Rankin's lemma of higher genus and explicit formulas for Hecke operators

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To dear Yuri Ivanovich Manin for his seventieth birthday with admiration

Summary. We develop explicit formulas for Hecke operators of higher genus in terms of spherical coordinates. Applications are given to summation of various generating series with coefficients in local Hecke algebra and in a tensor product of such algebras. In particular, we formulate and prove Rankin's lemma in genus two. An application to a holomorphic lifting from $GSp_2 \times GSp_2$ to GSp_4 is given using Ikeda-Miyawaki constructions.

1 Introduction: generating series for the Hecke operators

Let p be a prime. The Satake isomorphism [Sa63] relates p-local Hecke algebras of reductive groups over $\mathbb{Q}$ to certain polynomial rings. Then one can use a computer in order to find interesting identities between Hecke operators, between their eigenvalues, and relations to Fourier coefficients of modular forms of higher degree.

The purpose of the present paper is to extend Rankin's Lemma to the summation of Hecke series of higher genus using symbolic computations. We refer to [Ma-Pa77], where Rankin's Lemma was used in the elliptic modular case for multiplicative and additive convolutions of Dirichlet series. That work was further developped in [Pa87], [Pa02], see also [Ma-Pa05].

Recall that a classical method to produce L-functions for an algebraic group G over $\mathbb{Q}$ uses the generating series

$$\sum_{n=1}^{\infty} \lambda_f(n) n^{-s} = \prod_{p \text{ primes}} \sum_{\delta=0}^{\infty} \lambda_f(p^\delta) p^{-\delta s},$$

of the eigenvalues of Hecke operators on an automorphic form f on G. We study the generating series of Hecke operators $\mathbf{T}(n)$ for the symplectic group Sp_g, and $\lambda_f(n) = \lambda_f(\mathbf{T}(n))$.

Let $\Gamma = \mathrm{Sp}_g(\mathbb{Z}) \subset \mathrm{S}L_{2g}(\mathbb{Z})$ be the Siegel modular group of genus g, and $[\mathbf{p}]_g = p\mathbf{I}_{2g} = \mathbf{T}(\underbrace{p, \cdots, p}_{2g})$ be the scalar Hecke operator for Sp_g. According to Hecke and Shimura,

$$D_p(X) = \sum_{\delta=0}^{\infty} \mathbf{T}(p^\delta) X^\delta$$
$$= \begin{cases} \dfrac{1}{1 - \mathbf{T}(p)X + p[\mathbf{p}]_1 X^2}, & \text{if } g = 1 \\ & \text{(see [Hecke], and [Shi71], Theorem 3.21),} \\ \dfrac{1 - p^2[\mathbf{p}]_2 X^2}{1 - \mathbf{T}(p)X + \{p\mathbf{T}_1(p^2) + p(p^2+1)[\mathbf{p}]_2\}X^2 - p^3[\mathbf{p}]_2\mathbf{T}(p)X^3 + p^6[\mathbf{p}]_2^2 X^4} \\ & \text{if } g = 2 \text{ (see [Shi63], Theorem 2),} \end{cases}$$

where $\mathbf{T}(p)$, $\mathbf{T}_i(p^2)$ $(i = 1, \cdots, g)$ are the $g+1$ generators of the corresponding Hecke ring over $\mathbb{Z}$ for the symplectic group Sp_g, in particular, $\mathbf{T}_g(p^2) = [\mathbf{p}]_g$.

It was established by A. N. Andrianov, that there exist polynomials

$$E(X), F(X) \in \mathbb{Q}[\mathbf{T}(p), \mathbf{T}_1(p^2), \cdots, \mathbf{T}_g(p^2), X] \text{ such that}$$

$$D(X) = \sum_{\delta=0}^{\infty} \mathbf{T}(p^\delta) X^\delta = \frac{E(X)}{F(X)},$$

with a polynomial $F(X) = \sum_{j=0}^{2^g} \mathbf{q}_j X^j$ of degree 2^g, and such that $E(X) = \sum_{j=0}^{2^g-2} \mathbf{u}_j X^j$ is a polynomial of degree $2^g - 2$ with the leading term

$$(-1)^{g-1} p^{g(g+1)2^{g-2} - g^2} [\mathbf{p}]^{2^{g-1}-1} X^{2^g-2}$$

(as stated in Theorem 6 at p. 451 of [An70] and at p. 61 of § 1.3, [An74]).

In the present paper we study explicit formulas for Hecke operators in higher genus in terms of spherical coordinates. Applications are given to summation of various generating series with coefficients in local Hecke algebras. The question of computing such series explicitly was raised by Prof. S.Friedberg during first author's talk at the conference "Zeta Functions" (The Independent Moscow University, September 18-22, 2006).

In particular, we formulate and prove Rankin's lemma in genus two for generating series with coefficients in a tensor product of local Hecke algebras.

We prove in Theorem 4 that for $g = 2$,

$$\sum_{\delta=0}^{\infty} \mathbf{T}(p^\delta) \otimes \mathbf{T}(p^\delta)\, X^\delta = \frac{(1 - p^6[\mathbf{p}] \otimes [\mathbf{p}]X^2) \cdot R(X)}{S(X)},$$

for certain two polynomials

$R(X) = 1 + r_1 X + \cdots + r_{11} X^{11} + r_{12} X^{12}$ with $r_1 = r_{11} = 0$,

$S(X) = 1 + s_1 X + \cdots + s_{16} X^{16}$

with coefficients explicitly expressed through Hecke operators in Appendix. A motivic interpretation of the polynomial S is given in terms of the tensor product of motives.

For the group $GL(n)$, a simpler result could be obtained using the Tamagawa generating series (see [Tam]) and its expansion into simple fractions:

$$\frac{1}{(1 - x_1 X)\cdot \ldots \cdot(1 - x_n X)} = \sum_{i=1}^{n} \frac{\alpha_i}{1 - x_i X} \text{ where } \alpha_i = \frac{x_i^{n-1}}{\prod_{j\neq i}^{n}(x_j - x_i)},$$

see Remark 6.

Also, it is helpful for the reader to remember cases where the Hecke series do have simple numerators: Andrianov ($GSp(2)$, see ([An74]), Tamagawa ($GL(n)$, see [Tam]), Böcherer (for standard-L-function, see [BHam]).

2 Results

2.1 Preparation: a formula for the total Hecke operator $\mathbf{T}(p^\delta)$ of genus 2

We establish first the following useful formula (in spherical variables x_0, x_1, x_2):

$$\begin{aligned}
&\Omega_x^{(2)}(\mathbf{T}(p^\delta)) = \qquad (1)\\
&\quad p^{-1}\, x_0^\delta (p\, x_1^{(3+\delta)}\, x_2 - p\, x_1^{(2+\delta)} - p\, x_1^{(3+\delta)}\, x_2^{(2+\delta)} + p\, x_1^{(2+\delta)}\, x_2^{(3+\delta)}\\
&\quad - p\, x_1\, x_2^{(3+\delta)} + p\, x_2^{(2+\delta)} + p\, x_1 - p\, x_2 - x_1^{(2+\delta)}\, x_2^2 + x_1^{(1+\delta)}\, x_2\\
&\quad + x_1^{(2+\delta)}\, x_2^{(1+\delta)} - x_1^{(1+\delta)}\, x_2^{(2+\delta)} + x_1^2\, x_2^{(2+\delta)} - x_1\, x_2^{(1+\delta)} - x_1^2\, x_2 + x_1\, x_2^2)/\\
&\quad ((1 - x_1)\,(1 - x_2)\,(1 - x_1\, x_2)\,(x_1 - x_2))\\
&= - p^{-1}\, x_0^\delta ((1 - x_1\, x_2)\,(p\, x_1 - x_2)\, x_1^{(\delta+1)} + (1 - x_1\, x_2)\,(x_1 - p\, x_2)\, x_2^{(\delta+1)}\\
&\qquad - (1 - p\, x_1\, x_2)\,(x_1 - x_2)\,(x_1\, x_2)^{(\delta+1)} - (p - x_1\, x_2)(x_1 - x_2))/\\
&\qquad ((1 - x_1)\,(1 - x_2)\,(1 - x_1\, x_2)\,(x_1 - x_2)).
\end{aligned}$$

Andrianov's generating series

The expression (1) comes from the following Andrianov's generating series

$$\sum_{\delta=0}^{\infty} \Omega_x^{(2)}(\mathbf{T}(p^\delta))\, X^\delta = \frac{1 - \frac{x_0^2\, x_1\, x_2}{p}\, X^2}{(1 - x_0\, X)\,(1 - x_0\, x_1\, X)\,(1 - x_0\, x_2\, X)\,(1 - x_0\, x_1\, x_2\, X)}$$

after developing and a simplification using change of summation.

Note that the formula (1) makes it possible to treat higher generating series of the following type

$$D_{p,m}(X) = \sum_{\delta=0}^{\infty} \Omega_x^{(2)}(\mathbf{T}(p^{m\delta}))\, X^\delta \ \ (m = 2, 3, \cdots)$$

(in spherical variables x_0, x_1, x_2).

2.2 Rankin's generating series in genus 2

Let us use the spherical variables x_0, x_1, x_2 and y_0, y_1, y_2 for the Hecke operators.

Note that there are two types of convolutions: the first one is defined through the Fourier coefficients (it was used by [An-Ka] for the analytic continuation of the standard L- function), and the second one is defined through the eigenvalues of Hecke operators, and it is more suitable in order to treat the L-functions attached to tensor products of representations of the Langlands group. However, a link between the two types is known only for $g = 1$.

In order to state a multiplicative analogue of Rankin's lemma in genus two we need to write the corrseponding formula for Hecke operator $\mathbf{T}(p^\delta)$ (in spherical variables y_0, y_1, y_2).

$$\begin{aligned}\Omega_y^{(2)}(\mathbf{T}(p^\delta)) &= p^{-1}\, y_0^\delta (p\, y_1^{(3+\delta)}\, y_2 - p\, y_1^{(2+\delta)} - p\, y_1^{(3+\delta)}\, y_2^{(2+\delta)} + p\, y_1^{(2+\delta)}\, y_2^{(3+\delta)} \\ &\quad - p\, y_1\, y_2^{(3+\delta)} + p\, y_2^{(2+\delta)} + p\, y_1 - p\, y_2 - y_1^{(2+\delta)}\, y_2^2 + y_1^{(1+\delta)}\, y_2 \\ &\quad + y_1^{(2+\delta)}\, y_2^{(1+\delta)} - y_1^{(1+\delta)}\, y_2^{(2+\delta)} + y_1^2\, y_2^{(2+\delta)} - y_1\, y_2^{(1+\delta)} - y_1^2\, y_2 + y_1\, y_2^2)/ \\ &\quad ((1 - y_1)\,(1 - y_2)\,(1 - y_1\, y_2)\,(y_1 - y_2))\end{aligned}$$

Then we have that the product of the above polynomials is given by

$$
\begin{aligned}
&\Omega_x^{(2)}(\mathbf{T}(p^\delta)) \cdot \Omega_y^{(2)}(\mathbf{T}(p^\delta)) = \\
&\quad p^{-2}\, x_0^\delta\, y_0^\delta (p\, x_1^{(3+\delta)}\, x_2 - p\, x_1^{(2+\delta)} - p\, x_1^{(3+\delta)}\, x_2^{(2+\delta)} + p\, x_1^{(2+\delta)}\, x_2^{(3+\delta)} \\
&\quad - p\, x_1\, x_2^{(3+\delta)} + p\, x_2^{(2+\delta)} + p\, x_1 - p\, x_2 - x_1^{(2+\delta)}\, x_2^2 + x_1^{(1+\delta)}\, x_2 \\
&\quad + x_1^{(2+\delta)}\, x_2^{(1+\delta)} - x_1^{(1+\delta)}\, x_2^{(2+\delta)} + x_1^2\, x_2^{(2+\delta)} - x_1\, x_2^{(1+\delta)} - x_1^2\, x_2 + x_1\, x_2^2) \\
&\quad \cdot (p\, y_1^{(3+\delta)}\, y_2 - p\, y_1^{(2+\delta)} - p\, y_1^{(3+\delta)}\, y_2^{(2+\delta)} + p\, y_1^{(2+\delta)}\, y_2^{(3+\delta)} \\
&\quad - p\, y_1\, y_2^{(3+\delta)} + p\, y_2^{(2+\delta)} + p\, y_1 - p\, y_2 - y_1^{(2+\delta)}\, y_2^2 + y_1^{(1+\delta)}\, y_2 \\
&\quad + y_1^{(2+\delta)}\, y_2^{(1+\delta)} - y_1^{(1+\delta)}\, y_2^{(2+\delta)} + y_1^2\, y_2^{(2+\delta)} - y_1\, y_2^{(1+\delta)} - y_1^2\, y_2 + y_1\, y_2^2)/ \\
&\qquad ((1-x_1)(1-x_2)(1-x_1x_2)(x_1-x_2)(1-y_1)(1-y_2)(1-y_1y_2)(y_1-y_2))
\end{aligned}
$$

We wish to compute the generating series

$$\sum_{\delta=0}^{\infty} \Omega_x^{(2)}(\mathbf{T}(p^\delta)) \cdot \Omega_y^{(2)}(\mathbf{T}(p^\delta))\, X^\delta \in \mathbb{Q}[x_0, x_1, x_2, y_0, y_1, y_2][\![X]\!].$$

The answer is given by the following multiplicative analogue of Rankin's lemma in genus two:

Theorem 1. *The following equality holds*

$$
\begin{aligned}
&\sum_{\delta=0}^{\infty} \Omega_x^{(2)}(\mathbf{T}(p^\delta)) \cdot \Omega_y^{(2)}(\mathbf{T}(p^\delta))\, X^\delta = \qquad (2)\\
&- \frac{(p\, x_1 - x_2)\,(1 - p\, y_1\, y_2)\, x_1\, y_1\, y_2}{p^2\,(1-x_1)\,(1-x_2)\,(x_1-x_2)\,(1-y_1)\,(1-y_2)\,(1-y_1\,y_2)\,(1-x_0\,x_1\,y_0\,y_1\,y_2\,X)} \\
&+ \frac{x_2\, y_1\,(x_1 - p\, x_2)\,(p\, y_1 - y_2)}{p^2\,(1-x_1)\,(1-x_2)\,(x_1-x_2)\,(1-y_1)\,(1-y_2)\,(y_1-y_2)\,(1-x_0\,x_2\,y_0\,y_1\,X)} \\
&+ \frac{x_2\, y_2\,(x_1 - p\, x_2)(y_1 - p\, y_2)}{p^2\,(1-x_1)\,(1-x_2)\,(x_1-x_1)\,(1-y_1)\,(1-y_2)\,(y1-y_2)\,(1-x_0\,y_0\,x_2\,y_2\,X)} \\
&- \frac{x_2\, y_1\, y_2\,(x_1 - p\, x_2)\,(1 - p\, y_1\, y_2)}{p^2\,(1-x_1)\,(1-x_2)\,(x_1-x_2)\,(1-y_1)\,(1-y_2)\,(1-y_1\,y_2)\,(1-x_0\,x_2\,y_0\,y_1\,y_2\,X)} \\
&- \frac{x_1\,(p\, x_1 - x_2)\,(p - y_1\, y_2)}{p^2\,(1-x_1)\,(1-x_2)\,(x_1-x_2)\,(1-y_1)\,(1-y_2)\,(1-y_1\,y_2)\,(1-x_0\,x_1\,y_0\,X)} \\
&- \frac{x_1\, x_2\, y_1 (1 - p\, x_1\, x_2)\,(p\, y_1 - y_2)}{p^2\,(1-x_1)\,(1-x_2)\,(1-x_1\,x_2)\,(1-y_1)\,(1-y_2)\,(y_1-y_2)\,(1-x_0\,x_1\,x_2\,y_0\,y_1\,X)} \\
&- \frac{x_1\, x_2\, y_2\,(1 - p\, x_1\, x_2)\,(y_1 - p\, y_2)}{p^2\,(1-x_1)\,(1-x_2)\,(1-x_1\,x_2)\,(1-y_1)\,(1-y_2)\,(y_1-y_2)\,(1-x_0\,x_1\,x_2\,y_0\,y_2\,X)} \\
&+ \frac{y_1\, y_2\,(p - x_1\, x_2)\,(1 - p\, y_1\, y_2)}{p^2\,(1-x_1)\,(1-x_2)\,(1-x_1\,x_2)\,(1-y_1)\,(1-y_2)\,(1-y_1\,y_2)\,(1-x_0\,y_0\,y_1\,y_2\,X)} \\
&+ \frac{x_1\, x_2\,(1 - p\, x_1\, x_2)\,(p - y_1\, y_2)}{p^2\,(1-x_1)\,(1-x_2)\,(1-x_1\,x_2)\,(1-y_1)\,(1-y_2)\,(1-y_1\,y_2)\,(1-x_0\,x_1\,x_2\,y_0\,X)}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{x_1\, y_1\, (p\, x_1 - x_2)\, (p\, y_1 - y_2)}{p^2\, (1 - x_1)\, (1 - x_2)\, (x_1 - x_2)\, (1 - y_1)\, (1 - y_2)\, (y_1 - y_2)\, (1 - x_0\, x_1\, y_0\, y_1\, X)} \\
&+ \frac{x_1\, y_2\, (p\, x_1 - x_2)\, (y_1 - p\, y_2)}{p^2\, (1 - x_1)\, (1 - x_2)\, (x_1 - x_2)\, (1 - y_1)\, (1 - y_2)\, (y_1 - y_2)\, (1 - x_0\, x_1\, y_0\, y_2\, X)} \\
&- \frac{x_2\, (x_1 - p\, x_2)\, (p - y_1\, y_2)}{p^2\, (1 - x_1)\, (1 - x_2)\, (x_1 - x_2)\, (1 - y_1)\, (1 - y_2)\, (1 - y_1\, y_2)\, (1 - x_0\, x_2\, y_0\, X)} \\
&+ \frac{x_1\, x_2\, y_1\, y_2\, (1 - p\, x_1\, x_2)\, (1 - p\, y_1\, y_2)}{p^2\, (1 - x_1)\, (1 - x_2)\, (1 - x_1\, x_2)\, (1 - y_1)\, (1 - y_2)\, (1 - y_1\, y_2)\, (1 - x_0\, x_1\, x_2\, y_0\, y_1\, y_2\, X)} \\
&+ \frac{(p - x_1\, x_2)\, (p - y_1\, y_2)}{p^2\, (1 - x_1)\, (1 - x_2)\, (1 - x_1\, x_2)\, (1 - y_1)\, (1 - y_2)\, (1 - y_1\, y_2)\, (1 - x_0\, y_0\, X)} \\
&- \frac{y_1\, (p - x_1\, x_2)\, (p\, y_1 - y_2)}{p^2\, (1 - x_1)\, (1 - x_2)\, (1 - x_1\, x_2)\, (1 - y_1)\, (1 - y_2)\, (y_1 - y_2)\, (1 - x_0\, y_0\, y_1\, X)} \\
&- \frac{y_2\, (p - x_1\, x_2)\, (y_1 - p\, y_2)}{p^2\, (1 - x_1)\, (1 - x_2)\, (1 - x_1\, x_2)\, (1 - y_1)\, (1 - y_2)\, (y_1 - y_2)\, (1 - x_0\, y_0\, y_2\, X)}
\end{aligned}
$$

Remark 2 (On the denominator of series (2)). One finds using a computer that the polynomials not depending on X in the denominators of (2) cancel after the simplification in the ring $\mathbb{Q}[x_0, x_1, x_2, y_0, y_1, y_2][\![X]\!]$, so that the common denominator becomes

$$
\begin{aligned}
&(1 - x_0\, y_0 X)(1 - x_0\, y_0\, x_1\, X)(1 - x_0\, y_0\, y_1\, X)(1 - x_0\, y_0\, x_2\, X)(1 - x_0\, y_0\, y_2\, X) \\
&(1 - x_0\, y_0\, x_1\, y_1\, X)(1 - x_0\, y_0\, x_1\, x_2\, X)(1 - x_0\, y_0\, x_1\, y_2\, X)(1 - x_0\, y_0\, y_1\, x_2\, X) \\
&(1 - x_0\, y_0\, y_1\, y_2\, X)(1 - x_0\, y_0\, x_2\, y_2\, X)(1 - x_0\, y_0\, x_1\, y_1\, x_2\, X)(1 - x_0\, y_0\, x_1\, y_1\, y_2\, X) \\
&(1 - x_0\, y_0\, x_1\, x_2\, y_2\, X)(1 - x_0\, y_0\, y_1\, x_2\, y_2\, X)(1 - x_0\, y_0\, x_1\, y_1\, x_2\, y_2\, X).
\end{aligned}
$$

Remark 3 (Comparison with $g = 1$). It turns out by direct computation that the numerator of the rational fraction (2) is a product of the factor $1 - x_0^2 y_0^2\, x_1 y_1 x_2 y_2 X^2$ by a polynomial of degree 12 in X with coefficients in $\mathbb{Q}[x_0, x_1, x_2, y_0, y_1, y_2]$ with the constant term equal to 1 and the leading term

$$\frac{x_0^{12} y_0^{12} x_1^6 x_2^6 y_1^6 y_2^6}{p^2} X^{12}.$$

Moreover, the factor of degree 12 does not contain terms of degree 1 and 11 in X. The factor of degree 2 in X is very similar to one in the case $g = 1$ (this series was studied and used in [Ma-Pa77]):

$$
\begin{aligned}
&\sum_{\delta=0}^{\infty} \Omega_x^{(1)}(\mathbf{T}(p^\delta)) \cdot \Omega_y^{(1)}(\mathbf{T}(p^\delta))\, X^\delta = \sum_{\delta=0}^{\infty} \frac{x_0^\delta\, (1 - x_1^{(1+\delta)})}{1 - x_1} \cdot \frac{y_0^\delta\, (1 - y_1^{(1+\delta)})}{1 - y_1} X^\delta \\
&= \frac{1}{(1 - x_1)\, (1 - y_1)\, (1 - x_0\, y_0\, X)} - \frac{y_1}{(1 - x_1)\, (1 - y_1)\, (1 - x_0\, y_0\, y_1\, X)} \\
&- \frac{x_1}{(1 - x_1)\, (1 - y_1)\, (1 - x_0\, y_0\, x_1\, X)} + \frac{x_1\, y_1}{(1 - x_1)\, (1 - y_1)\, (1 - x_0\, y_0\, x_1\, y_1\, X)} \\
&= \frac{1 - x_0^2\, y_0^2\, x_1\, y_1\, X^2}{(1 - x_0\, y_0\, x_1\, y_1\, X)\, (1 - x_0\, y_0\, x_1\, X)\, (1 - x_0\, y_0\, y_1\, X)\, (1 - x_0\, y_0\, X)}
\end{aligned}
$$

2.3 Symmetric square generating series in genus 2

Using the same method, one can evaluate the symmetric square generating series and the cubic generating series of higher genus. Note that this series, written here in spherical variables x_0, x_1, x_2 is different from one studied by Andrianov-Kalinin, and has the form:

$$\sum_{\delta=0}^{\infty} \Omega_x^{(2)}(\mathbf{T}(p^{2\delta}))X^{\delta} = (1 - \frac{x_0^2 x_1 x_2}{p} X)\times$$
$$\times \frac{(1 + x_0^2 x_1 X + x_0^2 x_2 X + 2x_0^2 x_1 x_2 X + x_0^2 x_1 x_2^2 X + x_0^2 x_1^2 x_2 X + x_0^4 x_1^2 x_2^2 X^2)}{(1 - x_0^2\, x_1^2\, x_2^2\, X)(1 - x_0^2\, x_1^2\, X)(1 - x_0^2\, x_2^2\, X)(1 - x_0^2\, X)}$$

2.4 Cubic generating series in genus 2

The cubic generating series of higher genus, written here in spherical variables x_0, x_1, x_2 has the form:

$$\begin{aligned}
&\sum_{\delta=0}^{\infty} \Omega_x^{(2)}(\mathbf{T}(p^{3\,\delta}))\, X^{\delta} = p^{-1}(-p + x_0^6\, x_1^4\, x_2^2\, X^2 + x_0^6\, x_1^2\, x_2^4\, X^2 + 2\, x_0^6\, x_1^2\, x_2^3\, X^2 \\
&- p\, x_0^6\, x_1^4\, x_2^4\, X^2 - p\, x_0^6\, x_1^2\, x_2^4\, X^2 - 2\, p\, x_0^3\, x_1^2\, x_2\, X + x_0^6\, x_1\, x_2^3\, X^2 \\
&+ x_0^6\, x_1^3\, x_2\, X^2 + x_0^6\, x_1^3\, x_2^5\, X^2 + x_0^6\, x_1^5\, x_2^3\, X^2 + 3\, x_0^6\, x_1^3\, x_2^3\, X^2 \\
&+ x_0^6\, x_1^2\, x_2^2\, X^2 + 2\, x_0^6\, x_1^3\, x_2^2\, X^2 - p\, x_0^3\, x_1^2\, X - p\, x_0^3\, x_2^2\, X - p\, x_0^6\, x_1^4\, x_2^2\, X^2 \\
&- 2\, p\, x_0^3\, x_1\, x_2^2\, X - p\, x_0^6\, x_1^2\, x_2^2\, X^2 + x_0^3\, x_1^2\, x_2^2\, X + x_0^3\, x_1\, x_2\, X \\
&- p\, x_0^6\, x_1^2\, x_2^3\, X^2 - p\, x_0^6\, x_1^3\, x_2^2\, X^2 - p\, x_0^3\, x_1^2\, x_2^3\, X - p\, x_0^3\, x_1^3\, x_2^2\, X \\
&- 2\, p\, x_0^3\, x_1^2\, x_2^2\, X - p\, x_0^3\, x_1^3\, x_2\, X + x_0^3\, x_1^2\, x_2\, X + x_0^9\, x_1^4\, x_2^4\, X^3 \\
&- 2\, p\, x_0^6\, x_1^3\, x_2^3\, X^2 - 2\, p\, x_0^3\, x_1\, x_2\, X + x_0^9\, x_1^4\, x_2^5\, X^3 + x_0^9\, x_1^5\, x_2^4\, X^3 \\
&- p\, x_0^6\, x_1^3\, x_2^4\, X^2 + x_0^3\, x_1\, x_2^2\, X + x_0^9\, x_1^5\, x_2^5\, X^3 + x_0^6\, x_1^4\, x_2^4\, X^2 \\
&- p\, x_0^6\, x_1^4\, x_2^3\, X^2 - p\, x_0^3\, x_1\, x_2^3\, X - p\, x_0^3\, x_2\, X - p\, x_0^3\, x_1\, X + 2\, x_0^6\, x_1^4\, x_2^3\, X^2 \\
&+ 2\, x_0^6\, x_1^3\, x_2^4\, X^2)/((1 - x_0^3\, X)(1 - x_0^3 x_1^3\, X)(1 - x_0^3 x_2^3\, X)(1 - x_0^3 x_1^3 x_2^3 X)).
\end{aligned}$$

3 Proofs: formulas for the Hecke operators of Sp_g

3.1 Satake's spherical map Ω

Our result is based on the use of the Satake spherical map Ω, by applying the spherical map Ω to elements $\mathbf{T}(p^{\delta}) \in \mathcal{L}_{\mathbb{Z}}$ of Hecke ring $\mathcal{L}_{\mathbb{Z}} = \mathbb{Z}[\mathbf{T}(p), \mathbf{T}_1(p^2), \cdots, \mathbf{T}_n(p^2)]$ for the symplectic group, see [An87] chapter 3.

- Case $\mathbf{T}_1(p^2)$ In genus 2 (in spherical variables x_0, x_1, x_2), we obtain using Andrianov's formulas:

$$\Omega(\mathbf{T}_1(p^2)) = \frac{x_0^2\,((x_1^2\,x_2 + x_1\,x_2^2)\,p^2 + x_1\,x_2\,p^2 - x_1\,x_2 + (x_1 + x_2)\,p^2)}{p^3}.$$

- Cases $\mathbf{T}_2(p^2) = [\mathbf{p}]_2$ and $\mathbf{T}(p)$

$$\Omega(\mathbf{T}_2(p^2)) = \frac{x_0^2\,x_1\,x_2}{p^3}, \quad \Omega(\mathbf{T}(p)) = x_0(1 + x_1)(1 + x_2).$$

3.2 Use of Andrianov's generating series in genus 2

We refer to [An87], p.164, (3.3.75) for the following celebrated summation formula:

$$\sum_{\delta=0}^{\infty} \Omega^{(2)}(\mathbf{T}(p^\delta))\,X^\delta = \tag{3}$$
$$\frac{1 - \frac{x_0^2\,x_1\,x_2}{p}\,X^2}{(1 - x_0\,X)\,(1 - x_0\,x_1\,X)(1 - x_0\,x_2\,X)\,(1 - x_0\,x_1\,x_2\,X)}$$

gives after development and simplification the following formula

$$\begin{aligned}\Omega^{(2)}(\mathbf{T}(p^\delta)) &= p^{-1}\,x_0^\delta(p\,x_1^{(3+\delta)}\,x_2 - p\,x_1^{(2+\delta)} - p\,x_1^{(3+\delta)}\,x_2^{(2+\delta)} + p\,x_1^{(2+\delta)}\,x_2^{(3+\delta)}\\ &- p\,x_1\,x_2^{(3+\delta)} + p\,x_2^{(2+\delta)} + p\,x_1 - p\,x_2 - x_1^{(2+\delta)}\,x_2^2 + x_1^{(1+\delta)}\,x_2\\ &+ x_1^{(2+\delta)}\,x_2^{(1+\delta)} - x_1^{(1+\delta)}\,x_2^{(2+\delta)} + x_1^2\,x_2^{(2+\delta)} - x_1\,x_2^{(1+\delta)} - x_1^2\,x_2 + x_1\,x_2^2)/\\ &((1 - x_1)\,(1 - x_2)\,(1 - x_1\,x_2)\,(x_1 - x_2))\end{aligned}$$

Then we use two groups of variables: $x_0, \dots, x_n$ and $y_0, \dots, y_n$ in two copies Ω_x, Ω_y of the sperical map, in order to treat the tensor product of two local Hecke algebras.

Next, in order to carry out the summation of the series

$$\sum_{\delta=0}^{\infty} \Omega_x^{(2)}(\mathbf{T}(p^\delta)) \cdot \Omega_y^{(2)}(\mathbf{T}(p^\delta))\,X^\delta$$

on a computer, we used a subdivision of each summand (over δ) into smaller parts. These parts correspond to symbolic monomials in x_1^δ, y_1^δ, x_2^δ, y_2^δ, $(x_1x_2)^\delta$, $(y_1y_2)^\delta$.

3.3 Rankin's Lemma of genus 2 (compare with [Jia96])

Let us compute the series

$$D_p^{(1,1)}(X) = \sum_{\delta=0}^{\infty} \mathbf{T}(p^\delta) \otimes \mathbf{T}(p^\delta)\, X^\delta \in \mathcal{L}_{2,\mathbb{Z}} \otimes \mathcal{L}_{2,\mathbb{Z}}[\![X]\!]$$

in terms of the generators of Hecke's algebra $\mathcal{L}_{2,\mathbb{Z}} \otimes \mathcal{L}_{2,\mathbb{Z}}$ given by the following operators:

$$\mathbf{T}(p) \otimes 1, \mathbf{T}_1(p^2) \otimes 1, [\mathbf{p}] \otimes 1, 1 \otimes \mathbf{T}(p), 1 \otimes \mathbf{T}_1(p^2), 1 \otimes [\mathbf{p}] \in \mathcal{L}_{2,\mathbb{Z}} \otimes \mathcal{L}_{2,\mathbb{Z}}[\![X]\!].$$

Theorem 4. *For $g = 2$, we have the following explicit representation*

$$D_p^{(1,1)}(X) = \sum_{\delta=0}^{\infty} \mathbf{T}(p^\delta) \otimes \mathbf{T}(p^\delta)\, X^\delta = \frac{(1 - p^6[\mathbf{p}] \otimes [\mathbf{p}]X^2) \cdot R(X)}{S(X)}, \text{ where}$$
$$R(X), S(X) \in \mathcal{L}_{2,\mathbb{Z}} \otimes \mathcal{L}_{2,\mathbb{Z}}[X]$$

are given by the equalities (4) and (5):

$$R(X) = 1 + r_2X^2 + \cdots + r_{10}X^{10} + r_{12}X^{12} \in \mathcal{L}_{2,\mathbb{Z}} \otimes \mathcal{L}_{2,\mathbb{Z}}[X] \quad (4)$$
$$\text{with } r_1 = r_{11} = 0,$$
$$S(X) = 1 + s_1X + \cdots + s_{16}X^{16} \quad (5)$$
$$= 1 - (\mathbf{T}(p) \otimes \mathbf{T}(p))X + \cdots + (p^6[\mathbf{p}] \otimes [\mathbf{p}])^8 X^{16} \in \mathcal{L}_{2,\mathbb{Z}} \otimes \mathcal{L}_{2,\mathbb{Z}}[X],$$

with r_i and s_i given in Appendix. Moreover, there is an easy functional equation (similar to [An87], p.164, (3.3.79)):

$$s_{16-i} = (p^6[\mathbf{p}] \otimes [\mathbf{p}])^{8-i} s_i \quad (i = 0, \cdots, 8).$$

Remark 5 (Comparison with the case $g = 1$). The corresponding result in the case $g = 1$ written in terms of Hecke operators, looks as follows (see 3):

$$\sum_{\delta=0}^{\infty} \mathbf{T}(p^\delta) \otimes \mathbf{T}(p^\delta)\, X^\delta = (1 - p^2[\mathbf{p}] \otimes [\mathbf{p}]X^2)/$$
$$(1 - \mathbf{T}(p) \otimes \mathbf{T}(p)X + \big(p(\mathbf{T}(p)^2 \otimes [\mathbf{p}] + [\mathbf{p}] \otimes \mathbf{T}(p)^2) - 2p^2[\mathbf{p}] \otimes [\mathbf{p}]\big)X^2$$
$$- p^2\mathbf{T}(p)[\mathbf{p}] \otimes \mathbf{T}(p)[\mathbf{p}]X^3 + p^4[\mathbf{p}]^2 \otimes [\mathbf{p}]^2 X^4).$$

Indeed this follows directly from Remark 3.

Remark 6. For the group $GL(n)$, a simpler result could be obtained using the Tamagawa generating series (see [Tam]) and its expansion into simple fractions:

$$\frac{1}{(1-x_1X)\cdot\ldots\cdot(1-x_nX)}=\sum_{i=1}^{n}\frac{\alpha_i(x)}{1-x_iX}=\sum_{i=1}^{n}\sum_{r=0}^{\infty}\alpha_i(x)(x_iX)^r,$$
$$\frac{1}{(1-y_1X)\cdot\ldots\cdot(1-y_nX)}=\sum_{j=1}^{n}\frac{\alpha_j(y)}{1-y_jX}=\sum_{j=1}^{n}\sum_{r=0}^{\infty}\alpha_j(y)(y_jX)^r,$$
where $\alpha_i(x)=\dfrac{x_i^{n-1}}{\prod_{k\neq i}^{n}(x_k-x_i)},\alpha_j(y)=\dfrac{y_j^{n-1}}{\prod_{l\neq j}^{n}(y_l-y_j)}$, hence
$$\sum_{i,j=1}^{n}\sum_{r=0}^{\infty}\alpha_i(x)\alpha_j(y)(x_iy_jX)^r=\sum_{i,j=1}^{n}\frac{\alpha_i(x)\alpha_j(y)}{1-x_iy_jX}.$$

If $n=3$, this gives after simplification the following fraction:

$$\begin{aligned}&(y_2^2\,x_2^2\,y_1^2\,x_3^2\,y_3^2\,x_1^2\,X^6-y_2\,x_2\,y_1\,y_3^2\,x_1\,x_3^2\,X^4-y_2\,x_2^2\,y_1\,{y_3}^2\,x_1\,x_3\,X^4\\
&-y_2\,x_2\,y_1\,x_1^2\,x_3\,{y_3}^2\,X^4-y_2^2\,x_2\,y_1\,y_3\,x_1\,{x_3}^2\,X^4-y_2^2\,x_2^2\,y_1\,y_3\,x_1\,x_3\,X^4\\
&-y_2^2\,x_2\,y_1\,x_1^2\,x_3\,y_3\,X^4-y_2\,x_2^2\,y_1^2\,x_3\,y_3\,x_1\,X^4-y_2\,x_2\,y_1^2\,x_3^2\,y_3\,x_1\,X^4\\
&-y_2\,x_2\,y_1^2\,x_3\,x_1^2\,y_3\,X^4+y_2\,x_3\,y_1\,x_1^2\,y_3\,X^3+y_2\,x_2\,y_3^2\,x_1\,x_3\,X^3\\
&+y_2^2\,x_2\,y_3\,x_1\,x_3\,X^3+y_3\,x_2\,y_1^2\,x_3\,x_1\,X^3+y_2\,x_2\,y_1\,x_1^2\,y_3\,X^3\\
&+y_2\,x_2\,y_1\,y_3\,x_3^2\,X^3+y_2\,x_2^2\,y_1\,y_3\,x_3\,X^3+y_2^2\,x_2\,y_1\,x_1\,x_3\,X^3\\
&+4\,y_2\,x_2\,y_1\,y_3\,x_1\,x_3\,X^3+y_2\,x_2^2\,y_1\,y_3\,x_1\,X^3+y_2\,x_3^2\,y_1\,y_3\,x_1\,X^3\\
&+y_2\,x_2\,y_1^2\,x_3\,x_1\,X^3+x_2\,y_1\,y_3^2\,x_3\,x_1\,X^3-y_2\,x_2\,y_1\,x_3\,X^2-y_3\,x_2\,y_1\,x_1\,X^2\\
&-y_2\,x_2\,y_3\,x_1\,X^2-y_2\,x_3\,y_3\,x_1\,X^2-y_2\,x_3\,y_1\,x_1\,X^2-y_3\,x_3\,y_1\,x_1\,X^2\\
&-y_3\,x_2\,y_1\,x_3\,X^2-y_2\,x_2\,y_1\,x_1\,X^2-y_2\,x_2\,y_3\,x_3\,X^2+1)\Big/((1-x_3\,y_3\,X)\\
&(1-x_2\,y_3\,X)\,(1-x_1\,y_3\,X)(1-x_3\,y_2\,X)\,(1-x_2\,y_2\,X)\,(1-x_1\,y_2\,X)\\
&(1-x_3\,y_1\,X)\,(1-x_2\,y_1\,X)\,(1-x_1\,y_1\,X))\end{aligned}$$

4 Relations with L-functions and motives for Sp_n

The modest purpose of this section is to recall the motivic origin of the L-function produced by the denominator of our Hecke series.

L-functions, functional equation and motives for Sp_n (see [Pa94], [Yosh01])

One defines

- $Q_{f,p}(X)=(1-\alpha_0X)\prod_{r=1}^{n}\prod_{1\le i_1<\cdots<i_r\le n}(1-\alpha_0\alpha_{i_1}\cdots\alpha_{i_r}X)$,
- $R_{f,p}(X)=(1-X)\prod_{i=1}^{n}(1-\alpha_i^{-1}X)(1-\alpha_iX)\in\mathbb{Q}[\alpha_0^{\pm1},\cdots,\alpha_n^{\pm1}][X]$.

Then the spinor L-function $L(Sp(f), s)$ and the standard L-function $L(St(f), s, \chi)$ of f (for $s \in \mathbb{C}$, and for all Dirichlet characters) χ are defined as the Euler products

- $L(Sp(f), s, \chi) = \prod_p Q_{f,p}(\chi(p)p^{-s})^{-1}$
- $L(St(f), s, \chi) = \prod_p R_{f,p}(\chi(p)p^{-s})^{-1}$

Motivic *L*-functions

Following [Pa94] and [Yosh01], these functions are conjectured to be motivic for all $k > n$:

$$L(Sp(f), s, \chi) = L(M(Sp(f))(\chi), s), L(St(f), s) = L(M(St(f))(\chi), s), \text{ where}$$

and the motives $M(Sp(f)$ and $M(St(f)$ are *pure* if f is a genuine cusp form (not coming from a lifting of a smaller genus):

- $M(Sp(f))$ is a motive over $\mathbb{Q}$ with coefficients in $\mathbb{Q}(\lambda_f(n))_{n\in\mathbb{N}}$ of rank 2^n, of weight $w = kn - n(n+1)/2$, and of Hodge type $\oplus_{p,q} H^{p,q}$, with

$$\begin{aligned} p &= (k-i_1) + (k-i_2) + \cdots + (k-i_r), \\ q &= (k-j_1) + (k-j_2) + \cdots + (k-i_s), \text{ where } r+s=n, \\ &1 \le i_1 < i_2 < \cdots < i_r \le n, 1 \le j_1 < j_2 < \cdots < j_s \le n, \\ &\{i_1, \cdots, i_r\} \cup \{j_1, \cdots, i_s\} = \{1, 2, \cdots, n\}; \end{aligned} \tag{6}$$

- $M(St(f)$ is a motive over $\mathbb{Q}$ with coefficients in $\mathbb{Q}(\lambda_f(n))_{n\in\mathbb{N}}$ of rank $2n+1$, of weight $w = 0$, and of Hodge type $H^{0,0} \oplus_{i=1}^n (H^{-k+i,k-i} \oplus H^{k-i,-k+i})$.

A functional equation

Following general Deligne's conjecture [De79] on the motivic L-functions, the L-function satisfy a functional equation determined by the Hodge structure of a motive:

$$\Lambda(Sp(f), kn - n(n+1)/2 + 1 - s) = \varepsilon(f)\Lambda(Sp(f), s), \text{ where}$$

$\Lambda(Sp(f), s) = \Gamma_{n,k}(s)L(Sp(f), s), \varepsilon(f) = (-1)^{k2^{n-2}}$,

$\Gamma_{1,k}(s) = \Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$, $\Gamma_{2,k}(s) = \Gamma_{\mathbb{C}}(s)\Gamma_{\mathbb{C}}(s-k+2)$, and $\Gamma_{n,k}(s) = \prod_{p<q} \Gamma_{\mathbb{C}}(s-p)\Gamma_{\mathbb{R}}^{a_+}(s-(w/2))\Gamma_{\mathbb{R}}(s+1-(w/2))^{a_-}$ for some non-negative integers a_+ and a_-, with $a_+ + a_- = w/2$, and $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$

Motive of the Rankin product of genus $g = 2$

Let f and g be two Siegel cusp eigenforms of weights k and l, $k > l$, and let $M(Sp(f))$ and $M(Sp(g))$ be the spinor motives of f and g. Then $M(Sp(f))$ is a motive over $\mathbb{Q}$ with coefficients in $\mathbb{Q}(\lambda_f(n))_{n\in\mathbb{N}}$ of rank 4, of weight $w = 2k-3$, and of Hodge type $H^{0,2k-3} \oplus H^{k-2,k-1} \oplus H^{k-1,k-2} \oplus H^{2k-3,0}$, and $M(Sp(g))$ is a motive over $\mathbb{Q}$ with coefficients in $\mathbb{Q}(\lambda_g(n))_{n\in\mathbb{N}}$ of rank 4, of weight $w = 2l-3$, and of Hodge type $H^{0,2l-3} \oplus H^{l-2,l-1} \oplus H^{l-1,l-2} \oplus H^{2l-3,0}$.

The tensor product $M(Sp(f)) \otimes M(Sp(g))$ is a motive over $\mathbb{Q}$ with coefficients in $\mathbb{Q}(\lambda_f(n), \lambda_g(n))_{n\in\mathbb{N}}$ of rank 16, of weight $w = 2k+2l-6$, and of Hodge type

$$\begin{aligned}
&H^{0,2k+2l-6} \oplus H^{l-2,2k+l-4} \oplus H^{l-1,2k+l-5} \oplus H^{2l-3,2k-3}\\
&H^{k-2,k+2l-4} \oplus H^{k+l-4,k+l-2} \oplus H_{+}^{k+l-3,k+l-3} \oplus H^{k+2l-5,k-1}\\
&H^{k-1,k+2l-5} \oplus H_{-}^{k+l-3,k+l-3} \oplus H^{k+l-2,k+l-4} \oplus H^{k+2l-4,k-2}\\
&H^{2k-3,2l-3} \oplus H^{2k+l-5,l-1} \oplus H^{2k+l-4,l-2} \oplus H^{2k+2l-6,0}.
\end{aligned}$$

Motivic L-functions: analytic properties

Following Deligne's conjecture [De79] on motivic L-functions, applied for a Siegel cusp eigenform F for the Siegel modular group $\mathrm{Sp}_4(\mathbb{Z})$ of genus $n = 4$ and of weight $k > 5$, one has $\Lambda(Sp(F), s) = \Lambda(Sp(F), 4k-9-s)$, where

$$\begin{aligned}
&\Lambda(Sp(F), s) = \Gamma_{\mathbb{C}}(s)\Gamma_{\mathbb{C}}(s-k+4)\Gamma_{\mathbb{C}}(s-k+3)\Gamma_{\mathbb{C}}(s-k+2)\Gamma_{\mathbb{C}}(s-k+1)\\
&\times \Gamma_{\mathbb{C}}(s-2k+7)\Gamma_{\mathbb{C}}(s-2k+6)\Gamma_{\mathbb{C}}(s-2k+5)L(Sp(F), s),
\end{aligned}$$

(compare this functional equation with that given in [An74], p.115).

On the other hand, for $n = 2$ and for two cusp eigenforms f and g for $\mathrm{Sp}_2(\mathbb{Z})$ of weights k, l, $k > l+1$, $\Lambda(Sp(f) \otimes Sp(g), s) = \varepsilon(f,g)\Lambda(Sp(f) \otimes Sp(g), 2k+2l-5-s)$, $|\varepsilon(f,g)| = 1$, where

$$\begin{aligned}
&\Lambda(Sp(f) \otimes Sp(g), s) = \Gamma_{\mathbb{C}}(s)\Gamma_{\mathbb{C}}(s-l+2)\Gamma_{\mathbb{C}}(s-l+1)\Gamma_{\mathbb{C}}(s-k+2)\\
&\times \Gamma_{\mathbb{C}}(s-k+1)\Gamma_{\mathbb{C}}(s-2l+3)\Gamma_{\mathbb{C}}(s-k-l+2)\Gamma_{\mathbb{C}}(s-k-l+3)\\
&\times L(Sp(f) \otimes Sp(g), s).
\end{aligned}$$

We used here the Gauss duplication formula $\Gamma_{\mathbb{C}}(s) = \Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1)$. Notice that $a_+ = a_- = 1$ in this case, and the conjectural motive $M(Sp(f)) \otimes M(Sp(g))$ does not admit critical values.

5 A holomorphic lifting from $GSp_2 \times GSp_2$ to GSp_4: a conjecture

(compare with constructions in [BFG06], [BFG92], [Jia96] for generic automorphic forms).

Our computation makes it possible to compare the spinor Hecke series of genus 4 computed in [VaSp4] (in variables u_0, u_1, u_2, u_3, u_4) with the Rankin product of two Hecke series of genus 2 (in variables $x_0, x_1, x_2, y_0, y_1, y_2$). It follows from our computation that if we make the substitution $u_0 = x_0 y_0, u_1 = x_1, u_2 = x_2, u_3 = y_1, u_4 = y_2$ then the denominator of the series

$$\sum_{\delta=0}^{\infty} \Omega_u^{(4)}(\mathbf{T}(p^\delta))X^\delta$$

coincides with the denominator of the Rankin product

$$\sum_{\delta=0}^{\infty} \Omega_x^{(2)}(\mathbf{T}(p^\delta)) \cdot \Omega_y^{(2)}(\mathbf{T}(p^\delta))\, X^\delta \in \mathbb{Q}[x_0, x_1, x_2, y_0, y_1, y_2][\![X]\!].$$

On the basis of this equality we would like to push forward the following

Conjecture 7 (on a lifting from $GSp_2 \times GSp_2$ to GSp_4). *Let f and g be two Siegel modular forms of genus 2 and of weights $k > 4$ and $l = k - 2$. Then there exists a Siegel modular form F of genus 4 and of weight k with the Satake parameters*

$$\gamma_0 = \alpha_0\beta_0, \gamma_1 = \alpha_1, \gamma_2 = \alpha_2, \gamma_3 = \beta_1, \gamma_4 = \beta_2,$$

for a suitable choice of Satake's parameters $\alpha_0, \alpha_1, \alpha_2$ and $\beta_0, \beta_1, \beta_2$ of f and g.

Remark 8. An evidence for the conjecture comes from Ikeda-Miyawaki constructions ([Ike01], [Mur02], [Ike06]): let k be an even positive integer, $h \in S_{2k}(\Gamma_1)$ a normalized Hecke eigenform of weight $2k$, $F_2(h) \in S_{k+1}(\Gamma_2) = Maass(h)$ the Maass lift of h, and $F_{2n} \in S_{k+n}(\Gamma_{2n})$ the Ikeda lift of h (we assume $k \equiv n \bmod 2$, $n \in \mathbb{N}$).

Next let $f \in S_{k+n+r}(\Gamma_r)$ be an arbitrary Siegel cusp eigenform of genus r and weight $k + n + r$, with $n, r \geq 1$. Then according to Ikeda-Miyawaki (see [Ike06]) there exists a Siegel eigenform $\mathcal{F}_{h,f} \in S_{k+n+r}(\Gamma_{2n+r})$ such that

$$L(s, \mathcal{F}_{h,f}, St) = L(s, f, St) \prod_{j=1}^{2n} L(s + k + n - j, h) \tag{7}$$

(under a non-vanishing condition, see Theorem 2.3 at p.63 in [Mur02]). The form $\mathcal{F}_{h,f}$ is given by the integral

$$\mathcal{F}_{h,f}(Z) = \langle F_{2n+2r}(\mathrm{diag}(Z, Z'), f(Z')\rangle_{Z'}$$

If we take $n = 1, r = 2$, $k := k + 1$ then an example of the validity of the conjecture is given by $g = F_2(h)$,

$$(f, g) = (f, F_2(h)) \mapsto \mathcal{F}_{f,h} \in S_{k+3}(\Gamma_4),$$
$$(f, g) = (f, F_2(h)) \in S_{k+3}(\Gamma_2) \times S_{k+1}(\Gamma_2).$$

Remark 9. Notice that the Satake parameters of the Ikeda lift $F = F_{2m}(h)$ of h can be taken in the form $\beta_0, \beta_1, \cdots, \beta_{2m}$, where

$$\beta_0 = p^{mk-m(m+1)/2}, \beta_i = \alpha p^{i-1/2}, \quad \beta_{m+i} = \alpha^{-1} p^{i-1/2}, \ (i = 1, \cdots, m)$$

and

$$(1 - \alpha p^{k-1/2} X)(1 - \alpha^{-1} p^{k-1/2} X) = 1 - a(p)X + p^{2k-1}X^2, h = \sum_{n=1}^{\infty} a(n)q^n$$

see [Mur02].

The L-function of degree 16 in Conjecture 7 is related to the tensor product L-function as in [Jia96]. In the example of Remark 8 it coincides with the product of two shifted L-functions of degree 8 of Boecherer-Heim [BoeH06].

Conjecture 10 (on a lifting from $GSp_{2m} \times GSp_{2m}$ to GSp_{4m}).
Here is a version of Conjectire 7 for any even genus $r = 2m$. Let f and g be two Siegel modular forms of genus $2m$ and of weights $k > 2m$ and $l = k - 2m$. Then there exists a Siegel modular form F of genus $4m$ and of weight k with the Satake parameters $\gamma_0 = \alpha_0 \beta_0, \gamma_1 = \alpha_1, \gamma_2 = \alpha_2, \cdots, \gamma_{2m} = \alpha_{2m}, \gamma_{2m+1} = \beta_1, \cdots, \gamma_{4m} = \beta_{2m}$ for suitable choices $\alpha_0, \alpha_1, \cdots, \alpha_{2m}$ and $\beta_0, \beta_1, \cdots, \beta_{2m}$ of Satake's parameters of f and g.

One readily checks that the Hodge types of $M(Sp(f)) \otimes M(Sp(g))$ and $M(Sp(F))$ are again the same (of rank 2^{4m}) (it follows from the above description (6), and from Künneth's-type formulas).

An evidence for this version of the conjecture comes again from Ikeda-Miyawaki constructions ([Ike01], [Mur02], [Ike06]): let k be an even positive integer, $h \in S_{2k}(\Gamma_1)$ a normalized Hecke eigenform of weight $2k$, $F_{2n} \in S_{k+n}(\Gamma_{2n})$ the Ikeda lift of h of genus $2n$ (we assume $k \equiv n \bmod 2$, $n \in \mathbb{N}$).

Next let $f \in S_{k+n+r}(\Gamma_r)$ be an arbitrary Siegel cusp eigenform of genus r and weight $k + n + r$, with $n, r \geq 1$. If we take in (7) $n = m, r = 2m$, $k := k + m$, $k + n + r := k + 3m$, then an example of the validity of this version of the conjecture is given by

$$\begin{aligned}(f, g) &= (f, F_{2m}(h)) \mapsto \mathcal{F}_{h,f} \in S_{k+3m}(\Gamma_{4m}),\\ (f, g) &= (f, F_{2m}) \in S_{k+3m}(\Gamma_{2m}) \times S_{k+m}(\Gamma_{2m}).\end{aligned}$$

Another evidence comes from Siegel-Eiseinstein series

$$f = E_k^{2m} \text{ and } g = E_{k-2m}^{2m}$$

of even genus $2m$ and weights k and $k - 2m$: we have then

$$\begin{aligned}\alpha_0 &= 1, \alpha_1 = p^{k-2m}, \cdots, \alpha_{2m} = p^{k-1},\\ \beta_0 &= 1, \beta_1 = p^{k-4m}, \cdots, \beta_{2m} = p^{k-2m-1},\end{aligned}$$

then we have that

$$\gamma_0 = 1, \gamma_1 = p^{k-4m}, \cdots, \gamma_{2m} = p^{k-1},$$

are the Satake parameters of the Siegel-Eisenstein series $F = E_k^{4m}$.

Remark 11. If we compare the L-function of the conjecture (given by the Satake parameters $\gamma_0 = \alpha_0\beta_0, \gamma_1 = \alpha_1, \gamma_2 = \alpha_2, \cdots, \gamma_{2m} = \alpha_{2m}, \gamma_{2m+1} = \beta_1, \cdots, \gamma_{4m} = \beta_{2m}$ for suitable choices $\alpha_0, \alpha_1, \cdots, \alpha_{2m}$ and $\beta_0, \beta_1, \cdots, \beta_{2m}$ of Satake's parameters of f and g), we see that it corresponds to the tensor product of spinor L-functions, and is *not of the same type* as that of the Yoshida's lifting [Yosh81], which is a certain product of Hecke's L-functions.

We would like to mention in this context Langlands's functoriality: The denominators of our L-series belong to local Langlands L-factors (attached to representations of L-groups). If we consider the homomorphisms

$${}^L GSp_{2m} = GSpin(4m+1) \to GL_{2^{2m}}, \quad {}^L GSp_{4m} = GSpin(8m+1) \to GL_{2^{4m}},$$

we see that our conjecture is compatible with the homomorphism of L-groups

$$GL_{2^{2m}} \times GL_{2^{2m}} \to GL_{2^{4m}}, \ (g_1, g_2) \mapsto g_1 \otimes g_2, GL_n(\mathbb{C}) = {}^L GL_n.$$

However, it is unclear to us if Langlands's functoriality predicts a holomorphic Siegel modular form as a lift.

A Appendix: Coefficients of the polynomials $R(X)$ and $S(X)$

We give here explicit expressions for the coefficients of the polynomials $R(X)$ and $S(X)$ from Theorem 4. From these formulas one can observe some nice divisibility properties (by certain powers of p and the elements $[\mathbf{p}] \otimes [\mathbf{p}] \in \mathcal{L}_{2,\mathbb{Z}} \otimes \mathcal{L}_{2,\mathbb{Z}}$):

$$\begin{aligned} R(X) &= 1 + r_2 X^2 + \cdots + r_{10} X^{10} + r_{12} X^{12} \in \mathcal{L}_{2,\mathbb{Z}} \otimes \mathcal{L}_{2,\mathbb{Z}}[X] \\ &\quad \text{with } r_1 = r_{11} = 0, \\ S(X) &= 1 + s_1 X + \cdots + s_{16} X^{16} \\ &= 1 - (\mathbf{T}(p) \otimes \mathbf{T}(p))X + \cdots + (p^6[\mathbf{p}] \otimes [\mathbf{p}])^8 X^{16} \in \mathcal{L}_{2,\mathbb{Z}} \otimes \mathcal{L}_{2,\mathbb{Z}}[X], \end{aligned}$$

with r_i and s_i given as follows

$$\begin{aligned} r_2 &= p^2((2p-1)(p^2+1)[\mathbf{p}] \otimes [\mathbf{p}] - (p^2-p+1)(\mathbf{T}_1(p^2) \otimes [\mathbf{p}] + [\mathbf{p}] \otimes \mathbf{T}_1(p^2)) \\ &\quad - (\mathbf{T}_1(p^2) \otimes \mathbf{T}_1(p^2) + \mathbf{T}(p)^2 \otimes [\mathbf{p}] + [\mathbf{p}] \otimes \mathbf{T}(p)^2), \\ r_3 &= p^3(p+1)(2[\mathbf{p}] \otimes [\mathbf{p}] + \mathbf{T}_1(p^2) \otimes [\mathbf{p}] + [\mathbf{p}] \otimes \mathbf{T}_1(p^2))\mathbf{T}(p) \otimes \mathbf{T}(p), \end{aligned}$$

$$
\begin{aligned}
r_4 = {} & -p^5((p^7+2p^6-2p^5+6p^4+p^3+6p^2+p+2)[\mathbf{p}]^2\otimes[\mathbf{p}]^2\\
& -(p^2+1)(p^3-3p^2-p-3)(\mathbf{T}_1(p^2)\otimes[\mathbf{p}]+[\mathbf{p}]\otimes\mathbf{T}_1(p^2))[\mathbf{p}]\otimes[\mathbf{p}]\\
& +(p+4)(p^2+1)\mathbf{T}_1(p^2)[\mathbf{p}]\otimes\mathbf{T}_1(p^2)[\mathbf{p}]-(p^3-p^2-1)(\mathbf{T}_1(p^2)^2\otimes[\mathbf{p}]^2+[\mathbf{p}]^2\otimes\mathbf{T}_1(p^2)^2)\\
& +(\mathbf{T}_1(p^2)\otimes[\mathbf{p}]+[\mathbf{p}]\otimes\mathbf{T}_1(p^2))\mathbf{T}_1(p^2)\otimes\mathbf{T}_1(p^2)-p(p^3+2p^2-p+2)(\mathbf{T}(p)^2\otimes[\mathbf{p}]\\
& +[\mathbf{p}]\otimes\mathbf{T}(p)^2)[\mathbf{p}]\otimes[\mathbf{p}]-2p(\mathbf{T}(p)^2\otimes\mathbf{T}_1(p^2)+\mathbf{T}_1(p^2)\otimes\mathbf{T}(p)^2)[\mathbf{p}]\otimes[\mathbf{p}]\\
& +p^2(\mathbf{T}(p)^2\mathbf{T}_1(p^2)\otimes[\mathbf{p}]^2+[\mathbf{p}]^2\otimes\mathbf{T}(p)^2\mathbf{T}_1(p^2))+(p+2)\mathbf{T}(p)^2[\mathbf{p}]\otimes\mathbf{T}(p)^2[\mathbf{p}])\,,\\
r_5 = {} & -p^7(2(p+1)(2p^4-p^3+p^2-1)[\mathbf{p}]\otimes[\mathbf{p}]+(p+1)(p-2)(\mathbf{T}_1(p^2)\otimes[\mathbf{p}]+[\mathbf{p}]\otimes\mathbf{T}_1(p^2))\\
& -2\mathbf{T}_1(p^2)\otimes\mathbf{T}_1(p^2)-p(p+1)(\mathbf{T}(p)^2\otimes[\mathbf{p}]+[\mathbf{p}]\otimes\mathbf{T}(p)^2))\mathbf{T}(p)[\mathbf{p}]\otimes\mathbf{T}(p)[\mathbf{p}]\,,\\
r_6 = {} & -p^{10}\,(p\,(p^2+1)(p^5-2p^3-8p^2-p-4)[\mathbf{p}]^3\otimes[\mathbf{p}]^3\\
& -p\,(p^5+4p^4+2p^3+12p^2+p+6)(\mathbf{T}_1(p^2)\otimes[\mathbf{p}]+[\mathbf{p}]\otimes\mathbf{T}_1(p^2))[\mathbf{p}]^2\otimes[\mathbf{p}]^2\\
& +p\,(p-4)(p^2+1)\mathbf{T}_1(p^2)[\mathbf{p}]^2\otimes\mathbf{T}_1(p^2)[\mathbf{p}]^2\\
& -p\,(p+4)(p^2+1)(\mathbf{T}_1(p^2)^2\otimes[\mathbf{p}]^2+[\mathbf{p}]^2\otimes\mathbf{T}_1(p^2)^2)[\mathbf{p}]\otimes[\mathbf{p}]\\
& -p\,(\mathbf{T}_1(p^2)\otimes[\mathbf{p}]+[\mathbf{p}]\otimes\mathbf{T}_1(p^2))\mathbf{T}_1(p^2)[\mathbf{p}]\otimes\mathbf{T}_1(p^2)[\mathbf{p}]\\
& -p\,(\mathbf{T}_1(p^2)^3\otimes[\mathbf{p}]^3+[\mathbf{p}]^3\otimes\mathbf{T}_1(p^2)^3)\\
& -(p^5-4p^2-p-2)(\mathbf{T}(p)^2\otimes[\mathbf{p}]+[\mathbf{p}]\otimes\mathbf{T}(p)^2)[\mathbf{p}]^2\otimes[\mathbf{p}]^2\\
& +(p^2+3)(\mathbf{T}(p)^2\otimes\mathbf{T}_1(p^2)+\mathbf{T}_1(p^2)\otimes\mathbf{T}(p)^2)[\mathbf{p}]^2\otimes[\mathbf{p}]^2\\
& +(\mathbf{T}(p)^2[\mathbf{p}]\otimes\mathbf{T}_1(p^2)^2+\mathbf{T}_1(p^2)^2\otimes\mathbf{T}(p)^2[\mathbf{p}])[\mathbf{p}]\otimes[\mathbf{p}]\\
& +(p^3+3p^2+p+1)(\mathbf{T}(p)^2\mathbf{T}_1(p^2)\otimes[\mathbf{p}]^2+[\mathbf{p}]^2\otimes\mathbf{T}(p)^2\mathbf{T}_1(p^2))[\mathbf{p}]\otimes[\mathbf{p}]\\
& +(\mathbf{T}(p)^2\otimes[\mathbf{p}]+[\mathbf{p}]\otimes\mathbf{T}(p)^2)\mathbf{T}_1(p^2)[\mathbf{p}]\otimes\mathbf{T}_1(p^2)[\mathbf{p}]\\
& +(p^2+1)\mathbf{T}(p)^2[\mathbf{p}]^2\otimes\mathbf{T}(p)^2[\mathbf{p}]^2)\,,\\
r_7 = {} & -p^{13}\,(2(p+1)(p^3+p-1)[\mathbf{p}]\otimes[\mathbf{p}]-(p+1)(p^2-2p+2)(\mathbf{T}_1(p^2)\otimes[\mathbf{p}]+[\mathbf{p}]\otimes\mathbf{T}_1(p^2))\\
& -2\mathbf{T}_1(p^2)\otimes\mathbf{T}_1(p^2)-(p+1)(\mathbf{T}(p)^2\otimes[\mathbf{p}]+[\mathbf{p}]\otimes\mathbf{T}(p)^2))\mathbf{T}(p)[\mathbf{p}]^2\otimes\mathbf{T}(p)[\mathbf{p}]^2\,,\\
r_8 = {} & -p^{16}(p\,(2p^6+3p^5+6p^4-p^3+6p^2-p+2)[\mathbf{p}]^2\otimes[\mathbf{p}]^2\\
& +p\,(p^2+1)(p^3+3p^2-p+3)(\mathbf{T}_1(p^2)\otimes[\mathbf{p}]+[\mathbf{p}]\otimes\mathbf{T}_1(p^2))[\mathbf{p}]\otimes[\mathbf{p}]\\
& +p\,(p+4)(p^2+1)\mathbf{T}_1(p^2)[\mathbf{p}]\otimes\mathbf{T}_1(p^2)[\mathbf{p}]\\
& +p\,(p^2-p+1)(\mathbf{T}_1(p^2)^2\otimes[\mathbf{p}]^2+[\mathbf{p}]^2\otimes\mathbf{T}_1(p^2)^2)\\
& +p\,(\mathbf{T}_1(p^2)\otimes[\mathbf{p}]+[\mathbf{p}]\otimes\mathbf{T}_1(p^2))\mathbf{T}_1(p^2)\otimes\mathbf{T}_1(p^2)\\
& -p\,(2p^3+p^2+2p-1)(\mathbf{T}(p)^2\otimes[\mathbf{p}]+[\mathbf{p}]\otimes\mathbf{T}(p)^2)[\mathbf{p}]\otimes[\mathbf{p}]\\
& -2p^2(\mathbf{T}(p)^2\otimes\mathbf{T}_1(p^2)+\mathbf{T}_1(p^2)\otimes\mathbf{T}(p)^2)[\mathbf{p}]\otimes[\mathbf{p}]\\
& +p\,(\mathbf{T}(p)^2\mathbf{T}_1(p^2)\otimes[\mathbf{p}]^2+[\mathbf{p}]^2\otimes\mathbf{T}(p)^2\mathbf{T}_1(p^2))\\
& +(2p+1)\mathbf{T}(p)^2[\mathbf{p}]\otimes\mathbf{T}(p)^2[\mathbf{p}])[\mathbf{p}]^2\otimes[\mathbf{p}]^2\,,\\
r_9 = {} & p^{20}(p+1)(2[\mathbf{p}]\otimes[\mathbf{p}]+\mathbf{T}_1(p^2)\otimes[\mathbf{p}]+[\mathbf{p}]\otimes\mathbf{T}_1(p^2))\mathbf{T}(p)[\mathbf{p}]^3\otimes\mathbf{T}(p)[\mathbf{p}]^3\\
r_{10} = {} & p^{24}((p^2+1)(p^4+2p^3-p^2-1)[\mathbf{p}]\otimes[\mathbf{p}]+(p^3-p^2-1)(\mathbf{T}_1(p^2)\otimes[\mathbf{p}]+[\mathbf{p}]\otimes\mathbf{T}_1(p^2))\\
& -\mathbf{T}_1(p^2)\otimes\mathbf{T}_1(p^2)-p^2(\mathbf{T}(p)^2\otimes[\mathbf{p}]+[\mathbf{p}]\otimes\mathbf{T}(p)^2))[\mathbf{p}]^4\otimes[\mathbf{p}]^4\\
r_{11} = {} & 0,\\
r_{12} = {} & p^{34}[\mathbf{p}]^6\otimes[\mathbf{p}]^6,
\end{aligned}
$$

As for the coefficients of $S(X)$, one has

$$S(X) = 1 - (\mathbf{T}(p) \otimes \mathbf{T}(p))X + \cdots + (p^6[\mathbf{p}] \otimes [\mathbf{p}])^8 X^{16} \in \mathcal{L}_{2,\mathbb{Z}} \otimes \mathcal{L}_{2,\mathbb{Z}}[X],$$

where

$$\begin{aligned}
s_1 &= -\mathbf{T}(p) \otimes \mathbf{T}(p)\\
s_2 &= -p(2\,p\,(p^2+1)^2\,[\mathbf{p}] \otimes [\mathbf{p}] + 2\,p\,(p^2+1)\,(\mathbf{T}_1(p^2) \otimes [\mathbf{p}] + [\mathbf{p}] \otimes \mathbf{T}_1(p^2))\\
&\quad + 2\,p\,\mathbf{T}_1(p^2) \otimes \mathbf{T}_1(p^2) - (p^2+1)\,(\mathbf{T}(p)^2 \otimes [\mathbf{p}] + [\mathbf{p}] \otimes \mathbf{T}(p)^2)\\
&\quad - (\mathbf{T}(p)^2 \otimes \mathbf{T}_1(p^2) + \mathbf{T}_1(p^2) \otimes \mathbf{T}(p)^2))\\
s_3 &= p^2((2\,p^4 + 4\,p^2 - 1)\,[\mathbf{p}] \otimes [\mathbf{p}] + (2\,p^2 - 1)\,(\mathbf{T}_1(p^2) \otimes [\mathbf{p}] + [\mathbf{p}] \otimes \mathbf{T}_1(p^2))\\
&\quad - \mathbf{T}_1(p^2) \otimes \mathbf{T}_1(p^2) - p\,(\mathbf{T}(p)^2 \otimes [\mathbf{p}] + [\mathbf{p}] \otimes \mathbf{T}(p)^2))\mathbf{T}(p) \otimes \mathbf{T}(p))\\
s_4 &= p^4((p^8 + 12\,p^6 + 10\,p^4 + 4\,p^2 + 1)\,[\mathbf{p}]^2 \otimes [\mathbf{p}]^2\\
&\quad + 2\,(3\,p^6 + 5\,p^4 + 3\,p^2 + 1)\,(\mathbf{T}_1(p^2) \otimes [\mathbf{p}] + [\mathbf{p}] \otimes \mathbf{T}_1(p^2))[\mathbf{p}] \otimes [\mathbf{p}]\\
&\quad + 4\,(p^2+1)^2\,\mathbf{T}_1(p^2)[\mathbf{p}] \otimes \mathbf{T}_1(p^2)[\mathbf{p}]\\
&\quad + (3\,p^4 + 2\,p^2 + 1)\,(\mathbf{T}_1(p^2)^2 \otimes [\mathbf{p}]^2 + [\mathbf{p}]^2 \otimes \mathbf{T}_1(p^2)^2)\\
&\quad + 2\,(p^2+1)\,(\mathbf{T}_1(p^2) \otimes [\mathbf{p}] + [\mathbf{p}] \otimes \mathbf{T}_1(p^2))\mathbf{T}_1(p^2) \otimes \mathbf{T}_1(p^2)\\
&\quad + \mathbf{T}_1(p^2)^2 \otimes \mathbf{T}_1(p^2)^2\\
&\quad - 2\,p\,(p^4 + 4\,p^2 + 1)\,(\mathbf{T}(p)^2 \otimes [\mathbf{p}] + [\mathbf{p}] \otimes \mathbf{T}(p)^2)\,[\mathbf{p}] \otimes [\mathbf{p}]\\
&\quad - 4\,p\,(p^2+1)\,(\mathbf{T}(p)^2 \otimes \mathbf{T}_1(p^2) + \mathbf{T}_1(p^2) \otimes \mathbf{T}(p)^2)[\mathbf{p}] \otimes [\mathbf{p}]\\
&\quad - 2\,p\,(\mathbf{T}(p)^2[\mathbf{p}] \otimes \mathbf{T}_1(p^2)^2 + \mathbf{T}_1(p^2)^2 \otimes \mathbf{T}(p)^2[\mathbf{p}])\\
&\quad - 4\,p^3\,(\mathbf{T}(p)^2\mathbf{T}_1(p^2) \otimes [\mathbf{p}]^2 + [\mathbf{p}]^2 \otimes \mathbf{T}(p)^2\mathbf{T}_1(p^2))\\
&\quad + (p^2+2)\,\mathbf{T}(p)^2[\mathbf{p}] \otimes \mathbf{T}(p)^2[\mathbf{p}]\\
&\quad + (\mathbf{T}_1(p^2) \otimes [\mathbf{p}] + [\mathbf{p}] \otimes \mathbf{T}_1(p^2))\mathbf{T}(p)^2 \otimes \mathbf{T}(p)^2\\
&\quad + p^2\,(\mathbf{T}(p)^4 \otimes [\mathbf{p}]^2 + [\mathbf{p}]^2 \otimes \mathbf{T}(p)^4))\\
s_5 &= -p^6((6\,p^6 + 2\,p^4 - p^2 + 2)\,[\mathbf{p}]^2 \otimes [\mathbf{p}]^2\\
&\quad + (p^4 - p^2 + 3)\,(\mathbf{T}_1(p^2) \otimes [\mathbf{p}] + [\mathbf{p}] \otimes \mathbf{T}_1(p^2))[\mathbf{p}] \otimes [\mathbf{p}]\\
&\quad + (3\,p^2 + 4)\,\mathbf{T}_1(p^2)[\mathbf{p}] \otimes \mathbf{T}_1(p^2)[\mathbf{p}]\\
&\quad - (2\,p^2 - 1)\,(\mathbf{T}_1(p^2)^2 \otimes [\mathbf{p}]^2 + [\mathbf{p}]^2 \otimes \mathbf{T}_1(p^2)^2)\\
&\quad + (\mathbf{T}_1(p^2) \otimes [\mathbf{p}] + [\mathbf{p}] \otimes \mathbf{T}_1(p^2))\,\mathbf{T}_1(p^2) \otimes \mathbf{T}_1(p^2)\\
&\quad - p\,(2\,p^2 + 1)\,(\mathbf{T}(p)^2 \otimes [\mathbf{p}] + [\mathbf{p}] \otimes \mathbf{T}(p)^2)[\mathbf{p}] \otimes [\mathbf{p}]\\
&\quad - 2\,p\,(\mathbf{T}(p)^2 \otimes \mathbf{T}_1(p^2) + \mathbf{T}_1(p^2) \otimes \mathbf{T}(p)^2)[\mathbf{p}] \otimes [\mathbf{p}]\\
&\quad + p\,(\mathbf{T}(p)^2\mathbf{T}_1(p^2) \otimes [\mathbf{p}]^2 + [\mathbf{p}]^2 \otimes \mathbf{T}(p)^2\mathbf{T}_1(p^2))\\
&\quad + \mathbf{T}(p)^2[\mathbf{p}] \otimes \mathbf{T}(p)^2[\mathbf{p}])\mathbf{T}(p) \otimes \mathbf{T}(p))
\end{aligned}$$

$$
\begin{aligned}
s_6 = &-p^8(2\,p^2\,(p^8+6\,p^6+11\,p^4+8\,p^2+2)\,[\mathbf{p}]^3\otimes[\mathbf{p}]^3\\
&+2\,p^2\,(5\,p^4+12\,p^2+6)\,\mathbf{T}_1(p^2)[\mathbf{p}]^2\otimes\mathbf{T}_1(p^2)[\mathbf{p}]^2\\
&+(3\,p^4+10\,p^2-1)\,\mathbf{T}(p)^2[\mathbf{p}]^2\otimes\mathbf{T}(p)^2[\mathbf{p}]^2-\mathbf{T}(p)^2\mathbf{T}_1(p^2)[\mathbf{p}]\otimes\mathbf{T}(p)^2\mathbf{T}_1(p^2)[\mathbf{p}]\\
&+2\,p^2\,(3\,p^6+11\,p^4+12\,p^2+4)\,(\mathbf{T}_1(p^2)\otimes[\mathbf{p}]+[\mathbf{p}]\otimes\mathbf{T}_1(p^2))\,[\mathbf{p}]^2\otimes[\mathbf{p}]^2\\
&+6\,p^2\,(p^2+1)^2\,(\mathbf{T}_1(p^2)^2\otimes[\mathbf{p}]^2+[\mathbf{p}]^2\otimes\mathbf{T}_1(p^2)^2)[\mathbf{p}]\otimes[\mathbf{p}]\\
&+6\,p^2\,(p^2+1)\,(\mathbf{T}_1(p^2)\otimes[\mathbf{p}]+[\mathbf{p}]\otimes\mathbf{T}_1(p^2))\,\mathbf{T}_1(p^2)[\mathbf{p}]\otimes\mathbf{T}_1(p^2)[\mathbf{p}]\\
&+2\,p^2\,(p^2+1)\,(\mathbf{T}_1(p^2)^3\otimes[\mathbf{p}]^3+[\mathbf{p}]^3\otimes\mathbf{T}_1(p^2)^3)\\
&+2\,p^2\,(\mathbf{T}_1(p^2)^2\otimes[\mathbf{p}]^2+[\mathbf{p}]^2\otimes\mathbf{T}_1(p^2)^2)\mathbf{T}_1(p^2)\otimes\mathbf{T}_1(p^2)\\
&-p\,(5\,p^6+13\,p^4+10\,p^2+2)\,(\mathbf{T}(p)^2\otimes[\mathbf{p}]+[\mathbf{p}]\otimes\mathbf{T}(p)^2)[\mathbf{p}]^2\otimes[\mathbf{p}]^2\\
&-p\,(7\,p^4+12\,p^2+4)\,(\mathbf{T}(p)^2\otimes\mathbf{T}_1(p^2)+\mathbf{T}_1(p^2)\otimes\mathbf{T}(p)^2)[\mathbf{p}]^2\otimes[\mathbf{p}]^2\\
&-3p\,(\,p^2+1)\,(\mathbf{T}(p)^2[\mathbf{p}]\otimes\mathbf{T}_1(p^2)^2+\mathbf{T}_1(p^2)^2\otimes\mathbf{T}(p)^2[\mathbf{p}])[\mathbf{p}]\otimes[\mathbf{p}]\\
&-p\,(\mathbf{T}(p)^2[\mathbf{p}]^2\otimes\mathbf{T}_1(p^2)^3+\mathbf{T}_1(p^2)^3\otimes\mathbf{T}(p)^2[\mathbf{p}]^2)\\
&-2\,p\,(3\,p^4+4\,p^2+1)\,(\mathbf{T}(p)^2\mathbf{T}_1(p^2)\otimes[\mathbf{p}]^2+[\mathbf{p}]^2\otimes\mathbf{T}(p)^2\mathbf{T}_1(p^2))[\mathbf{p}]\otimes[\mathbf{p}]\\
&-2\,p\,(3\,p^2+1)\,(\mathbf{T}(p)^2\otimes[\mathbf{p}]+[\mathbf{p}]\otimes\mathbf{T}(p)^2)\mathbf{T}_1(p^2)[\mathbf{p}]\otimes\mathbf{T}_1(p^2)[\mathbf{p}]\\
&-p\,(p^2+1)\,(\mathbf{T}(p)^2\mathbf{T}_1(p^2)^2\otimes[\mathbf{p}]^3+[\mathbf{p}]^3\otimes\mathbf{T}(p)^2\mathbf{T}_1(p^2)^2)\\
&-p\,(\mathbf{T}(p)^2\mathbf{T}_1(p^2)\otimes[\mathbf{p}]^2+[\mathbf{p}]^2\otimes\mathbf{T}(p)^2\mathbf{T}_1(p^2))\mathbf{T}_1(p^2)\otimes\mathbf{T}_1(p^2)\\
&+(5\,p^2-1)\,(\mathbf{T}_1(p^2)\otimes[\mathbf{p}]+[\mathbf{p}]\otimes\mathbf{T}_1(p^2))\,\mathbf{T}(p)^2[\mathbf{p}]\otimes\mathbf{T}(p)^2[\mathbf{p}]\\
&+2\,p^2\,(p^2+1)\,(\mathbf{T}(p)^4\otimes[\mathbf{p}]^2+[\mathbf{p}]^2\otimes\mathbf{T}(p)^4)[\mathbf{p}]\otimes[\mathbf{p}]\\
&+2\,p^2\,(\mathbf{T}(p)^4\otimes\mathbf{T}_1(p^2)[\mathbf{p}]+\mathbf{T}_1(p^2)[\mathbf{p}]\otimes\mathbf{T}(p)^4)\,[\mathbf{p}]\otimes[\mathbf{p}]\\
&-p\,(\mathbf{T}(p)^4\otimes\mathbf{T}(p)^4[\mathbf{p}]+\mathbf{T}(p)^4[\mathbf{p}]\otimes\mathbf{T}(p)^4)\,[\mathbf{p}]\otimes[\mathbf{p}])\\
s_7 = &\,p^{11}(p\,(5\,p^6-2\,p^4+2)\,\mathbf{T}(p)[\mathbf{p}]^3\otimes\mathbf{T}(p)[\mathbf{p}]^3\\
&+8\,p\,\mathbf{T}(p)\mathbf{T}_1(p^2)[\mathbf{p}]^2\otimes\mathbf{T}(p)\mathbf{T}_1(p^2)[\mathbf{p}]^2\\
&+p\,\mathbf{T}(p)^3[\mathbf{p}]^2\otimes\mathbf{T}(p)^3[\mathbf{p}]^2\\
&-p\,(p^4-3)\,(\mathbf{T}_1(p^2)\otimes[\mathbf{p}]+[\mathbf{p}]\otimes\mathbf{T}_1(p^2))\,\mathbf{T}(p)[\mathbf{p}]^2\otimes\mathbf{T}(p)[\mathbf{p}]^2\\
&-p\,(\mathbf{T}_1(p^2)^2\otimes[\mathbf{p}]^2+[\mathbf{p}]^2\otimes\mathbf{T}_1(p^2)^2)\mathbf{T}(p)[\mathbf{p}]\otimes\mathbf{T}(p)[\mathbf{p}]\\
&+2\,p\,(\mathbf{T}_1(p^2)\otimes[\mathbf{p}]+[\mathbf{p}]\otimes\mathbf{T}_1(p^2))\,\mathbf{T}(p)\mathbf{T}_1(p^2)[\mathbf{p}]\otimes\mathbf{T}(p)\mathbf{T}_1(p^2)[\mathbf{p}]\\
&-p\,(\mathbf{T}_1(p^2)^3\otimes[\mathbf{p}]^3+[\mathbf{p}]^3\otimes\mathbf{T}_1(p^2)^3)\mathbf{T}(p)\otimes\mathbf{T}(p)\\
&-(3\,p^4-3\,p^2+2)\,(\mathbf{T}(p)^2\otimes[\mathbf{p}]+[\mathbf{p}]\otimes\mathbf{T}(p)^2)\mathbf{T}(p)[\mathbf{p}]^2\otimes\mathbf{T}(p)[\mathbf{p}]^2\\
&+(p^2-3)\,(\mathbf{T}(p)^2\otimes\mathbf{T}_1(p^2)+\mathbf{T}_1(p^2)\otimes\mathbf{T}(p)^2)\mathbf{T}(p)[\mathbf{p}]^2\otimes\mathbf{T}(p)[\mathbf{p}]^2\\
&-(\mathbf{T}(p)^2[\mathbf{p}]\otimes\mathbf{T}_1(p^2)^2+\mathbf{T}_1(p^2)^2\otimes\mathbf{T}(p)^2[\mathbf{p}])\mathbf{T}(p)[\mathbf{p}]\otimes\mathbf{T}(p)[\mathbf{p}]\\
&+(2\,p^2-1)\,(\mathbf{T}(p)^2\mathbf{T}_1(p^2)\otimes[\mathbf{p}]^2+[\mathbf{p}]^2\otimes\mathbf{T}(p)^2\mathbf{T}_1(p^2))\mathbf{T}(p)[\mathbf{p}]\otimes\mathbf{T}(p)[\mathbf{p}]\\
&-(\mathbf{T}(p)^2\otimes[\mathbf{p}]+[\mathbf{p}]\otimes\mathbf{T}(p)^2)\mathbf{T}(p)\mathbf{T}_1(p^2)[\mathbf{p}]\otimes\mathbf{T}(p)\mathbf{T}_1(p^2)[\mathbf{p}])
\end{aligned}
$$

$$\begin{aligned}
s_8 = {} & p^{14}(2\,p^2\,(2\,p^8+4\,p^6+14\,p^4+12\,p^2+3)\,[\mathbf{p}]^4\otimes[\mathbf{p}]^4 \\
& +4\,p^2\,(p^6+7\,p^4+9\,p^2+3)\,(\mathbf{T}_1(p^2)\otimes[\mathbf{p}]+[\mathbf{p}]\otimes\mathbf{T}_1(p^2))\,[\mathbf{p}]^3\otimes[\mathbf{p}]^3 \\
& +16\,p^2\,(p^2+1)^2\,\mathbf{T}_1(p^2)[\mathbf{p}]^3\otimes\mathbf{T}_1(p^2)[\mathbf{p}]^3 \\
& +2\,p^2\,(3\,p^4+10\,p^2+5)\,(\mathbf{T}_1(p^2)^2\otimes[\mathbf{p}]^2+[\mathbf{p}]^2\otimes\mathbf{T}_1(p^2)^2)\,[\mathbf{p}]^2\otimes[\mathbf{p}]^2 \\
& +8\,p^2\,(p^2+1)\,(\mathbf{T}_1(p^2)\otimes[\mathbf{p}]+[\mathbf{p}]\otimes\mathbf{T}_1(p^2))\,\mathbf{T}_1(p^2)[\mathbf{p}]^2\otimes\mathbf{T}_1(p^2)[\mathbf{p}]^2 \\
& +4\,p^2\,\mathbf{T}_1(p^2)^2[\mathbf{p}]^2\otimes\mathbf{T}_1(p^2)^2[\mathbf{p}]^2 \\
& +4\,p^2\,(p^2+1)\,(\mathbf{T}_1(p^2)^3\otimes[\mathbf{p}]^3+[\mathbf{p}]^3\otimes\mathbf{T}_1(p^2)^3)\,[\mathbf{p}]\otimes[\mathbf{p}] \\
& +p^2\,(\mathbf{T}_1(p^2)^4\otimes[\mathbf{p}]^4+[\mathbf{p}]^4\otimes\mathbf{T}_1(p^2)^4) \\
& -4\,p\,(2\,p^6+3\,p^4+4\,p^2+1)\,(\mathbf{T}(p)^2\otimes[\mathbf{p}]+[\mathbf{p}]\otimes\mathbf{T}(p)^2)[\mathbf{p}]^3\otimes[\mathbf{p}]^3 \\
& -8\,p\,(p^2+1)^2\,(\mathbf{T}(p)^2\otimes\mathbf{T}_1(p^2)+\mathbf{T}_1(p^2)\otimes\mathbf{T}(p)^2)[\mathbf{p}]^3\otimes[\mathbf{p}]^3 \\
& -4\,p\,(p^2+1)\,(\mathbf{T}(p)^2[\mathbf{p}]\otimes\mathbf{T}_1(p^2)^2+\mathbf{T}_1(p^2)^2\otimes\mathbf{T}(p)^2[\mathbf{p}])[\mathbf{p}]^2\otimes[\mathbf{p}]^2 \\
& -4\,p\,(p^4+4\,p^2+1)\,(\mathbf{T}(p)^2\mathbf{T}_1(p^2)\otimes[\mathbf{p}]^2+[\mathbf{p}]^2\otimes\mathbf{T}(p)^2\mathbf{T}_1(p^2))[\mathbf{p}]^2\otimes[\mathbf{p}]^2 \\
& -8\,p\,(p^2+1)\,(\mathbf{T}(p)^2\otimes[\mathbf{p}]+[\mathbf{p}]\otimes\mathbf{T}(p)^2)\mathbf{T}_1(p^2)[\mathbf{p}]^2\otimes\mathbf{T}_1(p^2)[\mathbf{p}]^2 \\
& -4\,p\,(\mathbf{T}(p)^2\otimes\mathbf{T}_1(p^2)+\mathbf{T}_1(p^2)\otimes\mathbf{T}(p)^2)\mathbf{T}_1(p^2)[\mathbf{p}]^2\otimes\mathbf{T}_1(p^2)[\mathbf{p}]^2 \\
& -4\,p^3\,(\mathbf{T}(p)^2\mathbf{T}_1(p^2)^2\otimes[\mathbf{p}]^3+[\mathbf{p}]^3\otimes\mathbf{T}(p)^2\mathbf{T}_1(p^2)^2)+[\mathbf{p}]\otimes[\mathbf{p}] \\
& +2\,(5\,p^4+2\,p^2+2)\,\mathbf{T}(p)^2[\mathbf{p}]^3\otimes\mathbf{T}(p)^2[\mathbf{p}]^3 \\
& +2\,(p^2+2)\,(\mathbf{T}_1(p^2)\otimes[\mathbf{p}]+[\mathbf{p}]\otimes\mathbf{T}_1(p^2))\,\mathbf{T}(p)^2[\mathbf{p}]^2\otimes\mathbf{T}(p)^2[\mathbf{p}]^2 \\
& +2\,\mathbf{T}(p)^2\mathbf{T}_1(p^2)[\mathbf{p}]^2\otimes\mathbf{T}(p)^2\mathbf{T}_1(p^2)[\mathbf{p}]^2 \\
& +(\mathbf{T}_1(p^2)^2\otimes[\mathbf{p}]^2+[\mathbf{p}]^2\otimes\mathbf{T}_1(p^2)^2)\mathbf{T}(p)^2[\mathbf{p}]\otimes\mathbf{T}(p)^2[\mathbf{p}] \\
& +(3\,p^4+2\,p^2+1)\,(\mathbf{T}(p)^4\otimes[\mathbf{p}]^2+[\mathbf{p}]^2\otimes\mathbf{T}(p)^4)[\mathbf{p}]^2\otimes[\mathbf{p}]^2 \\
& +2\,(p^2+1)\,(\mathbf{T}(p)^4\otimes\mathbf{T}_1(p^2)[\mathbf{p}]+\mathbf{T}_1(p^2)[\mathbf{p}]\otimes\mathbf{T}(p)^4)\,[\mathbf{p}]^2\otimes[\mathbf{p}]^2 \\
& +(\mathbf{T}(p)^4\otimes\mathbf{T}_1(p^2)^2+\mathbf{T}_1(p^2)^2\otimes\mathbf{T}(p)^4)\,[\mathbf{p}]^2\otimes[\mathbf{p}]^2 \\
& -2\,p\,(\mathbf{T}(p)^2\otimes[\mathbf{p}]+[\mathbf{p}]\otimes\mathbf{T}(p)^2)\mathbf{T}(p)^2[\mathbf{p}]^2\otimes\mathbf{T}(p)^2[\mathbf{p}]^2)
\end{aligned}$$

Then we find the remaining coefficients $s_9,\cdots,s_{16}$, using an easy functional equation (similar to [An87], p.164, (3.3.79)):

$$s_{16-i}=(p^6[\mathbf{p}]\otimes[\mathbf{p}])^{8-i}s_i \quad (i=0,\cdots,8).$$

To conclude with, we give the Newton polygons of $R(X)$ and $S(X)$ with respect to powers of p and X (see Figure 1). It follows from our comutation that all slopes are *integral*. We hope that these polygons could help to find some geometric objects attached to the polynomials $R(X)$ and $S(X)$, in the spirit of a recent work of C.Faber and G.Van Der Geer, [FVdG].

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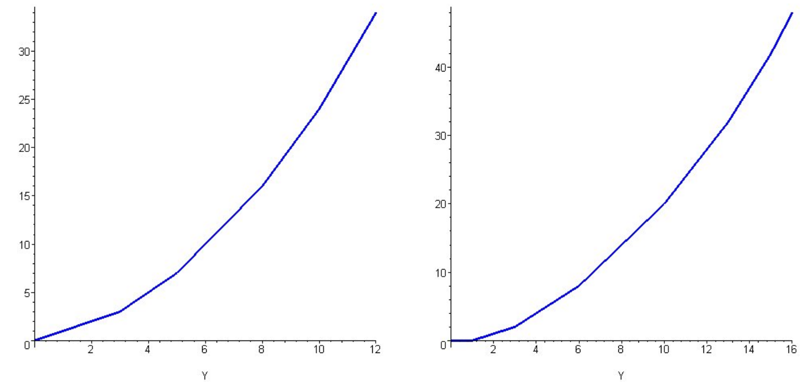
Fig. 1. Newton polygons of $R(X)$ and $S(X)$ with respect to powers of p and X, of heights 34 and 48, resp.



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