is entire unless $n = 1$, which gives rise to the only possible pole at $s = 1$.

Consider the automorphic L-function

$$L(s) = L(s, \Pi_{\tau,L'} \times \Pi^\vee_{\tau,L'} \otimes (\chi/\chi^c)) \tag{4.2.3}$$

Comparing this to (4.2.2) we find that

$$\begin{aligned} L(s) &= L(s, [\rho^n_{E,L'} \otimes (\chi \oplus \chi^c)] \otimes [\rho^{n,\vee}_{E,L'} \otimes (\chi^{-1} \oplus \chi^{c,-1})] \otimes [\chi/\chi^c]) \\ &= L(s, (\rho^n_{E,L'} \otimes \rho^{n,\vee}_{E,L'}) \otimes [(\chi/\chi^c) \oplus (\chi/\chi^c) \oplus (\chi/\chi^c)^2]) \cdot L(s, \rho^n_{E,L'} \otimes \rho^{n,\vee}_{E,L'}) \end{aligned} \tag{4.2.4}$$

Writing

$$\rho^n_{E,L'} \otimes \rho^{n,\vee}_{E,L'} = \oplus_{i=0}^{n-1} \rho^{2i+1}_{E,L'} \otimes \omega_\ell^{-i}$$

we see that the first factor of the last line of (4.2.4) is a product of invertible L-functions without poles at $s = 1$; the final factor of the last line has a simple pole at $s = 1$. Thus $L(s)$ has a simple pole at $s = 1$. But $L(s)$ is an automorphic L-function for $GL(n) \times GL(n)$. We rewrite $L(s)$ using (4.2.2):

$$L(s) = L(s, \Pi_1(\tau) \times \Pi_1(\tau)^\vee \otimes (\chi/\chi^c)) \cdot L(s, \Pi_1(\tau)^c \times \Pi_1(\tau)^{c,\vee} \otimes (\chi/\chi^c))$$
$$\cdot L(s, \Pi_1(\tau) \times \Pi_1(\tau)^{c,\vee} \otimes (\chi/\chi^c)) \cdot L(s, \Pi_1(\tau)^c \times \Pi_1(\tau)^\vee \otimes (\chi/\chi^c))$$

Applying the Jacquet-Shalika classification theorem we see that exactly one of the factors has a simple pole, and in that case the factor is necessarily of the form $L(s, \Pi \times \Pi^\vee)$. Since $(\chi/\chi^c)_\infty$ is a character of infinite order, neither of the first two factors can have a pole; we must therefore either have

$$\Pi_1(\tau)^\vee \xrightarrow{\sim} \Pi_1(\tau)^{c,\vee} \otimes (\chi/\chi^c)$$

or

$$\Pi_1(\tau)^{c,\vee} \xrightarrow{\sim} \Pi_1(\tau)^\vee \otimes (\chi/\chi^c).$$

In other words, up to exchanging $\Pi_1(\tau)$ with $\Pi_1(\tau)^c$, we have

$$\Pi_1(\tau) \otimes \chi \xrightarrow{\sim} (\Pi_1(\tau) \otimes \chi)^c;$$

hence $\Pi_1(\tau) \otimes \chi$ descends to a cuspidal automorphic representation of $GL(n, F'^{,+})$. This completes the proof. □

Now $(\Pi_\tau)_\infty$ is cohomological, hence $(\Pi_\tau)_{L',\infty}$ is also cohomological. But $(\Pi_\tau)_{L',\infty}$ is represented as a subquotient of the representation of $GL(2n, L'_\infty)$ induced from the representation $(\Pi_1(\tau))_\infty \otimes (\Pi_1(\tau)^c)_\infty$ of the Levi factor $GL(n, L'_\infty) \times GL(n, L'_\infty)$ of the relevant maximal parabolic, and it follows that $\Pi_1(\tau)_\infty$ is also cohomological. Thus $\Pi_1(\tau)$ satisfies condition (1) of Expected Theorem 1.2.

On the other hand, ρ_τ has a symplectic polarization with multiplier ω_ℓ^{1-2n}, by construction. It follows that the associated automorphic representation Π_τ is self dual. This property is preserved under base change to L'. It thus follows from (4.2.2) and the Jacquet-Shalika classification theorem that

$$\{\Pi_1(\tau), \Pi_1(\tau)^c\} = \{\Pi_1(\tau)^\vee, \Pi_1(\tau)^{c,\vee}\}$$

as sets. Thus either (a) $\Pi_1(\tau)$ satisfies condition (2) of Expected Theorem 1.2, or (b) $\Pi_1(\tau) \xrightarrow{\sim} \Pi_1(\tau)^\vee$.

Assume (a) holds. Then $\Pi_1(\tau)$ satisfies both conditions of Expected Theorem 1.2, hence is associated to an n-dimensional Galois representation $\rho_1(\tau)$ of $\Gamma_{L'}$. It follows from (4.2.2) that

$$\rho_1(\tau) \oplus \rho_1(\tau)^c \xrightarrow{\sim} \rho^n_{E,\ell} \otimes \chi_\ell \oplus \rho^n_{E,\ell} \otimes \chi^c_\ell \tag{4.3}$$

Since $\rho^n_{E,\ell}$ is irreducible, it follows that it must be equal to either $\rho_1(\tau) \otimes \chi_\ell$ or $\rho_1(\tau) \otimes \chi^c_\ell$. In either case, $\rho^n_{E,\ell}$ is associated to a cuspidal automorphic representation $\Pi_{n,L'}$ of $GL(n, L')$. Since $\rho^n_{E,\ell}$ descends to a representation of $\Gamma_{F',+}$, $\Pi_{n,L'}$ descends to a cuspidal automorphic representation Π_n of $GL(n, F'^{,+})$. Thus $\rho^n_{E,\ell}$ is automorphic over $F'^{,+}$.

Assume (b) holds. Then the central character ξ of $\Pi_1(\tau)$ is also self-dual, i.e. $\xi = \xi^{-1}$. It follows from Lemma . that

$$\Pi_1(\tau) \otimes \chi \xrightarrow{\sim} \Pi_1(\tau)^c \otimes \chi^c.$$

Combining this with (b), we have

$$\xi \cdot \chi \circ \det = \xi^c \cdot \chi^c \circ \det;$$

This implies that χ/χ^c is self-dual, ie.

$$(\chi^c)^2 = \chi^2.$$

But this is already false for the archimedean components. Thus (b) is impossible.

We have thus proved that $\rho^n_{E,\ell}$ is automorphic over $F^{\prime,+}$. More generally, Theorem 2.1 allows us to add new r_τ's of different dimensions $2n_i$, with n_i odd, to the list in Corollary 2.5. By adding $\rho^{2i+1}_{E,\ell} \otimes \tau_i$ to the list – we are free to vary the 2-dimensional τ_i if we like, we thus obtain our main theorem:

Theorem 4.4. *Assume the Expected Theorems of §1. Let F^+ be a totally real field, and let E be an elliptic curve over F^+. Let n be a positive integer. Then there is a finite totally real Galois extension $F^{\prime,+}/F$ and, for each positive integer $i \leq n$, a cuspidal automorphic representation Π_i of $GL(i, F^{\prime,+})$, satisfying conditions (a) and (b) of Expected Theorem 1.2, such that*

$$\rho^i_{E,\ell} \mid_{\Gamma_{F^{\prime,+}}} = \rho_{\Pi_i,\ell}.$$

In particular, if $i > 1$, $L^{norm}(s, \rho^i_{E,\ell,F^{\prime,+}}) = L(s, \Pi_i)$ is an entire function.

I repeat that we are not assuming that E have non-integral j-invariant; however, all statements are conditional on the Expected Theorems of §1.

5. Applications and generalizations

We continue to admit the Expected Theorems of §1. Let E and E' be two elliptic curves over F^+ without complex multiplication. Assume E and E' are not isogenous. It then follows from Faltings' isogeny theorem that $\rho_{E,\ell}$ and $\rho_{E',\ell}$ are not isomorphic as representations of Γ_{F^+} for all ℓ. Since the traces of $Frob_v$, for primes v of good reduction for E and E', are integers that determine $\rho_{E,\ell}$ and $\rho_{E',\ell}$ up to isomorphism, it follows that $\bar{\rho}_{E,\ell}$ and $\bar{\rho}_{E',\ell}$ are not isomorphic for sufficiently large ℓ. By Serre's theorem, if ℓ is sufficiently large, $Im(\bar{\rho}_{E,\ell}) = Im(\bar{\rho}_{E',\ell}) = GL(2, \mathbb{F}_\ell)$.

Let m, m' be two positive integers. Applying Theorem 2.1, we obtain the analogue of Corollary 2.5 for the collection of representations $r_i = \rho^{2i}_{E,\ell}$, $r'_j = \rho^{2j}_{E',\ell}$, $1 \leq i \leq m$, $1 \leq j \leq m'$, together with $r_\tau = \rho^m_{E,\ell} \otimes \tau$ and $r'_{\tau'} = \rho^{m'} \otimes \tau'$ if m or m' is odd, provided $\bar{\tau}$ and $\bar{\tau'}$ are "big enough". Bearing in mind the results of §3, we have the following statement:

Proposition 5.1. *Let L be as in §3, and define*

$$\tau_\ell = Ind_{\Gamma_L}^{\Gamma_\mathbb{Q}} \chi_\ell;\ \tau'_\ell = Ind_{\Gamma_L}^{\Gamma_\mathbb{Q}} \chi'_\ell$$

where χ (resp. χ') is a Hecke character with $\chi_\infty(z) = z^{-m}$, (resp. $\chi'_\infty(z) = z^{-m'}$) if m (resp. m') is odd. We assume

$$\chi \mid_{\mathbf{A}_\mathbb{Q}^\times} = \mid \bullet \mid_{\mathbf{A}}^{-m} \cdot \eta_L;\ \chi' \mid_{\mathbf{A}_\mathbb{Q}^\times} = \mid \bullet \mid_{\mathbf{A}}^{-m'} \cdot \eta_L$$

Suppose there is a prime $\ell > \sup(2m+1, 2m'+1)$, unramified in F^+, such that

(i) *E has good ordinary reduction at all primes w dividing ℓ*
(ii) *ℓ splits in L;*
(iii) *ℓ splits in $\mathbb{Q}(\zeta_{2i+1})$, $i = 1, \ldots, \sup(m-1, m'-1)$; in particular, $\ell \equiv 1 \pmod{mm'}$.*
(iv) $\frac{\ell-1}{m} > 2$, $\frac{\ell-1}{m'} > 2$.

Let M be an arbitrary extension of F^+. Define r_i, r'_j r_τ and $r_{\tau'}$ as above. Then there is a totally real Galois extension $F'^{,+}/F^+$, linearly disjoint from M over F^+, with the property that for $i = 1, \ldots, m$, (resp. $j = 1, \ldots, m'$) $r_{i,F',+} = r_i \mid_{\Gamma_F'^{,+}}$ (resp. $r'_{j,F',+}$) corresponds to an automorphic representation Π_i of $GL(2i, F'^{,+})$ ((resp. Π'_j of $GL(2j, F'^{,+})$) and such that r_τ (resp. $r'_{\tau'}$) corresponds to an automorphic representation Π_τ of $GL(2m, F'^{,+})$ (resp. $\Pi'_{\tau'}$ of $GL(2m', F'^{,+})$. If $F'/F'^{,+}$ is a CM quadratic extension, then the base change $\Pi_{i,F'}$ (resp. $\Pi_{\tau,F'}$) has archimedean constituent isomorphic to $\Pi_{\infty,0}(2i, F')$ (resp. $\Pi_{\infty,0}(2n, F')$), and likewise for $\Pi'_{j,F'}$, $\Pi'_{\tau',F'}$.

The discussion of §4 applies to both Π_τ and Π'_τ, and we obtain the following strengthening of Theorem 4.4.

Theorem 5.2. *Assume the Expected Theorems of §1. Let F^+ be a totally real field, let E and E' be elliptic curves over F^+, and assume E and E' do not become isogenous over an abelian extension of F^+. Let m and m' be positive integers. Then there is a finite totally real Galois extension $F'^{,+}/F$ and, for each positive integer $i \leq m$, (resp. $j \leq m'$) a cuspidal automorphic representation Π_i of $GL(i, F'^{,+})$ (resp. Π'_j of $GL(j, F'^{,+})$) satisfying conditions (a) and (b) of Expected Theorem 1.2, such that*

$$\rho^i_{E,\ell} \mid_{\Gamma_{F',+}} = \rho_{\Pi_i,\ell}; \rho^j_{E',\ell} \mid_{\Gamma_{F',+}} = \rho_{\Pi'_j,\ell}$$

In particular, if $m \cdot m' > 1$,

$$L^{norm}(s, \rho^m_{E,\ell,F',+} \otimes \rho^{m'}_{E',\ell,F',+}) = L(s, \Pi_m \times \Pi'_{m'})$$

is an entire function.

The Rankin-Selberg L-function has no poles if $m \neq m'$; if $m = m'$ it has a pole if and only if $\Pi'_{m'} \xrightarrow{\sim} \Pi_m^\vee$. which implies that the corresponding Galois representations $\rho^m_{E,\ell}$ and $\rho^m_{E',\ell}$ are isomorphic. The kernel of the map of the standard 2-dimensional representation of $GL(2)$ to its mth symmetric power is finite and contained in the center. If $\rho^m_{E,\ell} \xrightarrow{\sim} \rho^m_{E',\ell}$, it thus follows that the corresponding adjoint representations $ad\ \rho_{E,\ell}$ and $ad\ \rho_{E',\ell}$ are isomorphic, hence that there exists an abelian character η, necessarily finite, such that $\rho_{E',\ell} \xrightarrow{\sim} \rho_{E,\ell} \otimes \eta$. Thus $\rho_{E',\ell}$ and $\rho_{E,\ell}$ become isomorphic over a finite extension of F^+, hence E and E' are isogenous by Faltings' theorem.

Using Brauer's theorem, as in the proof of Theorem 4.2 of [HST], we then obtain:

Theorem 5.3. *Assume the Expected Theorems of §1. Let F^+ be a totally real field, let E and E' be elliptic curves over F^+, and assume E and E' do not become isogenous over an abelian extension of F^+. Let m and m' be positive integers. Then the L-function $L(s, \rho^m_{E,\ell} \otimes \rho^{m'}_{E',\ell})$ is invertible and satisfies the expected functional equation.*

Proof. This is obtained from Theorem 5.2 by applying Brauer's theorem, as in the proof of Theorem 4.2 of [HST]. It suffices to mention that the non-vanishing of the Rankin-Selberg L-function along the line $Re(s) = 1$ of a pair of cuspidal automorphic representations (with unitary central characters) is due in general to Shahidi [Sh]. □

Finally, here is the precise statement of the question of Mazur and Katz mentioned in the introduction, together with the affirmative response. Recall the notation k_v and q_v of §1.

Theorem 5.4. *Assume the Expected Theorems of §1. Let F^+ be a totally real field, let E and E' be elliptic curves over F^+, and assume E and E' do not become isogenous over an abelian extension of F^+. For any prime v of F^+ where E and E' both have good reduction, we let*

$$|E(k_v)| = (1 - q_v^{\frac{1}{2}} e^{i\phi_v})(1 - q_v^{\frac{1}{2}} e^{-i\phi_v})$$

$$|E'(k_v)| = (1 - q_v^{\frac{1}{2}} e^{i\psi_v})(1 - q_v^{\frac{1}{2}} e^{-i\psi_v})$$

where $\phi_v, \psi_v \in [0, \pi]$.

Then the pairs $(\phi_v, \psi_v) \in [0, \pi] \times [0, \pi]$ are uniformly distributed with respect to the measure

$$\frac{4}{\pi^2} sin^2\phi\ sin^2\psi\ d\phi d\psi.$$

Proof. Theorem 5.4 follows directly from Theorem 5.3 by the argument in [S], Appendix to §I. □

6. Concluding remarks

The author and Richard Taylor have independently noticed that tensoring with an induced representation from a Hecke character may be useful in other situations. For example, let f be an elliptic modular form of weight k, $\rho_{f,\ell} : \Gamma_{\mathbb{Q}} \to GL(2, \bar{\mathbb{Q}}_\ell)$ the associated two-dimensional Galois representation, and let $\rho^n_{f,\ell} = Sym^{n-1}\rho_{f,\ell}$. There is no hope of applying the potential modularity technique of [HST] to $\rho^n_{f,\ell}$ if $k > 2$: the series of Hodge-Tate weights at ℓ has gaps for all n, and Griffiths transversality implies it is impossible to obtain families of positive dimension of motives with such Hodge-Tate numbers. However, if k is odd, one can choose a Hecke character χ of an abelian CM extension $L/\mathbb{Q}$ of degree $k-1$ with infinity type so chosen that $\rho^n_{f,\ell} \otimes Ind_{\Gamma_L}^{\Gamma_{\mathbb{Q}}} \chi_\ell$ has an unbroken series of Hodge-Tate weights. (If k is even one takes L of degree $2(k-1)$.)

Two serious obstacles remain. In the first place, the constructions in §4 require that we know in advance that the symmetric power L-functions are invertible, and this information is not available *a priori* for $k > 2$. In the second place, the arguments for finding rational points on moduli spaces over number fields unramified at ℓ break down in higher weights.

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