Gerstenhaber and Batalin-Vilkovisky structures on Lagrangian intersections

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Summary. Let M and N be Lagrangian submanifolds of a complex symplectic manifold S. We construct a Gerstenhaber algebra structure on $\mathcal{T}or_*^{\mathcal{O}_S}(\mathcal{O}_M, \mathcal{O}_N)$ and a compatible Batalin-Vilkovisky module structure on $\mathcal{E}xt^*_{\mathcal{O}_S}(\mathcal{O}_M, \mathcal{O}_N)$. This gives rise to a de Rham type cohomology theory for Lagrangian intersections.

Introduction

We are interested in intersections of Lagrangian submanifolds of holomorphic symplectic manifolds. Thus we work over the complex numbers in the analytic category.

There are two main aspects of this paper we would like to explain in the introduction: categorification of intersection numbers, and Gerstenhaber and Batalin-Vilkovisky structures on Lagrangian intersections.

Categorification of Lagrangian intersection numbers

This paper grew out of an attempt to categorify Lagrangian intersection numbers. We will explain what we mean by this, and how we propose a solution to the problem. Our construction looks very promising, but is still conjectural.

Lagrangian intersection numbers: smooth case

Let S be a (complex) symplectic manifold and L, M Lagrangian submanifolds. Since L and M are half-dimensional, the expected dimension of their intersection is zero. Intersection theory therefore gives us the intersection number

$$\#(L \cap M)\,,$$

if the intersection is compact. In the general case, we get a class

$$[L \cap M]^{\mathrm{vir}} \in A_0(L \cap M)$$

in degree zero Borel-Moore homology, such that in the compact case

$$\#(L \cap M) = \deg[L \cap M]^{\mathrm{vir}} .$$

If the intersection $X = L \cap M$ is smooth,

$$[X]^{\mathrm{vir}} = c_{\top}(E) \cap [X] ,$$

where E is the excess bundle of the intersection, which fits into the exact sequence

$$0 \longrightarrow T_X \longrightarrow T_L|_X \oplus T_M|_X \longrightarrow T_S|_X \longrightarrow E \longrightarrow 0$$

of vector bundles on X. The symplectic form σ defines an isomorphism $T_S|_X = \Omega_S|_X$. Under this isomorphism, the subbundle $T_L|_X$ corresponds to the conormal bundle $N^{\vee}_{L/S}$. Thus we can rewrite our exact sequence as

$$0 \longrightarrow E^{\vee} \longrightarrow N^{\vee}_{L/S} \oplus N^{\vee}_{M/S} \longrightarrow \Omega_S|_X \longrightarrow \Omega_X \longrightarrow 0 ,$$

which shows that the excess bundle E is equal to the cotangent bundle Ω_X. Thus, in the smooth case,

$$[X]^{\mathrm{vir}} = c_{\top}(E) \cap [X] = c_{\top}(\Omega_X) \cap [X] = (-1)^n c_{\top}(T_X) \cap [X] ,$$

and in the smooth and compact case,

$$\#(L \cap M) = \deg[X]^{\mathrm{vir}} = (-1)^n \int_X c_{\top}(T_X) = (-1)^n \chi(X) ,$$

where $2n$ is the dimension of S and $\chi(X)$ is the topological Euler characteristic of X. This shows that we can make sense of the intersection number even if the intersection is not compact: define the intersection number to be the signed Euler characteristic.

Intersection numbers: singular case

In [1], it was shown how to make sense of the statement that Lagrangian intersection numbers are signed Euler characteristics in the case that the intersection X is singular. An integer invariant $\nu_X(P) \in \mathbb{Z}$ of the singularity of the analytic space X at the point $P \in X$ was introduced.

In the case of a Lagrangian intersection $X = L \cap M$, the number $\nu_X(P)$ can be described as follows. Locally around P, we can assume that S is equal to the cotangent bundle of M and $M \subset S$ is the zero section. Moreover, we can assume that L is the graph of a closed, even exact, 1-form ω on M. If $\omega = df$, for a holomorphic function $f : M \to \mathbb{C}$, defined near P, then

$$\nu_X(P) = (-1)^n \big(1 - \chi(F_P)\big) , \tag{1}$$

where $n = \dim M$ and F_P is the Milnor fibre of f at P.

The main theorem of [1] implies that if L and M are Lagrangian submanifolds of the symplectic manifold S, with compact intersection X, then

$$\#X = \deg[X]^{\mathrm{vir}} = \chi(X, \nu_X)\,,$$

the weighted Euler characteristic of X with respect to the constructible function ν_X, which is defined as

$$\chi(X, \nu_X) = \sum_{i \in \mathbb{Z}} i \cdot \chi(\{\nu_X = i\})\,.$$

In particular, arbitrary Lagrangian intersection numbers are always well-defined: the intersection need not be smooth or compact. The integer $\nu_X(P)$ may be considered as the contribution of the point P to the intersection $X = L \cap M$.

Categorifying intersection numbers: smooth case

To categorify the intersection number means to construct a cohomology theory such that the intersection number is equal to the alternating sum of Betti numbers. If X is smooth (not necessarily compact) a natural candidate is (shifted) holomorphic de Rham cohomology

$$\#X = (-1)^n \chi(X) = \sum (-1)^{i-n} \dim_{\mathbb{C}} \mathbb{H}^i\big(X, (\Omega_X^\bullet, d)\big)\,.$$

Here $(\Omega_X^\bullet, d)$ is the holomorphic de Rham complex of X and $\mathbb{H}^i$ its hypercohomology. Of course, by the holomorphic Poincaré lemma, hypercohomology reduces to cohomology.

Categorification: compact case

If the intersection $X = L \cap M$ is compact, but not necessarily smooth, we have

$$\begin{aligned}\#X &= \sum_i (-1)^{i-n} \dim_{\mathbb{C}} \mathrm{Ext}^i_{\mathcal{O}_S}(\mathcal{O}_L, \mathcal{O}_M)\\ &= \sum_{i,j} (-1)^i (-1)^{j-n} \dim_{\mathbb{C}} H^i\big(X, \mathcal{E}xt^j_{\mathcal{O}_S}(\mathcal{O}_L, \mathcal{O}_M)\big)\,.\end{aligned}$$

If X is smooth, $\mathcal{E}xt^j_{\mathcal{O}_S}(\mathcal{O}_L, \mathcal{O}_M) = \Omega_X^j$, so this reduces to Hodge cohomology

$$\#X = \sum_{i,j} (-1)^i (-1)^{j-n} \dim_{\mathbb{C}} H^i(X, \Omega_X^j)\,.$$

This justifies using the sheaves $\mathcal{E}xt^j_{\mathcal{O}_S}(\mathcal{O}_L, \mathcal{O}_M)$ as replacements for the sheaves Ω^j_X if X is not smooth any longer. To get finite-dimensional cohomology groups, we will construct de Rham type differentials

$$d : \mathcal{E}xt^j_{\mathcal{O}_S}(\mathcal{O}_L, \mathcal{O}_M) \longrightarrow \mathcal{E}xt^{j+1}_{\mathcal{O}_S}(\mathcal{O}_L, \mathcal{O}_M)\,,$$

so that the hypercohomology groups

$$\mathbb{H}^i\big(X, (\mathcal{E}xt^\bullet_{\mathcal{O}_S}(\mathcal{O}_L, \mathcal{O}_M), d)\big)$$

are finite dimensional, even if X is not compact. Returning to the compact case, for any such d, we necessarily have

$$\#X = \sum_i (-1)^{i-n} \dim_{\mathbb{C}} \mathbb{H}^i\big(X, (\mathcal{E}xt^\bullet_{\mathcal{O}_S}(\mathcal{O}_L, \mathcal{O}_M), d)\big)\,.$$

Categorification: local case

Every symplectic manifold S is locally isomorphic to the cotangent bundle Ω_N of a manifold N. The fibres of the induced vector bundle structure on S are Lagrangian submanifolds, and thus we have defined (locally on S) a foliation by Lagrangian submanifolds, i.e., a *Lagrangian foliation.* (Lagrangian foliations are also called *polarizations.*) We may assume that the leaves or our Lagrangian foliation of S are transverse to the two Lagrangians L and M whose intersection we wish to study. Then L and M turn into the graphs of 1-forms on N. The Lagrangian condition implies that these 1-forms on N are closed. Without loss of generality, we may assume that one of these 1-forms is the zero section of Ω_N and hence identify M with N. By making $M = N$ smaller if necessary, we may assume that the closed 1-form defined by L is exact. Then L is the graph of the 1-form df, for a holomorphic function f on M. Thus the intersection $L \cap M$ is now the zero locus of the 1-form df:

$$X = Z(df)\,.$$

This is the *local case.*

Multiplying by df defines a differential

$$\begin{aligned} s : \Omega^j_M &\longrightarrow \Omega^{j+1}_M \\ \omega &\longmapsto df \wedge \omega\,. \end{aligned}$$

Because df is closed, the differential s commutes with the de Rham differential $d : \Omega^j_M \to \Omega^{j+1}_M$. Thus the de Rham differential passes to cohomology with respect to s:

$$d : h^j(\Omega^\bullet_M, s) \longrightarrow h^{j+1}(\Omega^\bullet_M, s)\,,$$

where h^j denotes the cohomology sheaves, which are coherent sheaves of $\mathcal{O}_X$-modules. Let us denote these cohomology sheaves by

$$\mathcal{E}^j = h^j(\Omega_M^\bullet, s)\,.$$

We have thus defined a complex of sheaves on X

$$(\mathcal{E}^\bullet, d)\,, \tag{2}$$

where the $\mathcal{E}^i$ are coherent sheaves of $\mathcal{O}_X$-modules, and the differential d is $\mathbb{C}$-linear. It is a theorem of Kapranov [2], that the cohomology sheaves $h^i(\mathcal{E}^\bullet, d)$ are constructible sheaves on X and thus have finite dimensional cohomology groups. It follows that the hypercohomology groups

$$\mathbb{H}^i\big(X, (\mathcal{E}^\bullet, d)\big)$$

are finite-dimensional as well.

We conjecture that the constructible function

$$P \mapsto \sum_i (-1)^{i-n} \dim_{\mathbb{C}} \mathbb{H}^i_{\{P\}}\big(X, (\mathcal{E}, d)\big)\,,$$

of fiberwise Euler characteristic of $(\mathcal{E}, d)$ is equal to the function ν_X, from above (1). This would achieve the categorification in the local case. In particular, for the non-compact intersection numbers we would have

$$\chi(X, \nu_X) = \sum_i (-1)^{i-n} \dim_{\mathbb{C}} \mathbb{H}^i\big(X, (\mathcal{E}, d)\big)\,.$$

We remark that if f is a homogeneous polynomial (in a suitable set of coordinates), then this conjecture is true.

To make the connection with the compact case (and because this construction is of central importance to the paper), let us explain why

$$\mathcal{E}^i = \mathcal{E}xt^i_{\mathcal{O}_S}(\mathcal{O}_L, \mathcal{O}_M)\,.$$

Denote the projection $S = \Omega_M \to M$ by π. The 1-form on Ω_M which corresponds to the vector field generating the natural $\mathbb{C}^*$-action on the fibres we shall call α. Then $d\alpha = \sigma$ is the symplectic form on S. We consider the 1-form $s = \alpha - \pi^* df$ on S. Its zero locus in S is equal to the graph of df. Let us denote the subbundle of Ω_S annihilating vector fields tangent to the fibres of π by E. Then $s \in \Omega_S$ is a section of E and we obtain a resolution of the structure sheaf of $\mathcal{O}_L$ over $\mathcal{O}_S$:

$$\cdots \longrightarrow \Lambda^2 E^\vee \xrightarrow{\widetilde{s}} E^\vee \xrightarrow{\widetilde{s}} \mathcal{O}_S\ \ ,$$

where $\widetilde{s}$ denotes the derivation of the differential graded $\mathcal{O}_S$-algebra $\Lambda^\bullet E^\vee$ given by contraction with s. Taking duals and tensoring with $\mathcal{O}_M$, we obtain a complex of vector bundles $(\Lambda E|_M, s|_M)$ which computes $\mathcal{E}xt^i_{\mathcal{O}_S}(\mathcal{O}_L, \mathcal{O}_M)$. One checks that $(\Lambda E|_M, s|_M) = (\Omega_M, s)$.

Categorification: global case

We now come to the contents of this paper. let S be a symplectic manifold and L, M Lagrangian submanifolds with intersection X. Let us use the abbreviation $\mathcal{E}^i = \mathcal{E}xt^i_{\mathcal{O}_S}(\mathcal{O}_L, \mathcal{O}_M)$. The $\mathcal{E}^i$ are coherent sheaves of $\mathcal{O}_X$-modules. The main theorem of this paper is that the locally defined de Rham differentials (2) do not depend on the way we write S as a cotangent bundle, or, in other words, that d is independent of the chosen polarization of S. Thus, the locally defined d glue, and we obtain a globally defined canonical de Rham type differential

$$d : \mathcal{E}^i \to \mathcal{E}^{i+1} .$$

In the case that X is smooth, $\mathcal{E}^i = \Omega^i_X$, and d is the usual de Rham differential. We may call $(\mathcal{E}^\bullet, d)$ the *virtual de Rham complex* of the Lagrangian intersection X. Conjecturally, $(\mathcal{E}, d)$ categorifies Lagrangian intersection numbers in the sense that for the local contribution of the point $P \in X$ to the Lagrangian intersection we have

$$\nu_X(P) = \sum_i (-1)^{i-n} \dim_{\mathbb{C}} \mathbb{H}^i_{\{P\}}\big(X, (\mathcal{E}, d)\big) .$$

Hence, for the non-compact intersection numbers we should have

$$\chi(X, \nu_X) = \sum_i (-1)^{i-n} \dim_{\mathbb{C}} \mathbb{H}^i\big(X, (\mathcal{E}, d)\big) .$$

In particular, if the intersection is compact, $\#X = \chi(X, \nu_X)$ should be the alternating sum of the Betti numbers of the hypercohomology groups of the virtual de Rham complex.

Donaldson-Thomas invariants

Our original motivation for this research was a better understanding of Donaldson-Thomas invariants. It is to be hoped that the moduli spaces giving rise to Donaldson-Thomas invariants (spaces of stable sheaves of fixed determinant on Calabi-Yau threefolds) are Lagrangian intersections, at least locally. We have two reasons for believing this: first of all, the obstruction theory giving rise to the virtual fundamental class is *symmetric*, a property shared by the obstruction theories of Lagrangian intersections. Secondly, at least heuristically, these moduli spaces are equal to the critical set of the holomorphic Chern-Simons functional.

Our 'exchange property' should be useful for gluing virtual de Rham complexes if the moduli spaces are only local Lagrangian intersections.

In this way we hope to construct a virtual de Rham complex on the Donaldson-Thomas moduli spaces and thus categorify Donaldson-Thomas invariants.

Gerstenhaber and Batalin-Vilkovisky structures on Lagrangian intersections

The virtual de Rham complex $(\mathcal{E}^\bullet, d)$ is just one half of the story. There is also the graded sheaf of $\mathcal{O}_X$-algebras $\mathcal{A}^\bullet$ given by

$$\mathcal{A}^i = \mathcal{T}or_{-i}^{\mathcal{O}_S}(\mathcal{O}_L, \mathcal{O}_M)\,,$$

Locally, $\mathcal{A}^\bullet$ is given as the cohomology of $(\Lambda T_M, \widetilde{s})$, in the above notation. The Lie-Schouten-Nijenhuis bracket induces a $\mathbb{C}$-linear bracket operation

$$[\,,]: \mathcal{A}\bullet \otimes_{\mathbb{C}} \mathcal{A}^\bullet \longrightarrow \mathcal{A}^\bullet$$

of degree $+1$. We show that these locally defined brackets glue to give a globally defined bracket making $(\mathcal{A}^\bullet, \wedge, [\,])$ a sheaf of Gerstenhaber algebras.

Then $\mathcal{E}^\bullet$ is a sheaf of modules over $\mathcal{A}^\bullet$. (The module structure is induced by contraction.) The bracket on $\mathcal{A}^\bullet$ and the differential on $\mathcal{E}^\bullet$ satisfy a compatibility condition, see (5). We say that $(\mathcal{E}, d)$ is a *Batalin-Vilkovisky module* over the Gerstenhaber algebra $(\mathcal{A}, \wedge, [\,])$. (This structure has been called a *calculus* by Tamarkin and Tsygan in [4].)

In the case that L and M are oriented submanifolds, i.e., the highest exterior powers of the normal bundles have been trivialized, we have an identification

$$\mathcal{A}^i = \mathcal{E}^{n+i}\,.$$

Transporting the differential from $\mathcal{E}^\bullet$ to $\mathcal{A}^\bullet$ via this identification turns $(\mathcal{A}, \wedge, [\,], d)$ into a Batalin-Vilkovisky algebra.

To prove these facts we have to study *differential* Gerstenhaber algebras and *differential* Batalin-Vilkovisky modules over them. We will prove that locally defined differential Gerstenhaber algebras and their differential Batalin-Vilkovisky modules are quasi-isomorphic, making their cohomologies isomorphic and hence yielding the well-definedness of the bracket and the differential.

First order truncation

In this paper we are only interested in the Gerstenhaber and Batalin-Vilkovisky structures on $\mathcal{A}$ and $\mathcal{E}$. In other words, we only deal with the structures induced on cohomology. This amounts to a truncation of the full derived Lagrangian intersection. Because of our modest goal, we only need to study differential Gerstenhaber and Batalin-Vilkovisky structures *up to first order*. In future research, we hope to address the complete derived structure on Lagrangian intersections.

This would certainly involve studying the Witten deformation of the de Rham complex in more detail. Related work along these lines has been done by Kashiwara and Schapira [3].

Overview

1. Algebra

In this introductory section, we discuss algebraic preliminaries. We review the definitions of *differential Gerstenhaber algebra*, and *differential Batalin-Vilkovisky module*. This is mainly to fix our notation. There are quite a few definitions to keep track of; we apologize for the lengthiness of this section.

2. Symplectic geometry

Here we review a few basic facts about complex symplectic manifolds. In particular, the notions of *Lagrangian foliation*, *polarization*, and the *canonical partial connection* are introduced.

3. Derived Lagrangian intersections on polarized symplectic manifolds

On a *polarized* symplectic manifold, we define *derived intersections* of Lagrangian submanifolds. These are (sheaves of) Gerstenhaber algebras on the scheme theoretic intersection of two Lagrangian submanifolds. The main theorem we prove about these derived intersections is a certain invariance property with respect to *symplectic correspondences*. We call it the *exchange property*.

We repeat this program for *derived homs*, (the Batalin-Vilkovisky case), and *oriented* derived intersections (the oriented Batalin-Vilkovisky case).

4. The Gerstenhaber structure on $\mathcal{T}or$ and the Batalin-Vilkovisky structure on $\mathcal{E}xt$

In this section we use the exchange property to prove that, after passing to cohomology, we do not notice the polarization any more. The Gerstenhaber and Batalin-Vilkovisky structures are independent of the polarization chosen to define them.

This section closes with an example of a symplectic correspondence and the corresponding exchange property.

5. Further remarks

In this final section we define *virtual de Rham cohomology* of Lagrangian intersections. We speculate on what *virtual Hodge theory* might look like. We introduce a natural differential graded category associated to a complex symplectic manifold. (It looks like a kind of holomorphic, de Rham type analogue of the Fukaya category.) Finally, we mention the conjectures connecting the virtual de Rham complex to the perverse sheaf of vanishing cycles.

1 Algebra

Let M be a manifold. Regular functions, elements of $\mathcal{O}_M$, have degree 0. By ΛT_M we mean the graded sheaf of polyvector fields on M. We think of it as a sheaf of graded $\mathcal{O}_M$-algebras (the product being $\wedge$), concentrated in *non-positive* degrees, the vector fields having degree -1. By $\Omega^\bullet_M$ we denote the graded sheaf of differential forms on M. This we think of as a sheaf of graded $\mathcal{O}_M$-modules, concentrated in *non-negative* degrees, with 1-forms having degree $+1$. We will denote the natural pairing of T_M with Ω_M by $X \lrcorner\, \omega \in \mathcal{O}_S$, for $X \in T_M$ and $\omega \in \Omega_M$. The following is, of course, well known:

Lemma 1.1. *There exists a unique extension of $\lrcorner$ to an action of the sheaf of graded $\mathcal{O}_M$-algebras ΛT_M on the sheaf of graded $\mathcal{O}_S$-modules $\Omega^\bullet_M$, which satisfies*

(i) $f \lrcorner\, \omega = f\omega$, for $f \in \mathcal{O}_S$ and $\omega \in \Omega^\bullet_M$ (linearity over $\mathcal{O}_M$),

(ii) $X \lrcorner\,(\omega_1 \wedge \omega_2) = (X \lrcorner\, \omega_1) \wedge \omega_2 + (-1)^{\overline{\omega}_1}\omega_1 \wedge (X \lrcorner\, \omega_2)$, for $X \in T_M$ and $\omega_1, \omega_2 \in \Omega^\bullet_M$, (the degree -1 part acts by derivations),

(iii) $(X \wedge Y) \lrcorner\, \omega = X \lrcorner\,(Y \lrcorner\, \omega)$, for $X, Y \in \Lambda T_M$, $\omega \in \Omega^\bullet_M$ (action property).

Now turn things around and note that any section $s \in \Omega_M$ defines a derivation of degree $+1$ on ΛT_M, which we shall denote by $\widetilde{s}$. It is the unique derivation which extends the map $T_M \to \mathcal{O}_M$ given by $\widetilde{s}(X) = X \lrcorner\, s$, for all $X \in T_M$. (Note that this is not a violation of the universal sign convention, see Remark 1.3.)

Lemma 1.2. *The pair $(\Lambda T_M, \widetilde{s})$ is a sheaf of differential graded $\mathcal{O}_M$-algebras. Left multiplication by s defines a differential on $\Omega^\bullet_M$ and the pair $(\Omega^\bullet_M, s)$ is a sheaf of differential graded modules over $(\Lambda T_M, \widetilde{s})$.*

Proof. This amounts to the formula

$$s \wedge (X \lrcorner\, \omega) = \widetilde{s}(X) \lrcorner\, \omega + (-1)^{\overline{X}} X \lrcorner\,(s \wedge \omega) \tag{3}$$

for all $\omega \in \Omega^\bullet_M$ and $X \in \Lambda T_M$. □

Remark 1.3. Set $\langle X, \omega \rangle$ equal to the degree zero part of $X \lrcorner\, \omega$. This is a perfect pairing $\Lambda T_M \otimes_{\mathcal{O}_M} \Omega^\bullet_M \to \mathcal{O}_M$, expressing the fact that $\Omega^\bullet_M$ is the $\mathcal{O}_M$-dual of ΛT_M. According to Formula (3), we have, if $\deg X + \deg \omega + 1 = 0$,

$$\langle \widetilde{s}(X), \omega \rangle + (-1)^{\overline{X}} \langle X, s \wedge \omega \rangle = 0\,.$$

This means that the derivation $\widetilde{s}$ and left multiplication by s are $\mathcal{O}_S$-duals of one another. To explain the signs, note that we think of $\widetilde{s}$ and s as differentials on the graded sheaves ΛT_M and $\Omega^\bullet_M$, and for differentials of degree $+1$ the sign convention is

$$0 = D\langle X, \omega \rangle = \langle DX, \omega \rangle + (-1)^{\overline{X}} \langle X, D\omega \rangle\,.$$

In particular, for $\deg X = 1$ and $\omega = 1$ we get $\widetilde{s}(X) = \langle X, s \rangle = X \lrcorner\, s$.

Remark 1.4. We can summarize Formula (3) more succinctly as

$$[s, i_X] = i_{\widetilde{s}(X)}\,,$$

where $i_X : \Omega^\bullet_M \to \Omega^\bullet_M$ denotes the endomorphism $\omega \mapsto X \lrcorner\, \omega$.

1.1 Differential Gerstenhaber algebras

Let S be a manifold and A a graded sheaf of $\mathcal{O}_S$-modules.

Definition 1.5. A **bracket** on A of degree $+1$ is a homomorphism

$$[\,] : A \otimes_{\mathbb{C}} A \longrightarrow A$$

of degree $+1$ satisfying:

(i) $[\,]$ is a graded $\mathbb{C}$-linear derivation in each of its two arguments,

(ii) $[\,]$ is graded commutative (not anti-commutative).

If $[\,]$ satisfies, in addition, the Jacobi identity, we shall call $[\,]$ a **Lie bracket**.

The sign convention for brackets of degree $+1$ is that the comma is treated as carrying the degree $+1$, the opening and closing bracket as having degree 0. Thus, when passing an odd element past the comma, the sign changes. For example, the graded commutativity reads:

$$[Y, X] = (-1)^{\overline{X}\overline{Y} + \overline{X} + \overline{Y}}[X, Y]\,.$$

Definition 1.6. A **Gerstenhaber algebra** over $\mathcal{O}_S$ is a sheaf of graded $\mathcal{O}_S$-modules A, concentrated in non-positive degrees, endowed with

(i) a commutative (associative, of course) product $\wedge$ of degree 0 with unit, making A a sheaf of graded $\mathcal{O}_S$-algebras,

(ii) a Lie bracket $[\,]$ of degree $+1$ (see Definition 1.5).

In our cases, the underlying $\mathcal{O}_S$-module of A will always be coherent and $\mathcal{O}_S \to A^0$ will be a surjection of coherent $\mathcal{O}_S$-algebras. The main example is the following:

Example 1.7. Let $M \subset S$ be a submanifold and $A = \Lambda_{\mathcal{O}_M} T_M$ the polyvector fields on M. The bracket is the Schouten-Nijenhuis bracket.

Definition 1.8. A **differential Gerstenhaber algebra** is a Gerstenhaber algebra A over $\mathcal{O}_S$ endowed with an additional $\mathbb{C}$-linear map $\widetilde{s} : A \to A$ of degree $+1$ which satisfies

(i) $[\widetilde{s}, \widetilde{s}] = \widetilde{s}^2 = 0$,

(ii) $\widetilde{s}$ is a derivation with respect to $\wedge$, in particular it is $\mathcal{O}_S$-linear,

(iii) $\widetilde{s}$ is a derivation with respect to $[\,]$.

Thus, neglecting the bracket, a differential Gerstenhaber algebra is a sheaf of differential graded algebras over $\mathcal{O}_S$.

Lemma 1.9. *Let $(A, \widetilde{s})$ be a differential Gerstenhaber algebra. Let $I \subset A^0$ be the image of $\widetilde{s} : A^{-1} \to A^0$. This is a sheaf of ideals in A^0. Then the cohomology $h^*(A, \widetilde{s})$ is a Gerstenhaber algebra with $h^0(A, \widetilde{s}) = A^0/I$.*

Proof. This is clear: the fact that $\widetilde{s}$ is a derivation with respect to both products on A implies that the two products pass to $h^*(A, \widetilde{s})$. Then all the properties of the products pass to cohomology. □

Example 1.10. Let $M \subset S$ and $A = \Lambda T_M$ be as in Example 1.7. In addition, let $s \in \Omega_M$ be a *closed* 1-form. Then $(\Lambda T_M, \widetilde{s})$ with $\wedge$ and Schouten-Nijenhuis bracket $[\,]$ is a differential Gerstenhaber -algebra. The closedness of s makes $\widetilde{s}$ a derivation with respect to $[\,]$.

1.2 Morphisms of differential Gerstenhaber algebras

Definition 1.11. Let A and B be Gerstenhaber algebras over $\mathcal{O}_S$. A **morphism** of Gerstenhaber algebras is a homomorphism $\phi : A \to B$ of graded $\mathcal{O}_S$-modules (of degree zero) which is compatible with both $\wedge$ and $[\,]$:

(i) $\phi(X \wedge Y) = \phi(X) \wedge \phi(Y)$,

(ii) $\phi([X, Y]) = [\phi(X), \phi(Y)]$.

Definition 1.12. Let $(A, \widetilde{s})$ and $(B, \widetilde{t})$ be differential Gerstenhaber algebras over $\mathcal{O}_S$. A **(first order) morphism** of differential Gerstenhaber algebras is a pair $(\phi, \{\,\})$, where $\phi : A \to B$ is a degree zero homomorphism of graded $\mathcal{O}_S$-modules, and $\{\,\} : A \otimes_{\mathbb{C}} A \to B$ is a degree zero $\mathbb{C}$-bilinear map , such that

(i) $\phi(X \wedge Y) = \phi(X) \wedge \phi(Y)$ and $\phi(\widetilde{s}X) = \widetilde{t}\phi(X)$, so that $\phi : A \to B$ is a morphisms of differential graded $\mathcal{O}_S$-algebras,

(ii) $\{\,\}$ is symmetric, i.e., $\{Y, X\} = (-1)^{\overline{XY}}\{X, Y\}$,

(iii) $\{\,\}$ is a $\mathbb{C}$-linear derivation with respect to $\wedge$ in each of its arguments, where the A-module structure on B is given by ϕ, in other words,

$$\{X \wedge Y, Z\} = \phi(X) \wedge \{Y, Z\} + (-1)^{\overline{XY}}\phi(Y) \wedge \{X, Z\}\,,$$

and

$$\{X, Y \wedge Z\} = \{X, Y\} \wedge \phi(Z) + (-1)^{\overline{YZ}}\{X, Z\} \wedge \phi(Y)\,,$$

(iv) the default of ϕ to commute with $[\,]$ is equal to the default of the $\mathcal{O}_S$-linear differentials to behave as derivations with respect to $\{\,\}$,

$$\phi[X, Y] - [\phi(X), \phi(Y)] = (-1)^{\overline{X}}\widetilde{t}\{X, Y\} - (-1)^{\overline{X}}\{\widetilde{s}X, Y\} - \{X, \widetilde{s}Y\}\,. \quad (4)$$

Remark 1.13. We will always omit the qualifier 'first order', as we will not consider any 'higher order' morphisms in this paper. This is because, in the end, we are only interested in the cohomology of our differential Gerstenhaber algebras. To keep track of the induced structure on cohomology, first order morphisms suffice. We hope to return to 'higher order' questions in future research.

Remark 1.14. Suppose all conditions in Definition 1.12 except the last are satisfied. Then both sides of the equation in Condition (iv) are symmetric of degree one and $\mathbb{C}$-linear derivations with respect to $\wedge$ in each of the two arguments. Thus, to check Condition (iv), it suffices to check on $\mathbb{C}$-algebra generators for A.

Lemma 1.15. *A morphism of differential Gerstenhaber algebras*

$$(\phi, \{\,\}) : (A, \widetilde{s}) \longrightarrow (B, \widetilde{t})$$

induces a morphism of Gerstenhaber algebras on cohomology. In other words,

$$h^*(\phi) : h^*(A, \widetilde{s}) \longrightarrow h^*(B, \widetilde{t})$$

respects both $\wedge$ *and* $[\,]$.

Proof. Any morphism of differential graded $\mathcal{O}_S$-algebras induces a morphism of graded algebras when passing to cohomology. Thus $h^*(\phi)$ respects $\wedge$. The fact that $h^*(\phi)$ respects the Lie brackets, follows form Property (iv) of Definition 1.12. All three terms on the right hand side of said equation vanish in cohomology. □

Definition 1.16. A **quasi-isomorphism** of differential Gerstenhaber algebras is a morphism of differential Gerstenhaber algebras which induces an isomorphism of Gerstenhaber algebras on cohomology.

1.3 Differential Batalin-Vilkovisky modules

Definition 1.17. Let A be a Gerstenhaber algebra. A sheaf of graded $\mathcal{O}_S$-modules L, with an action $\lrcorner$ of A, making L a graded A-module, is called a **Batalin-Vilkovisky module** over A, if it is endowed with a $\mathbb{C}$-linear map $d : L \to L$ of degree $+1$ satisfying

(i) $[d, d] = d^2 = 0$,

(ii) For all $X, Y \in A$ and every $\omega \in L$ we have

$$\begin{aligned} d(X \wedge Y \lrcorner \omega) + (-1)^{\overline{X}+\overline{Y}} X \wedge Y \lrcorner d\omega + (-1)^{\overline{X}}[X, Y] \lrcorner \omega = \\ (-1)^{\overline{X}} X \lrcorner d(Y \lrcorner \omega) + (-1)^{\overline{XY}+\overline{Y}} Y \lrcorner d(X \lrcorner \omega)\,. \end{aligned} \tag{5}$$

Remark 1.18. Write i_X for the endomorphism $\omega \mapsto X \lrcorner\, \omega$ of L. Then Formula (5) can be rewritten as

$$[X,Y] \lrcorner\, \omega = \big[[i_X, d], i_Y\big](\omega)$$

or simply

$$i_{[X,Y]} = \big[[i_X, d], i_Y\big]\,. \tag{6}$$

Note also that $\big[[i_X, d], i_Y\big] = \big[i_X, [d, i_Y]\big]$.

The action property $(X \wedge Y) \lrcorner\, \omega = X \lrcorner\, (Y \lrcorner\, \omega)$ translates into $i_{X \wedge Y} = i_X \circ i_Y$.

In our applications, Batalin-Vilkovisky modules will always be coherent over $\mathcal{O}_S$. Note that there is no multiplicative structure on L, so there is no requirement for the differential d to be a derivation.

Example 1.19. Let $M \subset S$ and $A = \Lambda T_M$ be the Gerstenhaber algebra of polyvector fields on M, as in Example 1.7. Then $\Omega_M^\bullet$ with exterior differentiation d is a Batalin-Vilkovisky module over ΛT_M.

Definition 1.20. A **differential Batalin-Vilkovisky module** over the differential Gerstenhaber algebra $(A, \widetilde{s})$ is a Batalin-Vilkovisky module L for the underlying Gerstenhaber algebra A, endowed with an additional $\mathbb{C}$-linear map $s : L \to L$ of degree $+1$ satisfying:

(i) $[s, s] = s^2 = 0$,

(ii) (M, s) is a differential graded module over the differential graded algebra $(A, \widetilde{s})$, i.e., we have

$$s(X \lrcorner\, \omega) = \widetilde{s}(X) \lrcorner\, \omega + (-1)^{\overline{X}} X \lrcorner\, s(\omega)\,,$$

for all $X \in A$, $\omega \in L$. More succinctly: $[s, i_X] = i_{\widetilde{s}(X)}$.

(iii) $[d, s] = 0$.

Note that the differential s is necessarily $\mathcal{O}_S$-linear. This distinguishes it from d.

Lemma 1.21. *Let (L, s) be a differential Batalin-Vilkovisky module over the differential Gerstenhaber algebra $(A, \widetilde{s})$. Then $h^*(L, s)$ is a Batalin-Vilkovisky module for the Gerstenhaber algebra $h^*(A, \widetilde{s})$.*

Proof. First, $h^*(M, s)$ is a graded $h^*(A, \widetilde{s})$-module. The condition $[d, s] = 0$ implies that d passes to cohomology. Then the properties of d pass to cohomology as well. □

Example 1.22. Let $M \subset S$ be a submanifold and $s \in \Omega_M$ a closed 1-form. Then $(\Omega_M^\bullet, s)$ (see Lemma 1.2) is a differential Batalin-Vilkovisky module over the differential Gerstenhaber algebra $(\Lambda T_M, \widetilde{s})$ of Example 1.10.

1.4 Homomorphisms of differential Batalin-Vilkovisky modules

Definition 1.23. Let A and B be Gerstenhaber algebras and $\phi : A \to B$ a morphism of Gerstenhaber algebras. Let L be a Batalin-Vilkovisky module over A and M a Batalin-Vilkovisky module over B. A **homomorphism** of Batalin-Vilkovisky modules of degree n (covering ϕ) is a degree n homomorphism of graded A-modules $\psi : L \to M$ (where the A-module structure on M is defined via ϕ), which commutes with d:

(i) $\psi(X \lrcorner \omega) = (-1)^{n\overline{X}} \phi(X) \lrcorner \psi(\omega)$,

(ii) $\psi d_L(\omega) = (-1)^n d_M \psi(\omega)$.

We write the latter condition as $[\psi, d] = 0$.

Definition 1.24. Let $(A, \widetilde{s})$ and $(B, \widetilde{t})$ be differential Gerstenhaber algebras and $(\phi, \{\,\}) : (A, \widetilde{s}) \to (B, \widetilde{t})$ a morphism of differential Gerstenhaber algebras. Let (L, s) be a differential Batalin-Vilkovisky module over $(A, \widetilde{s})$ and (M, t) a differential Batalin-Vilkovisky module over $(B, \widetilde{t})$. A **(first order) homomorphism** of differential Batalin-Vilkovisky modules of degree n covering $(\phi, \{\,\})$ is a pair (ψ, δ), where $\psi : (L, s) \to (M, t)$ is a degree n homomorphism of differential graded $(A, \widetilde{s})$-modules, where the $(A, \widetilde{s})$-module structure on (M, t) is through ϕ. Moreover, $\delta : L \to M$ is a $\mathbb{C}$-linear map, also of degree n, satisfying

(i) the commutator property

$$\psi \circ d - (-1)^n d \circ \psi = -2(-1)^n t \circ \delta + 2\delta \circ s\,, \tag{7}$$

(ii) compatibility with the bracket $\{\,\}$ property

$$\begin{aligned} \delta(X \wedge Y \lrcorner \omega) + (-1)^{n(\overline{X}+\overline{Y})} \phi(X) \wedge \phi(Y) \lrcorner \delta\omega + (-1)^{n(\overline{X}+\overline{Y})} \{X, Y\} \lrcorner \psi(\omega) \\ = (-1)^{n\overline{X}} \phi(X) \lrcorner \delta(Y \lrcorner \omega) + (-1)^{\overline{XY}+n\overline{Y}} \phi(Y) \lrcorner \delta(X \lrcorner \omega)\,. \end{aligned} \tag{8}$$

Remark 1.25. The same comments as those in Remark 1.13 apply.

Remark 1.26. If we use the same letter s to denote the $\mathcal{O}_S$-linear differentials on L and M, we can rewrite the commutator conditions of Definition 1.24 more succinctly as

$$[\psi, s] = 0 \qquad [\psi, d] - 2[\delta, s] = 0\,.$$

The compatibility with bracket property can be rewritten as

$$\big[\iota_X, [\iota_Y, \delta]\big] = \iota_{\{X,Y\}} \circ \psi\,. \tag{9}$$

Note the absence of a condition on the commutator $[\delta, d]$. This would be a 'higher order' condition.

Remark 1.27. It is a formal consequence of properties of the commutator bracket that the left hand side of (9) is a $\mathbb{C}$-linear derivation in each of its two arguments X, Y. The same is true of the right hand side by assumption. Thus we have: if all properties of Definition 1.24, except for (i) and (ii) are satisfied, then to check that (ii) is satisfied, it suffices to do this for all X and Y belonging to a set of $\mathbb{C}$-algebra generators for A.

Remark 1.28. Suppose all properties of Definition 1.24 except for (i) are satisfied. Suppose also that L is free of rank one as an A-module on the basis $\omega^\circ \in L$. Then it suffices to prove Equation (7) applied to elements of the form $X \lrcorner \omega^0$, where X runs over a set of generators of A as an A^0-module.

Lemma 1.29. *Let $(\psi, \delta) : (L, s) \to (M, t)$ be a homomorphism of differential Batalin-Vilkovisky modules over the morphism $(\phi, \{\,\}) : (A, \widetilde{s}) \to (B, \widetilde{t})$ of differential Gerstenhaber algebras. Then $h^*(\psi) : h^*(L, s) \to h^*(M, t)$ is a homomorphism of Batalin-Vilkovisky modules over the morphism of Gerstenhaber algebras $h^*(\phi) : h^*(A, \widetilde{s}) \to h^*(B, \widetilde{t})$.*

Proof. Evaluating the right hand side of Equation 7 on s-cocycles in L, yields t-boundaries in M. □

1.5 Invertible differential Batalin-Vilkovisky modules

Definition 1.30. We call the Batalin-Vilkovisky module L over the Gerstenhaber algebra A **invertible**, if, locally in S, there exists a section ω° of L such that the evaluation homomorphism

$$\begin{aligned} \Psi^\circ : A &\longrightarrow L \\ X &\longmapsto (-1)^{\overline{X}\overline{\omega}^\circ} X \lrcorner \omega^\circ \end{aligned}$$

is an isomorphism of sheaves of $\mathcal{O}_S$-modules. Any such ω° will be called a (local) **orientation** for L over A.

Note that if the degree of an orientation ω° is n, then $L^k = 0$, for all $k > n$, by our assumption on A. Thus orientations always live in the top degree of L. Moreover, if orientations exist everywhere locally, L^n is an invertible sheaf over A^0.

Lemma 1.31. *Let L be an invertible Batalin-Vilkovisky module over the Gerstenhaber algebra A and assume that ω° is a (global) orientation for L over A. Then, transporting the differential d via Ψ° to A yields a $\mathbb{C}$-linear map of degree $+1$ which we will call $d^\circ : A \to A$. It is characterized by the formula*

$$d^\circ(X) \lrcorner \omega^\circ = d(X \lrcorner \omega^\circ)\,.$$

It squares to 0 and it satisfies:

$$(-1)^{\overline{X}}[X,Y] = d^\circ(X)\wedge Y + (-1)^{\overline{X}} X \wedge d^\circ(Y) - d^\circ(X\wedge Y)\,, \qquad (10)$$

for all $X, Y \in A$. In other words, d° is a **generator** *for the bracket $[\,]$, making A a* **Batalin-Vilkovisky algebra**.

Proof. Follows directly from Formula (5) upon noticing that because ω° is top-dimensional, it is automatically d-closed: $d\omega^\circ = 0$. □

Corollary 1.32. *If the Gerstenhaber algebra admits an invertible Batalin-Vilkovisky module it is locally a Batalin-Vilkovisky algebra.*

Example 1.33. The Batalin-Vilkovisky module $\Omega^\bullet_M$ over the Gerstenhaber algebra ΛT_M of Example1.10 is invertible. Any non-vanishing top-degree form $\omega^\circ \in \Omega^n_M$ is an orientation for $\Omega^\bullet_M$, where $n = \dim M$. Thus, the Schouten-Nijenhuis algebra ΛT_M is a Batalin-Vilkovisky algebra. For Calabi-Yau manifolds, i.e., $\Omega^n_M = \mathcal{O}_S$, a generator for the Batalin-Vilkovisky algebra is given.

Definition 1.34. Let (L, s) be a differential Batalin-Vilkovisky module over the differential Gerstenhaber algebra $(A, \widetilde{s})$. Then (L, s) is called **invertible**, if the underlying Batalin-Vilkovisky module L is invertible over the underlying Gerstenhaber algebra A. An **orientation** for $(A, \widetilde{s})$ is an orientation of the underlying L.

Proposition 1.35. *Let (L, s) be an invertible differential Batalin-Vilkovisky module over the differential Gerstenhaber algebra $(A, \widetilde{s})$. Then under the isomorphism Ψ° defined by an orientation ω° of L over A, the differential $\widetilde{s}$ corresponds to the differential s. In particular, the induced differential d° on A has the property*

$$[d^\circ, \widetilde{s}] = 0\,,$$

besides satisfying (10). Hence $(A, d^\circ, \widetilde{s})$ is a **differential Batalin-Vilkovisky algebra**.

Moreover, the cohomology $h^*(L, s)$ is an invertible Batalin-Vilkovisky module over the Gerstenhaber algebra $h^*(A, \widetilde{s})$. We have $h^n(L, s) = L^n/I$ and the image of any orientation of L over A under the quotient map $L^n \to L^n/I$ gives an orientation for $h^*(L, s)$ over $h^*(A, \widetilde{s})$.

Proof. The equation $s\circ\Psi^\circ = (-1)^{\overline{\omega}^\circ}\Psi^\circ\circ\widetilde{s}$ follows immediately from $[s, i_X] = i_{\widetilde{s}(X)}$ upon noticing that $s(\omega) = 0$. As Ψ° is therefore an isomorphism of differential graded $\mathcal{O}_S$-modules, the cohomology is isomorphic: $h^*(A, \widetilde{s}) \xrightarrow{\sim} h^*(L, s)$. The rest follows from this. □

Example 1.36. For a closed 1-form s on M, the differential Batalin-Vilkovisky module $(\Omega^\bullet_M, s)$ over the differential Gerstenhaber algebra $(\Lambda T_M, \widetilde{s})$ of Example 1.22 is invertible. Any trivialization of Ω^n_M defines an orientation.

1.6 Oriented homomorphisms of invertible Batalin-Vilkovisky modules

Definition 1.37. Let $\phi : A \to B$ be a morphism of Gerstenhaber algebras and $\psi : L \to M$ a homomorphism of invertible Batalin-Vilkovisky modules covering ϕ. Let ω_L° and ω_M° be orientations for L and M, respectively. The homomorphism $\psi : L \to M$ is said to **preserve the orientations** (or be oriented) if $\psi(\omega_L^\circ) = \omega_M^\circ$.

Lemma 1.38. *Suppose given oriented invertible Batalin-Vilkovisky modules L and M over the Gerstenhaber algebras A and B, making A and B into Batalin-Vilkovisky algebras. Suppose $\psi : L \to M$ is an oriented homomorphism of Batalin-Vilkovisky modules. Then under the identifications of L and M with A and B given by ω_L° and ω_M°, the map $\psi : L \to M$ corresponds to $\phi : A \to B$. Hence $\phi : A \to B$ commutes with d°. Thus ϕ is a morphism of Batalin-Vilkovisky algebras: it respects $\wedge$, $[\,]$ and d°.*

Definition 1.39. Let $(\psi, \delta) : (L, s) \to (M, t)$ be a homomorphism of invertible differentiable Batalin-Vilkovisky modules over $(\phi, \{\,\}) : (A, \widetilde{s}) \to (B, \widetilde{t})$. Let ω_L° and ω_M° be orientations for L and M, respectively. We call (ψ, δ) **oriented** if $\psi(\omega_L^\circ) = \omega_M^\circ$ and $\delta(\omega_L^\circ) = 0$.

Proposition 1.40. *Suppose $(\psi, \delta) : (L, s, \omega_L^\circ) \to (M, t, \omega_M^\circ)$ is an oriented homomorphism of oriented invertible differential Batalin-Vilkovisky modules over $(\phi, \{\,\}) : (A, \widetilde{s}) \to (B, \widetilde{t})$. Then $(A, \widetilde{s}, [\,], d^\circ)$ and $(B, \widetilde{t}, [\,], d^\circ)$ are differential Batalin-Vilkovisky algebras. Transporting $\delta : L \to M$ via the identifications of L and M with A and B to a map $\delta^\circ : A \to B$, satisfying*

$$\delta^\circ(X) \lrcorner\, \omega_M = (-1)^{\overline{\delta X}} \delta(X \lrcorner\, \omega_L)\,,$$

we get a triple

$$(\phi, \{\,\}, \delta^\circ) : (A, \widetilde{s}, [\,], d^\circ) \longrightarrow (B, \widetilde{t}, [\,], d^\circ)\,,$$

which satisfies the following conditions:

(i) $\phi : (A, \widetilde{s}) \to (B, \widetilde{t})$ *is a morphism of differential graded algebras,*

(ii) the commutator property

$$\phi \circ d^\circ - d^\circ \circ \phi = -2\,\widetilde{t} \circ \delta^\circ + 2\,\delta^\circ \circ \widetilde{s}\,,$$

or, by abuse of notation, $[\phi, d^\circ] - 2[\delta^\circ, \widetilde{s}] = 0$,

(iii) the map δ° is a generator for the bracket $\{\,\}$,

$$\{X, Y\} = \delta^\circ(X) \wedge \phi(Y) + \phi(X) \wedge \delta^\circ(Y) - \delta^\circ(X \wedge Y)\,,$$

(iv) the default of ϕ to preserve $[\,]$ equals the default of $\widetilde{s}$ to be a derivation with respect to $\{\,\}$, Equation (4).

Thus $(\phi, \{\,\}, \delta^\circ)$ *is a* **(first order) morphism of differential Batalin-Vilkovisky algebras**.

The Lie bracket $[\,]$ is determined by its generator d^0, and the bracket $\{\,\}$ is determined by its generator δ^0. Thus, in a certain sense, the two brackets are redundant. Moreover, Condition (iv) is implied by Conditions (ii) and (iii).

Remark 1.41. A morphism of differential Batalin-Vilkovisky algebras

$$(\phi, \{\,\}, \delta^\circ) : (A, \widetilde{s}, [\,], d^\circ) \longrightarrow (B, \widetilde{t}, [\,], d^\circ)$$

induces on cohomology

$$h^*(\phi) : \big(h^*(A, \widetilde{s}), [\,], d^\circ\big) \to \big(h^*(B, \widetilde{t}), [\,], d^\circ\big)$$

a morphism of Batalin-Vilkovisky algebras.

2 Symplectic geometry

Let (S, σ) be a *symplectic manifold,* i.e., a complex manifold S endowed with a closed holomorphic 2-form $\sigma \in \Omega^2_S$ which is everywhere non-degenerate, i.e., $X \to X \lrcorner\, \sigma$ defines an isomorphism of vector bundles $T_S \to \Omega_S$. The (complex) dimension of S is even, and we will denote it by $2n$.

A submanifold $M \subset S$ is *Lagrangian*, if the restriction of this isomorphism $T_S|_M \to \Omega_S|_M$ identifies $T_M \subset T_S|_M$ with $T_M^\perp \subset \Omega_S|_M$. An equivalent condition is that the restriction of σ to a 2-form on M vanishes and that $\dim M = n$. More generally, we define an *immersed Lagrangian*, to be an unramified morphism $i : M \to S$, where M is a manifold of dimension n, such that $i^*\sigma \in \Omega^2_M$ vanishes.

Holomorphic coordinates $x_1, \ldots, x_n, p_1, \ldots, p_n$ on S are called *Darboux coordinates*, if

$$\sigma = \sum_{i=1}^{n} dp_i \wedge dx_i\,.$$

Let us introduce one further piece of notation. For a subbundle $E \subset \Omega_S$ we consider the associated bundles $E^\perp$, $E^\vee$ and $E^\dagger$ defined the short exact sequences of vector bundles

$$0 \longrightarrow E^\perp \longrightarrow T_S \longrightarrow E^\vee \longrightarrow 0\,,$$

and

$$0 \longrightarrow E \longrightarrow \Omega_S \longrightarrow E^\dagger \longrightarrow 0\,.$$

2.1 Lagrangian foliations

Definition 2.1. A **Lagrangian foliation** on S is an integrable distribution $F \subset T_S$, where $F \subset T_S$ is a Lagrangian subbundle, i.e., $X \to X \lrcorner \sigma$ defines an isomorphism of vector bundles $F \to F^\perp \subset \Omega_S$.

All leaves of the Lagrangian foliation F are Lagrangian submanifolds of S. The Lagrangian foliation $F \subset T_S$ may be equivalently defined in terms of the subbundle $E = F^\perp \subset \Omega_S$. Usually, we find it more convenient to specify $E \subset \Omega_S$, rather than $F \subset T_S$. In terms of E, we have the following isomorphism of short exact sequences of vector bundles:

$$\begin{tikzcd}
0 \arrow[r] & E^\perp \arrow[r] \arrow[d, "\cong"'] & T_S \arrow[r] \arrow[d, "\cong"', "\lrcorner\sigma"] & E^\vee \arrow[r] \arrow[d, "\cong"'] & 0 \\
0 \arrow[r] & E \arrow[r] & \Omega_S \arrow[r] & E^\dagger \arrow[r] & 0
\end{tikzcd}$$

Definition 2.2. . A **polarized** symplectic manifold is a symplectic manifold endowed with a Lagrangian foliation.

The canonical partial connection

Any foliation $F \subset T_S$ defines a partial connection on the quotient bundle T_S/F:

$$\nabla : T_S/F \longrightarrow F^\vee \otimes T_S/F\,, \tag{11}$$

given by

$$\nabla_Y(X) = [Y, X]\,,$$

for $Y \in F$ and $X \in T_S/F$. This partial connection is flat. The dual bundle of T_S/F is $F^\perp \subset \Omega_S$. The dual connection

$$\nabla : F^\perp \longrightarrow F^\vee \otimes F^\perp$$

is given by

$$\nabla_Y(\omega) = Y \lrcorner d\omega$$

for $Y \in F$ and $\omega \in F^\perp \subset \Omega_S$.

Let us specialize to the case that F is Lagrangian. Then we can transport the partial connection from $F^\perp$ to F, via the isomorphism $F \cong F^\perp$. We obtain the **canonical partial flat connection**

$$\nabla : F \longrightarrow F^\vee \otimes F$$

characterized by

$$\nabla_Y(X) \lrcorner \sigma = Y \lrcorner d(X \lrcorner \sigma)\,,$$

for $Y \in F$ and $X \in F$. The dual of this partial connection is

$$\nabla : F^\vee \longrightarrow F^\vee \otimes F^\vee \,,$$

which is characterized by

$$\nabla_Y (X \lrcorner \sigma) = [Y, X] \lrcorner \sigma \,,$$

for $Y \in F$ and $X \in T_S$.

Oriented Lagrangian foliations

Definition 2.3. Let $F \subset T_S$ a Lagrangian foliation on S. An **orientation** of F is a nowhere vanishing global section

$$\theta \in \Gamma(S, \Lambda^n F) \,,$$

which is *flat* with respect to the canonical partial connection on F.

A polarized symplectic manifold is called **oriented**, if its Lagrangian foliation is endowed with an orientation.

Remark 2.4. If θ is an orientation of the Lagrangian foliation F, then we have $\nabla(\theta \lrcorner \sigma^n) = 0$. (Note that $\theta \lrcorner \sigma^n \in \Lambda^n F^\perp \subset \Lambda^n \Omega_S$.)

2.2 Polarizations and transverse Lagrangians

Let $E \subset \Omega_S$ define a Lagrangian foliation on S.

Lemma 2.5. *Let M be a Lagrangian submanifold of S which is everywhere transverse to E. Then there exists (locally near M) a unique section s of E, such that $ds = \sigma$ and $M = Z(s)$, i.e., M is the zero locus of s (as a section of the vector bundle E).*

Definition 2.6. We call s the **Euler form** of M with respect to E, or the **Euler section** of M in E.

Remark 2.7. Conversely, if s is any section of E such that $ds = \sigma$, then $Z(s)$ is a Lagrangian submanifold. Thus we have a canonical one-to-one correspondence between sections s of E such that $ds = \sigma$ and Lagrangian submanifolds of S transverse to E.

Lemma 2.8. *Let (S, F, σ, θ) be an oriented polarized symplectic manifold and $E = F^\perp$. Let $M \subset S$ be a Lagrangian submanifold, everywhere transverse to F. Then near every point of M there exists a set of Darboux coordinates $x_1, \ldots, x_n, p_n \ldots, p_n$ such that*

(i) $M = Z(p_1, \ldots, p_n)$,

(ii) $F = \langle \frac{\partial}{\partial p_1} \ldots, \frac{\partial}{\partial p_n} \rangle$

(iii) $\nabla(\frac{\partial}{\partial p_i}) = 0$, *for all* $i = 1, \ldots, n$,

(iv) $\theta = \frac{\partial}{\partial p_i} \wedge \ldots \wedge \frac{\partial}{\partial p_n}$,

Moreover, in these coordinates we have

(i) $E = \langle dx_1, \ldots, dx_n \rangle$,

(ii) the Euler form s of M inside E is given by $s = \sum p_i dx_i$.

3 Derived Lagrangian intersections on polarized symplectic manifolds

Definition 3.1. Let (S, E, σ) be a polarized symplectic manifold and L, M immersed Lagrangians of S which are both transverse to E. Then the **derived intersection**

$$L \Cap_S M$$

is the sheaf of differential Gerstenhaber algebras $(\Lambda T_M, \widetilde{t})$ on M, where $\widetilde{t}$ is the derivation on ΛT_M induced by the restriction to M of the Euler section $t \in E \subset \Omega_S$ of L.

Since $dt = \sigma$, and M is Lagrangian, the restriction of t to M is closed, and so $\widetilde{t}$ is a derivation with respect to the Schouten-Nijenhuis bracket on ΛT_M, making $(\Lambda T_M, \widetilde{t})$ a differential Gerstenhaber algebra.

Remark 3.2. After passing (locally in L) to suitable étale neighborhoods of L in S we can assume that L is embedded (not just immersed) in S and that L admits a globally defined Euler section t on S. This defines the derived intersection étale locally in M, and the global derived intersection is defined by gluing in the étale topology on M.

Remark 3.3. If we forget about the bracket, the underlying complex of $\mathcal{O}_S$-modules $(\Lambda T_M, \widetilde{t})$ represents the derived tensor product

$$\mathcal{O}_L \overset{L}{\otimes}_{\mathcal{O}_S} \mathcal{O}_M$$

in the derived category of sheaves of $\mathcal{O}_S$-modules.

Remark 3.4. The derived intersection $L \Cap_S M$ depends a priori on the polarization E. We will see later (see the proof of Theorem 4.2) that different polarizations lead to *locally* quasi-isomorphic derived intersections. (The quasi-isomorphism is not canonical, as it depends on the choice of a third polarization transverse to both of the polarizations being compared. It is not clear that such a third polarization can necessarily be found globally.)

Remark 3.5. The derived intersection does not seem to be symmetric. We will see below that $L \Cap_S M = M \Cap_{\overline{S}} L$, where $\overline{S} = (S, -\sigma)$, but only if $\overline{S}$ is endowed with a different polarization, transverse to E. Then the issue of change of polarization of Remark 3.4 arises.

Definition 3.6. Let S, L, M be as in Definition 3.1. Let M be *oriented*, i.e., endowed with a nowhere vanishing top-degree differential form ω_M°. (Since M is Lagrangian, this amounts to the same as a trivialization of the determinant of the normal bundle $N_{M/C}$.) We call the differential Batalin-Vilkovisky algebra $(\Lambda T_M, \widetilde{t}, [\,], d^\circ)$, where d° is induced by ω_M° as in Section 1.5, the **oriented derived intersection**, notation $L \Cap_S^\circ M$.

By a *local system* we mean a vector bundle (locally free sheaf of finite rank) endowed with a flat connection. Every local system P on a complex manifold M has an associated holomorphic de Rham complex $(P \otimes_{\mathcal{O}_M} \Omega_M^\bullet, d)$, where d denotes the covariant derivative.

Definition 3.7. Let (S, E, σ) be a polarized symplectic manifold and L, M immersed Lagrangians, both transverse to E. Let P be a local system on M and Q a local system on S. The **derived hom** from $Q|L$ to $P|M$ is the differential Batalin-Vilkovisky module

$$\mathbb{R}\mathcal{H}om_S\big(Q|L, P|M\big) = \big(\Omega_M^\bullet \otimes Q^\vee|_M \otimes P, t\big)$$

over the differential Gerstenhaber algebra

$$L \Cap_S M = (\Lambda T_M, \widetilde{t}\,)\,.$$

The tensor products are taken over $\mathcal{O}_M$. The closed 1-form $t \in \Omega_M$ is the restriction to M of the Euler section of L inside E. The $\mathcal{O}_M$-linear differential t is multiplication by t and the $\mathbb{C}$-linear differential d is covariant derivative with respect to the induced flat connection on $Q^\vee|_M \otimes P$.

Remark 3.8. If we forget about the $\mathbb{C}$-linear differential d and the flat connections on P and Q, the underlying complex of $\mathcal{O}_S$-modules $\mathbb{R}\mathcal{H}om_S(Q|L, P|M)$ represents the derived sheaf of homomorphisms $\mathbb{R}\mathcal{H}om_{\mathcal{O}_S}(Q|_L, P)$ in the derived category of sheaves of $\mathcal{O}_S$-modules.

3.1 The exchange property: Gerstenhaber case

Given two symplectic manifolds S', S, of dimensions $2n'$ and $2n$, a **symplectic correspondence** between S' and S is a manifold C of dimension $n + n'$, together with morphisms $\pi' : C \to S'$ and $\pi : C \to S$, such that

(i) $\pi^*\sigma = \pi'^*\sigma'$ (as sections of Ω_C),

(ii) $C \to S' \times S$ is unramified.

Thus a symplectic correspondence is an immersed Lagrangian of

$$\overline{S}' \times S = (S' \times S, \sigma - \sigma')\,.$$

Let $C \to S' \times S$ be a symplectic correspondence. We say that the immersed Lagrangian $L \to S$ is **transverse** to C, if

(i) for every $(Q,P) \in C \times_S L$ we have that

$$T_C|_Q \oplus T_L|_P \longrightarrow T_S|_{\pi(Q)}$$

is surjective, hence the pullback $L' = C \times_S L$ is a manifold of dimension n'

(ii) the natural map $L' \to S'$ is unramified (and hence L' is an immersed Lagrangian of S').

By exchanging the roles of S and S' we also get the notion of transversality to C for immersed Lagrangians of S'.

Exchange property setup

Let (S, E, σ) and (S', E', σ') be polarized symplectic manifolds. Let $E^\perp \subset T_S$ and $E'^\perp \subset T_{S'}$ be the corresponding Lagrangian foliations. Consider a **transverse** symplectic correspondence $C \to S' \times S$. This means that $C \to S' \times S$ is transverse to the foliation $E'^\perp \times E^\perp$ of $S' \times S$. In particular, the composition

$$T_C \longrightarrow \pi^* T_S \longrightarrow \pi^* E^\vee$$

is surjective. Hence the foliation $E^\perp \subset T_S$ pulls back to a foliation $F \subset T_C$ of rank n'. We have the exact sequence of vector bundles

$$0 \longrightarrow F \longrightarrow T_C \longrightarrow \pi^* E^\vee \longrightarrow 0\,. \tag{12}$$

Similarly, the foliation $E'^\perp \subset T_{S'}$ pulls back to a foliation $F' \subset T_C$ of rank n with the exact sequence

$$0 \longrightarrow F' \longrightarrow T_C \longrightarrow \pi'^* E'^\vee \longrightarrow 0\,.$$

Moreover, F and F' are transverse foliations of C and so we have

$$F \oplus F' = T_C = \pi'^* E'^\vee \oplus \pi^* E^\vee\,.$$

Even though it is not strictly necessary, we will make the assumption that $F \subset T_C$ descends to a Lagrangian foliation $\widetilde{F} \subset T_{S'}$ and $F' \subset T_C$ descends to a Lagrangian foliation $\widetilde{F'} \subset T_S$. This makes some of the arguments simpler.

Remark 3.9. The composition

$$F \longrightarrow T_C \longrightarrow \pi'^* T_{S'} \xrightarrow{\lrcorner \pi'^* \sigma'} \pi'^* \Omega_{S'} \longrightarrow \pi'^* E'^\dagger$$

defines an isomorphism of vector bundles $\beta : F \xrightarrow{\sim} \pi'^* E'^\dagger$, and its inverse $\eta : \pi'^* E'^\dagger \xrightarrow{\sim} F$. We can reinterpret these as perfect pairings $\beta : F \otimes_{\mathcal{O}_C} \pi'^* E'^\perp \to \mathcal{O}_C$ and $\eta : F^\vee \otimes_{\mathcal{O}_C} \pi'^* E'^\dagger \to \mathcal{O}_C$. These will be important in the proof below.

Now assume given immersed Lagrangians L of S and M' of S'. Assume both are transverse to C. Then we obtain manifolds L' and M by the pullback diagram

$$\begin{tikzcd} & L' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & L \arrow[d] \\ M \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & C \arrow[r, "\pi"] \arrow[d, "\pi'"] & S \\ M' \arrow[r] & S' & \end{tikzcd} \tag{13}$$

Then L' is an immersed Lagrangian of S' and M an immersed Lagrangian of S.

Finally, we assume that L and M are transverse to E and that M' and L' are transverse to E'. As a consequence, L' is transverse to F' and M is transverse to F.

Remark 3.10. As M is transverse to F, we have a canonical isomorphism $F|_M = N_{M/C}$. Also, since $\pi'^* N_{M'/S'} = N_{M/C}$, we have $\pi'^* E'^{\perp}|_M = N_{M/C}$. Thus, restricting the pairings β and η to M, we obtain: $\beta|_M : N_{M/C} \otimes_{\mathcal{O}_M} N_{M/C} \to \mathcal{O}_M$ and $\eta|_M : N^{\vee}_{M/C} \otimes_{\mathcal{O}_M} N^{\vee}_{M/C} \to \mathcal{O}_M$.

Lemma 3.11. *If $s \in E$ is the Euler section of M in E and s' the Euler section of M' in E', then the homomorphism $\beta|_M : N_{M/C} \to N^{\vee}_{M/C}$ fits into the commutative diagram*

$$\begin{tikzcd} T_C \arrow[rr, "\widetilde{s}-\widetilde{s}'"] \arrow[d] & & \mathcal{O}_C \arrow[r, "d"] & \Omega_C \arrow[d] \\ T_C|_M \arrow[r] & N_{M/C} \arrow[r, "\beta|_M"] & N^{\vee}_{M/C} \arrow[r] & \Omega_C|_M \end{tikzcd}$$

Proof. Let $P \in M \subset C$ be a point. It suffices to prove the claim locally near P. Let $\widetilde{F} \subset T_{S'}$ be the Lagrangian foliation on S', which pulls back to $F \subset T_C$. Then $\widetilde{F}$ is transverse to both E' and M'.

Choose holomorphic functions $x_1, \ldots, x_{n'}$ in a neighborhood of $\pi'(P)$ in S', such that $dx_1, \ldots, dx_{n'}$ is a basis for $E' \subset \Omega_{S'}$. Also, choose $y_1, \ldots, y_{n'}$, such that $dy_1, \ldots, dy_{n'}$ is a basis for $\widetilde{F}^{\perp} \subset \Omega_{S'}$. Then (x_i, y_j) is a set of coordinates for S' near $\pi'(P)$.

Let $\overline{s}$ be the Euler section of M' in $\widetilde{F}^{\perp}$ and f be the unique holomorphic function on S', defined in a neighborhood of $\pi'(P)$, such that $f(\pi'(P)) = 0$ and $df = \overline{s} - s'$. Then we have $\overline{s} = \sum_j \frac{\partial f}{\partial y_j} dy_j$ and $s' = -\sum_i \frac{\partial f}{\partial x_i} dx_i$. Moreover, $\sigma' = \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial y_j} dx_i \wedge dy_j$.

We remark that the composition $T_{S'} \stackrel{df}{\to} \mathcal{O}_{S'} \stackrel{d}{\to} \Omega_{S'}$ factors through $N_{M'/S'} \to N^{\vee}_{M'/S'}$, because f vanishes on M'. The resulting map is, in fact, the Hessian of f. Via our identifications, this Hessian agrees with the map $\widetilde{F}|_{M'} \to E'^{\dagger}|_{M'}$ induced by σ', because, with our choice of coordinates, $\widetilde{F}|_{M'} = N_{M'/S'}$ has basis $\frac{\partial}{\partial x_i}$ and $E'^{\dagger}|_{M'} = N^{\vee}_{M'/S'}$ has basis dy_j.

To transfer this result from S' to C, we remark that the pullback of $\overline{s}$ to C is necessary equal to the pullback of s to C. Thus the composition $d \circ (\widetilde{s} - \widetilde{s}')$ is equal to the Hessian of the pullback of f to C. This is, by what we proved above, equal to the pullback of the map induced by σ'. □

Theorem 3.12. *There are canonical quasi-isomorphisms of differential Gerstenhaber algebras*

$$(M' \times L) \Cap_{\overline{S}' \times S} C \longrightarrow L \Cap_S M\,,$$

and

$$(M' \times L) \Cap_{\overline{S}' \times S} C \longrightarrow M' \Cap_{\overline{S}'} L'\,.$$

In particular, the derived intersections $L \Cap_S M$ and $M' \Cap_{\overline{S}'} L'$ are canonically quasi-isomorphic.

Proof. Passing to étale neighborhoods of L in S and M' in S' will not change anything about either derived intersection $L \Cap_S M$ or $M' \Cap_{\overline{S}'} L'$, so we may assume, without loss of generality, that

(i) L is embedded (not just immersed) in S (and same for M' in S'),

(ii) L admits a global Euler section t with respect to E on S (and M' has the Euler section s' in E' on S')

Then the Euler section of M' with respect to E' on $\overline{S}'$ is $-s'$. Thus the derived intersection $L \Cap_S M$ is equal to $(\Lambda T_M, \widetilde{t})$ and the derived intersection $M' \Cap_{\overline{S}'} L'$ equals $(\Lambda T_{L'}, -\widetilde{s}')$.

Pulling back the 1-form t via π, we obtain a 1-form on C, which we shall, by abuse of notation, also denote by t. Similarly, pulling back s' via π' we get the 1-form s' on C. The difference $t - s'$ is closed on C, and thus we have the differential Gerstenhaber algebra $(\Lambda T_C, \widetilde{t} - \widetilde{s}')$. We remark that it is equal to $(M' \times L) \Cap_{\overline{S}' \times S} C$.

Recall that we have the identification $T_C = \pi'^* E'^{\vee} \oplus \pi^* E^{\vee}$. Under this direct sum decomposition $\widetilde{t} - \widetilde{s}'$ splits up into two components, $-\widetilde{s}'$ and $\widetilde{t}$. Hence we obtain the decomposition

$$(\Lambda T_C, \widetilde{t} - \widetilde{s}') = \pi'^*(\Lambda E'^{\vee}, -\widetilde{s}') \otimes \pi^*(\Lambda E^{\vee}, \widetilde{t})$$

of differential graded $\mathcal{O}_C$-algebras.

Recall that $\mathcal{O}_{S'} \to \mathcal{O}_{M'}$ induces a quasi-isomorphism of differential graded $\mathcal{O}_{S'}$-algebras $(\Lambda E'^{\vee}, -\widetilde{s}') \to \mathcal{O}_{M'}$. Because the pullback $M = M' \times_{S'} C$ is transverse, we get an induced quasi-isomorphism

$$\pi'^*(\Lambda E'^{\vee}, -\widetilde{s}') \longrightarrow \mathcal{O}_M$$

of differential graded $\mathcal{O}_C$-algebras. Tensoring with $\pi^*(\Lambda E^\vee, \widetilde{t})$, we obtain the quasi-isomorphism

$$(\Lambda T_C, \widetilde{t} - \widetilde{s}') \longrightarrow (\Lambda E^\vee, \widetilde{t})|_M \,.$$

Noting that $E^\vee|_M = T_M$, because M is an immersed submanifold in S transverse to E, we see that $(\Lambda E^\vee, \widetilde{t})|_M = (\Lambda T_M, \widetilde{t})$ and so we have a quasi-isomorphism of differential graded $\mathcal{O}_C$-algebras

$$\phi : (\Lambda T_C, \widetilde{t} - \widetilde{s}') \longrightarrow (\Lambda T_M, \widetilde{t}) \,. \tag{14}$$

For analogous reasons, we also have the quasi-isomorphism

$$\phi' : (\Lambda T_C, \widetilde{t} - \widetilde{s}') \longrightarrow (\Lambda T_{L'}, -\widetilde{s}') \,.$$

The proof will be finished, if we can enhance ϕ and ϕ' by brackets, making them morphisms of differential Gerstenhaber algebras. We will concentrate on ϕ. The case of ϕ' follows by symmetry.

Thus we shall define a bracket

$$\{\,\} : \Lambda T_C \otimes_{\mathbb{C}} \Lambda T_C \longrightarrow \Lambda T_M \,, \tag{15}$$

such that $(\phi, \{,\})$ becomes a morphism of differential Gerstenhaber algebras.

We use the foliation $F \subset T_C$. It defines (11) a partial flat connection

$$\nabla : T_C/F \longrightarrow F^\vee \otimes_{\mathcal{O}_C} T_C/F \,.$$

By the usual formulas we can transport ∇ onto the exterior powers of T_C/F. In our context, we obtain

$$\nabla : \pi^* \Lambda E^\vee \longrightarrow F^\vee \otimes_{\mathcal{O}_C} \pi^* \Lambda E^\vee \,.$$

To get the signs right, we will consider the elements of the factor $F^\vee$ in this expression to have degree zero.

Let us write the projection $\Lambda T_C \to \pi^* \Lambda E^\vee$ as ρ. We identify $F^\vee|_M$ with $N^\vee_{M/C}$ and $(\pi^* \Lambda E^\vee)|_M$ with ΛT_M. Then ϕ is the composition of ρ with restriction to M. We now define for $X, Y \in \Lambda T_C$

$$\{X, Y\} = \eta\big(\nabla(\rho X)|_M \wedge \nabla(\rho Y)|_M\big) \,. \tag{16}$$

In this formula, '$\wedge$' denotes the homomorphism (all tensors are over $\mathcal{O}_M$)

$$\begin{aligned}(N^\vee_{M/C} \otimes \Lambda T_M) \otimes (N^\vee_{M/C} \otimes \Lambda T_M) &\longrightarrow (N^\vee_{M/C} \otimes N^\vee_{M/C}) \otimes \Lambda T_M \\ v \otimes X \otimes w \otimes Y &\longmapsto v \otimes w \otimes X \wedge Y \,.\end{aligned}$$

There is no sign correction in this definition, because the elements of $N^\vee_{M/C}$ are considered to have degree zero, by our sign convention. We have also extended the map η linearly to

$$\eta : (N^\vee_{M/C} \otimes_{\mathcal{O}_M} N^\vee_{M/C}) \otimes_{\mathcal{O}_M} \Lambda T_M \longrightarrow \Lambda T_M \,.$$

Claim. The conditions of Definition 1.12 are satisfied by $(\phi, \{\,\})$.

All but the last condition follow easily from the definitions. Let us check Condition (iv). We use Remark 1.14. The $\mathbb{C}$-algebra ΛT_C is generated in degrees 0 and -1. As generators in degree -1, we may take the basic vector fields of a coordinate system for C. We choose this coordinate system such that M is cut out by a subset of the coordinates. Then, if we plug in generators of degree -1 for both X and Y in Formula (4), every term vanishes. Also, if we plug in terms of degree 0 for both X and Y, both sides of (4) vanish for degree reasons. By symmetry, we thus reduce to considering the case where X is of degree -1, i.e., a vector field on C, and Y is of degree 0, i.e., a regular function on C.

Hence we need to prove that for all $X \in T_C$ and $g \in \mathcal{O}_C$ we have

$$X(g)|_M - \rho(X)|_M\big(g|_M\big) = \{(\widetilde{t} - \widetilde{s}')X, g\} - \widetilde{t}\{X, g\}\,. \tag{17}$$

Let s denote the Euler section of M in $E \subset \Omega_S$, and its pullback to C. We will prove that

$$X(g)|_M - \rho(X)|_M\big(g|_M\big) = \{(\widetilde{s} - \widetilde{s}')X, g\} \tag{18}$$

and

$$\{(\widetilde{t} - \widetilde{s})X, g\} = \widetilde{t}\{X, g\}\,. \tag{19}$$

Equation (18) involves only M, not L, and Equation (19) involves only E, not E'. Together, they imply Equation (17).

All terms in these three equations are $\mathcal{O}_S$-linear in X and derivations in g, and may hence be considered as $\mathcal{O}_C$-linear maps $T_C \to \mathcal{D}er(\mathcal{O}_C, \mathcal{O}_M)$. As $\mathcal{D}er(\mathcal{O}_C, \mathcal{O}_M) = \mathcal{H}om_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_M) = T_C|M$, we may also think of them as $\mathcal{O}_C$-linear maps $T_C \to T_C|_M$.

For example, the $\mathcal{O}_C$-linear map

$$\begin{aligned} T_C &\longrightarrow T_C|_M \\ X &\longmapsto \{(\widetilde{s} - \widetilde{s}')X, \,\cdot\,\} \end{aligned} \tag{20}$$

Is equal to the composition

$$T_C \xrightarrow{\widetilde{s}-\widetilde{s}'} \mathcal{O}_C \xrightarrow{d} \Omega_C \longrightarrow F^\vee|_M \xrightarrow{\eta} F|_M \longrightarrow T_C|_M\ .$$

Then the commutative diagram (Lemma 3.11)

$$\begin{tikzcd}
T_C \arrow[r, "\widetilde{s}-\widetilde{s}'"] \arrow[d] & \mathcal{O}_C \arrow[r, "d"] & \Omega_C \arrow[d] \arrow[dr] & & & \\
T_C|_M \arrow[d] & & \Omega_C|_M \arrow[r] & F^\vee|_M \arrow[r, "\eta"] & F|_M \arrow[r] & T_C|_M \\
N_{M/C} \arrow[rr, "\beta"] & & N^\vee_{M/C} \arrow[u] \arrow[ur, "\mathrm{id}"'] & & &
\end{tikzcd}$$

and the fact that η is the inverse of β, proves that (20) is equal to the composition

$$T_C \longrightarrow T_C|_M \xrightarrow{p} T_C|_M \;,$$

where p is the projection onto the the second summand of the decomposition

$$T_C|_M = T_M \oplus N_{M/C}$$

given by the foliation F transverse to M in C. If we denote by q the projection onto the first summand, we see that the map

$$\begin{aligned} T_C &\longrightarrow T_C|_M \\ X &\longmapsto \rho(X)|_M(\,\cdot\,|_M) \end{aligned} \tag{21}$$

is equal to

$$T_C \longrightarrow T_C|_M \xrightarrow{q} T_C|_M \;.$$

Thus (20) and (21) sum up to the restriction map $T_C \to T_C|_M$, which is equal to the map given by $X \mapsto X(\,\cdot\,)|_M$. This proves (18).

Now, let us remark that for any closed 1-form u on C we have

$$\widetilde{u}[Y,X] = Y\big(\widetilde{u}(X)\big) - X\big(\widetilde{u}(Y)\big)\,.$$

If $u \in \pi^*E \subset \Omega_C$, then $\widetilde{u}(Y) = 0$, for all $Y \in F$. So if $Y \in F$ we have

$$\widetilde{u}[Y,X] = Y\big(\widetilde{u}(X)\big)\,.$$

We have $\widetilde{u}[Y,X] = \widetilde{u}\big(\nabla(X)(Y)\big)$ by definition of the partial connection ∇ and we can write $Y\big(\widetilde{u}(X)\big) = \langle Y, d\big(\widetilde{u}(X)\big)\rangle$. In other words, the diagram

$$\begin{tikzcd} T_C/F \arrow[rr, "\nabla"] \arrow[d, "\widetilde{u}"'] & & \mathcal{H}om(F, T_C/F) \arrow[d, "\_\circ\widetilde{u}"] \\ \mathcal{O}_C \arrow[r, "d"] & \Omega_C \arrow[r] & F^\vee \end{tikzcd}$$

commutes. Thus, the larger diagram

$$\begin{tikzcd} T_C \arrow[r, "\rho"] \arrow[dr, "\widetilde{u}"'] & \pi^*E^\vee \arrow[r, "\nabla"] \arrow[d, "\widetilde{u}"] & F^\vee \otimes \pi^*E^\vee \arrow[r] \arrow[d, "\mathrm{id}\otimes\widetilde{u}"] & F^\vee|_M \otimes T_M \arrow[r, "\eta\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\widetilde{u}|_M"] & F|_M \otimes T_M \arrow[d, "\mathrm{id}\otimes\widetilde{u}|_M"] \\ & \mathcal{O}_C \arrow[r, "d"] & F^\vee \arrow[r] & F^\vee|_M \arrow[r, "\eta"] & F|_M \arrow[d] \\ & & & & T_C|_M \end{tikzcd}$$

commutes as well. We can apply these considerations to $u = t - s$. Then $\widetilde{u} = \widetilde{t} - \widetilde{s}$ and $\widetilde{u}|_M = \widetilde{t}|_M$. Thus the upper composition in this diagram represents the right hand side of Equation (19), and the lower composition represents the left hand side of Equation (19). This exhibits that (19) holds and finishes the proof of the theorem. $\square$

3.2 The Batalin-Vilkovisky case

For the exchange property in the Batalin-Vilkovisky case, we require an *orientation* on the symplectic correspondence $C \to S' \times S$.

Definition 3.13. Let $\pi : C \to S$ be a morphism of complex manifolds, $F \subset T_C$ and $\widetilde{F} \subset T_S$ foliations. The foliations F, $\widetilde{F}$ are **compatible** (with respect to π), if $F \to T_C \to \pi^* T_S$ factors through $\pi^* \widetilde{F} \to \pi^* T_S$.

If F and $\widetilde{F}$ are compatible, then partial connections with respect to $\widetilde{F}$ pull back to partial connections with respect to F.

Now let (S, E, σ), (S', E', σ') and $C \to S' \times S$ be, as in Section 3.1, polarized symplectic manifolds with a transverse symplectic correspondence. Let F and F' be, as in 3.1, the inverse image foliations:

$$\begin{tikzcd} & F \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & E^{\perp} \arrow[d] \\ F' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & C \arrow[r, "\pi"] \arrow[d, "\pi'"] & S \\ E'^{\perp} \arrow[r] & S' & \end{tikzcd} \tag{22}$$

Furthermore, we suppose that $\widetilde{F} \subset T_{S'}$ is a Lagrangian foliation on S' compatible with F via π' and that $\widetilde{F}' \subset T_S$ is a Lagrangian foliation on S, compatible with F' via π. Since the composition $F \to \pi'^* T_{S'} \to \pi'^* E'^{\vee}$ is an isomorphism, the map $F \to \pi'^* T_{S'}$ identifies F with a subbundle of $\pi'^* T_{S'}$. As F and $\widetilde{F}$ have the same rank, it follows that $F \to \pi'^* \widetilde{F}$ is an isomorphism of subbundles of $\pi'^* T_{S'}$. Similarly, we have and identification $F' \to \pi^* \widetilde{F}'$ of subbundles of $\pi^* T_S$.

Thus we have two Lagrangian foliations on $\overline{S}' \times S$, namely $E'^{\perp} \times E^{\perp}$ and $\widetilde{F} \times \widetilde{F}'$. Both Lagrangian foliations are transverse to C, and they are transverse to each other near C.

Definition 3.14. If $\theta \in \Gamma(S', \Lambda^{n'} \widetilde{F})$ and $\theta' \in \Gamma(S, \Lambda^n \widetilde{F}')$ are orientations of the Lagrangian foliations $\widetilde{F}$ on S' and $\widetilde{F}'$ on S, we call the data $(\widetilde{F}, \theta, \widetilde{F}', \theta')$ an **orientation** of the symplectic correspondence $C \to S' \times S$.

We call the transverse symplectic correspondence of polarized symplectic manifolds $C \to S' \times S$ **orientable**, if it admits an orientation.

Exchange property setup

Let (S, E, σ) and (S', E', σ') be polarized symplectic manifolds and $C \to S' \times S$, $(\widetilde{F}, \theta, \widetilde{F}', \theta')$ an oriented transverse symplectic correspondence. Moreover, let $L \to S$ and $M' \to S'$ be, as in Section 3.1, immersed Lagrangians transverse to C, such that the induced M and L' are transverse to E and E', respectively. (This latter condition is satisfied if M' and L are transverse to $\widetilde{F}$ and $\widetilde{F}'$, respectively.) Pulling back θ to C gives us a trivialization of $\Lambda^{n'}F$ and restricting further to M gives a trivialization of the determinant of the normal bundle $\Lambda^{n'}N_{M/C}$, because of the canonical identification $F|_M = N_{M/C}$. Similarly, θ' gives rise to a trivialization of the determinant of the normal bundle $\Lambda^n N_{L'/C}$.

Finally, let P' be a local system on S' and Q a local system on S. Let $P = {\pi'}^* P'$ and $Q' = \pi^* Q$ be the pullbacks of these local systems to C.

Theorem 3.15. *There exists a canonical quasi-isomorphism of differential Batalin-Vilkovisky modules*

$$\mathbb{R}\mathcal{H}om_{\overline{S}'\times S}\big(\mathcal{O}|(M'\times L), (P\otimes {Q'}^\vee)|C\big) \longrightarrow \mathbb{R}\mathcal{H}om_S\big(Q|L, P|M\big)$$

of degree $-n'$, covering the corresponding canonical quasi-isomorphism of differential Gerstenhaber algebras of Theorem 3.12. Moreover, there is the quasi-isomorphism of differential Batalin-Vilkovisky modules

$$\mathbb{R}\mathcal{H}om_{\overline{S}'\times S}\big(\mathcal{O}|(M'\times L), (P\otimes {Q'}^\vee)|C\big) \longrightarrow \mathbb{R}\mathcal{H}om_{\overline{S}'}\big({P'}^\vee|M', {Q'}^\vee|L'\big)$$

of degree $-n$, covering the other canonical quasi-isomorphism of differential Gerstenhaber algebras of Theorem 3.12.

Thus, the derived homs $\mathbb{R}\mathcal{H}om_S(Q|L, P|M)$ and $\mathbb{R}\mathcal{H}om_{\overline{S}'}({P'}^\vee|M', {Q'}^\vee|L')$ are canonically quasi-isomorphic, up to a degree shift $n'-n$.

Proof. Let t and s' be as in the proof of Theorem 3.12. We need to construct quasi-isomorphisms of Batalin-Vilkovisky modules

$$(\psi,\delta) : (\Omega_C^\bullet \otimes P\otimes {Q'}^\vee, t-s') \longrightarrow (\Omega_M^\bullet\otimes P\otimes {Q'}^\vee, t)$$

and

$$(\psi',\delta') : (\Omega_C^\bullet \otimes P\otimes {Q'}^\vee, t-s') \longrightarrow (\Omega_{L'}^\bullet\otimes P\otimes {Q'}^\vee, -s')\,.$$

(Note that because the elements of P and ${Q'}^\vee$ have degree zero, it is immaterial in which order we write the two factors P and ${Q'}^\vee$.) The case of (ψ',δ') being analogous, we will discuss only (ψ,δ).

Let us start with ψ. Denote the pullback of the orientation $\theta\in\Lambda^{n'}\widetilde{F}$ to C by the same letter, thus giving us a trivialization $\theta\in\Lambda^{n'}F$. Note that contracting $\alpha\in\Omega_C^\bullet$ with θ gives a form $\theta\lrcorner\alpha$ in the subbundle $\pi^*\Lambda E\subset\Omega_C^\bullet$, see (12). Recall the non-degenerate symmetric bilinear form $\beta : N_{M/C}\otimes_{\mathcal{O}_M} N_{M/C}\to\mathcal{O}_M$ of Remark 3.10. Since $F|_M = N_{M/C}$, we may apply the discriminant of β to $\theta|_M\otimes\theta|_M$ to obtain the nowhere vanishing regular function

$$g = \det\beta\,(\theta|_M\otimes\theta|_M)\in\mathcal{O}_M \tag{23}$$

on M. The homomorphism ψ is now defined as the composition

$$\psi : \Omega^\bullet_C \xrightarrow{\theta \,\lrcorner\, \cdot} \pi^*\Lambda E \xrightarrow{\mathrm{res}\,|_M} \Lambda E|_M = \Omega^\bullet_M \xrightarrow{\cdot g} \Omega^\bullet_M \,.$$

Tensoring with $P \otimes Q'^\vee$, we obtain the quasi-isomorphism of differential graded modules

$$\psi : (\Omega^\bullet_C \otimes P \otimes Q'^\vee, t - s) \longrightarrow (\Omega^\bullet_M \otimes P \otimes Q'^\vee, t)$$

covering the morphism of differential graded algebras ϕ of (14). The formula for ψ is

$$\psi(\alpha) = g \cdot \lceil \theta \,\lrcorner\, \alpha \rceil_M \,. \tag{24}$$

('Ceiling brackets' denote restriction.) Note that $\deg \psi = -n'$

Let us next construct $\delta : \Omega^\bullet_C \to \Omega^\bullet_M$. Recall the canonical partial flat connection on the Lagrangian foliation $\widetilde{F}$ on S':

$$\widetilde{\nabla} : \widetilde{F} \longrightarrow \widetilde{F}^\vee \otimes_{\mathcal{O}_{S'}} \widetilde{F} \,,$$

defined by the requirement

$$\widetilde{\nabla}_{\widetilde{Y}}(\widetilde{X}) \,\lrcorner\, \sigma' = \widetilde{Y} \,\lrcorner\, d(\widetilde{X} \,\lrcorner\, \sigma') \,,$$

for $\widetilde{Y}, \widetilde{X} \in \widetilde{F}$. As F is compatible with $\widetilde{F}$ via π', we get the pullback partial flat connection

$$\nabla : \pi'^* \widetilde{F} \longrightarrow F^\vee \otimes_{\mathcal{O}_C} \pi'^* \widetilde{F} \,.$$

Making the identification $F = \pi'^* \widetilde{F}$ we rewrite this partial connection as

$$\nabla : F \longrightarrow F^\vee \otimes_{\mathcal{O}_C} F \,.$$

It is characterized by the formula

$$\nabla_Y(X) \,\lrcorner\, \sigma = Y \,\lrcorner\, d(X \,\lrcorner\, \sigma) \,,$$

for $Y, X \in F$. We have written σ for the restriction of the symplectic form σ' to C. The dual connection

$$\nabla : F^\vee \longrightarrow F^\vee \otimes_{\mathcal{O}_C} F^\vee \tag{25}$$

satisfies

$$\nabla_Y(X' \,\lrcorner\, \sigma) = [Y, X'] \,\lrcorner\, \sigma \,,$$

for $Y \in F$ and $X' \in T_C$.

Recall that we also have the partial connection

$$\nabla : \pi^* E \longrightarrow F^\vee \otimes_{\mathcal{O}_C} \pi^* E \tag{26}$$

defined by $\nabla_Y \omega = Y \,\lrcorner\, d\omega$, for $Y \in F$ and $\omega \in \pi^* E \subset \Omega_C$. We used the dual of this connection in the proof of Theorem 3.12.

Thus, we have partial flat connections on F and $\pi^* E$, in such a way that the canonical homomorphism $F \to \pi^* E$ given by $X \mapsto X \,\lrcorner\, \sigma$ is flat. We hope there will be no confusion from using the same symbol ∇ for both partial connections. As usual, we get induced partial connections on all tensor operations involving F and π^*E. We define

$$\nabla^2 : \pi^*\Lambda E \longrightarrow F^\vee \otimes F^\vee \otimes \pi^*\Lambda E$$

as the composition (all tensor products are over $\mathcal{O}_C$)

$$\pi^*\Lambda E \xrightarrow{\nabla} F^\vee \otimes \pi^*\Lambda E \xrightarrow{\nabla} F^\vee \otimes F^\vee \otimes \pi^*\Lambda E\,.$$

We will also need

$$\nabla^3 : \mathcal{O}_C \longrightarrow F^\vee \otimes F^\vee \otimes F^\vee\,.$$

To simplify notation, let us assume that the closed 1-form $s - s'$ on C is exact. Let I be the ideal of M in $\mathcal{O}_C$. Then there exists a unique regular function $f \in I^2$, such that $df = s - s'$. The fact that $f \in I^2$ follows because s and s' vanish in $\Omega_C|_M$, so df vanishes in $\Omega_C|_M$. Then the Hessian of f is a symmetric bilinear form $N_{M/C} \otimes_{\mathcal{O}_M} N_{M/C} \to \mathcal{O}_M$, and is equal to $\beta|_M$, by Lemma 3.11..

Finally, we define $\delta : \Omega_C^\bullet \to \Omega_M^\bullet$ as a certain $\mathbb{C}$-linear combination of the two compositions

$$\begin{tikzcd}
\Omega_C^\bullet \arrow[r, "\theta \lrcorner \cdot"] & \pi^*\Lambda E \arrow[r, "\nabla^2"] & F^\vee \otimes_{\mathcal{O}_C} F^\vee \otimes_{\mathcal{O}_C} \pi^*\Lambda E \arrow[d, "\mathrm{res}\,|_M"] & & \\
 & & N_{M/C}^\vee \otimes_{\mathcal{O}_M} N_{M/C}^\vee \otimes_{\mathcal{O}_M} \Omega_M^\bullet \arrow[r, "\eta"] & \Omega_M^\bullet \arrow[r, "\cdot g"] & \Omega_M^\bullet\,,
\end{tikzcd}$$

and

$$\begin{tikzcd}
\pi^*\Lambda E \arrow[r, "\nabla"] & F^\vee \otimes \pi^*\Lambda E \arrow[r, "\nabla^3(f)\otimes \mathrm{id}"] & (F^\vee)^{\otimes 4} \otimes \pi^*\Lambda E \arrow[d, "\mathrm{res}\,|_M"] & & \\
\Omega_C^\bullet \arrow[u, "\theta \lrcorner \cdot"] & & (N^\vee)^{\otimes 4} \otimes \Omega_M^\bullet \arrow[r, "\eta\otimes\eta"] & \Omega_M^\bullet \arrow[r, "\cdot g"] & \Omega_M^\bullet\,.
\end{tikzcd}$$

Here η and $\eta \otimes \eta$ are the linear extension of the map η from Remark 3.10. In fact, we define:

$$\delta(\alpha) = -\tfrac{1}{2} g \cdot \eta\big(\lceil \nabla^2(\theta \lrcorner \alpha)\rceil_M\big) + \tfrac{1}{2} g \cdot (\eta \otimes \eta)\big(\lceil \nabla^3(f) \otimes \nabla(\theta \lrcorner \alpha)\rceil_M\big)\,.$$

As P and $Q'^\vee$ have flat connections on them, their pullbacks to C do, too. In particular, we can *partially* differentiate. Thus, δ extends naturally to the map $\delta : \Omega_C^\bullet \otimes P \otimes Q'^\vee \to \Omega_M \otimes P \otimes Q'^\vee$.

We need to check Properties (i) and (ii) of Definition 1.24. To simplify notation, we will only spell out the case where $P = Q' = (\mathcal{O}_C, d)$, leaving the general case to the reader.

Proving (ii) is a straightforward but tedious calculation using the properties of the partial connections on $\pi^*\Lambda E$, $\pi^*\Lambda E^\vee$, F and $F^\vee$, in particular, compatibility with contraction. One can simplify this calculation by using Remark 1.27: choose $\mathbb{C}$-algebra generators for ΛT_C in such a way that the degree -1 generators are flat for the partial connection (see below). This reduces to checking (8) for the case where X and Y are of degree 0, i.e., regular functions x and y on C. The claim is:

$$\delta(xy\omega) + xy\delta(\omega) + \{x, y\}\psi(\omega) = x\delta(y\omega) + y\delta(x\omega)\,,$$

for all $\omega \in \Omega_C$. We leave the details to the reader, and only write down the terms containing $dx \otimes dy$ and only after canceling $g \cdot (\theta \lrcorner \omega)|_M$. In fact, from the term $\delta(xy\omega)$ we get the contribution

$$-\tfrac{1}{2}\eta\lceil dx \otimes dy + dy \otimes dx\rceil_M\,,$$

and from the term $\{x, y\}\psi(\omega)$ we get the contribution

$$\eta\lceil dx \otimes dy\rceil_M\,,$$

and these two expressions do, indeed, add up to 0, because η is symmetric.

To prove (i), we shall use Remark 1.28. We will carefully choose a local trivialization of the vector bundle T_C, as this will give local generators for ΛT_C as $\mathcal{O}_C$-algebra.

The equations we wish to prove can be checked locally. So we pick a point $P \in M$ and pass to a sufficiently small analytic neighborhood of P in C.

Choose holomorphic functions $p_1, \ldots, p_{n'}$ in a neighborhood of $\pi'(P)$ in S' satisfying

(i) $p_1, \ldots, p_{n'}$ cut out the submanifold $M' \subset S'$,

(ii) $dp_1, \ldots, dp_{n'}$ form a frame of $\widetilde{F}^\vee$,

(iii) $dp_1, \ldots, dp_{n'}$ are flat for the partial connection on $\widetilde{F}^\vee$,

(iv) $\theta \lrcorner (dp_1 \wedge \ldots \wedge dp_{n'}) = 1$.

Denote the pullbacks of these functions to C by the same letters. Then these functions on C cut out M, their differentials are flat for the partial connection (25) and form a frame for $F^\vee$. Also, the last property remains true as written. Such p_i exist by Lemma 2.8

Similarly, we choose holomorphic functions $x_1, \ldots, x_n$ in a neighborhood of $\pi(P)$ in S, such that $dx_1, \ldots, dx_n$ form a frame for the subbundle $E \subset \Omega_S$. Then the dx_i are automatically flat for the partial connection on E. Again, we denote the pullbacks to C of these functions by the same letters. For the functions $x_1, \ldots, x_n$ on C we have that their differentials $dx_1, \ldots, dx_n$ form a flat frame for the subbundle π^*E of Ω_C. In particular, the restrictions of $x_1, \ldots, x_n$ to $M \subset C$ form a set of coordinates for M near P.

Then the union of these two families $dp_1, \ldots, dp_{n'}, dx_1, \ldots, dx_n$ forms a basis for Ω_C. We denote the dual basis (as usual) by $\frac{\partial}{\partial p_1}, \ldots, \frac{\partial}{\partial p_{n'}}, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$. We have

$$\theta = (-1)^{\frac{1}{2}n'(n'-1)} \frac{\partial}{\partial p_1} \wedge \ldots \wedge \frac{\partial}{\partial p_{n'}} \,.$$

We define

$$\omega^\circ = dp_1 \wedge \ldots \wedge dp_{n'} \wedge dx_1 \wedge \ldots \wedge dx_n \,,$$

which is a basis for $\Omega^\bullet_C$ as a ΛT_C-module. Note that

$$\theta \lrcorner \omega^\circ = dx_1 \wedge \ldots \wedge dx_n \,.$$

We denote the restriction to M of $dx_1 \wedge \ldots \wedge dx_n$ by τ°. This is a basis for $\Omega^\bullet_M$ as a ΛT_M-module. We have

$$\lceil \theta \lrcorner \omega^\circ \rceil_M = \tau^\circ \,.$$

We now have to prove that for $X = \frac{\partial}{\partial p_i}$ and $X = \frac{\partial}{\partial x_j}$ we have

$$\begin{aligned} \psi\big(d(X \lrcorner \omega^\circ)\big) &- (-1)^{n'} d\big(\psi(X \lrcorner \omega^\circ)\big) \\ &= -2(-1)^{n'} t \wedge \delta(X \lrcorner \omega^\circ) + 2\,\delta\big((t-s') \wedge (X \lrcorner \omega^\circ)\big) \,. \end{aligned} \tag{27}$$

For any of our values for X we have $d(X \lrcorner \omega^\circ) = 0$. So the first term in (27) always vanishes. Similarly, the third term always vanishes, because every of our values for X, as well as θ and ω° are flat for the partial connection ∇. Thus only the second and fourth term of (27) contribute.

Consider the fourth term. We have

$$\begin{aligned} \delta\big((t-s') \wedge (X \lrcorner \omega^\circ)\big) &= \delta\big((t-s) \wedge (X \lrcorner \omega^\circ)\big) + \delta\big(df \wedge (X \lrcorner \omega^\circ)\big) \\ &= \delta\big(\big(X \lrcorner (t-s)\big)\omega^\circ\big) + \delta\big(X(f)\,\omega^\circ\big) \,. \end{aligned}$$

Since $t-s$ is a section of $\pi^* E$ and is also a closed 1-form, $t-s$ is flat with respect to the partial connection ∇. Since X is by assumption also flat with respect to ∇, it follows that $\nabla\big(X \lrcorner (t-s)\big) = 0$, and hence the fourth term of (27) is equal to

$$\delta\big((t-s') \wedge (X \lrcorner \omega^\circ)\big) = \delta\big(X(f)\,\omega^\circ\big) \,.$$

Thus, (27) reduces to

$$(-1)^{n'} d\big(\psi(X \lrcorner \omega^\circ)\big) = -2\,\delta\big(X(f)\,\omega^\circ\big) \,. \tag{28}$$

Let us first consider the case that $X = \frac{\partial}{\partial p_i}$. In this case $\theta \wedge X = 0$, so that the left hand side of (28) vanishes. The claim is therefore, that

$$\delta\big(\tfrac{\partial f}{\partial p_i}\,\omega^\circ\big) = 0 \,,$$

for all $i = 1, \ldots, n'$. This is equivalent to

$$\eta\big(\lceil \nabla^2 \tfrac{\partial f}{\partial p_i} \rceil_M\big) = (\eta \otimes \eta)\big(\lceil \nabla^3 f \rceil_M \otimes \lceil \nabla \tfrac{\partial f}{\partial p_i} \rceil_M\big) \,. \tag{29}$$

To check (29), let us write it out in coordinates. The right hand side is equal to

$$\begin{aligned}\sum_{k,l}\sum_{m,n}\eta_{kl}\eta_{mn}\left.\frac{\partial^3 f}{\partial p_k\partial p_l\partial p_m}\right|_{p=0}\left.\frac{\partial^2 f}{\partial p_n\partial p_i}\right|_{p=0} &= \sum_{k,l}\sum_{m}\eta_{kl}\delta^i_m\left.\frac{\partial^3 f}{\partial p_k\partial p_l\partial p_m}\right|_{p=0}\\ &= \sum_{k,l}\eta_{kl}\left.\frac{\partial^3 f}{\partial p_k\partial p_l\partial p_i}\right|_{p=0},\end{aligned}$$

which is, indeed, equal to the left hand side of (29).

Now let us consider the case $X = \frac{\partial}{\partial x_j}$. Recall that the ideal I defining M is given by $I = (p_1, \dots, p_{n'})$. Since $f \in I^2$, we still have $\frac{\partial f}{\partial x_j} \in I^2$ and hence $\nabla(\frac{\partial f}{\partial x_j})|_{p=0} = 0$. Thus, the right hand side of (28) is equal to

$$\begin{aligned}-2\,\delta(\tfrac{\partial f}{\partial x_j}\omega^\circ) &= g\cdot\eta\big(\lceil\nabla^2\tfrac{\partial f}{\partial x_j}\rceil_M\big)\,\tau^\circ + 0\\ &= g\cdot\operatorname{tr}\big(\eta\cdot\tfrac{\partial}{\partial x_j}H(f)\big)\,\tau^\circ,\end{aligned}$$

because differentiating with respect to x_j commutes with restriction to $M = \{p = 0\}$. (We have written $H(f)$ for the Hessian of f.) On the other hand, the left hand side of (28) is equal to

$$(-1)^{n'} d\big(\psi(X\lrcorner\,\omega^\circ)\big) = \tfrac{\partial g}{\partial x_j}\,\tau^\circ,$$

and thus, our final claim is equivalent to

$$\tfrac{\partial g}{\partial x_j} = g\cdot\operatorname{tr}\big(\eta\cdot\tfrac{\partial}{\partial x_i}H(f)\big).$$

Recalling that $g = \det H(f)$, and η is the inverse of $H(f)$, this claim follows from the following:

Claim. Let A be an invertible square matrix of regular functions on the manifold M. Then for every vector field X on M we have

$$(\det A)^{-1}X(\det A) = \operatorname{tr}\big(A^{-1}X(A)\big).$$

This last claim is both well-known and easy to check. □

3.3 The oriented Batalin-Vilkovisky case

The setup is exactly the same as in Section 3.2, with one additional ingredient; namely an orientation of C, i.e., a nowhere vanishing global section $\omega^\circ_C \in \Omega^{n+n'}_C$. We require ω°_C to be compatible with the orientation $(\widetilde{F}, \widetilde{F}', \theta, \theta')$ on the symplectic correspondence $C \to S' \times S$ in the following sense: namely we ask that

(i) $\nabla(\theta\lrcorner\,\omega^\circ_C) = 0$, where $\nabla : \pi^*E \to F^\vee \otimes \pi^*E$ is the partial connection (26) on $F^\perp = \pi^*E$ defined by the foliation F of C,

(ii) $\nabla'(\theta' \lrcorner \omega_C^\circ) = 0$, where $\nabla' : \pi'^* E' \to F'^\vee \otimes \pi'^* E'$ is the corresponding partial connection defined by the foliation F' of C.

Now, $(\theta \lrcorner \omega_C^\circ)|_M$ is an orientation of M (recalling that $(\pi^* E)|_M = \Omega_M$). We shall denote it by ω_M°. Similarly, $(\theta' \lrcorner \omega_C^\circ)|_{L'}$ is an orientation of L', which we shall denote by $\omega_{L'}^\circ$. This orients the three Lagrangian intersections in Theorem 3.12.

Theorem 3.16. *The quasi-isomorphisms of differential Gerstenhaber algebras of Theorem 3.12 are canonically enhanced to quasi-isomorphisms of differential Batalin-Vilkovisky algebras*

$$(M' \times L) \Cap^\circ_{S' \times S} C \longrightarrow L \Cap^\circ_S M\,,$$

and

$$(M' \times L) \Cap^\circ_{S' \times S} C \longrightarrow M' \Cap^\circ_{S'} L'\,.$$

In particular, the oriented derived intersections $L \Cap^\circ_S M$ and $M' \Cap^\circ_{S'} L'$ are canonically quasi-isomorphic.

Proof. In view of Theorems 3.12 and 3.15 and the results of Section 1.6, we only need to check that

(i) $\psi(\omega_C^\circ) = \omega_M^\circ$,
(ii) $\psi'(\omega_C^\circ) = \omega_{L'}^\circ$,
(iii) $\delta(\omega_C^\circ) = 0$,
(iv) $\delta'(\omega_C^\circ) = 0$,

where (ψ, δ) and (ψ', δ') are the homomorphisms of differential Batalin-Vilkovisky modules constructed in the proof of Theorem 3.15. But the first two follow from the above definitions and the last two from the above assumptions. $\square$

Remark 3.17. If $C = S$ and $C \to S \times S$ is the diagonal, a canonical choice for the orientation of C is $\omega_C^\circ = \sigma^n$, by Remark 2.4. In this case, we also have $\omega_M^\circ = \theta|_M$, via the identification $\Omega_M^n = \Lambda^n N_{M/S} = \Lambda^n F|_M$. Similarly, $\omega_{L'}^\circ = \theta'|_{L'}$.

4 The Gerstenhaber structure on $\mathcal{T}or$ and the Batalin-Vilkovisky structure on $\mathcal{E}xt$

4.1 The Gerstenhaber algebra structure on $\mathcal{T}or$

Let L and M be immersed Lagrangians in the symplectic manifold S. Write $\mathcal{T}or^i_{\mathcal{O}_S}(\mathcal{O}_L, \mathcal{O}_M) = \mathcal{T}or^{\mathcal{O}_S}_{-i}(\mathcal{O}_L, \mathcal{O}_M)$. The direct sum

$$\mathcal{T}or^\bullet_{\mathcal{O}_S}(\mathcal{O}_L, \mathcal{O}_M) = \bigoplus_i \mathcal{T}or^i_{\mathcal{O}_S}(\mathcal{O}_L, \mathcal{O}_M)$$

is a graded sheaf of $\mathcal{O}_S$-algebras, concentrated in non-positive degrees.

Remark 4.1. To be precise, we have to use the analytic étale topology on S to be able to think of $\mathcal{T}or^{\bullet}_{\mathcal{O}_S}(\mathcal{O}_L, \mathcal{O}_M)$ as a sheaf of $\mathcal{O}_S$-algebras. If L and M are embedded, not just immersed, we can use the usual analytic topology. Alternatively, introduce the fibered product $Z = L \times_S M$ and think of $\mathcal{T}or^{\bullet}_{\mathcal{O}_S}(\mathcal{O}_L, \mathcal{O}_M)$ as a sheaf of graded $\mathcal{O}_Z$-algebras.

Theorem 4.2. *There exists a unique bracket of degree* $+1$ *on* $\mathcal{T}or^{\bullet}_{\mathcal{O}_S}(\mathcal{O}_L, \mathcal{O}_M)$ *such that*

(i) $\mathcal{T}or^{\bullet}_{\mathcal{O}_S}(\mathcal{O}_L, \mathcal{O}_M)$ *is a sheaf of Gerstenhaber algebras,*

(ii) whenever E *is a (local) polarization of* S*, such that* L *and* M *are transverse to* E*, then this sheaf of Gerstenhaber algebras is obtained from the derived intersection* $L \Cap_S M$ *(defined with respect to* E*) by passing to cohomology.*

Proof. Without loss of generality, assume that L and M are submanifolds. For every point of S we can find an open neighborhood in S, over which we can choose a polarization E, which is transverse to L and M. This proves uniqueness.

For existence, we have to prove that any two polarizations E, E'' give rise to the same bracket on $\mathcal{T}or^{\bullet}_{\mathcal{O}_S}(\mathcal{O}_L, \mathcal{O}_M)$. This is a local question, so we may choose a third polarization E', which is transverse to both E and E'', and also to L and M.

We will apply the exchange property, Theorem 3.12, twice. First to the symplectic correspondence $\Delta : C = S \to S \times S$ between the polarized symplectic manifolds (S, E') and (S, E), then to the symplectic correspondence $\Delta : C = \overline{S} \to \overline{S} \times \overline{S}$ between $(\overline{S}, E'')$ and $(\overline{S}, E')$. We obtain the following diagram of quasi-isomorphisms of sheaves of differential Gerstenhaber algebras:

$$\begin{tikzcd}
 & (M \times L) \Cap^{E' \times E}_{\overline{S} \times S} S \arrow[r] \arrow[d] & L \Cap^{E}_{S} M \\
(L \times M) \Cap^{E'' \times E'}_{S \times \overline{S}} \overline{S} \arrow[r] \arrow[d] & M \Cap^{E'}_{\overline{S}} L & \\
L \Cap^{E''}_{S} M & &
\end{tikzcd} \tag{30}$$

We have included the polarizations defining the derived intersections in the notation.

Passing to cohomology sheaves, we obtain the following diagram of isomorphisms of sheaves of Gerstenhaber algebras:

$$
\begin{tikzcd}
 & \mathcal{T}or^{\bullet}_{\mathcal{O}_{S\times S}}(\mathcal{O}_{M\times L},\mathcal{O}_S) \arrow[r] \arrow[d] & \mathcal{T}or^{\bullet}_{\mathcal{O}_S}(\mathcal{O}_L,\mathcal{O}_M) \\
\mathcal{T}or^{\bullet}_{\mathcal{O}_{S\times S}}(\mathcal{O}_{L\times M},\mathcal{O}_S) \arrow[r] \arrow[d] & \mathcal{T}or^{\bullet}_{\mathcal{O}_S}(\mathcal{O}_M,\mathcal{O}_L) & \\
\mathcal{T}or^{\bullet}_{\mathcal{O}_S}(\mathcal{O}_L,\mathcal{O}_M) & &
\end{tikzcd}
$$

One checks that all these morphisms are the *canonical* ones, and hence that the composition of all four of them is the *identity* on $\mathcal{T}or^{\bullet}_{\mathcal{O}_S}(\mathcal{O}_L,\mathcal{O}_M)$. If the identity preserves the two brackets on $\mathcal{T}or^{\bullet}_{\mathcal{O}_S}(\mathcal{O}_L,\mathcal{O}_M)$ defined by E and E'', respectively, then the two brackets are equal. □

4.2 The Batalin-Vilkovisky structure on $\mathcal{E}xt$

Let L and M continue to denote immersed Lagrangians in the symplectic manifold S. Furthermore, let P be a local system on M and Q a local system on L. The direct sum

$$\mathcal{E}xt^{\bullet}_{\mathcal{O}_S}(Q,P)=\bigoplus_i \mathcal{E}xt^{i}_{\mathcal{O}_S}(Q,P)$$

is a graded sheaf of $\mathcal{T}or^{\bullet}_{\mathcal{O}_S}(\mathcal{O}_L,\mathcal{O}_M)$-modules.

Theorem 4.3. *There exists a unique $\mathbb{C}$-linear differential*

$$d:\mathcal{E}xt^{i}_{\mathcal{O}_S}(Q,P)\longrightarrow \mathcal{E}xt^{i+1}_{\mathcal{O}_S}(Q,P)\,,$$

(for all i) such that

(i) $\mathcal{E}xt^{\bullet}_{\mathcal{O}_S}(Q,P)$ is a sheaf of Batalin-Vilkovisky modules over the Gerstenhaber algebra $\mathcal{T}or^{\bullet}_{\mathcal{O}_S}(\mathcal{O}_L,\mathcal{O}_M)$,

(ii) whenever E is a (local) polarization of S, such that L and M are transverse to E, and $\overline{Q}$ is a local system on S restricting to Q on L, this sheaf of Batalin-Vilkovisky modules is obtained from the derived hom $\mathbb{R}\mathcal{H}om_S(\overline{Q}|L,P|M)$ (defined with respect to E) by passing to cohomology.

Proof. Uniqueness is clear. Let us prove existence. For this, we assume given two polarization E, E'', transverse to L and M, and two extensions $\overline{Q}$ and $\widehat{Q}$ of Q to S. To compare the derived homs $\mathbb{R}\mathcal{H}om^{E}_S(\overline{Q}|L,P|M)$ and $\mathbb{R}\mathcal{H}om^{E''}_S(\widehat{Q}|L,P|M)$, we choose (locally) a third polarization E', transverse to L and M, and E and E'', and an extension $\overline{P}$ of P to S. These choices make the five derived homs in Diagram (32), below, well-defined.

To define the homomorphisms of differential Batalin-Vilkovisky modules in (32), we orient the symplectic correspondence given by the diagonal of S in the canonical way, as in Remark 3.17, by σ^n. The corresponding symplectic correspondence given by the diagonal of $\overline{S}$ is hence oriented by $(-1)^n\sigma^n$. We also orient the three Lagrangian foliations F, F', F'' on S, by choosing $\theta \in \Lambda^n F$, $\theta' \in \Lambda^n F'$ and $\theta'' \in \Lambda^n F''$. (Note that $\widetilde{F} = F = E^\perp$, etc., in our case.) But the choice of θ, θ'' is not completely arbitrary. In fact, notice that both $F|_L$ and $F''|_L$ are complements to $T_L \subset T_S|_L$, so that we get a canonical identification $F|_L \xrightarrow{\sim} F''|_L$. We choose θ and θ'' in such a way that the composition

$$\mathcal{O}_L \xrightarrow{\theta|_L} \det F|_L \xrightarrow{\cong} \det F''|_L \xrightarrow{\theta''^\vee|_L} \mathcal{O}_L \tag{31}$$

is equal to the identity.

Now, by applying the exchange property, Theorem 3.15, twice, as in the proof of Theorem 4.2, we obtain the following diagram of quasi-isomorphisms of differential Batalin-Vilkovisky modules, covering Diagram (30) of differential Gerstenhaber algebras:

$$\begin{tikzcd}
\mathbb{R}\mathcal{H}om^{E'\times E}_{\overline{S}\times S}\big(\mathcal{O}|(M\times L),(\overline{P}\otimes\overline{Q}^\vee)|S\big) \arrow[r] & \mathbb{R}\mathcal{H}om^{E}_{S}(\overline{Q}|L,P|M) \arrow[dl, phantom] \\
\mathbb{R}\mathcal{H}om^{E''\times E'}_{S\times\overline{S}}\big(\mathcal{O}|(L\times M),(\widehat{Q}^\vee,\otimes\overline{P})|S\big) \arrow[r] \arrow[from=1-1, to=2-2, phantom] & \mathbb{R}\mathcal{H}om^{E'}_{\overline{S}}(\overline{P}^\vee|M,Q^\vee|L) \\
 & \mathbb{R}\mathcal{H}om^{E''}_{S}(\widehat{Q}|L,P|M)
\arrow[from=1-1, to=2-2]
\arrow[from=2-1, to=3-2]
\end{tikzcd} \tag{32}$$

When passing to cohomology, the first and the last item in this diagram are both equal to $\mathcal{E}xt^\bullet_{\mathcal{O}_S}(Q,P)$. We claim that the induced isomorphism on cohomology is equal to the identity. For simplicity, we will prove this for the case that P and Q are the trivial rank one local systems. Then the differential Batalin-Vilkovisky modules of Diagram (32) are invertible. We orient them using σ^n, $(-1)^n\sigma^n$, and $(\theta \lrcorner \sigma^n)|_M$, $(\theta' \lrcorner \sigma^n)|_L$ and $(\theta'' \lrcorner \sigma^n)|_M$, respectively, as in Section 3.3. Then the homomorphisms in Diagram (32) do not preserve orientations according to Definition 1.37, because of the presence of the functions g, defined in (23), entering into the definition of ψ, Equation (24).

Let us call these functions, from the top to the bottom, g_1, g_2, g_3, g_4. We also need more detailed notation for the various maps β of Remark 3.9 and introduce

$$\beta_{ij} : F^{(i)} \to T_S \xrightarrow{\lrcorner\sigma} \Omega_S \to F^{(j)^\vee},$$

where $i,j = 0,1,2$ denotes the number of primes on the letter F. Using similar notation, we introduce the functions

$$h_{ij} = (\theta^{(j)})^\vee \circ \det\beta_{ij} \circ \theta^{(i)}.$$

These are functions on S, invertible where they are defined.

On the submanifold M, we have canonical isomorphisms $\alpha_{ij} : F^{(i)}|_M \to F^{(j)}|_M$ and functions

$$a_{ij} = (\theta^{(j)})^{-1} \circ \det \alpha_{ij} \circ \theta^{(i)} .$$

Similarly, on L, we have canonical isomorphisms $\gamma_{ij} : F^{(i)}|_L \to F^{(j)}|_L$ and functions

$$c_{ij} = (\theta^{(j)})^{-1} \circ \det \alpha_{ij} \circ \theta^{(i)} .$$

With these notations, we now have:

$$\begin{aligned} g_1 &= h_{01} a_{01} , \\ g_2 &= h_{10} c_{10} , \\ g_3 &= h_{12} c_{12} , \\ g_4 &= h_{21} a_{21} . \end{aligned}$$

Hence the default of the maps in (32) to preserve orientations is given by the product

$$\frac{g_2\big((-1)^n g_4\big)}{g_1\big((-1)^n g_3\big)} = \frac{h_{10} c_{10} h_{21} a_{21}}{h_{01} a_{01} h_{12} c_{12}} = \frac{c_{10} a_{21}}{a_{01} c_{12}} ,$$

noting that h_{ij} is dual, and hence equal, to h_{ji}.

Now note that we have two orientations on M, namely $(\theta \lrcorner \sigma^n)|_M$ and $(\theta'' \lrcorner \sigma^n)|_M$. On $\mathcal{E}xt^\bullet_{\mathcal{O}_S}(\mathcal{O}_L, \mathcal{O}_M)$, this difference induces a factor of a_{20}. Thus, to prove our claim, we need to show that

$$a_{20} = \frac{c_{10} a_{21}}{a_{01} c_{12}} .$$

Now, it is clear that $a_{ij} a_{jk} = a_{ik}$, and $c_{ij} c_{jk} = c_{ik}$. Thus, our claim is equivalent to

$$c_{20} = 1 ,$$

which is true, because c_{02} is the homomorphism of Diagram (31), which is the identity, by assumption. □

4.3 Oriented case

Theorem 4.4. *Let L and M be immersed Lagrangians of the symplectic manifold S of dimension 2n. Then every orientation of M defines a generator for the bracket of Theorem 4.2. More precisely, every trivialization ω°_M of Ω^n_M defines a differential d° on $\mathcal{T}or^\bullet_{\mathcal{O}_S}(\mathcal{O}_L, \mathcal{O}_M)$ making the latter a sheaf of Batalin-Vilkovisky algebras.*

Proof. From Example 1.36 and Proposition 1.35, we get that ω°_M defines a differential d° on $\mathcal{T}or^\bullet_{\mathcal{O}_S}(\mathcal{O}_L, \mathcal{O}_M)$. We have to show that d° does not depend on the polarization. For this we repeat the proof of Theorem 4.3, making sure that all morphisms of differential Batalin-Vilkovisky modules preserve orientations. For this, we have to be more careful with our choices. Of course, F and F'', the two polarizations to be compared, are given. But we will choose F' in a special way, as follows.

First note that on L, both $F|_L$ and $F''|_L$ are complements to T_L inside $T_S|_L$. Thus we obtain an isomorphism $\phi : F|_L \to F''|_L$, characterized by $\phi(X)-X \in T_L$, for all $X \in F|_L$. There exists a canonical subbundle $H \subset T_S|_L$, such that H is complementary to $F|_L$, $F''|_L$ and T_L, and the isomorphism $\widetilde{\phi}$: $F|_L \to F''|_L$, characterized by $\widetilde{\phi}(X)-X \in H$ is equal to $-\phi$. (Essentially, H is obtained by negating the F''-components of the vectors in T_L, but preserving their F-components.) The subbundle $H \subset T_S|_L$ is isotropic, so we can extend it, at least locally, to a Lagrangian subbundle $F' \subset T_S$. With this choice we will have

$$\frac{h_{10}}{h_{12}} = \Big(\frac{h_{10}}{h_{12}}\Big)^{\vee} = (-1)^n c_{02}\,,$$

and hence

$$g_2 = (-1)^n g_3\,.$$

Now, finally, we choose, first an orientation ω_L° of L and then θ and θ' in such a way that

$$g_1(\theta \lrcorner \sigma^n)|_M = \omega_M^\circ\,,$$
$$g_2(\theta' \lrcorner \sigma^n)|_M = \omega_L^\circ\,.$$

Then, by the choice of F', we have

$$g_3\big(\theta' \lrcorner (-1)^n \sigma^n\big)|_M = \omega_L^\circ\,.$$

We choose θ'' in such a way that $c_{02} = 1$, as above. Then $h_{10} = (-1)^n h_{12}$, and hence $h_{01} = (-1)^n h_{21}$ and $h_{01}a_{01}a_{20} = (-1)^n h_{21}a_{21}$. In other words, $g_1 a_{20} = (-1)^n g_4$, or

$$g_4\big(\theta'' \lrcorner (-1)^n \sigma^n\big)|_M = g_1(\theta \lrcorner \sigma^n)|_M = \omega_M^\circ\,.$$

Now all four homomorphisms of Diagram (32) preserve orientations, and hence they are equal to the morphisms of Diagram (30). This finishes the proof. □

Corollary 4.5. *In the non-oriented case, the sheaf of Gerstenhaber algebras* $\mathcal{T}or^{\bullet}_{\mathcal{O}_S}(\mathcal{O}_L, \mathcal{O}_M)$ *is locally a Batalin-Vilkovisky algebra, albeit in a non-canonical way.*

4.4 The exchange property

Let S and S' be a complex symplectic manifolds, of dimension $2n$, $2n'$, respectively.

Definition 4.6. A symplectic correspondence $C \to S' \times S$ is called **regular**, if for every point $P \in C$ one can find polarizations $E \subset \Omega_S$, defined in a neighborhood of $\pi(P) \in S$, and $E' \subset \Omega_{S'}$, defined in a neighborhood of $\pi'(P) \in S'$, such that

(i) C is transverse to ${E'}^\perp \times E^\perp$ inside $S' \times S$,

(ii) the induced foliations F, F' on C descend to foliations $\widetilde{F}$, $\widetilde{F}'$ on S' and S, respectively, as in Section 3.2.

Theorem 4.7. *Let $C \to S' \times S$ be a regular symplectic correspondence. Let $L \to S$ be an immersed Lagrangian transverse to C and $M' \to S'$ an immersed Lagrangian transverse to C. Then there is a canonical isomorphism of sheaves of Gerstenhaber algebras*

$$\mathcal{T}or^\bullet_{\mathcal{O}_{S'}}(\mathcal{O}_{L'}, \mathcal{O}_{M'}) = \mathcal{T}or^\bullet_{\mathcal{O}_S}(\mathcal{O}_L, \mathcal{O}_M)\,,$$

with notation as in (13).

If L, M' and C are oriented, then this is an isomorphism of sheaves of Batalin-Vilkovisky algebras.

Proof. We apply the exchange property, Theorem 3.12, twice, first to $C \to S' \times S$, then to $S' \to S' \times S'$. □

Theorem 4.8. *Let $C \to S' \times S$ be a regular symplectic correspondence. Let $L \to S$ be an immersed Lagrangian transverse to C and $M' \to S'$ an immersed Lagrangian transverse to C. Let P' be a local system on M' and Q a local system on L. Then there is a canonical isomorphism of sheaves of Batalin-Vilkovisky modules*

$$\mathcal{E}xt^\bullet_{\mathcal{O}_{S'}}(Q|_{L'}, P') = \mathcal{E}xt^\bullet_{\mathcal{O}_S}(Q, P'|_M)\,,$$

covering the isomorphism of sheaves of Gerstenhaber algebras of Theorem 4.7.

Proof. We apply the exchange property, Theorem 3.15, twice, first to $C \to S' \times S$, then to $S' \to S' \times S'$. The details are similar to the proof of Theorem 4.3. □

An example

Let M be a complex manifold and $S = \Omega_M$ the cotangent bundle with its canonical symplectic structure. There are two typical examples of immersed Lagrangians:

(i) the graph of a closed 1-form $\omega \in \Gamma(M, \Omega_M)$ which we denote by $\Gamma_\omega \subset \Omega_M$. This is in fact embedded.

(ii) the conormal bundle $C_{Z/M} \to \Omega_M$, where $Z \to M$ is an immersion of complex manifolds, i.e., a holomorphic map, injective on tangent spaces.

Given one of each, we consider the Lagrangian intersection $\Gamma_\omega \cup C_{Z/M}$. Note that it is supported on $Z(\omega) \cap Z$. We use the notation

$$\mathcal{T}_M(\omega, Z) = \mathcal{T}or^{\bullet}_{\mathcal{O}_{\Omega_M}}(\mathcal{O}_{\Gamma_\omega}, \mathcal{O}_{C_{Z/M}}),$$

and

$$\mathcal{E}_M(\omega, Z) = \mathcal{E}xt^{\bullet}_{\mathcal{O}_{\Omega_M}}(\mathcal{O}_{\Gamma_\omega}, \mathcal{O}_{C_{Z/M}}).$$

Let $f : M \to N$ be a holomorphic map between complex manifolds M, N. Consider the symplectic manifolds $S' = \Omega_M$ and $S = \Omega_N$. The pullback vector bundle $f^*\Omega_N$ is then a symplectic correspondence C:

$$\begin{tikzcd} f^*\Omega_N \arrow[r] \arrow[d] & \Omega_N \\ \Omega_M & \end{tikzcd}$$

Let us assume that $f^*\Omega_N \to \Omega_M$ fits into a short exact sequence of vector bundles

$$0 \longrightarrow K \longrightarrow f^*\Omega_N \longrightarrow \Omega_M \longrightarrow \Omega_{M/N} \longrightarrow 0\,.$$

Then the symplectic correspondence $C = f^*\Omega_N$ is regular.

If $Z \to M$ is an immersion (i.e., injective on tangent spaces), such that the composition $Z \to N$ is also an immersion, then the conormal bundle $C_{Z/M} \to \Omega_M$ is an immersed Lagrangian transverse to $f^*\Omega_N$. The corresponding immersed Lagrangian of Ω_N is the conormal bundle $C_{Z/N}$.

If $\omega \in \Gamma(N, \Omega_N)$ is a closed 1-form, then its graph is a Lagrangian submanifold of Ω_N, which is automatically transverse to $f^*\Omega_N$. The corresponding Lagrangian submanifold of Ω_M is the graph of the pullback form $f * \omega$.

Corollary 4.9. *There is a canonical isomorphism of Gerstenhaber algebras with Batalin-Vilkovisky modules*

$$\mathcal{T}_N(\omega, Z) = \mathcal{T}_M(f^*\omega, Z)\,, \qquad \mathcal{E}_N(\omega, Z) = \mathcal{E}_M(f^*\omega, Z)\,.$$

5 Further remarks

5.1 Virtual de Rham cohomology

Let M and L be Lagrangian submanifolds of the complex symplectic manifold S. Let X be their scheme-theoretic intersection. Let $\mathcal{E} = \mathcal{E}xt^{\bullet}_{\mathcal{O}_S}(\mathcal{O}_L, \mathcal{O}_M)$ be endowed with the differential d from Section 4.2. The sheaf $\mathcal{E}$ is a coherent $\mathcal{O}_X$-module, the differential d is $\mathbb{C}$-linear.

Definition 5.1. We call $(\mathcal{E}, d)$ the **virtual de Rham complex** of X.

Theorem 5.2. *The complex $(\mathcal{E}, d)$ is constructible.*

Proof. The claim is local in X, so we may assume that the symplectic manifold S is the cotangent bundle of the manifold M, that the first Lagrangian M is the zero section and the second Lagrangian L is the graph of an exact 1-form df, where $f : M \to \mathbb{C}$ is a holomorphic function. In this form theorem was proved by Kapranov; see the remarks towards the bottom of page 72 in [2]. □

Corollary 5.3. *The hypercohomology group $\mathbb{H}^p\big(X,(\mathcal{E},d)\big)$ is finite dimensional. Moreover, for $Z \subset X$ Zariski closed, $\mathbb{H}^p_Z\big(X,(\mathcal{E},d)\big)$ is also finite dimensional.*

By abuse of notation, we will write $\mathbb{H}^p(X,\mathcal{E})$ and $\mathbb{H}^p_Z(X,\mathcal{E})$, instead of $\mathbb{H}^p\big(X,(\mathcal{E},d)\big)$ and $\mathbb{H}^p_Z\big(X,(\mathcal{E},d)\big)$, respectively.

Definition 5.4. We call the hypercohomology group $\mathbb{H}^p(X,\mathcal{E})$ the p-th **virtual de Rham cohomology** group of the Lagrangian intersection X.

Corollary 5.5. *The function*

$$P \longmapsto \sum_i (-1)^i \dim_{\mathbb{C}} \mathbb{H}^i_{\{P\}}(X,\mathcal{E})$$

is a constructible function $\chi : X \to \mathbb{Z}$.

We may think of $\chi : X \to \mathbb{Z}$ as the fiberwise Euler characteristic of the constructible complex $(\mathcal{E},d)$.

5.2 A speculation in Hodge theory

Remark 5.6. There is the standard spectral sequence of hypercohomology

$$E_1^{pq} = H^q(X,\mathcal{E}^p) \Longrightarrow \mathbb{H}^{p+q}\big(X,(\mathcal{E},d)\big)\,. \tag{33}$$

This should be viewed as a generalization of the Hodge to de Rham spectral sequence.

There is also the usual local do global spectral sequence

$$E_2^{p,q} = H^p(X,\mathcal{E}^q) \Longrightarrow \operatorname{Ext}^{p+q}_{\mathcal{O}_S}(\mathcal{O}_L,\mathcal{O}_M)\,. \tag{34}$$

One may speculate to what extent these spectral sequences degenerate.

Example 5.7. For example, if M is a manifold and $S = \Omega_M$ the cotangent bundle with its standard symplectic structure, and we consider the intersection of M (the zero section) with itself, we get:

$$\mathcal{E}xt^p_{\mathcal{O}_S}(\mathcal{O}_M,\mathcal{O}_M) = \Omega^p_M\,.$$

Moreover, $(\mathcal{E},d) = (\Omega^\bullet_M,d)$ is the de Rham complex of M and Lagrangian intersection cohomology is equal to de Rham cohomology of M. Thus the spectral sequence (33) is the usual Hodge to de Rham spectral sequence:

$$E_1^{pq} = H^q(M, \Omega^p) \Longrightarrow \mathbb{H}^{p+q}\big(M, (\Omega_M^\bullet, d)\big) .$$

On the other hand, we have

$$\operatorname{Ext}^i_{\mathcal{O}_S}(\mathcal{O}_M, \mathcal{O}_M) = \bigoplus_{p+q=i} H^i(M, \Omega^q) ,$$

in other words, the E_2-term of the spectral sequence (34) is equal to the abutment. Thus $\operatorname{Ext}^i_{\mathcal{O}_S}(\mathcal{O}_M, \mathcal{O}_M)$ is equal to Hodge cohomology of M.

We may, then, rewrite the Hodge to de Rham spectral sequence (33) as

$$\operatorname{Ext}^i_{\mathcal{O}_S}(\mathcal{O}_M, \mathcal{O}_M) \Longrightarrow \mathbb{H}^i\big(M, (\mathcal{E}, d)\big) .$$

This, of course, degenerates if M is proper and gives the equality

$$\operatorname{Ext}^i_{\mathcal{O}_S}(\mathcal{O}_M, \mathcal{O}_M) = \mathbb{H}^i\big(M, (\mathcal{E}, d)\big) , \tag{35}$$

if M is Kähler, by Hodge theory.

The following conjecture is thus a generalization of Hodge theory:

Conjecture 5.8. *Under sufficiently strong hypotheses, including certainly that the intersection $X = L \cap M$ is* complete *and some analogue of the Kähler condition, for example that X is projective, we we have*

$$\mathbb{H}^p\big(X, (\mathcal{E}, d)\big) = \operatorname{Ext}^p_{\mathcal{O}_S}(\mathcal{O}_L, \mathcal{O}_M) .$$

5.3 A differential graded category

Let S be a symplectic variety and $\mathfrak{U} = \{U_i\}$ an affine open cover of S. Construct a category $\mathcal{A}$ as follows: objects of $\mathcal{A}$ are pairs (M, P), where M is a Lagrangian submanifold of S and P is a local system on M. We do not assume that $M \to S$ is a closed immersion, it suffices that this map be affine. We often omit the first component of such a pair (M, P) from the notation.

For objects (M, P) and (L, Q) of $\mathcal{A}$ we define $\operatorname{Hom}_{\mathcal{A}}(Q, P)$ to be the total complex associated to the double complex

$$C^\bullet\big(\mathfrak{U}, (\mathcal{E}^\bullet, d)\big)$$

given by Cech cochains with respect to the cover $\mathfrak{U}$ with values in the virtual de Rham complex $\mathcal{E} = \mathcal{E}xt^\bullet_{\mathcal{O}_S}(Q, P)$ (endowed with the differential from Theorem 4.3).

Theorem 5.9. *This defines a differential graded category.*

Proof. For simplicity of notation, we deal only with Lagrangian submanifolds M, N and L, leaving the generalization to local systems to the reader. There are natural Yoneda pairings (all tensors and $\mathcal{E}xt$'s are over $\mathcal{O}_S$)

$$\mathcal{E}xt^i(\mathcal{O}_M,\mathcal{O}_N)\otimes\mathcal{E}xt^j(\mathcal{O}_N,\mathcal{O}_L)\longrightarrow\mathcal{E}xt^{i+j}(\mathcal{O}_M,\mathcal{O}_L)\,.$$

We need to show that these are compatible with the canonical differential of Section 4.2. This is a local question, so we may assume that S has three Lagrangian foliations F, F' and F'', all transverse to each other, and all transverse to M, N, L. Let s, t', u'' be the Euler sections of M, N, L with respect to F, F' and F'', respectively. Then we can represent $\mathcal{E}xt^\bullet(\mathcal{O}_M,\mathcal{O}_N)$ as the cohomology of $(\Omega^\bullet_S, s-t')$, and $\mathcal{E}xt^\bullet(\mathcal{O}_N,\mathcal{O}_L)$ as the cohomology of $(\Omega^\bullet_S, t'-u'')$, and $\mathcal{E}xt^\bullet(\mathcal{O}_M,\mathcal{O}_L)$ as the cohomology of $(\Omega^\bullet_S, s-u'')$. So the claim will follow if we can produce a morphism of complexes

$$(\Omega^\bullet_S, s-t')\otimes(\Omega^\bullet_S, t'-u'')\longrightarrow(\Omega^\bullet_S, s-u'')\,.$$

But this is easy: just take the cup product. □

Remark 5.10. The cohomology groups of the hom-spaces in $\mathcal{A}$ are the virtual de Rham cohomology groups of Lagrangian intersections.

Remark 5.11. The category $\mathcal{A}$ does not depend on the affine cover $\mathfrak{U}$ in any essential way.

Remark 5.12. Of course, it is tempting to speculate on relations of $\mathcal{A}$ to the Fukaya category of S. We will leave this to future research.

5.4 Relation to vanishing cycles

Let S be a complex symplectic manifold of dimension $2n$. Let L, M Lagrangian submanifolds, and $X = L\cap M$ their intersection.

In [1], we introduced for any scheme X a constructible function $\nu_X : X \to \mathbb{Z}$. The value $\nu_X(P)$ is an invariant of the singularity (X,P). In our context, the singularity (X,P) is the critical set of a holomorphic function $f : M \to \mathbb{C}$, locally defined near $P \in M$. Hence (see [1]), the invariant $\nu_X(P)$ is equal to the Milnor number of f at P, i.e., we have

$$\nu_X(P) = (-1)^n\big(1-\chi(F_P)\big)\,,$$

where F_P is the Milnor fibre of f at the point P.

Conjecture 5.13. *We have $\chi(P) = \nu_X(P)$.*

This conjecture would follow from Remark 2.12 (b) of [2]. Note that Kapranov refers to this as a *fact*, which is *not obvious*, although *probably not very difficult*.

Conjecture 5.14. *If the intersection X is compact, so that the intersection number $\#^{\mathrm{vir}}(X)$ is well-defined, we have*

$$\#^{\mathrm{vir}}(X) = \sum_i (-1)^i \dim \mathbb{H}^i(X,\mathcal{E})\,,$$

i.e., the intersection number if equal to the **virtual Euler characteristic** *of X, defined in terms of virtual de Rham cohomology.*

To see that Conjecture 5.13 implies Conjecture 5.14, recall from [1], that the intersection X has a symmetric obstruction theory. The main result of [1] implies that $\#^{\mathrm{vir}}(X) = \chi(X, \nu_X)$. But if $\chi = \nu_X$, then $\chi(X, \nu_X) = \chi(X, \chi) = \sum_i (-1)^i \dim \mathbb{H}^i(X, \mathcal{E})$.

Remark 5.15. If S is the cotangent bundle of M, and L is the graph of df, where $f : M \to \mathbb{C}$ is a holomorphic function, then the Lagrangian intersection $X = L \cap M$ is the critical set of f. Thus X carries the perverse sheaf of vanishing cycles Φ_f. In [2], Kapranov constructs, at least conjecturally, a spectral sequence whose E_2-term is $(\mathcal{E}, d)$ and whose abutment is, in some sense, Φ_f.

Conjecture 5.16. *In the general case of a Lagrangian intersection $X = L \cap M$ inside a complex symplectic manifold S, we conjecture the existence of a natural perverse sheaf on X, which locally coincides with the perverse sheaf of vanishing cycles of Remark 5.15. There should be a spectral sequence relating $(\mathcal{E}, d)$ to this perverse sheaf of vanishing cycles. We believe that [3] may be related to this question. This conjecture, in some sense, categorifies Conjecture 5.13.*

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