Exact Category of Modules of Constant Jordan type

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To Yuri Manin with admiration

Summary. For a finite group scheme G, we continue our investigation of those finite dimensional kG-modules which are of constant Jordan type. We introduce a Quillen exact category structure $\mathcal{C}(kG)$ on these modules and investigate $K_0(\mathcal{C}(kG))$. We study which Jordan types can be realized as the Jordan types of (virtual) modules of constant Jordan type. We also briefly consider thickenings of $\mathcal{C}(kG)$ inside the triangulated category $\mathrm{stmod}(kG)$.

1 Introduction

Together with Julia Pevtsova, the authors introduced in [6] an intriguing class of modules for a finite group G (or, more generally, for an arbitrary finite group scheme), the kG-modules of constant Jordan type. This class includes projective modules and endotrivial modules. It is closed under taking direct sums, direct summands, k-linear duals, and tensor products. We have several methods for constructing modules of constant Jordan type, typically using cohomological techniques. In addition, Andrei Suslin has introduced several interesting constructions which associate modules of constant Jordan type to an arbitrary finite dimensional kG-module in the special case that $G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.

What strikes us as remarkable is how challenging is the problem of classifying modules of constant Jordan type even for relatively simple finite group schemes. In this paper we address two other aspects of the theory that also present formidable challenges. The first is the realization problem of determining which Jordan types can actually occur for modules of constant Jordan type. The second question concerns stratification of the entire module category by modules of constant Jordan type.

To consider realization, we give the class of kG-modules of constant Jordan type the structure of a Quillen exact category $\mathcal{C}(kG)$ using "locally split short exact sequences." This structure suggests itself naturally once kG-modules are treated from the point of view of π-points as in [10], a point of view necessary to even define modules of constant Jordan type. With respect to this exact category structure, the Grothendieck group $K_0(\mathcal{C}(kG))$ arises as a natural invariant. There are natural Jordan type functions JType, $\overline{\text{JType}}$ defined on $K_0(\mathcal{C}(kG))$ which are useful to formulate questions of realizability of (virtual) modules of constant Jordan type. The reader will find several results concerning the surjectivity of these functions.

A seemingly very difficult goal is the classification of kG-modules of constant Jordan type, or at least the determination of $K_0(\mathcal{C}(kG))$. In this paper, we provide a calculation of $K_0(\mathcal{C}(kG))$ for two very simple examples: the Klein four group and the first infinitesimal kernel of SL_2.

The category $\mathcal{C}(kG)$ possesses many closure properties. However, the complexity of this category is reflected in the observation that an extension of modules of constant Jordan type need not be of constant Jordan type. We conclude this paper by a brief consideration of a stratification of the stable module category stmod(kG) by "thickenings" of $\mathcal{C}(kG)$.

We are very grateful to Julia Pevtsova and Andrei Suslin for many discussions. This paper is part of a longer term project which will reflect their ideas and constructions.

2 The exact category $\mathcal{C}(kG)$

As shown in [6], there is a surprising array of kG-modules of constant Jordan type. One evident way to construct new examples out of old is to use locally split extensions (see, for example, Proposition 2.4). In order to focus on examples which seem more essential, we introduce in Definition 2.3 the Quillen exact category $\mathcal{C}(kG)$ of modules of constant Jordan type whose admissible short exact sequences are those which are locally split.

We begin by recalling the definition of a π-point of a finite group scheme over k. This is a construction that is necessary to formulate the concept of modules of constant Jordan type.

Definition 2.1. *Let G be a finite group scheme with group algebra kG (the linear dual of the coordinate algebra $k[G]$). A π-point of G is a map of K-algebras $\alpha_K : K[t]/t^p \to KG_K$ which is left flat and which factors through some abelian unipotent subgroup scheme $U_K \subset G_K$; here K is an arbitrary field extension of k and G_K is the base extension of G along K/k.*

Two π-points $\alpha_K,\ \beta_L$ of G are said to be equivalent (denoted $\alpha_K \sim \beta_L$) provided that for every finite dimensional kG-module M the $K[t]/t^p$-module $\alpha_K^*(M_K)$ is projective if and only if the $L[t]/t^p$-module $\beta_L^*(M_L)$ is projective.

In [10], it is shown that the set of equivalence classes of π-points of G admits a scheme structure which is defined in terms of the representation theory of G and which is denoted $\Pi(G)$. Moreover, it is verified that this scheme is isomorphic to the projectivization of the affine scheme of $\mathrm{H}^\bullet(G,k)$,

$$\Pi(G) \;\cong\; Proj\; \mathrm{H}^\bullet(G,k).$$

Here, $\mathrm{H}^\bullet(G,k)$ is the finitely generated commutative k-algebra defined to be the cohomology algebra $\mathrm{H}^*(G,k)$ if $p = char(k)$ equals 2 and to be the subalgebra of $\mathrm{H}^*(G,k)$ generated by homogeneous classes of even degree if $p > 2$.

We next introduce admissible monomorphisms and admissible epimorphisms, formulated in terms of π-points.

Definition 2.2. *Let G be a finite group scheme. A short exact sequence of kG-modules $0 \to M_1 \to M_2 \to M_3 \to 0$ is said to be locally split if its pull-back via any π-point $\alpha_K : K[t]/t^p \to KG$ is split as a short exact sequence of $K[t]/t^p$-modules. We shall frequently refer to such a locally split short exact sequence as an admissible sequence.*

Moreover, we say that a monomorphism $f : M_1 \to M_2$ of kG-modules is an admissible monomorphism if it can be completed to a locally split short exact sequence. Similarly, an epimorphism $g : M_2 \to M_3$ is said to be an admissible epimorphism if it can be completed to a locally split short exact sequence.

We typically identify an admissible monomorphism with the inclusion of the image of the injective map $f : M \to N$.

The objects of our study are kG-modules of constant Jordan type as defined below (and introduced in [6]). We recall that a finite dimensional $K[t]/t^p$ module M of dimension N is isomorphic to

$$a_p[p] + \cdots + a_i[i] \cdots + a_1[1], \quad \sum_i a_i \cdot i = N,$$

where $[i] = K[t]/t^i$ is the indecomposable $K[t]/t^p$-module of dimension i. We refer to the p-tuple $(a_p, \ldots, a_1)$ as the Jordan type of M and designate this Jordan type by JType(M); the $(p-1)$-tuple $(a_{p-1}, \ldots, a_1)$ will be called the stable Jordan type of M.

Definition 2.3. *A finite dimensional module M for a finite group scheme G is said to be of* **constant Jordan type** *if the Jordan type of $\alpha_K^*(M_K)$ is independent of the choice of the π-point α_K of G.*

In the following proposition, we see that the class of modules of constant Jordan type is closed under locally split extensions.

Proposition 2.4. *Let G be a finite group scheme and let $\mathcal{E}$ denote a short exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ of finite dimensional kG-modules.*

1. *Assume $\mathcal{E}$ is locally split. Then M_1 and M_3 are of constant Jordan type if and only if M_2 is of constant Jordan type.*

2. If M_1 and M_3 are modules of constant Jordan type, then $\mathcal{E}$ is locally split if and only if $\alpha_K^*(M_{2,K}) \simeq \alpha_K^*(M_{1,K}) \oplus \alpha_K^*(M_{3,K})$ for some representative α_K of each generic point of $\Pi(G)$.

Proof. Observe that a short exact sequence of finite dimensional $K[t]/t^p$-modules $0 \to N_1 \to N_2 \to N_3 \to 0$ is split if and only if $\mathrm{JType}(N_2) = \mathrm{JType}(N_1) + \mathrm{JType}(N_3)$. Thus, if $\mathcal{E}$ is locally split and if $M_1,\ M_3$ have constant Jordan type, then M_2 does as well.

If $\mathcal{E}$ is locally split and M_2 has constant Jordan type, then the proof that both M_1 and M_3 have constant Jordan type is verified using the same argument as that of [6, 3.7] using [6, 3.5]; the point is that the Jordan type of M_1 at some representative of a generic point of $\Pi(G)$ must be greater or equal to the Jordan type of M_1 at some representative of any specialization. The same applies to M_2 and M_3. As shown in [11, 4.2], the Jordan type of any finite dimensional kG-module at a generic point of $\Pi(G)$ is independent of the choice of π-point representing that generic point.

If $0 \to V_1 \to V_2 \to V_3 \to 0$ is a short exact sequence of $k[t]/t^p$-modules, then the Jordan type of V_2 must be greater or equal to the sum of the Jordan types of V_1 and V_3. Thus, the observation of the preceding paragraph verifies the following: assume that $\mathcal{E}$ is a short exact sequence of kG-modules with M_1, M_3 of constant Jordan type; if M_3 has the minimal possible Jordan type (namely the sum of the Jordan types of M_1 and M_3) at each generic π-point of G, then M_3 must have constant Jordan type. $\square$

As the following proposition asserts, admissible monomorphisms and admissible epimorphisms as in Definition 2.2 are associated to structures of exact categories (in the sense of Quillen [17]) on the category $\mathrm{mod}(kG)$ of finite dimensional kG-modules and the full subcategory $\mathcal{C}(kG)$ of modules of constant Jordan type.

Proposition 2.5. *The collection $\underline{E}$ of locally split short exact sequences of finite dimensional kG-modules constitutes a class of admissible sequences providing* $\mathrm{mod}(kG)$ *with the structure of an exact category in the sense of Quillen (cf. [17]).*

Similarly, the class $\underline{E}_{\mathcal{C}}$ of locally split short exact sequences of kG-modules of constant Jordan type also constitutes a class of admissible sequences, thereby providing $\mathcal{C}(kG)$ with the structure of exact subcategory of $\mathrm{mod}(kG)$.

Proof. According to Quillen, to verify that $\underline{E}$ provides $\mathrm{mod}(kG)$ with the structure of an exact category we must verify three properties. The first property consists of the conditions that any short exact sequence isomorphic to one in $\underline{E}$ is itself in $\underline{E}$; that $\underline{E}$ contains all split short exact sequences; and that if $0 \to M' \xrightarrow{i} M \xrightarrow{j} M'' \to 0$ is a short exact sequence in $\underline{E}$, then $i : M' \to M$ is a kernel for $j : M \to M''$ and $j : M \to M''$ is a cokernel for $i : M' \to M$. These conditions are essentially immediate.

The second property consists of the conditions that the class of admissible monomorphisms (i.e., monomorphisms occurring in an exact sequence of $\underline{E}$) is closed under composition and closed under push-out with respect to any map of $\mathrm{mod}(kG)$; similarly that the class of admissible epimorphisms is closed under composition and pull-back. This follows from the observation that these properties hold for split exact sequences, thus also for those exact sequences split at every π-point.

The third property asserts that any map $f : M' \to M$ of kG-modules with the property that there exists some map $g : M \to Q$ of kG-modules such that the composition $g \circ f : M' \to M \to Q$ is an admissible monomorphism is itself an admissible monomorphism; and the analogous statement for admissible epimorphisms. This is clear, for any splitting of the composition $\alpha_K^*(g \circ f) : \alpha_K^*(M'_K) \to \alpha_K^*(M_K) \to \alpha_K^*(Q_K)$ gives a splitting of $\alpha_K^*(f) : \alpha_K^*(M'_K) \to \alpha_K^*(M_K)$.

In view of Proposition 2.4, the preceding discussion for $\underline{E}$ applies equally to the class $\underline{E}_{\mathcal{C}}$ of locally split short exact sequences of $\mathcal{C}(kG)$. $\square$

The following proposition makes the evident points that not every short exact sequence of modules of constant Jordan type is locally split, that some non-split short exact sequences are locally split, and that a non-locally split extension of modules of constant Jordan type might not have a middle term which is of constant Jordan type.

Proposition 2.6. *Let E be an elementary abelian p-group and let $I \subset kE$ be the augmentation ideal. Then the short exact sequence of kE-modules of constant Jordan type*

$$0 \longrightarrow I^i/I^j \longrightarrow I^i/I^\ell \longrightarrow I^j/I^\ell \longrightarrow 0$$

is not locally split for any $i < j < \ell \leq p$.

If the rank of E is at least 2, then a non-trivial negative Tate cohomology class $\xi \in \hat{\mathrm{H}}^n(G,k)$ determines a locally split (but not split) short exact sequence of the form

$$0 \to k \to E \to \Omega^{n-1}(k) \to 0.$$

In contrast, if the rank of E is at least 2 and if $0 \neq \zeta \in \mathrm{H}^1(E,k)$, then the associated short exact sequence $0 \to k \to M \to k \to 0$ of kE-modules is such that M does not have constant Jordan type.

Proof. By [6, 2.1], each I^i/I^ℓ is of constant Jordan type. The fact that the sequence $0 \to I^i/I^j \to I^i/I^\ell \to I^j/I^\ell \to 0$ is not locally split follows from Proposition 2.4 and an easy computation of Jordan types. The second assertion is a special case of [6, 6.3].

Finally, the extension of k by k determined by $\zeta \in \mathrm{H}^1(E,k)$ is split when pulled-back via the π-point α_K if and only if $\alpha_K^*(\zeta_K) = 0 \in \mathrm{H}^1(K[t]/t^p, K)$.

The subset of $\Pi(E)$ consisting of those $[\alpha_K]$ satisfying $\alpha_K^*(\zeta_K) = 0$ is a hyperplane of $\Pi(E) \cong \mathbb{P}^{r-1}$ where r is the rank of E. Consequently, the Jordan type of $\alpha_K^*(M_K)$ is $2[1]$ on a hyperplane of $\Pi(E)$ and is $[2]$ otherwise. $\square$

3 The Grothendieck group $K_0(\mathcal{C}(kG))$

For any exact category $\mathcal{E}$ specified by a class $\underline{E}$ of admissible exact sequences, we denote by $K_0(\mathcal{E})$ the Grothendieck group given as the quotient of the free abelian group of isomorphism classes of objects of $\mathcal{E}$ modulo the relations generated by $[M_1] - [M_2] + [M_3]$ whenever $0 \to M_1 \to M_2 \to M_3 \to 0$ is an admissible sequence in $\underline{E}$. In this section, we begin a consideration of $K_0(\mathcal{C}(kG))$, the Grothendieck group of modules of constant Jordan type (with respect to locally split short exact sequences).

Following Quillen [17], we could further consider $K_i(\mathcal{C}(kG))$ for $i > 0$. Granted the current state of our understanding, the challenge of investigating $K_0(\mathcal{C}(kG))$ is sufficiently daunting that we postpone any consideration of $K_i(\mathcal{C}(kG)), i > 0$.

Proposition 3.1. *For any finite group scheme G, there are natural embeddings of exact categories*

$$\mathcal{P}(G) \longrightarrow \mathcal{C}(kG), \qquad \mathcal{C}(kG) \longrightarrow \mathrm{mod}(kG).$$

The first is from the exact category $\mathcal{P}(G)$ of finitely generated projective kG-modules into the category $\mathcal{C}(kG)$ of modules of constant Jordan type; the second is from $\mathcal{C}(kG)$ into the category $\mathrm{mod}(kG)$ *of all finitely generated kG-modules. These embeddings induce homomorphisms*

$$K_0(kG) \equiv K_0(\mathcal{P}(G)) \longrightarrow K_0(\mathcal{C}(kG)) \longrightarrow K_0(\mathrm{mod}(kG)).$$

Moreover, these homomorphisms are contravariantly functorial with respect to a closed immersion $i : H \to G$ of finite group schemes.

Proof. Since every short exact sequence of projective modules is split, we conclude that the full embedding $\mathcal{P}(G) \to \mathcal{C}(kG)$ of the category of finite dimensional projective kG-modules into the category of kG-modules of constant Jordan type (with admissible short exact sequences being the locally split short exact sequences) is an embedding of exact categories. The embedding $\mathcal{C}(kG) \subseteq \mathrm{mod}(kG)$ is clearly an embedding of exact categories.

If $i : H \to G$ is a closed embedding, then kG is projective as a kH-module (cf. [15, 8.16]) so that any projective kG-module restricts to a projective kH-module. By [6, 1.9], restriction of a kG-module of constant Jordan type to kH is again of constant Jordan type. Since restriction is exact and preserves locally split sequences (because every π-point of H when composed with i becomes a π-point of G), we obtain the asserted naturality with respect to $i : H \to G$. $\square$

Remark 3.2. If a module $M \in \mathcal{C}(kG)$ admits an admissible filtration (meaning the inclusion maps are admissible monomorphisms)

$$M_0 \subset M_1 \subset \cdots \subset M_n = M,$$

then $[M] = \sum_{i=1}^{n} [M_i/M_{i-1}] \in K_0(\mathcal{C}(kG))$.

As an immediate corollary of Proposition 3.1, we have the following.

Corollary 3.3. *If G is a finite group, then $K_0(\mathcal{P}(G)) \to K_0(\mathcal{C}(kG))$ is injective. More generally, if the Cartan matrix for the finite group scheme G is non-degenerate, then $K_0(\mathcal{P}(G)) \to K_0(\mathcal{C}(kG))$ is injective.*

Proof. A basis for $K_0(\mathcal{P}(G))$ is given by the classes of the indecomposable projective modules, while a basis $K_0(\mathrm{mod}(kG))$ is given by the classes of the irreducible modules. The Cartan matrix for kG represents the natural map $K_0(\mathcal{P}(G))$ to $K_0(\mathrm{mod}(kG))$ with respect to these bases. By Proposition 3.1, this map factors through $K_0(\mathcal{C}(kG))$. This proves the second statement. A theorem of Brauer (cf. [3, I.5.7.2]) says that if G is a finite group then the Cartan matrix for kG is non-singular. □

We consider a few elementary examples.

Example 3.4. Let G be the cyclic group $\mathbb{Z}/p$. Then

$$K_0(\mathcal{C}(k\mathbb{Z}/p)) \quad \simeq \quad \mathbb{Z}^p.$$

The map $K_0(\mathcal{P}(G)) \to K_0(\mathcal{C}(k\mathbb{Z}/p))$ is identified with the map $\mathbb{Z} \to \mathbb{Z}^p, \quad a \mapsto (a, 0, \ldots, 0)$. The map $K_0(\mathcal{C}(k\mathbb{Z}/p)) \to K_0(\mathrm{mod}(k\mathbb{Z}/p))$ is identified with the map $\mathbb{Z}^p \to \mathbb{Z}, \quad (a_p, \ldots, a_1) \mapsto \sum_i ia_i$.

Proposition 2.4 and the universal property of the Grothendieck group $K_0(\mathcal{C}(kG))$ immediately imply the following proposition.

Proposition 3.5. *Sending a kG-module M of constant Jordan type to the Jordan type of $\alpha_K^*(M_K)$ and to the stable Jordan type of $\alpha_K^*(M_K)$ for any π-point α_K of G determines homomorphisms*

$$\mathrm{JType} : K_0(\mathcal{C}(kG)) \longrightarrow \mathbb{Z}^p, \qquad \overline{\mathrm{JType}} : K_0(\mathcal{C}(kG)) \longrightarrow \mathbb{Z}^{p-1}. \tag{1}$$

We view an element of $K_0(\mathcal{C}(kG))$ as the class of a *virtual* kG-module of constant Jordan type. In the next section, we shall investigate to what extent the homomorphisms JType, $\overline{\mathrm{JType}}$ are surjective; in other words, the realizability of Jordan types by virtual kG-modules of constant Jordan type.

The function JType is not injective: for example, a module of constant Jordan type and its k-linear dual have the same Jordan type. The example of $\mathbb{Z}/2 \times \mathbb{Z}/2$ given in Proposition 3.6 provides a more explicit example of non-injectivity of JType.

We can achieve the next example because we know exactly what are the indecomposable kE-modules for the Klein-four group $E = \mathbb{Z}/2 \times \mathbb{Z}/2$ (cf. [2], [13]).

Proposition 3.6. *For any field k of characteristic 2, the group algebra kE of the Klein four group $E = \mathbb{Z}/2 \times \mathbb{Z}/2$ satisfies*

$$K_0(\mathcal{C}(kE)) \simeq \mathbb{Z}^3.$$

Proof. We recall from [6, 6.2] that any kE-module of constant Jordan type is of the form $kE^e \bigoplus(\oplus_i \Omega^{n_i}(k))$. One easily checks there is an admissible (i.e., locally split, short) exact sequence of kE-modules of the following form:

$$0 \longrightarrow \Omega^2(k) \longrightarrow \Omega^1(k) \oplus \Omega^1(k) \longrightarrow k \longrightarrow 0 \tag{2}$$

Hence, $[\Omega^2(k)] = 2[\Omega^1(k)] - [k]$ in $K_0(\mathcal{C}(kE))$. Consecutive applications of the Heller shift to the sequence (2) thus imply that $K_0(\mathcal{C}(kE))$ is generated by the classes of the three kE-modules: $kE,\ k,\ \Omega^1(k)$.

We define a function σ on the class of modules of constant Jordan type by sending $M \simeq kE^e \bigoplus(\oplus_i \Omega^{n_i}(k))$ to $\sigma(M) = \sum_i n_i$. We proceed to show that σ is additive on admissible sequences, and, hence, induces a homomorphism $\sigma : K_0(\mathcal{C}(kE)) \to \mathbb{Z}$.

Let $\xi : 0 \to M \to L \to N \to 0$ be an admissible sequence of kE-modules of constant Jordan type, and assume $N = N_1 \oplus N_2$. Since $\mathrm{Ext}^1_E(N, M) = \mathrm{Ext}^1_E(N_1, M) \oplus \mathrm{Ext}^1_E(N_2, M)$, we have $\xi = \xi_1 + \xi_2$ where $\xi_i \in \mathrm{Ext}^1_E(N_i, M)$. Moreover, both ξ_1, ξ_2 are admissible, and the additivity of σ on ξ_1, ξ_2 implies additivity of σ on ξ. Hence, we may assume that N is an indecomposable non-projective module of constant Jordan type, that is, $N \simeq \Omega^n(k)$. Similarly, we may assume $M \simeq \Omega^m(k)$.

Thus, we may assume that ξ has the form

$$0 \longrightarrow \Omega^m(k) \longrightarrow kE^e \bigoplus(\oplus_i \Omega^{n_i}(k)) \longrightarrow \Omega^n(k) \longrightarrow 0$$

By [6, 6.9], $\dim \Omega^{2a}(k) = 4a+1$; hence, $\dim \Omega^{2a+1}(k) = 4(a+1)+1$ (using the short exact sequence $0 \to \Omega^{2a+1}(k) \to P_{2a} \to \Omega^{2a}(k) \to 0$). Comparing dimensions, we conclude that σ is additive on an admissible sequence ξ if and only if σ is additive on $\Omega^1(\xi)$. (We are implicitly using [6, 1.8] asserting that M has constant Jordan type if and only if $\Omega^1(M)$ has constant Jordan type.) Thus, we may further assume that $n = 0$. Hence, ξ has the form

$$0 \longrightarrow \Omega^m(k) \longrightarrow kE^e \bigoplus(\oplus_i \Omega^{a_i}(k)) \longrightarrow k \longrightarrow 0$$

Since ξ is split on restriction to any π-point, there exists i such that the restriction $\Omega^{a_i}(k) \to k$ is not the zero map. Hence, the map $\Omega^{a_i}(k) \to k$ is surjective. Since the left end of the sequence ξ is projective free, we conclude that $e = 0$. Because the Jordan type of the middle term is the sum of the Jordan type of the ends, the middle term must have exactly two summands. Hence, the sequence has the form

$$0 \longrightarrow \Omega^m(k) \longrightarrow \Omega^a(k) \oplus \Omega^b(k) \longrightarrow k \longrightarrow 0$$

Because ξ represents an element in $\mathrm{H}^1(G, \Omega^m(k)) \simeq \hat{\mathrm{H}}^{1-m}(G, k)$ that vanishes on restriction along any π-point, $1 - m < 0$ by [6, 6.3]; in other words, m is positive. Likewise, a and b are not negative. That is, the map $\Omega^a(k) \to k$ represents an element in $\hat{\mathrm{H}}^a(G, k)$ which does not vanish on restriction to some π-point. This can only happen if $a \geq 0$. The same argument shows that $b \geq 0$.

Comparing dimensions (see [6, 6.9]), we get $2m + 1 + 1 = \dim \Omega^m(k) + \dim k = \dim \Omega^a(k) + \dim \Omega^b(k) = 2a + 1 + 2b + 1$. Hence, $m + 0 = a + b$, and we conclude that σ is additive.

We consider the map

$$\Psi = (\mathrm{JType}, \sigma) : K_0(\mathcal{C}(kE)) \longrightarrow \mathbb{Z}^2 \oplus \mathbb{Z}.$$

This is well defined by Proposition 3.5 and the observation that σ is additive as shown above. To prove the proposition, it suffices to show that Ψ has image a subgroup of finite index inside $\mathbb{Z}^2 \oplus \mathbb{Z}$. This follows from the observation that the vectors $\Psi(k) = (0, 1, 0)$, $\Psi(\Omega^1(k)) = (1, 1, 1)$, and $\Psi(kE) = (2, 0, 0)$ are linearly independent over $\mathbb{Q}$. $\square$

The preceding proof shows that the admissible sequences for the Klein four group have a generating set consisting of sequences of the form

$$0 \longrightarrow \Omega^{m+n+a}(k) \longrightarrow \Omega^{m+a}(k) \oplus \Omega^{n+a}(k) \oplus kE^e \longrightarrow \Omega^a(k) \longrightarrow 0, \tag{3}$$

where $m, n > 0$. Here,

$$2e \;=\; |m + n + a| + |a| - |m + a| - |n + a|;$$

hence, $e = 0$ unless a is negative and $m + n + a$ is positive. This fact is of use in the discussion to follow.

There are examples of group schemes for which the Cartan matrix is singular so that we can not apply Corollary 3.3 in these examples. Perhaps the simplest is the first Frobenius kernel $G = \mathfrak{G}_1$ of the algebraic group $\mathfrak{G} = SL_2$ with $p > 2$. In this case, kG is isomorphic to the restricted enveloping algebra of the restricted p-Lie algebra sl_2. It is known (cf. [12, 2.4]) that kG has $(p+1)/2$ blocks one of which has only a single projective irreducible module. Each of the other $(p-1)/2$ blocks has two non-isomorphic irreducible modules. Now suppose that B is one of these blocks. It has irreducible modules S and T. The projective covers Q_S of S and Q_T of T have the forms

$$Q_S = \begin{tikzcd} & S \arrow[dl] \arrow[dr] & \\ T \arrow[dr] & & T \arrow[dl] \\ & S & \end{tikzcd}, \qquad Q_T = \begin{tikzcd} & T \arrow[dl] \arrow[dr] & \\ S \arrow[dr] & & S \arrow[dl] \\ & T. & \end{tikzcd}$$

In particular, for any module M in B, $\mathrm{Rad}^3(M) = 0$, and moreover, if $\mathrm{Rad}^2(M) \neq \{0\}$, then M contains a projective direct summand. That is, if M is an indecomposable non-projective B-module then $\mathrm{Rad}^2(M) = \{0\}$.

We see from the above that the Cartan matrix of kG is a $p \times p$ matrix consisting of $(p+1)/2$ block matrices along the diagonal, $(p-1)/2$ of which are the Cartan matrices of blocks B of the group algebra as above. The Cartan matrix of such a block is a 2×2 matrices with 2 in every entry. The other block matrix of the Cartan matrix of kG is a 1×1 identity matrix. Hence the natural map from $K_0(\mathcal{P}(G))$ to $K_0(\mathrm{mod}(kG))$ is not injective. However, as we see below, $K_0(\mathcal{P}(G))$ still injects into $K_0(\mathcal{C}(kG))$.

Proposition 3.7. *Let $G = \mathfrak{G}_1$ where $\mathfrak{G} = SL_2$ and assume that $p \geq 3$. Then*

$$K_0(\mathcal{C}(kG)) \;\simeq\; \mathbb{Z}^{3p-2}.$$

Moreover, the map $K_0(\mathcal{P}(G)) \;\to\; K_0(\mathcal{C}(kG))$ is injective.

Proof. We begin by recalling that rational SL_2-modules are kG-modules of constant Jordan type by [6, 2.5]; in particular, every simple kG-module is a module of constant Jordan type.

As stated above, there are $(p-1)/2$ blocks $B_1, \dots, B_{(p-1)/2}$ as above. In addition there is another block B' containing only a single indecomposable module which is projective. Therefore,

$$K_0(\mathcal{C}(kG)) \;\simeq\; K_0(\mathcal{C}(B')) \oplus \sum_{i=1}^{(p-1)/2} K_0(\mathcal{C}(B_i)).$$

We know that $K_0(\mathcal{C}(kB')) \;\simeq\; \mathbb{Z}$. Consequently, it suffices to prove that $K_0(\mathcal{C}(B_i)) \simeq \mathbb{Z}^6$ for each i.

We consider $B = B_i$ with simple modules S and T. We may assume that S and T have stable constant Jordan types $1[i]$ and $1[p-i]$ respectively. The indecomposable B-modules can be classified using standard methods similar to those of [18] or the diagrammatic methods of [5]. Every non-simple, non-projective indecomposable B-module M has the property that $\mathrm{Soc}(M) = \mathrm{Rad}(M)$ is a direct sum of t copies of one of the simple modules S or T. The quotient $M/\mathrm{Rad}(M)$ is a direct sum of r copies of the other simple T or S. Moreover, we must have that r is one of $t-1$, t, or $t+1$, that is, $|t-r| \leq 1$. In the case that $r = t$, it is evident that the module M has periodic cohomology.

Or, at least, it is clear that the dimensions of $\Omega^n(M)$ are bounded for all n. This means that the module M must have proper non-trivial support variety in $\Pi(G)$, as the annihilator of its cohomology must be non-trivial. Consequently, such a module M can not have constant Jordan type. Thus, we conclude that $|t-r| = 1$, and that every indecomposable module of constant Jordan type is a syzygy of an irreducible module (cf. (7), (8) and (9), below). This last fact can also be deduced from recent results of Benson [4] on algebras with radical cube zero.

Using the diagrammatic methods, sketched below (as can also be done in the situation of Proposition 3.6), we first verify that the non-trivial admissible sequences are generated by sequences of the form

$$0 \to \Omega^{2(m+n)+a}(\mathfrak{X}) \to \Omega^{2m+a}(\mathfrak{X}) \oplus \Omega^{2n+a}(\mathfrak{X}) \oplus P \to \Omega^a(\mathfrak{X}) \to 0 \quad (4)$$

and

$$0 \to \Omega^{2(m+n-1)+a}(\mathfrak{X}) \to \Omega^{2m-1+a}(\mathfrak{Y}) \oplus \Omega^{2n-1+a}(\mathfrak{Y}) \oplus Q \to \Omega^a(\mathfrak{X}) \to 0 \quad (5)$$

and

$$0 \to \Omega^{2(m+n)-1+a}(\mathfrak{Y}) \to \Omega^{2m+a}(\mathfrak{X}) \oplus \Omega^{2n-1+a}(\mathfrak{Y}) \oplus R \to \Omega^a(\mathfrak{X}) \to 0 \quad (6)$$

where $\mathfrak{X}$ and $\mathfrak{Y}$ are either S or T and $\mathfrak{Y}$ is not the same as $\mathfrak{X}$. Here a can be any integer and $m, n > 0$. The modules P, Q and R are projective modules which are required for the exactness. In any of these sequences, the projective module P, Q or R is a eumber of copies of the projective cover of a simple module in the socle of the left-most term of the sequence, or in the top of the right-most term of the sequence. In what follows it might be helpful to note that, for dimension reasons, the projective module P in sequence (4) is zero except in the cases that $2(m+n)+a$ is positive and a is negative. Likewise, Q in sequence (5) is zero except when $2(m+n-1)+a$ is positive and a is negative. And a similar thing happens for sequence (6). To be very specific, P in (4) is a sum of copies of Q_T if $\mathfrak{X} = S$ and a is even or if $\mathfrak{X} = T$ and a is odd. Otherwise, it is a sum of copies of Q_S. The module Q in (5) and R in (6) are sum of copies of Q_T in the cases that $\mathfrak{X} = S$, $\mathfrak{Y} = T$ and a is even and $\mathfrak{X} = T$, $\mathfrak{Y} = S$ and a is odd. Otherwise, they are sum of copies of Q_S. The verification that these sequences, (4), (5) and (6), generate the collection of admissible sequences prodeeds as follows.

As in the proof of Proposition 3.6, we are looking for sequences representing elements of $\mathrm{Ext}^1_{kG}(-,-)$ which vanish on restriction along any π-point. Because Ext^1_{kG} distributes over direct sums, and any such distribution still satisfies the vanishing condition, we can restrict our consideration to sequences which have indecomposable end terms. Consequently, the right end term can be considered to be $\Omega^a(\mathfrak{X})$ for $\mathfrak{X}$ either S or T. Suppose that $\mathfrak{X}$ is S. We can translate the sequence by Ω^{-a} so that the right term is isomorphic to S. Note also that because the two end terms each have only one single non-projective Jordan block on restriction to any Π-point, there are at exactly two nonprojective direct summands in the middle term of the sequence.

Now we consider the diagrams for the syzygies of S amd T. For example,

$$\Omega^{-1}(S) = \begin{tikzcd} & S \arrow[dl] \arrow[dr] & \\ T & & T \end{tikzcd}, \qquad \Omega^{-1}(T) = \begin{tikzcd} & T \arrow[dl] \arrow[dr] & \\ S & & S \end{tikzcd} \tag{7}$$

$$\Omega(S) = \begin{tikzcd} T \arrow[dr] & & T \arrow[dl] \\ & S & \end{tikzcd}, \qquad \Omega(T) = \begin{tikzcd} S \arrow[dr] & & S \arrow[dl] \\ & T. & \end{tikzcd} \tag{8}$$

$$\Omega^2(S) = \begin{tikzcd} S \arrow[dr] & & S \arrow[dl] \arrow[dr] & & S \arrow[dl] \\ & T & & T & \end{tikzcd}, \quad \Omega^2(T) = \begin{tikzcd} T \arrow[dr] & & T \arrow[dl] \arrow[dr] & & T \arrow[dl] \\ & S & & S. & \end{tikzcd} \tag{9}$$

The diagram for an arbitrary syzygy of either S or T is merely an elongation of these diagrams.

If we have a locally split sequence whose right hand term is S (or T) and if $\Omega^m(\mathfrak{X})$ occurs in the middle term (with $\mathfrak{X}$ either S or T), then m must be nonnegative. The reason is that otherwise ($m < 0$) the Jordan type of the kernel of such a map has too many non projective blocks at any Π-point. In addition, such a map could not be right split at *any* π-point, because the socle of the kernel would have more irreducible constituents than the head.

To finish the verification that the non-trivial admissible sequences are generated by sequences of the form (4), (5), or (6), we suppose that there is a locally split sequence whose right hand term is S. The middle term must consist of two terms of the form $\Omega^m(\mathfrak{X})$ and $\Omega^n(\mathfrak{Y})$), with $m, n > 0$ (if either m or n is 0, then the sequence splits). Because both of these terms must map surjectively to S, we must have that $\mathfrak{X} \simeq S$ if m is even, and $\mathfrak{X} \simeq T$ if m is odd. The same happens for $\mathfrak{Y}$ and n. Finally, we notice that the left term of the sequence is determined entirely by its composition factors. A complete analysis, which we leave to the reader, reveals that the sequence must look like one of (4), (5), or (6).

Consider the following two collections of indecomposable B-modules:

$$\mathcal{U}_1 = \{\Omega^m(S),\ m \text{ even}\} \cup \{\Omega^n(T),\ n \text{ odd}\} \cup \{Q_T\}$$

and

$$\mathcal{U}_2 = \{\Omega^m(S),\ m \text{ odd}\} \cup \{\Omega^n(T),\ n \text{ even}\} \cup \{Q_S\}.$$

Observe that each of the sequences (4), (5) or (6) for any values of a, m, n, involves either modules all of which are isomorphic to elements of $\mathcal{U}_1$ or modules isomorphic to elements of $\mathcal{U}_2$.

Let $\mathcal{E}(B)$ be the exact category of all B-modules of constant Jordan type with the admissible sequences being the split exact sequences. Thus, $K_0(\mathcal{E}(B))$ is the Green ring $\mathbb{Z}[B]$ of B and is a free $\mathbb{Z}$-module on the classes of elements in the union of $\mathcal{U}_1$ and $\mathcal{U}_2$. Moreover factoring out the relators coming from the sequences (4) and (5) defines $K_0(\mathcal{C}(B))$ as a quotient of $\mathbb{Z}[B]$. As in the proof of Proposition 3.6, we conclude that the images of classes in $\mathcal{U}_1$ (respectively, $\mathcal{U}_2$) in $K_0(\mathcal{C}(B))$ are generated by the images of the classes $[S]$, $[\Omega^1(T)]$ and $[Q_T]$ (resp., $[T]$, $[\Omega^1(S)]$ and $[Q_S]$).

Now let $\mathcal{M}_1$ be the subgroup of $K_0(\mathcal{E}(B))$ generated by the classes of modules in $\mathcal{U}_1$, and let $\mathcal{M}_2$ be the subgroup generated by the classes of modules in $\mathcal{U}_2$. Then $K_0(\mathcal{E}(B)) \simeq \mathcal{M}_1 \oplus \mathcal{M}_2$. Let $\mathcal{K}$ be the subgroup generated by the relators determined by sequences (4) (5) and (6). That is, $\mathcal{K}$ is the kernel of the homomorphism of $K_0(\mathcal{E}(B))$ onto $K_0(\mathcal{C}(B))$. Notice that $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$ where each $\mathcal{K}_i = \mathcal{M}_i \cap \mathcal{K}$ is generated by the relators determined by those sequences of the form (4) (5) and (6) which only involve modules from $\mathcal{U}_i$. Consequently, we have that

$$K_0(\mathcal{C}(B)) \simeq K_0(\mathcal{E}(B))/\mathcal{K} \simeq \mathcal{M}_1/\mathcal{K}_1 \oplus \mathcal{M}_2/\mathcal{K}_2$$

Therefore, to complete the proof of the proposition, it suffices to exhibit homomorphisms $\mathcal{M}_1/\mathcal{K}_1 \to \mathbb{Z}^3$ and $\mathcal{M}_2/\mathcal{K}_2 \to \mathbb{Z}^3$ whose images are cofinite. We do this by showing that there are isomorphism $\theta_i : \mathcal{M}_i/\mathcal{K}_i \simeq K_0(\mathcal{C}(k'E))$ for $i = 1, 2$ and appealing to Proposition 3.6; here, $E = \mathbb{Z}/2 \times \mathbb{Z}/2$ is the Klein four group and k' is a field of characteristic 2.

For this purpose we define maps $\gamma_1 : \mathcal{M}_1 \to K_0(\mathcal{E}(k'E))$ and $\gamma_2 : \mathcal{M}_2 \to K_0(\mathcal{E}(k'E))$ as follows:

$$\gamma_1([\Omega^n(S)]) = [\Omega^n(k')] \text{ (for } n \text{ even)}, \qquad \gamma_2([\Omega^n(S)]) = [\Omega^n(k')] \text{ (for } n \text{ odd)},$$

$$\gamma_1([\Omega^n(T)]) = [\Omega^n(k')] \text{ (for } n \text{ odd)}, \qquad \gamma_2([\Omega^n(T)]) = [\Omega^n(k')] \text{ (for } n \text{ even)},$$

$$\gamma_1([Q_T]) = [k'E], \qquad \gamma_2([Q_S]) = [k'E].$$

The reader should remember that γ_i is only defined on the classes of modules in $\mathcal{U}_i$. Moreover, each γ_i is clearly surjective.

The important thing to note is that γ_1 takes the relators coming from those sequences (4), (5) and (6) involving only the modules of $\mathcal{U}_1$ bijectively to the relators coming from the sequences (3), which are then zero in $K_0(\mathcal{E}(k'E))$. To see this we need only replace the module $\mathfrak{X}$, $\mathfrak{Y}$ in sequences (4), (5) and (6) by k' and the projective modules P and Q by the appropriate sum of copies of $k'E$.

Moreover, relators coming from the sequences (3) generate the kernel of the natural quotient map from $K_0(\mathcal{E}(k'E))$ to $K_0(\mathcal{C}(k'E))$. Consequently, γ_1 induces an isomorphism θ_1 from $\mathcal{M}_1/\mathcal{K}_1$ to $K_0(\mathcal{C}(k'E))$. Likewise γ_2 induces in isomorphism $\theta_2 : \mathcal{M}_2/\mathcal{K}_2 \simeq K_0(\mathcal{C}(k'E))$. $\square$

4 Realization of Jordan types

In this section, we initiate the investigation of the images of the Jordan type functions, JType, $\overline{\text{JType}}$, introduced in Proposition 3.5. Our first proposition establishes the surjectivity of JType for an elementary abelian p-group of arbitrary rank. The reader is cautioned that this establishes realizability of Jordan types by virtual modules of constant Jordan type. Even for the rank 2 elementary abelian p-group $E = \mathbb{Z}/p \times \mathbb{Z}/p$ with $p > 3$, we do not know whether there is a kE-module M of constant Jordan type with $\overline{\text{JType}}(M) = [2]$, whereas Proposition 4.1 shows that we can realize the Jordan type [2] in a virtual module.

Proposition 4.1. *The map*

$$\text{JType} : K_0(\mathcal{C}(kE)) \longrightarrow \mathbb{Z}^p$$

of Proposition 3.5 is surjective provided that E is an elementary abelian p-group.

Proof. It suffices to verify that $M = kE/I^i$ has constant Jordan type of the form $1[i] + a_{i-1}[i-1] \cdots + a_1[1]$ for each $i, 1 \leq i \leq p$, where I is the augmentation ideal of kE. The fact that M has constant Jordan type is verified in [6, 2.1]. The fact that $a_j = 0$ for $j > i$ follows from the fact that $I^i \cdot M = 0$. The fact that $a_i = 1$ follows from the observation that the generator of M is not annihilated by $(g-1)^i \in I$ for any generator g of E whereas any element of I/I^i is annihilated by $(g-1)^i$ for every generator g of E. $\square$

The Jordan type of a direct sum of $k[t]/t^p$-modules is the sum of the Jordan types. The stable Jordan type of the Heller shift of the $k[t]/t^p$-module $\sum_{i=1}^{p-1} a_i[i]$ equals $\sum_{i=1}^{p-1} a_{p-i}[i]$. The Jordan type of a tensor product is given by the following proposition.

Proposition 4.2. *(cf. [6, 10.2]) Let $[i]$ be an indecomposable $k[t]/t^p$-module of dimension i for $1 \leq i \leq p$. Then if $j \geq i$, we have that*

$$[i] \otimes [j] = \begin{cases} [j-i+1] + [j-i+3] + \ldots + [j+i-3] + [j+i-1] & \text{if } j+i \leq p \\ \\ [j-i+1] + \ldots + [2p-1-i-j] + (j+i-p)[p] & \text{if } j+i > p. \end{cases}$$

The following proposition strongly restricts the possible images of the stable Jordan type function $\overline{\text{JType}} : K_0(\mathcal{C}(kG)) \to \mathbb{Z}^{p-1}$ of Proposition 3.5. We say that a subset $S \subset \mathbb{Z}^{p-1}$ is closed under Heller shifts if $\{a_1, \ldots, a_{p-1}\}$ is in S whenever $\sigma = \{a_{p-1}, \ldots a_1\} \in S$. Similarly, we say that $S \subset \mathbb{Z}^{p-1}$ is closed under direct sums (respectively, tensor products) if $\sigma + \tau \in S$ (resp. $\sigma \otimes \tau \in S$) whenever $\sigma,\ \tau \in S$. Here, $\sigma \otimes \tau$ is defined by the formula of Proposition 4.2 if both σ, τ have a single non-zero entry and is defined more generally by imposing biadditivity.

Proposition 4.3. *Let $S \subset \mathbb{Z}^{p-1}$ be a set of stable Jordan types. Moreover,*

1. *If S has the form $\overline{\mathrm{JType}}(\mathcal{C}(kG))$ for some finite group scheme G, then S is closed under Heller shifts, direct sums, and tensor products.*
2. *If $S = \{m[1] + n[p-1]; m, n \in \mathbb{Z}\}$, then S is closed under Heller shifts, direct sums, and tensor products.*
3. *If $[1],\ [2] \in S$ and S is closed under Heller shifts, direct sums, and tensor products, then $S = \mathbb{Z}^{p-1}$.*
4. *Similarly, if $[1],\ [3] \in S$ with $p > 2$ and S is closed under Heller shifts, direct sums, and tensor products, then $S = \mathbb{Z}^{p-1}$.*
5. *On the other hand, the assumption that $[1],\ [4] \in S$ and S is closed under Heller shifts, direct sums,and tensor products does not imply that $S = \mathbb{Z}^{p-1}$ for $p = 11$.*

Proof. Statement (1) is a consequence of [6, 1.8].

Statement (2) easily follows from the fact that $[p-1] \otimes [p-1] = [1] + (p-2)[p]$.

To prove (3), we observe that $[i] \otimes [2] = [i+1] + [i-1]$ for $i, 2 \leq i < p$. Thus, $[1], [2] \otimes [2] \in S$ implies that $[3] \in S$. Proceeding by induction, we see that $[i], [i-1], [i] \otimes [2] \in S$ implies that $[i+1] \in S$.

For (4), we may assume that $p > 5$. Then we obtain $[3] \otimes [3] = [1] + [3] + [5]$ which implies that $[5] \in S$; $[5] \otimes [3] = [3] + [5] + [7]$, so that $[7] \in S$. Continuing, we conclude that $[2i-1] \in S$, $1 \leq [2i-1] \leq p-2$. Applying Heller shifts to $[2i-1]$, we conclude the stable type $[p-2i+1] \in S$ for $1 \leq [2i-1] \leq p-2$, so that $[j] \in S$ for all $j, 1 \leq j \leq [p-1]$.

Finally, let $p = 11$ so that $[4] \otimes [4] = [1] + [3] + [5] + [7]$ and $p - 4 = 7$. Moreover, $[7] \otimes [4] = [7] + [9] + [11]$. We conclude that S contains the span of $\{[1], [4], [7], [10], [3] + [5], [8] + [6], [7] + [9]\}$. On the other hand, further tensor products with these classes never yield a Jordan type $\sum a_i[i]$ with $a_3 \neq a_5,\ a_6 \neq a_8$ or $a_7 \neq a_9$. □

The applicability of Proposition 4.3 to the question of realizability of stable Jordan types is reflected in the following proposition and its corollary.

Proposition 4.4. *Suppose that G is a finite group which has a normal abelian Sylow p-subgroup. There exists a kG-module with constant stable Jordan type $[2] + n[1]$ for some n. Moreover, the stable Jordan type function*

$$\overline{\mathrm{JType}} : \ K_0(\mathcal{C}(kG)) \longrightarrow \mathbb{Z}^{p-1}$$

is surjective.

Proof. Let $E \subseteq G$ be the subgroup consisting of all elements of order dividing p. This is the unique maximal elementary abelian subgroup of G. Let $M = k_E^{\uparrow G}$ be the induced module from the trivial module on E. Notice that because E is normal in G, E acts trivially on M and hence M is a module of constant Jordan type $s[1]$ where s is the index of E in G.

For any $t > 0$ we have that

$$\mathrm{Ext}^t_{kG}(k, M) \;\cong\; \mathrm{H}^t(G, k_E^{\uparrow G}) \;\cong\; \mathrm{H}^t(E, k).$$

by the Eckmann-Shapiro Lemma. In particular, $\mathrm{Ext}^1_{kG}(k, M) \cong \mathrm{H}^1(E, k)$ has a k-basis consisting of elements $\gamma_1, \ldots, \gamma_r$, where r is the rank of E. The elements have the property that for any π-point $\alpha_K : K[t]/(t^p) \to KG$ the restriction of some $\alpha_K^*(\gamma_i)$ is not zero for some i. It follows that the tuple $\zeta = (\gamma_1, \ldots, \gamma_r)$ is an element of $\mathrm{Ext}^1_{kG}((k_E^{\uparrow G})^r, k)$ which does not vanish when restricted along any π-point. Then ζ represents a sequence

$$\zeta : \qquad 0 \longrightarrow (k_E^{\uparrow G})^r \longrightarrow B \longrightarrow k \longrightarrow 0$$

which is not split on restriction along any π-point. As the first term in the sequence has constant Jordan type $rs[1]$, the middle term of the sequence must have constant Jordan type $[2] + (rs-1)[1]$.

The surjectivity of $\overline{\mathrm{JType}}$ now follows from Proposition 4.3(2). □

Corollary 4.5. *Let G be a finite unipotent abelian group scheme. Then the stable Jordan type function*

$$\overline{\mathrm{JType}} : \; K_0(\mathcal{C}(kG)) \longrightarrow \mathbb{Z}^{p-1}$$

is surjective.

Proof. Because G is a unipotent abelian group scheme, its group algebra kG is isomorphic as an algebra (though not as a Hopf algebra) to the group algebra of a finite abelian p-group. Hence, the previous proposition applies, since kG has a module of constant Jordan type $[2]+n[1]$ as constructed in the proof. □

We next briefly consider the question of realizability of stable Jordan types by indecomposable kG-modules by utilizing the Auslander-Reiten theory of almost split sequences (cf. [1]). In the following proposition, $\tau : \mathrm{stmod}(kG) \to \mathrm{stmod}(kG)$ denotes the Auslander-Reiten translation. This is a functor on the stable category and if G is a finite group or if kG is a symmetric algebra, then $\tau(M) = \Omega^2(M)$ for any finite dimensional kG-module M.

Proposition 4.6. *Let Θ be a connected component of the Auslander-Reiten quiver which has tree class A_∞. Let $\hat{K}$ be the subgroup of $K_0(\mathcal{C}(kG))$ generated by the classes of the modules in Θ. Let X_0 denote a kG-module in Θ that becomes an initial node of the tree of Θ once projectives are deleted. Then, $\hat{K}$ is generated by the classes $\{[\tau^n(X_0)] \mid n \in \mathbb{Z}\}$ and, if Θ contains a projective module P, by the class $[P]$ of that projective module.*

Proof. It suffices to show that the class $[M]$ of any indecomposable module M in Θ can be written as a linear combination of the classes of elements on the bottom row of the stable part of Θ (see diagram that follows). This is obvious if M lies on the bottom row of Θ. So assume that M is not on the bottom row. Moreover, to prove this for M it suffices to do so for $\tau^n(M)$ for some $n \in \mathbb{Z}$. This is because the functor τ is additive.

The stable part of the component Θ has the form

$$\begin{tikzcd}
 & & & & \cdots & & \\
 & \cdots \arrow[dr] & & X_2 \arrow[ur] \arrow[dr] & & \tau^{-1}(X_2) & \\
\cdots \arrow[dr] & & X_1 \arrow[ur] \arrow[dr] & & \tau^{-1}(X_1) \arrow[ur] \arrow[dr] & & \cdots \\
 & X_0 \arrow[ur] & & \tau^{-1}(X_0) \arrow[ur] & & \tau^{-2}(X_0) \arrow[ur] &
\end{tikzcd}$$

Without loss of generality, we can assume that $M = X_{n+1}$ for some n. In the case that $n = 0$, the almost split sequence for M has the form

$$0 \longrightarrow X_0 \longrightarrow X_1 \oplus \epsilon \longrightarrow \tau^{-1}(X_0) \longrightarrow 0$$

where ϵ is either the zero module or the projective module P. This is an admissible sequence by [6]. Hence we have that $[M] = [X_1] = [X_0] + [\tau^{-1}(X_0)] - [\epsilon]$ in $K_0(\mathcal{C}(kG))$.

If $n > 0$, then there is an almost split sequence having the form

$$0 \longrightarrow X_n \longrightarrow X_{n+1} \oplus \tau^{-1}(X_{n-1}) \longrightarrow \tau^{-1}(X_n) \longrightarrow 0.$$

Again this sequence is admissible, and we have that $[X_n] + [\tau^{-1}(X_n)] = [X_{n+1}] + [\tau^{-1}(X_{n-1})]$. The proposition now follows by induction. □

As an example of the applicability of the following corollary, recall that a finite p-group G has a single block and this block has wild representation type provided $\Pi(G)$ has dimension at least 1 and G is not a dihedral, quaternion or semi-dihedral 2-group. The proof of this corollary is essentially a verbatim repetition of the proof of [6, 8.8] granted Proposition 4.6.

Corollary 4.7. *Let G be a finite group and assume that k is algebraically closed. Assume that there exists an indecomposable kG-module of constant Jordan type with stable Jordan type $\underline{a} = \sum_{i=1}^{p-1} a_i[i]$ which lies in a block of wild representation type. Then there exists an indecomposable kG-module of constant Jordan type with stable Jordan type $n\underline{a}$ for any $n > 0$.*

Proof. By Erdmann's Theorem [7], the connected component of the Auslander-Reiten quiver of such a module has tree class A_∞. In the notation of the proof of Proposition 4.6, if X_0 has stable constant Jordan type $\underline{a}$, then because τ commutes with restrictions along π-points, so also does $\tau^n(X_0)$. Thus, by the relations developed in the proof, X_1 has stable constant Jordan type $2\underline{a}$, and inductively, X_n has stable constant Jordan type $(n+1)\underline{a}$. □

For an arbitrary finite group scheme, Corollary 4.7 would appear to remain valid in view of work of R. Farnsteiner [8], [9].

We next give a cohomological criterion for the realizability of all stable Jordan types by virtual modules of constant Jordan type.

Theorem 4.8. *Let G be a finite group scheme with the property that there exist odd dimensional classes $\zeta_1 \in \mathrm{H}^{2d_1-1}(G,k)$, $\ldots$, $\zeta_m \in \mathrm{H}^{2d_m-1}(G,k)$ such that $\cap_{i=1}^m V(\beta(\zeta_i)) = 0$, where $\beta : \mathrm{H}^{odd}(G,k) \to \mathrm{H}^{ev}(G,k)$ is the Bockstein cohomological operation of degree $+1$. Let*

$$L_{\zeta_1,\ldots,\zeta_m} \;=\; \ker\{ \textstyle\sum \tilde{\zeta}_i : \sum_{i=1}^m \Omega^{2d_i-1}(k) \longrightarrow k \}.$$

If $p > 2$, then $L_{\zeta_1,\ldots,\zeta_m}$ is a module of constant Jordan type whose stable Jordan type has the form $(m-1)[p-1] + 1[p-2]$. Consequently, for such G,

$$\overline{\mathrm{JType}} : K_0(\mathcal{C}(kG)) \;\longrightarrow\; \mathbb{Z}^{p-1}$$

is surjective.

Proof. Because $\cap_{i=1}^m V(\beta(\zeta_i)) = 0$, there does not exist an equivalence class of π-points $[\alpha_K] \in \Pi(G)$ with $\alpha_K^*(\beta(\zeta_{i,K})) = 0 \in \mathrm{H}^{2d_i}(K[t]/t^p, K)$ for all $i, 1 \leq i \leq m$. Hence, for each α_K of G there exists some i such that $\alpha_K^*(\zeta_{i,K}) \neq 0 \in \mathrm{H}^{2d_i-1}(K[t]/t^p, K)$. Hence, Proposition [6, 6.7] implies that $L_{\zeta_1,\ldots,\zeta_m}$ is of constant Jordan type with stable Jordan type $(m-1)[p-1] + 1[p-2]$.

The second assertion follows from the first by Proposition 4.3(2). □

We proceed to verify (in Proposition 4.11) below that the condition of Theorem 4.8 is satisfied by many finite groups. To do so, we use the following theorem of D. Quillen which asserts that $\mathrm{H}^*(GL(n,\mathbb{F}_\ell),k)$ maps isomorphically onto the invariants of the cohomology of a direct product of cyclic groups. Specifically, we have the following.

Theorem 4.9. *(D. Quillen [16, §8]) Assume $p > 2$, let $\mathbb{F}_\ell$ be a finite field with $(\ell, p) = 1$ and let r be the least integer such that p divides the order of the units $\mathbb{F}_{\ell^r}^*$ of $\mathbb{F}_{\ell^r}$. Let $\pi = Gal(\mathbb{F}_{\ell^r}/\mathbb{F}_\ell)$. Then the restriction map*

$$\mathrm{H}^*(GL(n,\mathbb{F}_\ell),k) \;\to\; \mathrm{H}^*((\mathbb{F}_{\ell^r}^*)^{\times m},k)^{\pi^{\times m} \rtimes \Sigma_m}$$

is an isomorphism, where $n = mr+e$, $0 \leq e < r$. In particular, $\mathrm{H}^*(GL(n,\mathbb{F}_\ell),k)_{red}$ is a polynomial algebra on generators $x_i \in \mathrm{H}^{2ri}(GL(n,\mathbb{F}_\ell),k)), 1 \leq i \leq m$.

The following is an easy consequence of Quillen's theorem.

Proposition 4.10. *We retain the hypotheses and notation of Theorem 4.9. Assume furthermore that p^2 does not divide the order of the units of $\mathbb{F}_{\ell^r}$. Then the cohomology $\mathrm{H}^*((\mathbb{F}_{\ell^r}^*)^{\times m},k)^{\pi^{\times m}}$ is an exterior algebra on generators*

$t_1, \ldots, t_m$ in degrees $2r-1$ tensor a symmetric algebra on the Bocksteins of the t_i's, $u_1 = \beta(t_1), \ldots, u_m = \beta(t_m)$ in degrees $2r$.

Set ω_i^s to be the class

$$\sum_{j_1 < \ldots < j_i} t_{j_s} \cdot u_{j_1} \cdots u_{j_{s-1}} \cdot u_{j_{s+1}} \cdots u_{j_i} \;\in\; \mathrm{H}^{2ri-1}((\mathbb{F}_{\ell^r}^*)^{\times m}, k)^{\pi^{\times m}}$$

for any $s, 1 \leq s \leq i$, and set ζ_i to be the class

$$\sum_{1 \leq s \leq i} \omega_i^s \in \; (\mathrm{H}^{2ri-1}((\mathbb{F}_{\ell^r}^*)^{\times m}, k)^{\pi^{\times m}})^{\Sigma_m} \;\cong\; \mathrm{H}^{2ri}(GL(n, \mathbb{F}_\ell), k).$$

Then the Bockstein applied to ζ_i equals i times x_i,

$$\beta(\zeta_i) \;=\; i \cdot x_i \in \mathrm{H}^{2ri}(GL(n, \mathbb{F}_\ell), k).$$

Proof. The Bockstein of each ω_i^s equals $\sum_{j_1 < \ldots < j_i} u_{j_1} \cdots u_{j_r} \in \mathrm{H}^{2ri}(T(n, \mathbb{F}_\ell), k)$, for any s, where $T(n, \mathbb{F}_\ell)$ is the torus which we can take to consist of the diagonal matrices. This element is equal to the restriction of $x_i \in \mathrm{H}^{2ri}(GL(n, \mathbb{F}_\ell), k)$. Thus, $i \cdot x_i$ and $\beta(\zeta_i) \in \mathrm{H}^{2ri}(GL(n, \mathbb{F}_\ell), k)$ both restrict to $i \cdot \sum_{j_1 < \ldots j_i} u_{j_1} \cdots u_{j_i} \in \mathrm{H}^{2ri}(T(n, \mathbb{F}_\ell), k)$. Since the restriction map $\mathrm{H}^{2ri}(GL(n, \mathbb{F}_\ell), k) \to \mathrm{H}^{2ri}(T(n, \mathbb{F}_\ell), k)$ is injective, the proposition follows. □

Proposition 4.11. *Assume $p > 2$, let $\mathbb{F}_\ell$ be a finite field with $(\ell, p) = 1$ and let r be the least integer such that p divides the order of the units $\mathbb{F}_{\ell^r}^*$ of $\mathbb{F}_{\ell^r}$. Let $\pi = Gal(\mathbb{F}_{\ell^r}/\mathbb{F}_\ell)$. Assume further that p^2 does not divide the order of the units $\mathbb{F}_{\ell^r}^*$ of $\mathbb{F}_{\ell^r}$. If $n = mr + e, 0 \leq e < r$ and if $m < p$, then any finite group G admitting an embedding $G \subset GL(n, \mathbb{F}_\ell)$ satisfies the hypothesis of Theorem 4.8.*

In particular, for any such finite group G,

$$\overline{\mathrm{JType}} : K_0(\mathcal{C}(kG)) \longrightarrow \mathbb{Z}^{p-1}$$

is surjective.

Proof. By Theorem 4.8, we may construct a $kGL(n, \mathbb{F}_\ell)$-module $L_{\zeta_1, \ldots, \zeta_m}$ (see 4.8) of constant Jordan type with stable Jordan type $m[p-1] + 1[p-2]$. The restriction of $L_{\zeta_1, \ldots, \zeta_m}$ to kG remains a module of constant Jordan type with the same stable Jordan type. Thus, the corollary follows by applying Proposition 4.3(2). □

5 Stratification by $\mathcal{C}(kG)$

The modules of constant Jordan type do not form a triangulated subcategory of the stable module category stmod(kG). On the other hand, $\mathcal{C}(kG)$ enjoys many good properties: as recalled earlier (from [6]), $\mathcal{C}(kG)$ is closed under direct sums, tensor products, k-linear duals, retracts, and Heller shifts. In $\mathrm{stmod}(kG)$, $M \mapsto \Omega^{-1}(M)$ is the translation functor on the triangulated category $\mathrm{stmod}(kG)$. Hence, $\mathcal{C}(kG)$ is a full subcategory of $\mathrm{stmod}(kG)$ closed under translations.

In this section, we briefly consider "thickenings" of $\mathcal{C}(kG) \subset \mathrm{stmod}(kG)$, adopting terminology of [14]. The question here is what modules can be assembled by successive extensions of modules of constant Jordan type. This is motivated partly by the observation that if a kG-module M is an extension of two modules of constant Jordan type then there is lower bound on the Jordan type (and perhaps even a minimal Jordan type) that can occur at any π-point, namely, the sum of Jordan types of the two extending modules. A well known class of examples are the modules L_ζ for $\zeta \in \mathrm{H}^{2m}(G,k)$ for some $m > 0$. Such a module is defined by a sequence

$$0 \longrightarrow L_\zeta \longrightarrow \Omega^{2m}(k) \xrightarrow{\tilde{\zeta}} k \longrightarrow 0$$

where the map $\tilde{\zeta}$ represents the cohomology class ζ. For any π-point α_K of G, $\alpha_K^*(L_{\zeta,K})$ has Jordan type $s[p]$ for some $s > 0$ if $\alpha_K^*(\zeta_K) \neq 0$ and Jordan type $s[p] + [p-1] + [1]$ if $\alpha_K^*(\zeta_K) = 0$.

Definition 5.1. *Set* $\mathrm{thick}^1(\mathcal{C})$ *equal to* $\mathcal{C}(kG)$*. For* $n > 1$*, set* $\mathrm{thick}^n(\mathcal{C})$ *equal to the smallest full subcategory of* $\mathrm{stmod}(kG)$ *which is closed under retracts and which contains all finite dimensional* kG*-modules* M *fitting in a distinguished triangle* $M' \to M \to N \to \Omega^{-1}(M')$ *with* $M' \in \mathrm{thick}^{n-1}(\mathcal{C})$ *and* N *a* kG*-module of constant Jordan type (i.e.,* $M \in \mathrm{thick}^1(\mathcal{C})$*).*

Furthermore, define

$$\mathrm{Thick}(\mathcal{C}) \ \equiv \ \cup_n \mathrm{thick}^n(\mathcal{C}) \ \subset \ \mathrm{stmod}(kG),$$

the smallest thick subcategory of $\mathcal{C}(kG)$ *containing* $\mathcal{C}(kG)$*.*

Any finite dimensional kG-module has a finite filtration with associated graded module a direct sum of irreducible modules (obtained by successively considering socles of quotient modules). Since the only isomorphism class of irreducible modules for a finite p-group is that of the trivial module and since every irreducible $k(SL_2)_1$-module has constant Jordan type by [6], we immediately conclude the following proposition.

Proposition 5.2. *Let* G *be a finite group scheme with the property that every irreducible* kG*-module has constant Jordan type. Then*

$$\mathrm{Thick}(\mathcal{C}) \ = \ \mathrm{stmod}(kG).$$

In particular, if G *is either a finite* p*-group or the first infinitesimal kernel* $(SL_2)_1$ *of* SL_2*, then* $\mathrm{Thick}(\mathcal{C}) \ = \ \mathrm{stmod}(kG)$*.*

The socle filtration employed in the proof of the above filtration might not be a good measure of the "level" of a kG-module as we shall see in Proposition 5.9.

Recall that a block B of kG is said to have defect group a given p-subgroup $P \subset G$ if P is the smallest subgroup of G such that every module in B is a direct summand of a kG-module obtained as the induced module of a kP-module. As shown in [11, 4.12], this implies that any kG-module in B has support in $i_*(\Pi(P)) \subset \Pi(G)$, where $i : \Pi(P) \to \Pi(G)$ is induced by the inclusion $P \subset G$.

Proposition 5.3. *Let G be a finite group and let B be a block of kG with defect group $P \neq \{1\}$. If $i_* : \Pi(P) \to \Pi(G)$ is not surjective, then no non-projective module in B has constant Jordan type.*

Moreover, if B is such a block, then no non-projective module in B is in $\mathrm{Thick}(\mathcal{C})$.

Proof. If M is a kG-module in B, then the support variety $\Pi(G)_M$ is contained in $i_*(\Pi(P)) \subset \Pi(G)$, a proper subvariety of $\Pi(G)$. Consequently, M does not have constant Jordan type. Thus the only modules of constant Jordan type in B are the projective modules. If a B-module were in $\mathrm{Thick}(\mathcal{C})$, then it would have to be an extension of projective B-modules and hence projective. □

Remark 5.4. The hypothesis of Proposition 5.3 is satisfied in numerous examples. For example the alternating group A_9 on nine letters and the first Janko group J_1 have such blocks in characteristic 2. The Mathieu group M_{12} has such a block in characteristic 3.

We conclude this investigation of modules of constant Jordan type by returning to the special case $G = E$ an elementary abelian p-group. We utilize a class of modules of constant Jordan type discovered by Andrei Suslin, those with the "constant image property." The widespread prevalence of such modules reveals that $\mathcal{C}(kE)$ must necessarily be large and complicated.

We consider an elementary abelian p-group E of rank r, and we consistently use the notation $kE = k[x_1, \ldots, x_r]/(x_1^p, \ldots, x_r^p)$. We denote by I_E the augmentation ideal of kE, and by $\overline{k}$ some choice of algebraic closure of k.

The following definition is similar to those definitions found in [11, §1]

Definition 5.5. *A finite dimensional kE-module M is said to have the constant image property if for any $0 \neq w_\alpha = \alpha_1 x_1 + \cdots + \alpha_r x_r \in \overline{k}E$,*

$$w_\alpha M_{\overline{k}} \;=\; \mathrm{Rad}\, M_{\overline{k}}.$$

Example 5.6. The module

$$M \;=\; \mathrm{Rad}^{(r-1)(p-1)}(kE) \equiv I_E^{(r-1)(p-1)}$$

has the constant image property.

To verify this, first observe that $M_{\overline{k}} = \mathrm{Rad}^{(r-1)(p-1)}(\overline{k}E)$, so that we may assume that k is algebraically closed. Every non-zero monomial of degree $(r-1)(p-1)+1$ in $k[x_1, \dots, x_r]$ has the form $x_1^{a_1} \cdots x_r^{a_r}$ for $0 \le a_i < p$ for all i. But as $a_1 + \cdots + a_r = (r-1)(p-1)+1$, it must be that $a_1 > 0$; i.e., $I_E^{(r-1)(p-1)+1} = x_1 I_E^{(r-1)(p-1)}$. Hence,

$$\mathrm{Rad}\, M \equiv I_E M = \mathrm{Rad}^{(r-1)(p-1)+1}(kE) = x_1 M$$

and we have verified the condition of Definition 5.5 for $w_\alpha = x_1$. This is sufficient, for given any $w_\alpha \neq 0$ there is an automorphism of kE which takes w_α to x_1 and takes I_E^n to I_E^n for all n.

Remark 5.7. As the reader can easily verify, the direct sum of modules with the constant image property and any quotient of a module with constant image property again have the constant image property. Thus, starting with Example 5.6, we obtain many additional modules with constant image property. Moreover, a simple induction argument implies that if the kE-module M has the constant image property, then for any $n > 0$ and any $0 \neq w_\alpha = \alpha_1 x_1 + \cdots + \alpha_r x_r \in \overline{k}E$,

$$w_\alpha^n M_{\overline{k}} = \mathrm{Rad}^n M_{\overline{k}}.$$

Proposition 5.8. *Suppose that M is a kE-module with the constant image property. Then M has constant Jordan type.*

Proof. For any $\alpha = (\alpha_1, \dots, \alpha_r) \neq \underline{0} \in \overline{k}^r$, the Jordan type of w_α on $M_{\overline{k}}$ is equivalent to the data consisting of the sequence of dimensions

$$\mathrm{Dim}\, M_{\overline{k}}, \quad \mathrm{Dim}\, w_\alpha M_{\overline{k}}, \quad \mathrm{Dim}\, w_\alpha^2 M_{\overline{k}}, \quad \dots, \quad \mathrm{Dim}\, w_\alpha^{p-1} M_{\overline{k}}.$$

Thus, Remark 5.7 implies that each of the w_α have the same Jordan type on $M_{\overline{k}}$. According to the original definition of maximal Jordan type [11, 1.4], some w_α has maximal Jordan type for M. Thus, the complement of the non-maximal variety of M contains all $\overline{k}$-rational points of $\mathbb{A}^r$, and hence the non-maximal variety of M is empty. This is equivalent to the assertion that M has constant Jordan type. $\square$

Proposition 5.9. *Suppose that E is an elementary abelian p-group of rank r. Then*

$$\mathrm{thick}^{2r}(\mathcal{C}) = \mathrm{stmod}(kE).$$

Proof. We first assume that $\mathrm{Rad}^p(M) = I_E^p M = 0$ and proceed to verify that M is in $\mathrm{thick}^2(\mathcal{C})$. Namely, let $Q = kE^t$ be the injective hull of M. Then we have an injection $\varphi : M \longrightarrow kE^t$. Because $I_E^p M = 0$, we must have that $\varphi(M) \subseteq I_E^{(r-1)(p-1)} kE^t = (I_E^{(r-1)(p-1)})^t$. This yields the exact sequence

$$0 \longrightarrow M \longrightarrow (I_E^{(r-1)(p-1)})^t \longrightarrow N \longrightarrow 0$$

where N is the quotient. Hence in $\mathrm{stmod}(kG)$ there is a distinguished triangle

$$M \longrightarrow (I_E^{(r-1)(p-1)})^t \longrightarrow N \longrightarrow \Omega^{-1}(M) \longrightarrow \cdots$$

Consequently, applying Proposition 5.8, we conclude that $M \in \mathrm{thick}^2(\mathcal{C})$.

In general, the modules

$$M/Rad^p(M),\ \mathrm{Rad}^p(M)/\,\mathrm{Rad}^{2p}(M),\ \ldots,\ \mathrm{Rad}^{(r-1)p}(M)/\,\mathrm{Rad}^{rp}(M)$$

each belong to $\mathrm{thick}^2(\mathcal{C})$, because they are all annihilated by I_E^p. Because $I_E^{rp}(M) = 0$, this readily implies that M is in $\mathrm{thick}^{2r}(\mathcal{C})$ as asserted. □

References

1. M. Auslander, I. Reiten, and S. Smalo, Representation Theory of Artin Algebras, Cambridge Studies in Advanced Mathematics 36, 1995.
2. V. Basev, *Representations of the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ in a field of characteristic 2*, (Russian) Dokl. Akad. Nauk. USSR **141** (1961), 1015- 1018.
3. D. Benson, Representations and Cohomology, vols. I and II, Cambridge University Press, 1991.
4. D. Benson, *Resolutions over symmetric algebras with radical cube zero*, (preprint).
5. D. J. Benson and J. F. Carlson, *Diagrammatic methods for group representations and cohomology*, Comm. in Algebra, **15**(1987), 53-121.
6. J. Carlson, E. Friedlander, and J. Pevtsova, Modules of constant Jordan type, To appear in J. fur die reine und ang. Math. (2007).
7. K. Erdmann, *On Auslander-Reiten components of group algebras*, J. Pure Appl. Algebra **104** (1995), 149-160.
8. R. Farnsteiner, *On the Auslander-Reiten quiver of an infinitesimal group*, Nagoya Math. J. **160** (2000), 103-121.
9. R. Farnsteiner, *Support spaces and Auslander-Reiten components*, to appear in Composit. Math.
10. E. Friedlander and J. Pevtsova, *Π-supports for modules for finite group schemes*, To appear in the Duke Math J.
11. E. Friedlander, J. Pevtsova, and A. Suslin, *Maximal and generic Jordan types*, To appear in Inventiones Math.
12. E. Friedlander and B. Parshall, *Modular representation theory of Lie algebras*, American J. Math **110** (1988), 1055-1094.
13. A. Heller, I Reiner, *Indecomposable representations*, Illinois J. Math 5 (1961), 314-323.
14. S. Iyengar, *Levels in triangulated categories*, Lecture given at Leeds.
15. J. C. Jantzen, Representations of Algebraic Groups, Academic Press (1987).
16. D. Quillen, *On the cohomology and K-theory of the general linear groups over a finite field*, Annals of Math **96** (1972), 552-586.
17. D. Quillen, Higher K-theory, I; in Algebraic K-theory I, SLN 341 (1973), 77 –139.
18. C. M. Ringel, *The indecomposable representations of the dihedral 2-groups*, Math. Ann. 214 (1975), 19–34.