Fields of u-invariant $2^r + 1$

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Summary. In this article we provide a uniform construction of fields with all known u-invariants. We also obtain the new values for the u-invariant: $2^r + 1$, for $r > 3$. The main tools here are the new discrete invariant of quadrics (so-called, *elementary discrete invariant*), and the methods of [14] (which permit to reduce the questions of *rationality* of elements of the Chow ring over the base field to that over bigger field - the generic point of a quadric).

1 Introduction

The u-invariant of a field is defined as the maximal dimension of anisotropic quadratic form over it. The problem to describe values of this invariant is one of major open problems in the theory of quadratic forms. Using elementary methods it is easy to establish that the u-invariant can not take values 3, 5 and 7. The conjecture of Kaplansky (1953) suggested that the only possible values are the powers of 2 (by that time, the examples of fields with u-invariant being any power of 2 were known). This conjecture was disproved by A.Merkurjev in 1991, who constructed fields with all even u-invariants. The next challenge was to find out if fields with odd u-invariant > 1 are possible at all. The breakthrough here was made by O.Izhboldin who in 1999 constructed a field of u-invariant 9 - see [4]. Still the question of other possible values remained open. This paper suggests a new uniform method of constructing fields with various u-invariants. In particular, we get fields with any even u-invariant without using the *index reduction formula* of Merkurjev. We also construct fields with u-invariant $2^r + 1$, for all $r \geqslant 3$. It should be mentioned that O.Izhboldin conjectured the existence of fields with such u-invariant, and suggested ideas to prove the conjecture. However, this paper employes very different new ideas. One can see the difference on the example of u-invariant 9. I would say that our method uses substantially more coarse invariants (like *generic discrete invariant of quadrics*), while the original construction used very subtle ones (like the *cokernel on the unramified cohomology*, etc. ...).

Thus, this paper amply demonstrates that the u-invariant questions can be solved just with the help of "coarse" invariants. The method is based on the new, so-called, *elementary discrete invariant* of quadrics (introduced in the paper). This invariant contains important piece of information about the particular quadric, and, at the same time, is quite handy to operate. The field with the given u-invariant is constructed using the standard field tower of A.Merkurjev. And the central problem is to control the behavior of the elementary discrete invariant while passing from the base field to the generic point of a (sufficiently large) quadric. This is done using the general statement from [14] concerning the question of rationality of small-codimensional classes in the Chow ring of arbitrary smooth variety under similar passage. The driving force behind all of this comes from the *symmetric operations* in Algebraic Cobordism ([12],[15]).

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2 Elementary discrete invariant

In this section we assume that the base field k has characteristic different from 2. We will fix an algebraic closure $\overline{k}$ of k.

For the nondegenerate quadratic form q we will denote by the capital letter Q the respective smooth projective quadric. The same applies to forms $p, p', q', \ldots$. Dimension of a quadric Q will be denoted as N_Q, and if there is no ambiguity, simply as N. We also denote $d_Q := [N_Q/2]$ (respectively, $d := [N/2]$). For the smooth variety X we will denote as $\mathrm{CH}^*(X)$ the Chow ring of algebraic cycles modulo rational equivalence on X, and as $\mathrm{Ch}^*(X)$ the Chow ring modulo 2 (see [3] for details).

To each smooth projective quadric Q/k of dimension N one can assign the so-called *generic discrete invariant* $GDI(Q)$ - see [13], which is defined as the collection of subrings

$$GDI(Q,i) := image(\mathrm{Ch}^*(F(Q,i)) \to \mathrm{Ch}^*(F(Q,i)|_{\overline{k}})),$$

for all $0 \leqslant i \leqslant d$, where $F(Q,i)$ is the Grassmannian of i-dimensional projective subspaces on Q, and the map is induced by the restriction of scalars $k \to \overline{k}$. Note, that $F(Q,0)$ is quadric Q itself, and $F(Q,d)$ is the last Grassmannian.

For $J \subset I \subset \{0, \ldots, d\}$ let us denote the natural projection between partial flag varieties $F(Q, I) \to F(Q, J)$ as π with subindex I with J underlined inside it. In particular, we have projections:

$$F(Q, i) \overset{\pi_{(0,\underline{i})}}{\longleftarrow} F(Q, 0, i) \overset{\pi_{(\underline{0},i)}}{\longrightarrow} Q.$$

The Chow ring of a split quadric is a free $\mathbb{Z}$-module with the basis h^s, l_s, $0 \leqslant s \leqslant d$, where $l_s \in \mathrm{CH}_s(Q|_{\overline{k}})$ is the class of a projective subspace of dimension s, and $h^s \in \mathrm{CH}^s(Q|_{\overline{k}})$ is the class of plane section of codimension s - see [11, Lemma 8].

In $\mathrm{CH}^*(F(Q, i)|_{\overline{k}})$ we have special classes: $Z_j^{\boxed{i-d}} \in \mathrm{CH}^j, N - d - i \leqslant j \leqslant N - i$, and $W_j^{\boxed{i-d}} \in \mathrm{CH}^j, 0 \leqslant j \leqslant d - i$, defined by:

$$Z_j^{\boxed{i-d}} := (\pi_{(0,\underline{i})})_*(\pi_{(\underline{0},i)})^*(l_{N-i-j}); \qquad W_j^{\boxed{i-d}} := (\pi_{(0,\underline{i})})_*(\pi_{(\underline{0},i)})^*(h^{i+j}).$$

Let us denote as $z_j^{\boxed{i-d}}$ and $w_j^{\boxed{i-d}}$ the same classes in Ch^*. We will call classes $z_j^{\boxed{i-d}}$ *elementary*. Notice, that the classes $w_j^{\boxed{i-d}}$ always belong to $GDI(Q, i)$.

Let $\mathcal{T}_i$ be the tautological $(i + 1)$-dimensional vector bundle on $F(Q, i)$. The following Proposition explains the meaning of our classes.

Proposition 2.1. *For any $0 \leqslant i \leqslant d$, and $N - d - i \leqslant j \leqslant N - i$,*

$$c_\bullet(-\mathcal{T}_i) = \sum_{j=0}^{d-i} W_j^{\boxed{i-d}} + 2 \sum_{d-i<j\leqslant N-i} Z_j^{\boxed{i-d}}.$$

Proof. Since $\mathcal{T}_0 = \mathcal{O}(-1)$ on Q, the statement is true for $i = 0$. Consider the projections

$$F(Q, i) \overset{\pi_{(0,\underline{i})}}{\longleftarrow} F(Q, 0, i) \overset{\pi_{(\underline{0},i)}}{\longrightarrow} Q.$$

Notice, that $F(Q, 0, i)$ is naturally identified with the projective bundle $\mathbb{P}_{F(Q,i)}(\mathcal{T}_i)$, and the sheaf $\pi^*_{(\underline{0},i)}(\mathcal{T}_0)$ is naturally identified with $\mathcal{O}(-1)$. Thus,

$$(\pi_{(0,\underline{i})})_*(\pi_{(\underline{0},i)})^*(c_\bullet(-\mathcal{T}_0)) = (\pi_{(0,\underline{i})})_*(c_\bullet(-\mathcal{O}(-1))) = c_\bullet(-\mathcal{T}_i).$$

□

Remark 2.2. In particular, for $i = d$ we get another proof of [13, Theorem 2.5(3)].

Definition 2.3. *Define the elementary discrete invariant $EDI(Q)$ as the collection of subsets $EDI(Q, i)$ consisting of those j that $z_j^{\boxed{i-d}} \in GDI(Q, i)$.*

One can visualise $EDI(Q)$ as the coordinate $d \times d$ square, where some integral nodes are marked, each row corresponds to particular grassmannian, and the codimension of a "node" is decreasing up and right. The lower row corresponds to the quadric itself, and the upper one - to the last grassmannian. The South-West corner is marked if and only if Q is isotropic.

Example 2.4. The $EDI(Q)$ for the 10-dimensional excellent form looks as:

$$\begin{array}{ccccc}
\bullet & \bullet & \bullet & \bullet & \circ \\
\bullet & \bullet & \circ & \bullet & \circ \\
\bullet & \bullet & \circ & \circ & \circ \\
\circ & \bullet & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ
\end{array}$$

The following statement puts serious constraints on possible markings.

Proposition 2.5. *Let $0 \leqslant i < d$, and $j \in EDI(Q,i)$. Then $j, j-1 \in EDI(Q, i+1)$.*

This can be visualised as:

$$\begin{tikzcd}
\bullet & \bullet \\
 & \bullet \arrow[ul] \arrow[u]
\end{tikzcd}$$

Proof. The Proposition easily follows from the next Lemma. Let us temporarily denote $\pi_{(\underline{i},i+1)}$ as α, and $\pi_{(i,\underline{i+1})}$ as β.

Lemma 2.6.

$$\alpha^*(Z_j^{\boxed{i-d}}) = \beta^*(Z_j^{\boxed{i+1-d}}) + c_1(\mathcal{O}(1)) \cdot \beta^*(Z_{j-1}^{\boxed{i+1-d}});$$
$$\alpha^*(W_j^{\boxed{i-d}}) = \beta^*(W_j^{\boxed{i+1-d}}) + c_1(\mathcal{O}(1)) \cdot \beta^*(W_{j-1}^{\boxed{i+1-d}}), 0 \leqslant j < d-i;$$
$$\alpha^*(W_{d-i}^{\boxed{i-d}}) = 2\beta^*(Z_{d-i}^{\boxed{i+1-d}}) + c_1(\mathcal{O}(1)) \cdot \beta^*(W_{d-i-1}^{\boxed{i+1-d}}),$$

where $\mathcal{O}(1)$ is the standard sheaf on the projective bundle

$$F(Q, i, i+1) = \mathbb{P}_{F(Q,i+1)}(\mathcal{T}_{i+1}^{\vee}),$$

for the vector bundle dual to the tautological one.

Proof. By definition, $Z_j^{\boxed{i-d}}, W_j^{\boxed{i-d}}$ have the form $(\pi_{(0,\underline{i})})_*(\pi_{(\underline{0},i)})^*(x)$, for certain $x \in \mathrm{CH}^*(Q|_{\overline{k}})$. Since the square

$$\begin{tikzcd}
F(Q,i,i+1) \arrow[d, "\pi_{(\underline{i},i+1)}"'] & F(Q,0,i,i+1) \arrow[l, "\pi_{(0,\underline{i},\underline{i+1})}"'] \arrow[d, "\pi_{(\underline{0},\underline{i},i+1)}"] \\
F(Q,i) & F(Q,0,i) \arrow[l, "\pi_{(0,\underline{i})}"]
\end{tikzcd}$$

is transversal Cartesian, $(\pi_{(\underline{i},i+1)})^*$ of these elements is equal to

$$(\pi_{(0,\underline{i},\underline{i+1})})_*(\pi_{(\underline{0},\underline{i},i+1)})^*(\pi_{(\underline{0},i)})^*(x) = (\pi_{(0,\underline{i},\underline{i+1})})_*(\pi_{(\underline{0},i,\underline{i+1})})^*(\pi_{(\underline{0},i+1)})^*(x).$$

Variety $F(Q,0,i,i+1)$ is naturally a divisor D on the transversal product $F(Q,0,i+1)\times_{F(Q,i+1)} F(Q,i,i+1)$ with $\mathcal{O}(D) = \pi^*_{(\underline{0},i,i+1)}(\mathcal{O}(h))\otimes \pi^*_{(0,\underline{i},\underline{i+1})}(\mathcal{O}(1))$, where $\mathcal{O}(h)$ is the sheaf given by the hyperplane section on Q. Then

$$\begin{aligned}(\pi_{(0,\underline{i},\underline{i+1})})_*(\pi_{(\underline{0},i,\underline{i+1})})^*(\pi_{(\underline{0},i+1)})^*(x) =\\ c_1(\mathcal{O}(1))\cdot(\pi_{(i,\underline{i+1})})^*(\pi_{(0,\underline{i+1})})_*(\pi_{(\underline{0},i+1)})^*(x)+\\ (\pi_{(i,\underline{i+1})})^*(\pi_{(0,\underline{i+1})})_*(\pi_{(\underline{0},i+1)})^*(h\cdot x).\end{aligned}$$

It remains to plug in the appropriate x. □

Notice, that the projective bundle theorem

$$\mathrm{Ch}^*(\mathbb{P}_{F(Q,i+1)}(\mathcal{T}^\vee_{i+1})) = \oplus_{l=0}^{i} c_1(\mathcal{O}(1))^l\cdot \mathrm{Ch}^*(F(Q,i+1))$$

implies that the element of this group is defined over k if and only if all of it's coordinates are. Since the cycle $z_j^{\boxed{i-d}}$ is defined over k, by Lemma 2.6 the cycles $z_j^{\boxed{i+1-d}}$ and $z_{j-1}^{\boxed{i+1-d}}$ are defined too. □

The following Proposition describes the EDI of the isotropic quadric.

Proposition 2.7. *Let $p' = p\perp\mathbb{H}$ be isotropic quadratic form (here $\mathbb{H}$ is a 2-dimensional hyperbolic form $\langle 1,-1\rangle$). Then EDI of P and P' are related as follows: for any $N_P - d_P - i \leqslant j \leqslant N_P - i$*

$$z_j^{\boxed{i-d_P}}(P)\ \textit{ is defined }\ \Leftrightarrow z_j^{\boxed{i+1-d_{P'}}}(P')\ \textit{ is defined.}$$

In other words, $EDI(P)$ fits well into $EDI(P')$, if we glue their N-E corners.

Proof. The quadric P can be identified with the quadric of lines on P' passing through the given rational point x. Then we have natural regular embedding $e: F(P,i)\to F(P',i+1)$, with $e^*(z_j^{\boxed{i+1-d_{P'}}}) = z_j^{\boxed{i-d_P}}$ and ($\Leftarrow$) follows.

We have natural maps $F(P,i)\xleftarrow{f}(F(P',i+1)\backslash F(P,i))\xrightarrow{g} F(P',i+1)$, where the map f sends $(i+1)$-dimensional plane π_{i+1} to $T_{x,P'}\cap(x+\pi_{i+1}))$ (expression in the projective space), and g is an open embedding. It is an exercise for the reader, to show that $f^*(z_j^{\boxed{i-d_P}}(P)) = g^*(z_j^{\boxed{i+1-d_{P'}}}(P'))$. It remains to observe that $F(P,i)$ has codimension $(N_P - i + 1) > j$ inside $F(P', i+1)$, and thus g^* performes an isomorphism on Ch^j. This proves ($\Rightarrow$). □

If we have a codimension 1 subquadric P of a quadric Q, the EDI's of them are related by the following.

Proposition 2.8.

$$z_j^{\boxed{i-d_Q}}(Q) \text{ is defined} \Rightarrow z_j^{\boxed{i-d_P}}(P) \text{ is defined;}$$

$$z_j^{\boxed{i-d_P}}(P) \text{ is defined} \Rightarrow z_j^{\boxed{i+1-d_Q}}(Q) \text{ is defined.}$$

Proof. Consider the natural embedding: $e : F(P, i) \to F(Q, i)$. Then, it follows from the definition that $e^*(z_j^{\boxed{i-d_Q}}) = z_j^{\boxed{i-d_P}}$. To prove the second statement just observe that Q is a codimension 1 subquadric in P', where $p' = p \perp \mathbb{H}$ (if $q = p \perp \langle a \rangle$, then $p' = q \perp \langle -a \rangle$), and apply Proposition 2.7. $\square$

Unfortunately, the Steenrod operations (see [1, 17]), in general, do not act on the $EDI(Q, i)$, since they do not preserve *elementary classes*. But they act in the lower and the upper row: for the quadric itself, and for the last grassmannian. Also, it follows from [13, Main Theorem 5.8] that $EDI(Q, d)$ carries the same information as $GDI(Q, d)$. The same is true about $EDI(Q, 0)$ and $GDI(Q, 0)$ by the evident reasons.

The action of the Steenrod operations on the *elementary classes* can be described as follows.

Proposition 2.9. *Let $0 \leqslant i \leqslant d$, and $N - d - i \leqslant j \leqslant N - i$, then*

$$S^m(z_j^{\boxed{i-d}}) = \sum_{k=0}^{d-i} \binom{j-k}{m-k} z_{j+m-k}^{\boxed{i-d}} \cdot w_k^{\boxed{i-d}},$$

where elementary classes of codimension more than $(N - i)$ are assumed to be 0.

Proof. We recall from [1] that on the Chow groups modulo 2 of smooth variety X one has the action of Steenrod operations $S^\bullet$ and $S_\bullet$, where the former commute with the pull-backs for all morphisms, and the latter commute with the push-forwards for proper morphisms. The relation between the upper and the lower operations is given by

$$S^\bullet = S_\bullet \cdot c_\bullet(T_X).$$

From this (and the description of the tangent bundle for the quadric and the projective space) one gets that $S^\bullet(l_s) = (1 + h)^{N-s+1} l_s$. Since $(\pi_{\underline{0},i})^*(\mathcal{O}(1))$ is the sheaf $\mathcal{O}(1)$ on $F(Q, 0, i) = \mathbb{P}_{F(Q,i)}(\mathcal{T}_i)$,

$$S_\bullet(\pi_{(\underline{0},i)})^*(l_{N-i-j}) = c_\bullet(-T_{F(Q,0,i)}) \cdot (1 + H)^{i+j+1} \cdot (\pi_{(\underline{0},i)})^*(l_{N-i-j}),$$

where $H = c_1(\mathcal{O}(1))$. Since $S_\bullet$ commutes with the push-forward morphisms,

$$S^\bullet(z_j^{\boxed{i-d}}) = S^\bullet(\pi_{(0,\underline{i})})_*(\pi_{(\underline{0},i)})^*(l_{N-i-j}) =$$
$$(\pi_{(0,\underline{i})})_*(c_\bullet(-T_{fiber})\cdot(1+H)^{i+j+1}\cdot(\pi_{(\underline{0},i)})^*(l_{N-i-j})).$$

Recall that if $\mathcal{V}$ is a virtual vector bundle of virtual dimension M, and $c_\bullet(\mathcal{V})(t) := \sum_{k\geqslant 0} c_k(\mathcal{V})\cdot t^{M-k}$, then for the divisor H, $c_\bullet(\mathcal{V}\otimes\mathcal{O}(H))(t) = c_\bullet(\mathcal{V})(t+H)$, and $c_\bullet(\mathcal{V}\otimes\mathcal{O}(H)) = c_\bullet(\mathcal{V}\otimes\mathcal{O}(H))(1) = c_\bullet(\mathcal{V})(1+H)$.

Since $c_\bullet(-T_{fiber}) = c_\bullet(-\mathcal{T}_i\otimes\mathcal{O}(1))$, by Proposition 2.1, $(\,mod\,2)$ this is equal to $\sum_{k=0}^{d-i} w_k^{\boxed{i-d}}(1+H)^{-i-1-k}$. Thus,

$$S^\bullet(z_j^{\boxed{i-d}}) = (\sum_{k=0}^{d-i} w_k^{\boxed{i-d}})(\pi_{(0,\underline{i})})_*(\pi_{(\underline{0},i)})^*((1+h)^{j-k}l_{N-i-j}) =$$
$$\sum_{r\geqslant 0}\sum_{k=0}^{d-i}\binom{j-k}{r-k} z_{j+r-k}^{\boxed{i-d}} w_k^{\boxed{i-d}}.$$

$\square$

Remark 2.10. In particular, for $i=d$ we get a new proof of [13, Theorem 4.1].

The following fact is well-known (see, for example, [2]). We will give an independent proof below.

Proposition 2.11. *The ring* $\mathrm{CH}^*(F(Q,i)|_{\overline{k}})$ *is generated by the classes*

$$Z_j^{\boxed{i-d}}, N-d-i\leqslant j\leqslant N-i, \ \textit{and}\ W_j^{\boxed{i-d}}, 0\leqslant j\leqslant d-i.$$

Proof. For $0\leqslant l\leqslant i$, let us denote the pull back of $Z_j^{\boxed{l-d}}$ to $F(Q,0,\dots,i)$ by the same symbol. On this flag variety we have natural line bundles $\mathcal{L}_k := \mathcal{T}_k/\mathcal{T}_{k-1}$. Let us denote $h_k := c_1(\mathcal{L}_k^{-1})$.

Lemma 2.12. *Let E/k be some field extension. Suppose that $Q|_E$ is split. Then the ring* $\mathrm{CH}^*(F(Q,0,\dots,i)|_E)$ *is generated by $W_j^{\boxed{l-d}}$, $0\leqslant l\leqslant i$, $1\leqslant j\leqslant d-l$, and $Z_j^{\boxed{l-d}}$, $0\leqslant l\leqslant i$, $N-d-l\leqslant j\leqslant N-l$.*

Proof. Induction on i. For $i=0$ the statement is evident.

Statement 2.13. *Let $\pi: Y\to X$ be a smooth morphism to a smooth variety X. For $x\in X^{(r)}$, Y_x be the fiber over the point x. Let ζ denote the generic point of X, and s_x :* $\mathrm{CH}^*(Y_\zeta)\to\mathrm{CH}^*(Y_x)$ *be the specialisation map. Let $B\subset\mathrm{CH}^*(Y)$ be a subgroup. Suppose:*

(a) *the map $B\to\mathrm{CH}^*(Y_\zeta)$ is surjective;*
(b) *all the maps s_x are surjective.*

Then $\mathrm{CH}^*(Y) = B \cdot \pi^*(\mathrm{CH}^*(X))$.

Proof. On $\mathrm{CH}^*(Y)$ we have decreasing filtration $F^\bullet$, where F^r consists of classes, having a representative with the image under π of codimension $\geqslant r$. This gives the surjection:

$$\oplus_r \oplus_{x \in X^{(r)}} \mathrm{CH}^*(Y_x) \to gr_{F^\bullet} \mathrm{CH}^{r+*}(Y).$$

Let $[x] \in \mathrm{CH}^r(X)$ be the class represented by the closure of x. Clearly, the image of $\pi^*([x]) \cdot B$ covers the image of $\mathrm{CH}^*(Y_x)$ in F^r/F^{r+1}. $\square$

Consider the projection

$$\pi_{(\underline{0},\dots,\underline{i-1},i)} : F(Q, 0, \dots, i-1, i) \to F(Q, 0, \dots, i-1).$$

Let $Q_{\{i\},x}/E(x)$ be the fiber of this projection over the point x. It is a split quadric of dimension $N - 2i$. Thus, the condition (b) of the Statement 2.13 is satisfied. Since $[\mathcal{T}_i|_{Q_{\{i\},\zeta}}] = [\mathcal{L}_i] + i \cdot [\mathcal{O}] = [\mathcal{O}(-h_i)] + i \cdot [\mathcal{O}]$ in $K_0(Q_{\{i\},\zeta})$, it follows from Proposition 2.1 that

$$Z_j^{\boxed{i-d}}|_{Q_{\{i\},\zeta}} = l_{N-2i-j}, \qquad W_j^{\boxed{i-d}}|_{Q_{\{i\},\zeta}} = h_i^j.$$

We can take B additively generated by $Z_j^{\boxed{i-d}}, N - d - i \leqslant j \leqslant N - 2i$, and $W_j^{\boxed{i-d}}, 0 \leqslant j \leqslant d - i$. Then the condition (a) will be satisfied too. The induction step follows. $\square$

Lemma 2.6 implies that the $Z_j^{\boxed{l-d}}, W_j^{\boxed{l-d}}$, for $l < i$ are expressible in terms of $Z_k^{\boxed{i-d}}, W_k^{\boxed{i-d}}$ and $h_m, 0 \leqslant m \leqslant i$. Let $A \subset \mathrm{CH}^*(F(Q, i))$ be the subring generated by $Z_j^{\boxed{i-d}}, W_j^{\boxed{i-d}}$. Since $F(Q, 0, \dots, i)$ is a variety of complete flags of subspaces of the vector bundle $\mathcal{T}_i$ on $F(Q, i)$, the ring $\mathrm{CH}^*(F(Q, 0, \dots, i))$ is isomorphic to

$$\mathrm{CH}^*(F(Q, i))[h_0, \dots, h_i]/(\sigma_r(h) - c_r(\mathcal{T}_i^\vee), 1 \leqslant r \leqslant i + 1),$$

where $\sigma_r(h)$ are elementary symmetric functions on h_k. But $c_r(\mathcal{T}_i^\vee) \in A$, by Proposition 2.1. Since A and $h_m, 0 \leqslant m \leqslant i$ generate $\mathrm{CH}^*(F(Q, 0, \dots, i))$, A must coincide with $\mathrm{CH}^*(F(Q, i))$. $\square$

In particular, since the cycles $W_j^{\boxed{i-d}}$ are defined over k, we have:

Corollary 2.14. *The graded part of* $\mathrm{CH}^*(F(Q, i)|_{\overline{k}})$ *of degree less or equal* $(d - i)$ *consists of classes which are defined over* k.

Notice, that for $i = d$, $\mathrm{Ch}^*(F(Q,d)|_{\overline{k}})$ is generated as a ring by $z_j^{\boxed{0}}$, and moreover, $GDI(Q,d)$ is always generated as a ring by the subset of $z_j^{\boxed{0}}$ contained in it - see [13, Main Theorem 5.8].

We will need one more simple fact.

Statement 2.15. *The class of a rational point on $F(Q,i)|_{\overline{k}}$ is given by the product*

$$\prod_{j=i}^{2i} Z_{N-j}^{\boxed{i-d}}$$

Proof. Use induction on N. Let x be a fixed rational point on $Q|_{\overline{k}}$. Then we have a natural regular embedding $e : F(P,i-1) \to F(Q,i)$, where P is the $(N-2)$-dimensional quadric of lines on Q passing through x, with $e_*(1) = Z_{N_Q-i}^{\boxed{i-d_Q}}(Q)$, and $e^*(Z_{N_Q-j}^{\boxed{i-d_Q}}(Q)) = Z_{N_P-j+1}^{\boxed{i-1-d_P}}(P)$. Thus the induction step follows from the projection formula. The base of induction is trivial. $\square$

3 Generic points of quadrics and Chow groups

Everywhere below we will assume that the base-field k has characteristic 0. Although, many things work for odd characteristics as well, the use of Algebraic Cobordism theory of M.Levine-F.Morel will require such an assumption.

In this section I would like to remind the principal result of [14]. Let Q be a smooth projective quadric, Y be a smooth quasiprojective variety, and $\overline{y} \in \mathrm{Ch}^m(Y|_{\overline{k}})$. This will be our main tool in the construction of fields with various u-invariants.

Theorem 3.1. ([14, Corollary 3.5],[15, Theorem 4.3].)
Suppose $m < N_Q - d_Q$. Then

$$\overline{y}|_{\overline{k(Q)}} \text{ is defined over } k(Q) \;\Leftrightarrow\; \overline{y} \text{ is defined over } k.$$

Example 3.2. Let $\alpha = \{a_1, \dots, a_n\} \in \mathrm{K}_n^M(k)/2$ be a nonzero pure symbol, and Q_α be the respective anisotropic Pfister quadric. Then in $EDI(Q_\alpha)$ the marked nodes will be exactly those ones which live above the main (N-W to S-E) diagonal.

$$\begin{array}{cccccc}
\circ & \bullet & \bullet & \cdots & \bullet & \bullet \\
\circ & \circ & \bullet & \cdots & \bullet & \bullet \\
\circ & \circ & \circ & \cdots & \bullet & \bullet \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\circ & \circ & \circ & \cdots & \circ & \bullet \\
\circ & \circ & \circ & \cdots & \circ & \circ
\end{array}$$

Really, over own generic point $k(Q_\alpha)$ quadric Q_α becomes hyperbolic, and so, all the elementary cycles are defined there. Then the cycles above the main diagonal got to be defined already over the base field, since their codimension is smaller than $N_{Q_\alpha} - d_{Q_\alpha}$. On the other hand, the N-W corner could not be defined over k, since otherwise all the elementary cycles on the last grassmannian of Q_α would be defined over k, but, by the Statement 2.15, the product of all these cycles is the class of a rational point on this grassmannian. Since Q_α is not hyperbolic over k, this is impossible. The rest of the picture follows from Proposition 2.5.

The proof of Theorem 3.1 uses Algebraic Cobordisms of M.Levine-F.Morel. Let me say few words about the latter.

3.1 Algebraic Cobordisms

In [7] M.Levine and F.Morel have constructed the universal oriented generalised cohomology theory Ω^* on the category of smooth quasiprojective varieties over the field k of characteristic 0, called *Algebraic Cobordism*.

For any smooth quasiprojective X over k, the additive group $\Omega^*(X)$ is generated by the classes $[v : V \to X]$ of projective maps from smooth varieties V subject to certain relations, and the upper grading is the codimensional one. There is natural morphism of theories $pr : \Omega^* \to \mathrm{CH}^*$. The main properties of Ω^* are:

(1) $\Omega^*(\mathrm{Spec}(k)) = \mathbb{L} = \mathrm{MU}(pt)$ - the Lazard ring, and the isomorphism is given by the topological realisation functor;
(2) $\mathrm{CH}^*(X) = \Omega^*(X)/\mathbb{L}^{<0} \cdot \Omega^*(X)$.

On Ω^* there is the action of the Landweber-Novikov operations (see [7, Example 4.1.25]). Let $R(\sigma_1, \sigma_2, \ldots) \in \mathbb{L}[\sigma_1, \sigma_2, \ldots]$ be some polynomial, where we assume $\deg(\sigma_i) = i$. Then $S^R_{L-N} : \Omega^* \to \Omega^{*+\deg(R)}$ is given by:

$$S^R_{L-N}([v : V \to X]) := v_*(R(c_1, c_2, \ldots) \cdot 1_V),$$

where $c_j = c_j(\mathcal{N}_v)$, and $\mathcal{N}_v := -T_V + v^*T_X$ is the *virtual normal bundle*.

If $R = \sigma_i$, we will denote the respective operation simply as S^i_{L-N}. The following statement follows from the definition of Steenrod and Landweber-Novikov operations - see P.Brosnan [1], A.Merkurjev [10], and M.Levine [6]

Proposition 3.3. *There is commutative square:*

$$\begin{array}{ccc} \Omega^*(X) & \xrightarrow{S^i_{L-N}} & \Omega^{*+i}(X) \\ \downarrow & & \downarrow \\ \mathrm{Ch}^*(X) & \xrightarrow[S^i]{} & \mathrm{Ch}^{*+i}(X), \end{array}$$

where S^i is the Steenrod operation ($mod\,2$) ([17, 1]).

In particular, using the results of P.Brosnan on S^i (see [1]), we get:

Corollary 3.4. (1) *$pr \circ S^i_{L-N}(\Omega^m) \subset 2 \cdot \mathrm{CH}^{i+m}$, if $i > m$;*
(2) *$pr \circ (S^m_{L-N} - \square)(\Omega^m) \in 2 \cdot \mathrm{CH}^{2m}$, where $\square$ is the square operation.*

This implies that (modulo $2-$torsion) we have well defined maps $\frac{pr\circ S^i_{L-N}}{2}$ and $\frac{pr\circ(S^m_{L-N}-\square)}{2}$. In reality, these maps can be lifted to a well defined, so-called, *symmetric operations $\Phi^{t^{i-m}}$* $: \Omega^m \to \Omega^{m+i}$ - see [15]. Since over algebraically closed field all our varieties are cellular, and thus, the Chow groups of them are torsion-free, we will not need such subtleties, but we will keep the notation from [15], and denote our maps as $\phi^{t^{i-m}}$.

3.2 Beyond the Theorem 3.1

Below we will need to study the relation between the rationality of $\overline{y}$ and $\overline{y}|_{\overline{k(Q)}}$ for $\mathrm{codim}(\overline{y})$ slightly bigger than $N_Q - d_Q$. The methods involved are just the same as are employed for the proof of Theorem 3.1.

Let Y be smooth quasiprojective variety, Q be smooth projective quadric. Let $v \in \mathrm{Ch}^*(Y \times Q)$ be some element, and $w \in \Omega^*(Y \times Q)$ be it's arbitrary lifting via pr. Over $\overline{k}$, quadric Q becomes a cellular variety with basis of Chow groups and Cobordisms given by the set $\{l_i, h^i\}_{0\leqslant i\leqslant d_Q}$ of projective subspaces and plane sections. This implies that

$$\mathrm{CH}^*(Y \times Q|_{\overline{k}}) = \oplus_{i=0}^{d_Q}(\mathrm{CH}^*(Y|_{\overline{k}}) \cdot l_i \oplus \mathrm{CH}^*(Y|_{\overline{k}}) \cdot h^i), \text{ and}$$

$$\Omega^*(Y \times Q|_{\overline{k}}) = \oplus_{i=0}^{d_Q}(\Omega^*(Y|_{\overline{k}}) \cdot l_i \oplus \Omega^*(Y|_{\overline{k}}) \cdot h^i)$$

- see [16, Section 2]. In particular,

$$\overline{v} = \sum_{i=0}^{d_Q}(\overline{v}^i \cdot h^i + \overline{v}_i \cdot l_i), \quad \text{and} \quad \overline{w} = \sum_{i=0}^{d_Q}(\overline{w}^i \cdot h^i + \overline{w}_i \cdot l_i).$$

Denote as $\widetilde{\mathrm{Ch}}^*$ the ring $\mathrm{CH}^*/(2, 2-torsion)$.

Proposition 3.5. *Let Q be a smooth projective quadric of dimension $\geqslant 4n - 1$, Y be a smooth quasiprojective variety, and $v \in \mathrm{Ch}^{2n+1}(Y \times Q)$ be some element. Then the class*

$$\overline{v}^0 + S^1(\overline{v}^1) + \overline{v}^1 \cdot \overline{v}_{N_Q-2n} + \overline{v}^0 \cdot \overline{v}_{N_Q-2n-1}$$

in $\widetilde{\mathrm{Ch}}^{2n+1}(Y|_{\overline{k}})$ is defined over k.

Corollary 3.6. *Let Q be a smooth projective quadric of dimension $\geqslant 4n-1$, Y be a smooth quasiprojective variety, and $\overline{y} \in \mathrm{Ch}^{2n+1}(Y|_{\overline{k}})$ is defined over $k(Q)$. Then, either*

(a) $z_{2n+1}^{\boxed{-d_Q}}(Q|_{k(Y)})$ *is defined; or*
(b) *for certain* $\overline{v}^1 \in \mathrm{Ch}^{2n}(Y|_{\overline{k}})$, *and for certain divisor* $\overline{v}_{N_Q-2n} \in \mathrm{Ch}^1(Y|_{\overline{k}})$, *the element*

$$\overline{y} + S^1(\overline{v}^1) + \overline{v}^1 \cdot \overline{v}_{N_Q-2n}$$

in $\widetilde{\mathrm{Ch}}^{2n+1}(Y|_{\overline{k}})$ *is defined over* k.

Proof. Since $\overline{y}$ is defined over $k(Q)$, there is $x \in \mathrm{Ch}^{2n+1}(Y|_{k(Q)})$ such that $\overline{x} = \overline{y}|_{\overline{k(Q)}}$. Using the surjection $\mathrm{CH}^*(Y \times Q) \twoheadrightarrow \mathrm{CH}^*(Y|_{k(Q)})$ lift the x to an element $v \in \mathrm{Ch}^{2n+1}(Y \times Q)$. Then $\overline{v} = \sum_{i=0}^{d_Q}(\overline{v}^i \cdot h^i + \overline{v}_i \cdot l_i)$, where $\overline{v}^0|_{\overline{k(Q)}} = \overline{y}|_{\overline{k(Q)}}$. But for any extension of fields $F/\overline{k}$ (with smaller one algebraically closed), the restriction morphism on Chow groups (with any coefficients) is injective by the specialisation arguments. Thus, $\overline{v}^0 = \overline{y}$. It remains to apply the Proposition 3.5, and observe that if $\overline{v}_{N_Q-2n-1} \in \mathrm{Ch}^0(Y|_{\overline{k}}) = \mathbb{Z}/2 \cdot 1$ is nonzero, then the class $l_{N_Q-2n-1} = z_{2n+1}^{\boxed{-d_Q}}$ is defined over $k(Y)$. Really, this class is just equal to $\overline{v}|_{\overline{k(Y)}}$. □

Remark 3.7. One can get rid of factoring $(2-torsion)$ in the statements above by using the genuine *symmetric operations* (see [15], cf. [14]) instead of the Landweber-Novikov operations. As was explained above, for our purposes it is irrelevant.

Before proving the Proposition let us study a bit some special power series. Denote as $\gamma(t) \in \mathbb{Z}/2[[t]]$ the power series $1 + \sum_{i \geqslant 0} t^{2^i}$. Then $\gamma(t)$ satisfies the equation:

$$\gamma^2 - \gamma = t,$$

and generates the quadratic extension of $\mathbb{Z}/2(t)$. In particular, for any $m \geqslant 0$, $\gamma^m = a_m\gamma + b_m$ for certain unique $a_m, b_m \in \mathbb{Z}/2(t)$. The following statement is clear.

Observation 3.8. (1) $a_{m+1} = a_m + b_m, \quad b_{m+1} = ta_m$
(2) a_m *and* b_m *are polynomials in* t *of degree* $\leqslant [m-1/2]$ *and* $[m/2]$, *respectively.*

For the power series $\beta(t)$ let us denote as $(\beta)_{\leqslant l}$ the polynomial $\sum_{j=0}^{l} \beta_j t^j$, and as $(\beta)_{>l}$ - the remaining part $\beta - (\beta)_{\leqslant l}$.

Lemma 3.9.

$$a_m = (\gamma^m)_{\leqslant [m/2]} = (\gamma^m)_{\leqslant [m-1/2]}$$

Proof. Let $m = 2^k + m_1$, where $0 \leqslant m_1 < 2^k$. Then $\gamma^m = \gamma^{2^k} \cdot \gamma^{m_1} = (a_{m_1}\gamma + b_{m_1}) + O(t^{2^k}) = (1 + \sum_{i=0}^{k-1} t^{2^i})a_{m_1} + b_{m_1} + O(t^{2^k})$. Observation 3.8 implies that $(\gamma^m)_{\leqslant [m/2]} = (1 + \sum_{i=0}^{k-1} t^{2^i})a_{m_1} + b_{m_1}$. On the other hand, $\gamma^{2^k} = \gamma + (\sum_{j=1}^{k-1} t^{2^j})$, thus γ^m is equal to

$$(\gamma + (\sum_{j=1}^{k-1} t^{2^j}))(a_{m_1}\gamma + b_{m_1}) =$$

$$a_{m_1}\gamma + a_{m_1}t + (\sum_{j=1}^{k-1} t^{2^j})a_{m_1}\gamma + b_{m_1}\gamma + (\sum_{j=1}^{k-1} t^{2^j})b_{m_1} =$$

$$((1 + \sum_{i=0}^{k-1} t^{2^i})a_{m_1} + b_{m_1})\gamma + (ta_{m_1} + (\sum_{j=1}^{k-1} t^{2^j})b_{m_1}).$$

Hence, $a_m = ((1 + \sum_{i=0}^{k-1} t^{2^i})a_{m_1} + b_{m_1}) = (\gamma^m)_{\leqslant[m/2]}$. The second equality follows from Observation 3.8(2) $\square$

$\square$

Lemma 3.9 implies that

$$\gamma^m = (\gamma^m)_{\leqslant[m-1/2]} \cdot \gamma + t(\gamma^{m-1})_{\leqslant[m-2/2]}.$$

Lemma 3.10.

$$(\gamma^m)_{>[m/2]} = t^m(1 + mt) + O(t^{m+2})$$

Proof. Use induction on m and on the number of 1's in the binary presentation of m. For $m = 2^k$ the statement is clear. Let now $m = 2^k + m_1$, where $0 < m_1 < 2^k$. We have: $(\gamma^m)_{>[m/2]} = ((\gamma^{m+1})_{\leqslant[m/2]})_{>[m/2]} + t((\gamma^m)_{\leqslant[m-1/2]} \cdot \gamma^{-1})_{>([m/2]-1)} = t(a_m \cdot \gamma^{-1})_{>[m/2]-1}$.

$a_m = (\gamma^m)_{\leqslant[m/2]} = (\gamma^{2^k} \cdot \gamma^{m_1})_{\leqslant[m/2]} = (\gamma^{m_1})_{\leqslant[m/2]} = (a_{m_1}\gamma + b_{m_1})_{\leqslant[m/2]}$, and since degrees of a_{m_1} and b_{m_1} are no more than $[m_1/2]$, this expression should be equal to $\gamma^{m_1} + a_{m_1}t^{2^k} + O(t^{2^{k+1}})$. Then

$$a_m \cdot \gamma^{-1} = \gamma^{m_1-1} + a_{m_1}\gamma^{-1}t^{2^k} + O(t^{2^{k+1}}) =$$

$$(a_{m_1-1}\gamma + b_{m_1-1}) + a_{m_1}\gamma^{-1}t^{2^k} + O(t^{2^{k+1}}) =$$

$$(a_{m_1-1}(1 + \sum_{i=0}^{k-1} t^{2^i}) + b_{m_1-1}) + t^{2^k}(a_{m_1-1} + a_{m_1}\gamma^{-1}) + O(t^{2^{k+1}}).$$

Since the degree of a_{m_1-1} is no more than $[m_1/2] - 1$, using Observation 3.8(1), we get:

$$(a_m \cdot \gamma^{-1})_{>[m/2]-1} = t^{2^k}(a_{m_1-1} + a_{m_1}\gamma^{-1}) + O(t^{2^{k+1}}) =$$

$$t^{2^k}(\gamma^{m_1-2} + a_{m_1-1}\gamma^{-1}) + O(t^{2^{k+1}}) =$$

$$t^{2^k}\gamma^{-1}(\gamma^{m_1-1} + (\gamma^{m_1-1})_{\leqslant[m_1-1/2]}) + O(t^{2^{k+1}}).$$

Consequently, $(\gamma^m)_{>[m/2]} = t^{2^k+1}(\gamma^{m_1-1})_{>[m_1-1/2]} \cdot \gamma^{-1} + O(t^{2^{k+1}})$. And, by the inductive assumption, this is equal to

$$t^{2^k+1}(t^{m_1-1}(1 + (m_1-1)t))\gamma^{-1} + O(t^{2^k+m_1+2}) = t^m(1 + mt) + O(t^{m+2}).$$

□

□

Corollary 3.11.

$$(a_m \cdot \gamma^{-1})_{>[m/2]-1} = t^{m-1}(1+mt) + O(t^{m+1})$$

Observe now that $\gamma^{-1}(t) = \sum_{i\geqslant 0} t^{2^i-1}$. Denote as $\delta(t)$ the polynomial $a_{2n+1}(t)$. Then

$$\delta(t)\gamma^{-1}(t) = \alpha(t) + t^{2n} + t^{2n+1} + O(t^{2n+2}), \tag{1}$$

where $\delta(t)$ and $\alpha(t)$ are polynomials of degree $\leqslant n$. Observation 3.8(1) shows that $\delta = 1 + t + \ldots$. For us it will be important that $\delta(t)\gamma^{-1}(t)$ does not contain monomials of degrees from $(n+1)$ to $(2n-1)$, but contains t^{2n} and t^{2n+1}.

Proof (of Proposition 3.5). The idea of the proof is the following: having some element $v \in \mathrm{Ch}^{2n}(Y \times Q)$ we first lift it via *pr* to some $w \in \Omega^*(Y \times Q)$, then restrict w to $Y \times Q_s$ for various subquadrics Q_s of Q, and apply to these restrictions the combination of the *symmetric operations* ϕ^{t^i} and $(\pi_{Y,s})_*$ (see below) in different order. The point is, that by adding the results with the appropriate coefficients one can get the expression in question. In particular, all the choices made while lifting to Ω^* will be cancelled out. And the needed coefficients are provided by the power series $\delta(t)$ above.

The case of $\dim(Q) \geqslant 4n-1$ can be reduced to that of $\dim(Q) = 4n-1$ by considering arbitrary subquadric $Q' \subset Q$ of dimension $4n-1$, and restricting v to $Y \times Q'$. So, we will assume that $\dim(Q) = 4n-1$.

Let $Q_s \stackrel{e_s}{\to} Q$ be arbitrary smooth subquadric of Q of dimension s. Denote as $w(s)$ the class $(id \times e_s)^*(w) \in \Omega^{2n+1}(Q_s \times Y)$. Then

$$\overline{w(s)} = \sum_{0\leqslant i\leqslant \min(2n-1,s)} \overline{w}^i \cdot h^i + \sum_{4n-s-1\leqslant j\leqslant 2n-1} \overline{w}_j \cdot l_{j-4n+s+1}.$$

Let $\pi_{Y,s} : Q_s \times Y \to Y$ be the natural projections.

Consider the element

$$u := (\pi_{Y,2n+1})_*\phi^{t^0}(w(2n+1)) + \sum_{p=n+1}^{2n+1} \delta_{2n+1-p}\phi^{t^{2p-(2n+1)}}(\pi_{Y,p})_*(w(p))$$

in $\mathrm{Ch}^{2n+1}(Y)$, where δ_j are the coefficients of the power series δ above. Let us compute $\overline{u}$. Since we are computing modulo 2-torsion, it is sufficient to compute $2\overline{u}$, which is equal to the Chow-trace of

$$(\pi_{Y,2n+1})_*(S^{2n+1}_{L-N} - \Box)(\overline{w}(2n+1)) + \sum_{p=n+1}^{2n+1} \delta_{2n+1-p} S^p_{L-N}(\pi_{Y,p})_*(\overline{w}(p)).$$

Using multiplicative properties of the Landweber-Novikov operations:

$$S^a_{L-N}(x \cdot y) = \sum_{b+c=a} S^b_{L-N}(x) S^c_{L-N}(y),$$

and Proposition 3.3, we get (*modulo* 4):

$$pr(\pi_{Y,2n+1})_* S^{2n+1}_{L-N}(\overline{w}(2n+1)) = \sum_{j=0}^{2n-1} \binom{j}{2n-j+1} \cdot 2 \cdot S^j(\overline{v}^j) +$$
$$pr(\binom{2n+1}{1} \cdot S^{2n}_{L-N}(\overline{w}_{2n-1}) + \binom{2n+2}{0} \cdot S^{2n+1}_{L-N}(\overline{w}_{2n-2})).$$

Codimension of $\overline{v}^j$ is $2n+1-j$, thus either $\binom{j}{2n-j+1}$ is zero, or $S^j(\overline{v}^j)$ is, and our expression is equal to $pr(S^{2n}_{L-N}(\overline{w}_{2n-1}) + S^{2n+1}_{L-N}(\overline{w}_{2n-2}))$. Also, (*modulo* 4),

$$pr(\pi_{Y,2n+1})_* \square(\overline{w}(2n+1)) = 2 \cdot pr(\overline{w}^0 \overline{w}_{2n-2} + \overline{w}^1 \overline{w}_{2n-1}).$$

In the same way, (*modulo* 4),

$$prS^p_{L-N}(\pi_{Y,p})_*(\overline{w}(p)) = \sum_{j=0}^{min(2n-1,p)} \binom{-(p+2-j)}{p-j} \cdot 2 \cdot S^j(\overline{v}^j) +$$
$$pr(\sum_{i=0}^{p-2n} \binom{-(i+1)}{i} S^{p-i}_{L-N}(\overline{w}_{i+4n-1-p})).$$

Observe, that the second sum is empty for $p < 2n$, is equal (*modulo* 4), to $prS^{2n}_{L-N}(\overline{w}_{2n-1})$ for $p = 2n$, and to $prS^{2n+1}_{L-N}(\overline{w}_{2n-2})$ for $p = 2n+1$ (we used here Corollary 3.4).

Since the coefficient $\binom{-(l+2)}{l}$ is odd if and only if $l = 2^k - 1$, for some k, the first sum is equal to:

$$2 \sum_{j=0}^{min(2n-1,p)} (\gamma^{-1})_{p-j} \cdot S^j(\overline{v}^j).$$

Taking into account that $\delta(t) = 1 + t + \ldots$, we get:

$$pr \sum_{p=n+1}^{2n+1} \delta_{2n+1-p} S^p_{L-N}(\pi_{Y,p})_*(\overline{w}(p)) =$$
$$2 \sum_{p=n+1}^{2n+1} \sum_{j=0}^{min(2n-1,p)} \delta_{2n+1-p}(\gamma^{-1})_{p-j} \cdot S^j(\overline{v}^j) +$$
$$(prS^{2n}_{L-N}(\overline{w}_{2n-1}) + prS^{2n+1}_{L-N}(\overline{w}_{2n-2})) =$$
$$2 \sum_{j=0}^{2n-1} (\delta \cdot \gamma^{-1})_{2n+1-j} S^j(\overline{v}^j) + (prS^{2n}_{L-N}(\overline{w}_{2n-1}) + prS^{2n+1}_{L-N}(\overline{w}_{2n-2})) =$$
$$2(\overline{v}^0 + S^1(\overline{v}^1)) + (prS^{2n}_{L-N}(\overline{w}_{2n-1}) + prS^{2n+1}_{L-N}(\overline{w}_{2n-2})),$$

in the light of formula (1) and Corollary 3.4.

Putting things together (and again using Corollary 3.4), we obtain:

$$2\overline{u} = 2(\overline{v}^0 + S^1(\overline{v}^1) + \overline{v}^1 \cdot \overline{v}_{2n-1} + \overline{v}^0 \cdot \overline{v}_{2n-2}).$$

Since u is defined over the base-field k, the Proposition is proven. □

There is another result which extends a bit Theorem 3.1.

Proposition 3.12. ([14, Statement 3.8]) *Let Y be smooth quasiprojective variety, Q smooth projective quadric over k. Let $\overline{y} \in \mathrm{Ch}^m(Y|_{\overline{k}})$. Suppose $z_{N_Q-d_Q}^{\boxed{0}}(Q)$ is defined. Then for $m \leqslant N_Q - d_Q$,*

$$\overline{y}|_{\overline{k(Q)}} \text{ is defined over } k(Q) \ \Leftrightarrow\ \overline{y} \text{ is defined over } k.$$

Proposition 3.12 extends Theorem 3.1 in the direction of the following:

Conjecture 3.13. ([14, Conjecture 3.11]) *In the notations of Theorem 3.1, suppose $z_l^{\boxed{N_Q-d_Q-l}}(Q)$ is defined. Then for any $m \leqslant l$,*

$$\overline{y}|_{\overline{k(Q)}} \text{ is defined over } k(Q) \ \Leftrightarrow\ \overline{y} \text{ is defined over } k.$$

This conjecture is known for $l = N_Q - d_Q, N_Q - 1, N_Q$.

3.3 Some auxiliary facts

For our purposes it will be important to be able (under certain conditions) to get rid of the last term in the formula from Proposition 3.5. For this we will need the following facts.

Proposition 3.14. *Let $0 \leqslant i \leqslant d_R$, and $F(R,i) \xleftarrow{\alpha} F(R,0,i) \xrightarrow{\beta} R$ be the natural projections. Let $z_{N_R-i}^{\boxed{i-d_R}}$ is defined. Let $t \in \mathrm{Ch}_{N_R-i}(F(R,i))$ be such that $\beta_*\alpha^*(t) = 1 \in \mathrm{Ch}^0(R)$. Then $\beta_*\alpha^*(t \cdot z_{N_R-i}^{\boxed{i-d_R}}) = l_i \in \mathrm{Ch}_i(R)$.*

Proof. Really, by the definition, $z_{N_R-i}^{\boxed{i-d_R}} = \alpha_*\beta^*(l_0)$. By the projection formula,

$$\alpha_*\beta^*(t \cdot z_{N_R-i}^{\boxed{i-d_R}}) = \beta_*\alpha^*(t \cdot z_{N_R-i}^{\boxed{i-d_R}}) = \beta_*\alpha^*\alpha_*(\alpha^*(t) \cdot \beta^*(l_0)).$$

Again, by the projection formula, $\beta_*(\alpha^*(t) \cdot \beta^*(l_0)) = l_0$. Thus, $\alpha^*(t) \cdot \beta^*(l_0)$ is a zero-cycle of degree 1 on $F(R,0,i)$, and $\alpha_*(\alpha^*(t) \cdot \beta^*(l_0))$ is a zero cycle of degree 1 on $F(R,i)$. Proposition follows. □

□

Let $v \in \mathrm{Ch}^m(Y \times Q)$ be some element. Then

$$\overline{v} = \sum_{i=0}^{d_Q} (\overline{v}^i \cdot h^i + \overline{v}_i \cdot l_i).$$

Statement 3.15. *Suppose* $z_m^{\boxed{N_Q-m-d}}(Q)$ *is defined. Then for any* v *as above, there exists* $u \in \mathrm{Ch}^m(Y \times Q)$ *such that* $\overline{u}^0 = \overline{v}^0$*, and* $\overline{u}_{N_Q-m} = 0$.

Proof. If $\overline{v}_{N_Q-m} = 0$, there is nothing to prove. Otherwise, the class $l_{N_Q-m} \in \mathrm{Ch}_{N_Q-m}(Q|_{k(Y)})$ is defined. Indeed, let

$$\rho_X : \mathrm{Ch}^*(Y \times X) \twoheadrightarrow \mathrm{Ch}^*(X|_{k(Y)})$$

be the natural restriction. Then $\rho_Q(\overline{v}) = l_{N_Q-m}$ plus $\lambda \cdot h^m$, if $2m = N_Q$ (notice, that $\overline{v}_{N_Q-m} \in \mathrm{Ch}^0$). This implies that the class $l_{N_Q-m} = z_{N_Q-m}^{\boxed{-d_Q}}$ is defined on $Q|_{k(Y)}$. Using Proposition 2.5 and Statement 2.15, we get that the class of a rational point is defined on $F(Q, N_Q - m)|_{k(Y)}$ (this proof is somewhat longer than the standard one, but it does not use the Theorem of Springer (see [5])!). Let $x \in \mathrm{Ch}_{\dim(Y)}(F(Q, N_Q - m) \times Y)$ be arbitrary lifting of this class with respect to $\rho_{F(Q,N_Q-m)}$. Let

$$F(Q, N_Q - m) \xleftarrow{\alpha} F(Q, 0, N_Q - m) \xrightarrow{\beta} Q$$

be the natural projections. Consider $u' := (\beta \times id)_*(\alpha \times id)^*(x) \in \mathrm{Ch}^m(Q \times Y)$. Proposition 3.14 implies that the (defined over k) cycle

$$u" := \pi_Y^*(\pi_Y)_*((h^{N_Q-m} \times 1_Y) \cdot (\beta \times id)_*(\alpha \times id)^*(x \cdot z_m^{\boxed{N_Q-m-d}}(Q)))$$

satisfy: $\overline{u"}^0 = \overline{u'}^0$, and (evidently) $\overline{u"}_{N_Q-m} = 0$. Since $\overline{u'}_{N_Q-m} = 1 = \overline{v}_{N_Q-m}$, it remains to take: $u := v - u' + u"$. □

□

4 Even u-invariants

The fields of any given even u-invariant were constructed by A.Merkurjev in [9] using his *index-reduction formula* for central simple algebras. The idea of such construction is based on the, so-called, *Merkurjev tower of fields*, which was first used in [8]. In our case, this tower is constructed as follows: let F be any field, and let S_F be the set of all (isomorphism classes of) quadrics over F of dimesion $> (M - 2)$. Let $F' := \lim_{I \subset S_F} F(\times_{i \in I} Q_i)$, where the limit is taken over all finite subsets of S_F via the natural restriction maps. Then starting from arbitrary field k one constructs the tower of fields

$k = k_0 \to k_1 \to \ldots \to k_r \to \ldots$, where $k_{r+1} := (k_r)'$. One gets huge field $k_\infty := \lim_r k_r$ having the property that all forms of dimension $> M$ over it are isotropic, and thus $u(k_\infty) \leqslant M$. But to get a field whose u-invariant is exactly M, one has to start with some special field k, and since one wants to have some anisotropic M-dimensional form p over k_∞, better to have it already over k, and then check that p will not become isotropic while passing from k to k_∞. Of course, to be able to control this, we need to know something interesting about p. That is, we need to control some other property which implies ours. More precisely, for a given base-field k, on the set of field extensions E/k we should define two properties: A and B, where

$$A(E) \text{ is satisfied } \Leftrightarrow \ p|_E \text{ is anisotropic,}$$

so that the following conditions are satisfied:

(1) $B \Rightarrow A$;
(2) $B(E) \Rightarrow B(E(Q))$, for arbitrary quadric Q/E of dimension $> \dim(P)$;
(3) Let $\{E_j\}_{j\in J}$ be the directed system of field extensions with the limit E_∞. Then $B(E_j)$ for all j implies $B(E_\infty)$.

Then $B(k) \Rightarrow A(k_\infty)$. So, if one finds quadratic form p of dimension M over k and some property B satisfying the above conditions (and such that $B(k)$ is satisfied), then k_∞ will have u invariant M.

In the case $M = 2n$ is even A.Merkurjev takes $p \in I^2$ and the following property B:

$$B(E) \text{ is satisfied } \Leftrightarrow \ C_0^+(p|_E) \text{ is a division algebra,}$$

where $C_0^+(p)$ is the "half" of the *even Clifford algebra* $C_0(p) = C_0^+(p) \times C_0^-(p)$ (both factors are isomorphic here). Of course, for $B(k)$ to be satisfied one has to start with some form p for which C_0^+ is division over the base field. The *generic* form from I^2 (that is, the form $\langle a_1, \ldots, a_{2n-1}, (-1)^n a_1 \cdot \ldots \cdot a_{2n-1}\rangle / k = k_0(a_1, \ldots, a_{2n-1})$) will do the job. The condition (1) is satisfied since the isotropy of $p|_E$ gives the matrix factor in $C_0^+(p|_E)$ ([5]), and the condition (3) is clear. The only nontrivial fact here is the condition (2), which follows from the index reduction formula of Merkurjev, claiming that over the generic point of a quadric Q the index of a division algebra D can drop at most by the factor 2, and the latter happens if and only if $C_0(q)$ can be mapped to D. Indeed, if p is of dimension $2n$, then $C_0^+(p)$ is a central simple algebra of rank 2^{n-1}, and $C_0(q)$ is either a simple algebra, or a product of two simple algebras of large rank (this will not be true for odd-dimensional p!), thus there is no ring homomorphisms $C_0(q) \to C_0^+(p)$, and the index of $C_0^+(p|_E)$ is equal to that of $C_0^+(p|_{E(Q)})$.

Let me give another construction, which does not use the *index reduction formula*. Instead, I will use the North-West corner of the *EDI* and the property:

$$B(E) \text{ is satisfied } \Leftrightarrow \ z^{\boxed{0}}_{N_P - d_P}(P|_E) \text{ is not defined,}$$

Statement 4.1. *Let $M = \dim(p)$ is even. Then our property B satisfies the above conditions* $(1) - (3)$.

Proof. The condition (1) follows from Proposition 2.5, since the property $A(E)$ is satisfied $\Leftrightarrow$ the South-West corner of the EDI is not defined for $P|_E$. The condition (3) is clear, since $\mathrm{CH}^*(X|_{E_\infty}) = \lim_j \mathrm{CH}^*(X|_{E_j})$. Finally, suppose $z_{n-1}^{\boxed{0}}(P|_E)$ is not defined. Then, by Theorem 3.1, for any form q of dimension $> M$, $z_{n-1}^{\boxed{0}}(P|_{E(Q)})$ is not defined as well (will not work for M - odd). Thus, the condition (2) is satisfied. □

□

Corollary 4.2. (A.Merkurjev, [9]) *For each $M = 2n$ there is a field of u-invariant M.*

Proof. Take any form p/k of dimension M such that $z_{n-1}^{\boxed{0}}(P)$ is not defined. One can use the generic form - see [14, Statement 3.6]. Then $B(k)$ is satisfied and, hence, $A(k_\infty)$ is satisfied too. □

□

5 Odd u-invariants

Let us analyse a bit the above construction. Instead of working with the cycle $z_{N_P}^{\boxed{-d_P}}$ - the class of a rational point on a quadric P, we worked with the (smaller codimensional!) cycle $z_{N_P-d_P}^{\boxed{0}}$, and used the fact that rationality of the former implies rationality of the latter (Proposition 2.5).

Unfortunately, for odd-dimensional forms we can not use the class $z_{N_P-d_P}^{\boxed{0}}$. Really, if p is any such form, then for $q := p \perp \langle det_\pm(p)\rangle$, $z_{N_P-d_P}^{\boxed{0}}(P|_{k(Q)})$ will be defined, since the rationality of this class is equivalent to the rationality of $z_{N_Q-d_Q}^{\boxed{0}}(Q|_{k(Q)})$ - observe that

$$G(Q, d_Q) = G(P, d_P) \coprod G(P, d_P),$$

and the rationality of the latter follows from the rationality of the class $z_{N_Q}^{\boxed{-d_Q}}(Q|_{k(Q)})$ (isotropy of $Q|_{k(Q)}$). So, even if we start from the form, where our class is not defined, over the generic point of some bigger-dimensional form it will become rational, and we can not control anisotropy of P.

But the rationality of $z_{N_P}^{\boxed{-d_P}}$ implies rationality not just of $z_{N_P-d_P}^{\boxed{0}}$, but of all the West edge $z_{N_P-d_P+s}^{\boxed{-s}}$, $0 \leqslant s \leqslant d_P$. So, let us use these other cycles.

Let the form p has dimension 2^r+1. In this case, one can use previous to the last grassmannian, and the class $z^{\boxed{-1}}_{2^{r-1}+1}$ on it.

Theorem 5.1. *Let* $\dim(p) = 2^r + 1$*,* $r \geqslant 3$*, and* $EDI(P)$ *looks as*

$$\begin{array}{cccc} ? & \circ & \cdots & \circ \\ \circ & \circ & \cdots & \circ \\ \cdots & \cdots & \cdots & \cdots \\ \circ & \circ & \cdots & \circ \end{array}$$

Let $\dim(q) > \dim(p)$*. Then* $EDI(P|_{k(Q)})$ *has the same property.*

Corollary 5.2. *For any* $r \geqslant 3$ *there is a field of* u*-invariant* $2^r + 1$*.*

Proof. Start with the *generic form* p over $k = k_0(a_1, \ldots, a_{2^r+1})$ and the property

$$B(E) \text{ is satisfied} \quad \Leftrightarrow \quad EDI(P|_E) \text{ is as in the Theorem 5.1,}$$

Then $EDI(P)$ is empty. This follows from Proposition 2.5 and [14, Statement 3.6]. Thus, $B(k)$ is satisfied. The condition (1) is satisfied by the definition of A and B. The condition (3) is satisfied since $\mathrm{CH}^*(X|_{E_\infty}) = \lim_j \mathrm{CH}^*(X|_{E_j})$. And the condition (2) is equivalent to the Theorem 5.1. Then, as we know, $A(k_\infty)$ is satisfied, and $u(k_\infty) = 2^r + 1$. □

□

Proof (of Theorem 5.1). Let $d := d_P = 2^{r-1} - 1$. It follows from Theorem 3.1 that the cycles $z^{\boxed{0}}_j(P|_{k(Q)})$, $1 \leqslant j \leqslant d$ are not defined. That is, we have $\circ$'s to the right of ?. In the light of Proposition 2.5 it remains only to treat the case of $z^{\boxed{-1}}_{d+2}(P|_{k(Q)})$ (that is, the node just below the ?).

This is done as follows. If this cycle is defined over $k(Q)$, we can lift it to a cycle v on $F(P, d-1) \times Q$. Over $\overline{k}$ quadric Q becomes cellular, and our cycle decomposes in a standard way, producing coordinates $\overline{v}^i, \overline{v}_i$. Using the fact that $dim(q) = 2^r + 1, r \geqslant 3$ one can correct v in such a way that the "last" of these coordinates will be zero. This is done by some play with elementary classes using Theorem 3.1, Proposition 3.12, Proposition 2.9 and other major statements, and is the most delicate part of the proof (in particular, it is the only place where the high specific of the dimension is used - everything else works for $dim(q) \equiv 1\,(mod\,4)$). After this is achieved, one can apply Proposition 3.5, and get a k-rational class on $F(P, d-1)$ given by the sum of 4 terms, the last of which will be zero because of our choice of v. Now, from the knowledge of the action of the Steenrod operations it is not difficult to prove that the obtained k-rational class is nonzero. Finally, we use the information about elementary classes on $F(P, d)$ and the main result of [13] to conclude that our nonzero k-rational class on $F(P, d-1)$ should be $z_{d+2}^{\boxed{-1}}$. This contradiction proves the Theorem.

In more details, suppose, $z_{d+2}^{\boxed{-1}}(P|_{k(Q)})$ is defined. We clearly can assume that $\dim(q) = \dim(p) + 1 = 2^r + 2$. Let us denote $F(P, d-1)$ temporarily as Y. We have $y \in \mathrm{Ch}^{d+2}(Y|_{k(Q)})$ such that $\overline{y} = z_{d+2}^{\boxed{-1}} \in \mathrm{Ch}^{d+2}(Y|_{\overline{k(Q)}})$. Let us lift it to $v \in \mathrm{Ch}^{d+2}(Y \times Q)$ via the natural projection $\mathrm{Ch}^{d+2}(Y \times Q) \overset{\rho_Y}{\twoheadrightarrow} \mathrm{Ch}^{d+2}(Y|_{k(Q)})$.

Statement 5.3. *There exists such $v \in \mathrm{Ch}^{d+2}(Y \times Q)$ that $\overline{v}^0 = \overline{y}$, and $\overline{v}_d = 0 \in \mathrm{Ch}^0(Y|_{\overline{k}})$.*

Proof. Let v be arbitrary lifting of y with respect to ρ_Y. If $\overline{v}_d = 0$, there is nothing to prove. Otherwise, let us show that $z_{2^{r-1}+1}^{\boxed{-1}}(Q)$ is defined over k.

Suppose $\overline{v}_d \neq 0$, then it is equal to $1 \in \mathrm{Ch}^0(Y|_{\overline{k}})$. Then $\overline{\rho_Q(v)} = l_d \in \mathrm{Ch}_d(Q|_{\overline{k(Y)}}) = \mathbb{Z}/2 \cdot l_d$. Thus, $z_{d+2}^{\boxed{-d_Q}}(Q|_{k(Y)})$ is defined. By Proposition 2.5, $z_3^{\boxed{-2}}(Q|_{k(Y)})$ is defined too (it lives in the same column above). We want to show that $z_3^{\boxed{-2}}(Q)$ is defined.

Consider the two towers of fibrations:

$$\begin{aligned} &\mathrm{Spec}(k) \leftarrow P \leftarrow \ldots \leftarrow F(P, 0, 1, \ldots, d-1); \\ &\mathrm{Spec}(k) \leftarrow Q \leftarrow \ldots \leftarrow F(Q, 0, 1, \ldots, d-1), \end{aligned}$$

with the generic fibers - quadrics $P = P_1, \ldots, P_d$, $Q = Q_1, \ldots, Q_d$ of dimension $2d+1, 2d-1, \ldots, 3$, and $2d+2, 2d, \ldots, 4$, respectively. Let us denote $k_{a,b} := k(F(P, 0, \ldots, a-1) \times F(Q, 0, \ldots, b-1))$. Then

$$k_{a+1,b} = k_{a,b}(P_a) \quad \text{and} \quad k_{a,b+1} = k_{a,b}(Q_b).$$

Since we have embeddings of fields

$$k \subset k(Y) = k(F(P, d-1)) \subset k(F(P, 0, \ldots, d-1)) = k_{d,0},$$

$z_3^{\boxed{-2}}(Q|_{k_{d,0}})$ is defined. Then by Proposition 2.5, $z_1^{\boxed{0}}(Q|_{k_{d,0}})$ is defined. By Theorem 3.1, $z_1^{\boxed{0}}(Q)$ and $z_2^{\boxed{0}}(Q) = (z_1^{\boxed{0}}(Q))^2 = S^1(z_1^{\boxed{0}}(Q))$ are defined.

It follows from Corollary 2.14 that for arbitrary elements $\overline{\alpha}, \overline{\beta} \in \mathrm{Ch}^*(F(Q, d-1)|_{\overline{k_{a-1,0}}})$ of codimension 2 and 1, respectively, the class $S^1(\overline{\alpha}) + \overline{\alpha} \cdot \overline{\beta}$ is defined over any field, where Q is defined, in particular, over $k_{a-1,0}$. It follows from Corollary 3.6 that

$$z_3^{\boxed{-2}}(Q|_{k_{a,0}}) \text{ is defined } \Rightarrow \begin{cases} \text{either} & z_3^{\boxed{-2}}(Q|_{k_{a-1,0}}) \text{ is defined;} \\ \text{or} & z_3^{\boxed{-d_{P_a}}}(P_a|_{k_{a-1,d}}) \text{ is defined,} \\ & \text{and } d_{P_a} \leqslant 3. \end{cases}$$

Let us show that the second case is impossible. Really, by Proposition 2.5,

$$z_3^{\boxed{-d_{P_a}}}(P_a|_{k_{a-1,d}}) \text{ is def. } \Rightarrow z_{3-d_{P_a}}^{\boxed{0}}(P_a|_{k_{a-1,d}}) \text{ is def.}$$

Since $3 - d_{P_a} \leqslant 2$, $\dim(Q_b) \geqslant 4$, and $z_2^{\boxed{0}}(Q_d)$ is defined, by Proposition 3.12 and Theorem 3.1,

$$z_{3-d_{P_a}}^{\boxed{0}}(P_a|_{k_{a-1,b}}) \text{ is def. } \Rightarrow z_{3-d_{P_a}}^{\boxed{0}}(P_a|_{k_{a-1,b-1}}) \text{ is def.}$$

Then $z_{3-d_{P_a}}^{\boxed{0}}(P_a/k_{a-1,0})$ is defined, and $z_{3-d_{P_a}}^{\boxed{0}}(P)$ is defined (by Theorem 3.1). This contradicts to the conditions of our Theorem (here we are using the fact that $r \geqslant 3$). Thus,

$$z_3^{\boxed{-2}}(Q|_{k_{a,0}}) \text{ is defined } \Rightarrow z_3^{\boxed{-2}}(Q|_{k_{a-1,0}}) \text{ is defined,}$$

and, consequently, $z_3^{\boxed{-2}}(Q)$ is defined. By Proposition 2.5, $z_3^{\boxed{-1}}(Q)$ is defined too. Proposition 2.9 implies that

$$z_{2^{r-1}+1}^{\boxed{-1}}(Q) = S^{2^{r-2}} S^{2^{r-3}} \dots S^2(z_3^{\boxed{-1}}(Q))$$

is also defined over k. Since $z_{2^{r-1}+1}^{\boxed{-1}}(Q)$ is defined, everything follows from Statement 3.15. $\square$

$\square$

Consider $v \in \mathrm{Ch}^{d+2}(Y \times Q)$ satisfying the conditions of Statement 5.3. As above, $\overline{v} = \sum_{i=0}^{2^{r-1}} (\overline{v}^i \cdot h^i + \overline{v}_i \cdot l_i)$. Then, by Proposition 3.5, the class $\overline{v}^0 + S^1(\overline{v}^1) + \overline{v}^1\overline{v}_{2^{r-1}} + \overline{v}^0\overline{v}_{2^{r-1}-1}$ is defined over k. But $\overline{v}_{2^{r-1}-1} = 0$. Thus on Y we have the class $\overline{v}^0 + S^1(\overline{v}^1) + \overline{v}^1\overline{v}_{2^{r-1}}$ defined over k.

Now it is time to use the specific of Y and v. Our Y is a grassmannian $F(P, d-1)$. In particular, it is a geometrically cellular variety, and the map $\mathrm{Ch}^*(Y|_{\overline{k}}) \to \mathrm{Ch}^*(Y|_{\overline{k(Q)}})$ is an isomorphism. Thus, $\overline{v}^0 = z_{d+2}^{\boxed{-1}}$. On the other hand, $\overline{v}_{2^{r-1}} = \overline{v}_{d+1}$ belongs to $\mathrm{Ch}^1(Y|_{\overline{k}})$, and so is equal either to 0, or to $w_1^{\boxed{-1}}$. So, on $F(P, d-1)$ we have class either of the form $z_{d+2}^{\boxed{-1}} + S^1(\overline{v}^1)$, or of the form $z_{d+2}^{\boxed{-1}} + S^1(\overline{v}^1) + \overline{v}^1 w_1^{\boxed{-1}}$ defined over k. The following Statement shows that such class should be nonzero.

Statement 5.4. *Let R be a smooth projective quadric of dimension $4n-1$. Then $z^{\boxed{-1}}_{N_R-d_R+1}(R)$ belongs to the image of neither of two maps:*

$$S^1, (S^1 + w^{\boxed{-1}}_1 \cdot \quad) : \mathrm{Ch}^{N_R-d_R}(F(R, d_R-1)) \to \mathrm{Ch}^{N_R-d_R+1}(F(R, d_R-1))$$

Proof. We can assume that $k = \overline{k}$. Consider the natural projections:

$$F(R, d_R-1) \stackrel{\alpha}{\leftarrow} F(R, d_R-1, d_R) \stackrel{\beta}{\rightarrow} F(R, d_R).$$

The map β provides $F(R, d_R-1, d_R)$ with the structure of the projective bundle $\mathbb{P}_{F(R,d_R)}(\mathcal{T}^{\vee}_{d_R})$, and the Chern classes of $\mathcal{T}_{d_R}$ are divisible by 2 - see Proposition 2.1 (and [13]). Thus, $\mathrm{Ch}^*(F(R, d_R-1, d_R)) = \mathrm{Ch}^*(F(R, d_R))[h]/(h^{d_R+1})$, where $h = c_1(\mathcal{O}(1))$. By Lemma 2.6,

$$\alpha^*(z^{\boxed{-1}}_{N_R-d_R+1}) = \beta^*(z^{\boxed{0}}_{N_R-d_R}) \cdot h, \text{ and } h = \alpha^*(w^{\boxed{-1}}_1).$$

The first fact now is simple, since

$$S^1(\alpha^*(z^{\boxed{-1}}_{N_R-d_R+1})) = S^1(\beta^*(z^{\boxed{0}}_{N_R-d_R}) \cdot h) = \beta^*(z^{\boxed{0}}_{N_R-d_R}) \cdot h^2,$$

by Proposition 2.9, and the latter element is nonzero. Thus, even $\alpha^*(z^{\boxed{-1}}_{N_R-d_R+1})$ can not be in the image of S^1, since $S^1 \circ S^1 = 0$.

To prove the second fact, observe that

$$\alpha^*(z^{\boxed{-1}}_{N_R-d_R+1}) = \beta^*(z^{\boxed{0}}_{N_R-d_R}) \cdot h = (S^1 + h \cdot \quad)(\beta^*(z^{\boxed{0}}_{N_R-d_R})).$$

Let $u \in \mathrm{Ch}^{N_R-d_R}(F(R, d_R-1))$ be such that $(S^1 + w^{\boxed{-1}}_1 \cdot \quad)(u) = z^{\boxed{-1}}_{N_R-d_R+1}$. Then $(S^1 + h \cdot \quad)(\beta^*(z^{\boxed{0}}_{N_R-d_R}) - \alpha^*(u)) = 0$. Since d_R is odd, the differential $(S^1 + h \cdot \quad)$ acts without cohomology on $\mathrm{Ch}^*(F(R, d_R))[h]/(h^{d_R+1})$. Consequently, $(\beta^*(z^{\boxed{0}}_{N_R-d_R}) - \alpha^*(u)) = (S^1 + h \cdot \quad)(w)$, for certain $w \in \mathrm{Ch}^{N_R-d_R-1}(F(R, d_R-1, d_R))$. This implies

$$\alpha_*\beta^*(z^{\boxed{0}}_{N_R-d_R}) = \alpha_*(S^1 + h \cdot \quad)(w),$$

since $\alpha_*\alpha^* = 0$. Notice that $\alpha : F(R, d_R-1, d_R) \to F(R, d_R-1)$ is a conic bundle with relative tangent sheaf $\alpha^*(\mathcal{O}(w^{\boxed{-1}}_1)) = \mathcal{O}(h)$. Thus, $\alpha_*(S^1 + h \cdot \quad)(w) = S^1(\alpha_*(w))$, and $\alpha^*\alpha_*(\beta^*(z^{\boxed{0}}_{N_R-d_R})) = S^1(\alpha^*\alpha_*(w))$. But $\alpha^*\alpha_*(\beta^*(z^{\boxed{0}}_{N_R-d_R})) = h^{N_R-d_R-1} = h^{d_R}$, and this element is not in the image of S^1, as one can easily see. The contradiction shows that u as above does not exist. □

□

It follows from the Statement 5.4 that in $\mathrm{Ch}^{d+2}(F(P, d-1)|_{\overline{k}})$ we have nonzero class x defined over k. Then $\alpha^*(x) \in \mathrm{Ch}^{d+2}(F(P, d-1, d)|_{\overline{k}})$ will be also nonzero class defined over k. But the subring of k-rational classes in $\mathrm{Ch}^*(F(P, d-1, d)|_{\overline{k}})$ is $GDI(P, d)[h]/(h^{d+1})$, and by the main result of [13], $GDI(P, d)$ as a ring is generated by the elementary classes $z_j^{\boxed{0}}$ contained in it. By the conditions of our Theorem, among such classes only $z_{d+1}^{\boxed{0}}$ could be defined over k. Then the degree $= (d+2)$ component of the subring of k-rational classes in $\mathrm{Ch}^*(F(P, d-1, d)|_{\overline{k}})$ is contained in $\mathbb{Z}/2 \cdot (\beta^*(z_{d+1}^{\boxed{0}}) \cdot h) = \mathbb{Z}/2 \cdot \alpha^*(z_{d+2}^{\boxed{-1}})$. Thus, if x is nonzero, it got to be $z_{d+2}^{\boxed{-1}}$. But this class is not defined over k by the condition of the Theorem. And the contradiction shows that the class $z_{d+2}^{\boxed{-1}}$ is not defined over $k(Q)$ as well. Theorem 5.1 is proven. $\square$

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