Quotients of Calabi-Yau varieties

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Let X be a Calabi-Yau variety over $\mathbb{C}$, that is, a projective variety with canonical singularities whose canonical class is numericaly trivial. Let G be a finite group acting on X and consider the quotient variety X/G. The aim of this paper is to determine the place of X/G in the birational classification of varieties. That is, we determine the Kodaira dimension of X/G and decide when it is uniruled or rationally connected.

If G acts without fixed points, then $\kappa(X/G) = \kappa(X) = 0$, thus the interesting case is when G has fixed points. We answer the above questions in terms of the action of the stabilizer subgroups near the fixed points.

The answer is especially nice if X is smooth. In the introduction we concentrate on this case. The precise general results are formulated later.

Definition 1. *Let V be a complex vector space and $g \in \mathrm{GL}(V)$ an element of finite order. Its eigenvalues (with multiplicity) can be written as $e(r_1), \dots, e(r_n)$, where $e(x) := e^{2\pi i x}$ and $0 \le r_i < 1$. Following [IR96, Rei02], we define the age of g as*

$$\mathrm{age}(g) := r_1 + \cdots + r_n.$$

Let G be a finite group and (ρ, V) a finite dimensional complex representation of G. We say that $\rho : G \to \mathrm{GL}(V)$ satisfies the (local) Reid-Tai condition if $\mathrm{age}(\rho(g)) \ge 1$ for every $g \in G$ for which $\rho(g)$ is not the identity (cf. [Rei80, 3.1]).

Let G be a finite group acting on a smooth projective variety X. We say that the G-action satisfies the (global) Reid-Tai condition if for every $x \in X$, the stabilizer representation $\mathrm{Stab}_x(G) \to \mathrm{GL}(T_x X)$ satisfies the (local) Reid-Tai condition.

Our first result relates the uniruledness of X/G to the Reid-Tai condition.

Theorem 2. *Let X be a smooth projective Calabi-Yau variety and G a finite group acting on X. The following are equivalent:*

1. $\kappa(X/G) = 0$.
2. X/G *is not uniruled.*
3. *The G-action satisfies the global Reid-Tai condition.*

The equivalence of (1) and (3) is essentially in [Rei80, Sec.3]. It is conjectured in general that being uniruled is equivalent to having Kodaira dimension $-\infty$. The main point of Theorem 2 is to establish this equivalence for varieties of the form X/G.

It can happen that X/G is uniruled but not rationally connected. The simplest example is when $X = X_1 \times X_2$ is a product, G acts trivially on X_1 and X_2/G is rationally connected. Then $X/G \cong X_1 \times (X_2/G)$ is a product of the Calabi–Yau variety X_1 and of the rationally connected variety X_2/G. We show that this is essentially the only way that X/G can be uniruled but not rationally connected. The key step is a description of rational maps from Calabi-Yau varieties to lower dimensional nonuniruled varieties.

Theorem 3. *Let X be a smooth, simply connected projective Calabi-Yau variety and $f : X \dashrightarrow Y$ a dominant map such that Y is not uniruled. Then one can write*

$$f : X \xrightarrow{\pi} X_1 \overset{g}{\dashrightarrow} Y$$

where π is a projection to a direct factor of $X \cong X_1 \times X_2$ and $g : X_1 \dashrightarrow Y$ is generically finite.

Note: C. Voisin pointed out that the smooth case dicussed above also follows from the Beauville-Bogomolov-Yau structure theorem, see [Bea83b, Bea83a]. The structure theorem is conjectured to hold also for singular Calabi-Yau varieties. In any case, we prove the singular version Theorem 14 in Section 2 by other methods.

Applying this to the MRC-fibration of X/G, we obtain the following.

Corollary 4. *Let X be a smooth, simply connected projective Calabi-Yau variety which is not a nontrivial product of two Calabi-Yau varieties. Let G be a finite group acting on X. The following are equivalent:*

1. X/G *is uniruled.*
2. X/G *is rationally connected.*
3. *The G-action does not satisfy the global Reid-Tai condition.*

Next we turn our attention to a study of the local Reid-Tai condition. For any given representation it is relatively easy to decide if the Reid-Tai condition is satisfied or not. It is, however, quite difficult to get a good understanding of all representations that satisfy it. For instance, it is quite tricky to determine all $\leq$ 4-dimensional representations of cyclic groups that satisfy the Reid-Tai condition, cf. [MS84, Mor85, MMM88, Rei87]. These turn out to be rather special.

By contrast, we claim that every representation of a "typical" nonabelian group satisfies the Reid-Tai condition. The groups which have some representation violating the Reid-Tai condition are closely related to complex reflection groups. In the second part of the paper we provide a kit for building all of them, using basic building blocks, all but finitely many of which are (up to projective equivalence) reflection groups.

Let G be a finite group and (ρ, V) a finite dimensional complex representation of G such that (ρ, V) does not satisfy the (local) Reid-Tai condition. That is, G has an element g such that $0 < \mathrm{age}(\rho(g)) < 1$. We say that such a pair (G, V) is a *non-RT pair* and g is an *exceptional element*. There is no essential gain in generality in allowing $\rho\colon G \to \mathrm{GL}(V)$ not to be faithful. We therefore assume that ρ is faithful, and remove it from the notation, regarding G as a subgroup of $\mathrm{GL}(V)$ (which is to be classified up to conjugation). If the conjugacy class of g does not generate the full group G, it must generate a normal subgroup H of G such that (H, V) is again a non-RT pair. After classifying the cases for which the conjugacy class of g generates G, we can take the normalization of each such G in $\mathrm{GL}(V)$; all finite subgroups intermediate between G and this normalizer give further examples. If V is reducible, then for every irreducible factor V_i of V on which g acts non-trivially, (G, V_i) is again a non-RT pair with exceptional element g. Moreover, if the conjugacy class of g generates G, then g must be an exceptional element for every non-trivial factor V_i of V. These reduction steps motivate the following definition:

Definition 5. *A basic non-RT pair is an ordered pair (G, V) consisting of a finite group G and a faithful irreducible representation V such that the conjugacy class of any exceptional element $g \in G$ generates G.*

Given a basic non-RT pair (G, V) and a positive integer n, we define $G_n = G \times \mathbb{Z}/n\mathbb{Z}$ and let V_n denote the tensor product of V with the character of $\mathbb{Z}/n\mathbb{Z}$ sending 1 to $e(1/n)$. Then V_n is always an irreducible representation of G_n and is faithful if n is prime to the order of G. Also if the conjugacy class of $g \in G$ generates the whole group, $a \in \mathbb{Z}$ is relatively prime to n, and n is prime to the order of G, then the conjugacy class of $(g, \overline{a}) \in G_n$ generates G_n. Finally, if g is an exceptional element, $(a, n) = 1$, and $a/n > 0$ is sufficiently small, then $(g, \overline{a})$ is exceptional. Thus for each basic non-RT pair (G, V), there are infinitely many other basic non-RT pairs which are projectively equivalent to it. To avoid this complication, we seek to classify basic non-RT pairs only up to projective equivalence:

Definition 6. *Pairs (G_1, V_1) and (G_2, V_2) are* projectively equivalent *if there exists an isomorphism $\mathrm{PGL}(V_1) \to \mathrm{PGL}(V_2)$ mapping the image of G_1 in $\mathrm{PGL}(V_1)$ isomorphically to the image of G_2 in $\mathrm{PGL}(V_2)$.*

We recall that a *pseudoreflection* $g \in \mathrm{GL}(V)$ is an element of finite order which fixes a codimension 1 subspace of V pointwise. A (complex) reflection group is a finite subgroup of $\mathrm{GL}(V)$ which is generated by pseudoreflections. We say that (G, V) is a reflection group if G is a reflection group in $\mathrm{GL}(V)$.

Definition 7. *We say that a basic non-RT pair* (G, V) *is of reflection type if* (G, V) *is projectively equivalent to some reflection group* (G', V').

The reflection groups are classified in [ST54]. Note that every pseudoreflection is of exceptional type, so every reflection group is a non-RT pair. On the other hand, there may be elements of exceptional type in a reflection group which are *not* pseudoreflections. Moreover, not every irreducible reflection group gives rise to a basic non-RT pair; it may happen that a particular conjugacy class of pseudoreflections fails to generate the whole group, since some irreducible reflection groups have multiple conjugacy classes of pseudoreflections.

We are interested in the other direction, however, and here we have the following theorem.

Theorem 8. *Up to projective equivalence, there are only finitely many basic non-RT pairs which are* not *of reflection type.*

We give an example in §5 below. Guralnick and Tiep [GT07] have given two other examples, also in dimension 4. In principle, our proof provides an effective way (via the classification of finite simple groups) to determine all examples. It seems likely that the final version of [GT07] will give such a classification

Finally, we study in detail the case when $X = A$ is an Abelian variety. In fact, this case was the starting point of our investigations. For Abelian varieties, the induced representations $\operatorname{Stab}_x G \to V = T_x A$ have the property that $V + V^*$ is isomorphic to the rational representation of $\operatorname{Stab}_x G$ on $H^1(A, \mathbb{Q})$. This property substantially reduces the number of cases that we need to consider and allows us to show that a basic non-RT pair arising from an Abelian variety which is not of reflection type is of the unique known type. A more precise statement is given in Theorem 36 below. Unlike the proof of Theorem 8, the proof of this theorem does not make use of the classification of finite simple groups.

A consequence of the analysis which ultimately gives Theorem 36 is the following easier statement.

Theorem 9. *Let* A *be a simple Abelian variety of dimension* ≥ 4 *and* G *a finite group acting on* A. *Then* A/G *has canonical singularities and* $\kappa(A/G) = 0$.

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1 Uniruled quotients

Let X be a projective Calabi–Yau variety and $G \subset \operatorname{Aut}(X)$ a finite group of automorphism. Our aim is to decide when is the quotient variety X/G

uniruled or rationally connected. Our primary interest is in the case when X is smooth, but the proof works without change whenever X has canonical singularities and K_X is numerically trivial. (Note that by [Kaw85, 8.2], K_X numerically trivial implies that K_X is torsion.)

Theorem 10. *Let X be a projective Calabi–Yau variety and G a finite group acting on X. The following are equivalent:*

1. *G acts freely outside a codimension ≥ 2 set and X/G has canonical singularities.*
2. *$\kappa(X/G) = 0$.*
3. *X/G is not uniruled.*

Proof. Set $Z := X/G$ and let $D_i \subset Z$ be the branch divisors of the quotient map $\pi : X \to Z$ with branching index e_i. Set $\Delta := \sum\bigl(1 - \frac{1}{e_i}\bigr)D_i$. By the Hurwitz formula,

$$K_X \sim_{\mathbb{Q}} \pi^*\bigl(K_Z + \Delta\bigr),$$

where $\sim_{\mathbb{Q}}$ means that some nonzero integral multiples of the two sides are linearly equivalent. Thus $K_Z + \Delta \sim_{\mathbb{Q}} 0$ and hence Theorem 10) is a special case of Theorem 11. □

Theorem 11. *Let Z be a projective variety and Δ an effective divisor on Z such that $K_Z + \Delta \sim_{\mathbb{Q}} 0$. The following are equivalent:*

1. *$\Delta = 0$ and Z has canonical singularities.*
2. *$\kappa(Z) = 0$.*
3. *Z is not uniruled.*

Proof. If $\Delta = 0$ then K_Z is numerically trivial. Let $g : Y \to Z$ be a resolution of singularities and write

$$K_Y \sim_{\mathbb{Q}} g^*K_Z + \sum a_i E_i \sim_{\mathbb{Q}} \sum a_i E_i$$

where the E_i are g-exceptional and $a_i \geq 0$ for every i iff Z has canonical singularities.

Thus if (1) holds then $K_Y \sim_{\mathbb{Q}} \sum a_i E_i$ is effective and so $\kappa(Y) \geq 0$. Since $\sum a_i E_i$ is exceptional, no multiple of it moves, hence $\kappa(Y) = \kappa(Z) = 0$.

The implication (2) $\Rightarrow$ (3) always holds (cf. [Kol96, IV.1.11]). It is conjectured that in fact (2) is equivalent to (3), but this is known only in dimensions ≤ 3 (cf. [KM98, 3.12–13]).

Thus it remains to prove that if (1) fails then Z is uniruled. We want to use the Miyaoka-Mori criterion [MM86] to get uniruledness. That is, a projective variety Y is uniruled if an open subset of it is covered by projective curves $C \subset Y$ such that $K_Y \cdot C < 0$ and $C \cap \operatorname{Sing} Y = \emptyset$.

If $\Delta \neq 0$ then we can take $C \subset Z$ to be any smooth complete intersection curve which does not intersect the singular locus of Z.

Thus we can assume from now on that $\Delta = 0$ and, as noted before, $K_Y \sim_{\mathbb{Q}} \sum a_i E_i$ where the E_i are g-exceptional and $a_i < 0$ for some i since Z does not have canonical singularities by assumption. For notational convenience assume that $a_1 < 0$.

Ideally, we would like to find curves $C \subset Y$ such that C intersects E_1 but no other E_i. If such a C exists then

$$(K_Y \cdot C) = (\sum a_i E_i \cdot C) = a_1 (E_1 \cdot C) < 0.$$

We are not sure that such curves exist. (The condition $K_Z \equiv 0$ puts strong restrictions on the singularities of Z and creates a rather special situation.)

Fortunately, it is sufficient to find curves C such that $(C \cdot E_1)$ is big and the other $(C \cdot E_i)$ are small. This is enough to give $K_Y \cdot C < 0$.

Lemma 12. *Let Z be a normal projective variety over a field of arbitrary characteristic, $g : Y \to Z$ a birational morphism and $E = \sum a_i E_i$ a noneffective g-exceptional Cartier divisor. Then Y is covered by curves C such that $(E \cdot C) < 0$.*

Proof. Our aim is to reduce the problem to a carefully chosen surface $S \to Y$ and then construct such curves directly on S. At each step we make sure that S can be chosen to pass through any general point of Y, so if we can cover these surfaces S with curves C, the resulting curves also cover Y.

Assume that $a_1 < 0$. If $\dim g(E_1) > 0$ then we can cut Y by pull backs of hypersurface sections of Z and use induction on the dimension.

Thus assume that $g(E_1)$ is a point. The next step would be to cut with hypersurface sections of Y. The problem is that in this process some of the divisors E_i may become nonexceptional, even ample. Thus first we need to kill all the other E_i such that $\dim g(E_i) > 0$.

To this end, construct a series of varieties $\sigma_i : Z_i \to Z$ starting with $\sigma_0 : Z_0 = Z$ as follows. Let $W_i \subset Z_i$ be the closure of $(\sigma_i^{-1} \circ g)_*(E_1)$, $\pi_i : Z_{i+1} \to Z_i$ the blow up of $W_i \subset Z_i$ and $\sigma_{i+1} = \sigma_i \circ \pi_i$.

By Abhyankar's lemma (in the form given in [KM98, 2.45]), there is an index j such that if $Y' \subset Z_j$ denotes the main component and $g' := \sigma_j : Y' \to Z$ the induced birational morphism then the following hold:

1. g' is an isomorphism over $Z \setminus g(E_1)$, and
2. $h := g^{-1} \circ g' : Y' \dashrightarrow Y$ is a local isomorphism over the generic point of E_1.

Thus $h^*(\sum a_i E_i)$ is g'-exceptional and not effective (though it is only guaranteed to be Cartier outside the indeterminacy locus of h).

Now we can cut by hypersurface sections of Y' to get a surface $S' \subset Y'$. Let $\pi : S \to S'$ be a resolution such that $h \circ \pi : S \to Y$ is a morphism and $f := (g' \circ \pi) : S \to T := g'(S') \subset Z$ the induced morphism. Then $(h \circ \pi)^* E$ is exceptional over T and not effective. Thus it is sufficient to prove Lemma 12 in case $S = Y$ is a smooth surface.

Fix an ample divisor H on S such that

$$H^1(S, \mathcal{O}_S(K_S + H + L)) = 0$$

for every nef divisor L. (In characteristic 0 any ample divisor works by the Kodaira vanishing theorem. In positive characteristic, one can use for instance [Kol96, Sec.II.6] to show that any H such that $(p-1)H - K_S - 4$(some ample divisor) is nef has this property.)

Assume next that H is also very ample and pick $B \in |H|$. Using the exact sequence

$$0 \to \mathcal{O}_S(K_S + 2H + L) \to \mathcal{O}_S(K_S + 3H + L) \to \mathcal{O}_B(K_B + 2H|_B + L|_B) \to 0,$$

we conclude that $\mathcal{O}_S(K_S + 3H + L)$ is generated by global sections.

By the Hodge index theorem, the intersection product on the curves E_i is negative definite hence nondegenerate. Thus we can find a linear combination $F = \sum b_i E_i$ such that $F \cdot E_1 > 0$ and $F \cdot E_i = 0$ for every $i \neq 1$. Choose H_Z ample on Z such that $F + g^* H_Z$ is nef.

Thus the linear system $|K_S + 3H + m(F + g^* H_Z)|$ is base point free for every $m \geq 0$. Let $C_m \in |K_S + 3H + m(F + g^* H_Z)|$ be a general irreducible curve. Then

$$\begin{aligned} (C_m \cdot E_1) &= m(F \cdot E_1) + (\text{constant}), \quad \text{and} \\ (C_m \cdot E_i) &= (\text{constant}) \quad \text{for } i > 1. \end{aligned}$$

Thus $(C_m \cdot E) \to -\infty$ as $m \to \infty$. □

The following consequence of Lemma 12 is of independent interest. In characteristic zero, Corollary 13 is equivalent to Lemma 12 by [BDPP], thus one can use Corollary 13 to give an alternate proof of Lemma 12. The equivalence should also hold in any characteristic.

Corollary 13 (Lazarsfeld, unpublished). *Let Z be a normal, projective variety, $g : Y \to Z$ a birational morphism and $E = \sum a_i E_i$ a g-exceptional Cartier divisor. Then E is pseudo-effective iff it is effective.* □

2 Maps of Calabi-Yau varieties

Every variety has lots of different dominant rational maps to projective spaces, but usually very few dominant rational maps whose targets are not unirational. The main result of this section proves a version of this for Calabi-Yau varieties.

Theorem 14. *Let X be a projective Calabi-Yau variety and $g : X \dashrightarrow Z$ a dominant rational map such that Z is not uniruled.*

Then there are

1. a finite Calabi-Yau cover $h_X : \tilde{X} \to X$,
2. an isomorphism $\tilde{X} \cong \tilde{F} \times \tilde{Z}$ *where* $\tilde{F}, \tilde{Z}$ *are Calabi-Yau varieties and* π_Z *denotes the projection onto* $\tilde{Z}$ *, and*
3. a generically finite map $g_Z : \tilde{Z} \dashrightarrow Z$

such that $g \circ h_X = g_Z \circ \pi_Z$.

Remark 15. 1. De-Qi Zhang pointed out to us that in general g_Z can not be chosen to be Galois (contrary to our original claim). A simple example is given as follows. Let A be an Abelian surface, $K = A/\{\pm 1\}$ the corresponding smooth Kummer surface and $m > 1$. Then multiplication by m on A descends to a rational map $K \dashrightarrow K$, but it is not Galois.

2. Standard methods of the Iitaka conjecture (see especially [Kaw85]) imply that for any dominant rational map $X \dashrightarrow Z$, either $\kappa(Z) = -\infty$ or π is an étale locally trivial fiber bundle with Calabi–Yau fiber over an open subset of Z. Furthermore, [Kaw85, Sec.8] proves Theorem 14 for the Albanese morphism. The papers [Zha96, Zha05] also contain related results and techniques.

First we explain how to modify the standard approach to the Iitaka conjecture to replace $\kappa(Z) = -\infty$ with Z uniruled.

The remaining steps are more subtle since we have to construct a suitable birational model of Z and then to extend the fiber bundle structure from the open set to everywhere, at least after a finite cover.

Proof. As we noted before, there is a finite Calabi–Yau cover $X' \to X$ such that $K_{X'} \sim 0$. In particular, X' has canonical singularities. We can further replace Z by its normalization in $\mathbb{C}(X)$. Thus we can assume to start with that $K_X \sim 0$, X has canonical singularities and g has irreducible general fibers.

If g is not a morphism along the closure of the general fiber of g then Z is uniruled. If X is smooth, this is proved in [Kol96, VI.1.9]; the general case follows from [HM05]. Thus there are open subsets $X^* \subset X$ and $Z^* \subset Z$ such that $g : X^* \to Z^*$ is proper. We are free to shrink Z^* in the sequel if necessary.

Let us look at a general fiber $F \subset X$ of g. It is a local complete intersection subvariety whose normal bundle is trivial. So, by the adjunction formula, the canonical class K_F is also trivial. F has canonical singularities by [Rei80, 1.13].

Choose smooth birational models $\sigma : X' \to X$ and $Z' \to Z$ such that the corresponding $g' : X' \to Z'$ is a morphism which is smooth over the complement of a simple normal crossing divisor $B' \subset Z'$. We can also assume that the image of every divisor in $X \setminus X^*$ is a divisor in Z'. Thus we can choose smooth open subvarieties $X^0 \subset X$ and $Z^* \subset Z^0 \subset Z'$ such that

1. $X \setminus X^0$ has codimension ≥ 2 in X, and
2. $g_0 := g'|_{X^0} : X^0 \to Z^0$ is flat and surjective (but not proper).

The proof proceeds in 3 steps.

First, we show that $\omega_{X'/Z'}|_{X^0}$ is the pull back of a line bundle L from Z^0 which is $\mathbb{Q}$-linearly equivalent to 0.

Second we prove that there is an étale cover $Z^1 \to Z^0$ and a birational map $Z^1 \times F \dashrightarrow X^0 \times_{Z^0} Z^1$ which is an isomorphism in codimension 1.

Third, we show that if $\tilde{X} \to X$ is the corresponding cover, then $\tilde{X}$ is a product of two Calabi-Yau varieties, as expected.

In order to start with Step 1, we need the following result about algebraic fiberspaces for which we could not find a convenient simple reference.

Proposition 16. *Notation as above. Then $g'_*\omega_{X'/Z'}$ is a line bundle and one can write the corresponding Cartier divisor*

$$\text{divisor class of } \left(g'_*\omega_{X'/Z'}\right) \sim_{\mathbb{Q}} J_g + B_g,$$

where

1. *B_g is an effective $\mathbb{Q}$-divisor supported on B', and*
2. *J_g is a nef $\mathbb{Q}$-divisor such that*
   a) *either $J_g \sim_{\mathbb{Q}} 0$ and g is an étale locally trivial fiber bundle over some open set of Z^*,*
   b) *or $(J_g \cdot C) > 0$ for every irreducible curve $C \subset Z'$ which is not contained in B' and is not tangent to a certain foliation of Z^*.*

Proof. Over the open set where g' is smooth, the results of [Gri70, Thm.5.2] endow $g'_*\omega_{X'/Z'}$ with a Hermitian metric whose curvature is semipositive. This metric degenerates along B' but this degeneration is understood [Fuj78, Kaw81], giving the decomposition $J_g + B_g$ where J_g is the curvature term and B_g comes from the singularities of the metric along B'.

Set $d = \dim F$, where F is a general fiber of g. If F is smooth, we can assume that F is also a fiber of g'. In this case the curvature is flat in the directions corresponding to the (left) kernel of the Kodaira–Spencer map

$$H^1(F, T_F) \times H^0(F, \Omega_F^d) \to H^1(F, \Omega_F^{d-1}).$$

If $\Omega_F^d \cong \mathcal{O}_F$ then this is identified with the Serre duality isomorphism

$$\left(H^{d-1}(F, \Omega_F^1)\right)^* \cong H^1(F, \Omega_F^{d-1}),$$

hence the above (left) kernel is zero. Thus $(J_g \cdot C) = 0$ iff the deformation of the fibers $g^{-1}(C) \to C$ is trivial to first order over every point of $C \setminus B'$. This holds iff the fibers of g are all isomorphic to each other over $C \setminus B'$.

The corresponding result for the case when F has canonical singularities is worked out in [Kaw85, Sec.6]. $\square$

Let us now look at the natural map

$$g'^*\left(\omega_{Z'} \otimes g'_*\omega_{X'/Z'}\right) \to \omega_{X'}$$

which is an isomorphism generically along F, thus nonzero. Hence there is an effective divisor D_1 such that

$$g'^*\big(\omega_{Z'} \otimes g'_*\omega_{X'/Z'}\big) \cong \omega_{X'}(-D_1).$$

Write $\omega_{X'} \cong \sigma^*\omega_X(D_2) \cong \mathcal{O}_{X'}(D_2)$, where D_2 is σ-exceptional.

Let $C \subset X^0$ be a general complete intersection curve. Then σ^{-1} is defined along C and setting $C' := \sigma^{-1}(C)$ we get that

$$\deg_{C'} g'^*\big(\omega_{Z'} \otimes g'_*\omega_{X'/Z'}\big) = \deg_{C'} \omega_{X'}(-D_1) = \big(C' \cdot (D_2 - D_1)\big) = -\big(C' \cdot D_1\big).$$

By the projection formula this implies that

$$\big(g_0(C) \cdot K_{Z'}\big) + \big(g_0(C) \cdot J_g\big) + \big(g_0(C) \cdot B_g\big) + (C \cdot \sigma_*(D_1)) = 0. \qquad (*)$$

If $\big(g_0(C) \cdot K_{Z'}\big) < 0$ then Z' is uniruled by the Miyaoka-Mori criterion [MM86], contrary to our assumptions. Thus all the terms on the left hand side are nonnegative, hence they are all zero.

Since C is a general curve, it can be chosen to be not tangent to any given foliation. Therefore J_g is torsion in $\mathrm{Pic}(Z')$ and $X^* \to Z^*$ is an étale locally trivial fiber bundle for a suitable Z^*. A general complete intersection curve intersects every divisor in X, thus $(C \cdot \sigma_*(D_1)) = 0$ implies that $\sigma_*(D_1) = 0$, that is, D_1 is σ-exceptional.

Similarly, $g_0(C)$ intersects every irreducible component of $Z^0 \setminus Z^*$. Thus $(g_0(C) \cdot B_g) = 0$ implies that $B_g|_{Z^0} = 0$. These together imply that $L := \big(g'_*\omega_{X'/Z'}\big)|_{Z^0}$ is $\mathbb{Q}$-linearly equivalent to 0 and $\omega_{X'/Z'}|_{X^0} \cong g_0^*L$. This completes the first step.

Now to Step 2. Apply Lemma 17 to $X^* \to Z^*$. We get a finite cover $\pi : Z^1 \to Z^0$ such that $X' \times_{Z^0} Z^1$ is birational to $F \times Z^1$. By shrinking Z^0, we may assume that Z^1 is also smooth. Eventually we prove that π is étale over Z^0, but for now we allow ramification over $Z^0 \setminus Z^*$.

Let $n : X^1 \to X' \times_{Z^0} Z^1$ be the normalization. We compare the relative dualizing sheaves

$$\omega_{F \times Z^1/Z^1} \cong \mathcal{O}_{F \times Z^1} \quad \text{and} \quad \omega_{X^1/Z^1}.$$

Let $X^1 \xleftarrow{u} Y \xrightarrow{v} F \times Z^1$ be a common resolution. We can then write

$$\omega_{Y/Z^1} \cong v^*\omega_{F \times Z^1/Z^1}(E_v) \cong \mathcal{O}_Y(E_v)$$

for some divisor E_v supported on $\mathrm{Ex}(v)$ and also

$$\omega_{Y/Z^1} \cong u^*\omega_{X^1/Z^1}(E'_u) \cong (g' \circ n \circ u)^*L(E_u)$$

for some divisors E'_u, E_u. Since $g^*L|_{X^0}$ is $\mathbb{Q}$-linearly equivalent to zero, we conclude from these that

$$u_*(E_u - E_v)|_{X^0} \sim_{\mathbb{Q}} 0.$$

Next we get some information about E_u and E_v. Since $F \times Z^1$ has canonical singularities, every irreducible component of E_v is effective. Furthermore, an irreducible component of $\mathrm{Ex}(v)$ appears with positive coefficient in E_v unless it dominates Z^0. Thus we see that $u_*(E_v)|_{X^0}$ is supported in $X^0 \setminus X^*$ and an irreducible component of $X^0 \setminus X^*$ appears with positive coefficient in $u_*(E_v)$, unless $v \circ u^{-1}$ is a local isomorphism over its generic point.

On the other hand, since $\omega_{X'/Z'}|_{X^0}$ is the pull back of L, $\omega_{X' \times_{Z^0} Z^1/Z^1}|_{X^0}$ is the pull back of $\pi^* L$. As we normalize, we subtract divisors corresponding to the nonnormal locus. Every other irreducible component of E_u is u-exceptional, hence gets killed by u_*. Thus we obtain that $u_*(E_u)|_{X^0}$ is also contained in $X^0 \setminus X^*$ and an irreducible component of $X^0 \setminus X^*$ either appears with negative coefficient or $X' \times_{Z'} Z^1$ is normal over that component and the coefficient is 0.

Thus $u_*(E_u - E_v)|_{X^0}$ is a nonpositive linear combination of the irreducible components of $X^0 \setminus X^*$ and it is also $\mathbb{Q}$-lineraly equivalent to 0. Since $X \setminus X^0$ has codimension ≥ 2, we conclude that $u_*(E_u - E_v)|_{X^0} = 0$. That is, $X^0 \times_{Z^0} Z^1$ is normal in codimension 1 and isomorphic to $F \times Z^1$, again only in codimension 1.

We may as well assume that $Z^1 \to Z^0$ is Galois with group G. We then have a corresponding G-action on $F \times Z^1$ for a suitable G-action on F. By taking the quotient, we obtain a birational map

$$\phi : W := (F \times Z^1)/G \to Z^0 \dashrightarrow X^0$$

which is an isomorphism in codimension 1. In particular,

$$\omega_{W/Z^0} \cong \phi^* \omega_{X^0/Z^0} \cong g_W^* L$$

where $g_W : W \to Z^0$ is the quotient of the projection map to Z^1. Using (18.1) we conclude that $g_W : W \to Z^0$ is in fact an étale locally trivial fiber bundle with fiber F, at least outside a codimension 2 set.

Then Lemma 17 shows that $Z^1 \to Z^0$ is also étale at every generic point of $Z^0 \setminus Z^*$, thus it is a finite étale cover.

Let now $\tilde{X} \to X$ be the normalization of X in the function field of X^1. Since $Z^1 \to Z^0$ is étale, we see that $\tilde{X} \to X$ is étale over X^0. Thus $\tilde{X} \to X$ is étale outside a codimension ≥ 2 set. In particular, $\tilde{X}$ is a Calabi-Yau variety.

Furthermore, the birational map $\phi : \tilde{X} \to F \times Z^1$ is an open embedding outside a codimension ≥ 2 set. That is, ϕ does not contract any divisor. This completes Step 2.

By Proposition 18, $\tilde{X}$ is itself a product $\tilde{F} \times \tilde{Z}$. Note that $F = \tilde{F}$ since ϕ is an isomorphism along $F \cong \pi^{-1}(z)$ for $z \in Z^*$. Thus $\tilde{X} \cong F \times \tilde{Z}$ and there is a generically finite map $\tilde{Z} \dashrightarrow Z$. □

Lemma 17. *Let $f : U \to V$ be a projective morphism between normal varieties. V smooth. Assume that f is an étale locally trivial fiber bundle with typical fiber F which is a Calabi-Yau variety. Then there is a finite étale cover $V' \to V$ such that the pull back $U \times_V V' \to V'$ is globally trivial. Moreover, we can choose $V' \to V$ such that its generic fiber depends only on the generic fiber of f.*

Proof. Let H be an ample divisor on U. Let $\pi : \mathrm{Isom}(F\times V, U, H) \to V$ denote the V-scheme parametrizing V-isomorphisms $\phi : F \times V \to U$ such that $\phi^* H$ is numerically equivalent to H_F. The fiber of $\mathrm{Isom}(F \times V, U, H) \to V$ over $v \in V$ is the set of isomorphisms $\phi : F \to U_v$ such that $\phi^*(H|_{U_v})$ is numerically equivalent to H_F.

Note that $\mathrm{Isom}(F \times V, U, H) \to V$ is an étale locally trivial fiber bundle with typical fiber $\mathrm{Aut}_H(F)$. Any étale multisection of π gives a required étale cover $V' \to V$.

Thus we need to find an étale multisection of a projective group scheme (in characteristic 0). The Stein factorization of π gives an étale cover $V_1 \to V$ and if we pull back everything to V_1, then there is a well defined identity component. Thus we are reduced to the case when $\pi : I \to V$ is torsor over an Abelian scheme $A \to V$.

Let I_g be the generic fiber and let $P \in I_g$ be a point of degree d. Let $S_d \subset I_g$ be the set of geometric points p such that $dp - P = 0 \in A_g$. Then S_d is defined over $k(V)$ and it is a principal homogeneous space over the subgroup of d-torsion points of A_g. We claim that the closure of S_d in I is finite and étale over V. Indeed, it is finite over codimension 1 points and also étale over codimension 1 points since the limit of nonzero d-torsion points cannot be zero. Thus it is also étale over all points by the purity of branch loci. $\square$

The $K_X = 0$ of the following lemma is proved in [Pet94, Thm.2].

Proposition 18. *Let X, U, V be normal projective varieties. Assume that X has rational singularities. Let $\phi : X \dashrightarrow U \times V$ be a birational map which does not contract any divisor. Then there are normal projective varieties U' birational to U and V' birational to V such that $X \cong U' \times V'$.*

Proof. We can replace U, V by resolutions, thus we may assume that they are smooth.

Let $X \xleftarrow{p} Y \xrightarrow{q} U \times V$ be a factorization of g. By assumption, $\mathrm{Ex}(q) \subset \mathrm{Ex}(p)$. Let H be a very ample divisor on X and $\phi_* H = q_* p^* H$ its birational transform. Then $|q_* p^* H| = |p^* H + m\,\mathrm{Ex}(q)|$ for $m \gg 1$. On the other hand,

$$|H| = |p^* H| \subset |p^* H + m\,\mathrm{Ex}(q)| \subset |p^* H + m\,\mathrm{Ex}(p)| = |H|.$$

Thus $|H| = |\phi_* H|$.

Assume that there are divisors H_U on U and H_V on V such that $\phi_* H \sim \pi_U^* H_U + \pi_V^* H_V$. Then

$$H^0(U \times V, \mathcal{O}_{U\times V}(\phi_* H)) = H^0(U, \mathcal{O}_U(H_U)) \otimes H^0(V, \mathcal{O}_V(H_V)).$$

Since X is the closure of the image of $U \times V$ under the linear system $|\phi_* H|$, we see that $X \cong U' \times V'$ where U' is the image of U under the linear system $|H_U|$ and V' is the image of V under the linear system $|H_V|$.

If $H^1(U, \mathcal{O}_U) = 0$, then $\operatorname{Pic}(U \times V) = \pi_U^* \operatorname{Pic}(U) + \pi_V^* \operatorname{Pic}(V)$, and we are done. In general, however, $\operatorname{Pic}(U \times V) \supsetneq \pi_U^* \operatorname{Pic}(U) + \pi_V^* \operatorname{Pic}(V)$, and we have to change H.

Fix points $u \in U$ and $v \in V$ and let $D_U := \phi_* H|_{U \times \{v\}}$ and $D_V := \phi_* H|_{\{u\} \times V}$. Set $D' := \phi_* H - \pi_U^* D_U - \pi_V^* D_V$. Then D' restricted to any $U \times \{v'\}$ is in $\operatorname{Pic}^0(U)$ and D' restricted to any $\{u'\} \times V$ is in $\operatorname{Pic}^0(V)$. Thus there is a divisor B on $\operatorname{Alb}(U \times V)$ such that $D' = \operatorname{alb}_{U \times V}^* B$, where, for a variety Z, $\operatorname{alb}_Z : Z \to \operatorname{Alb}(Z)$ denotes the Albanese map.

Choose divisors B_U on $\operatorname{Alb}(U)$ and B_V on $\operatorname{Alb}(V)$ such that $\pi_U^* B_U + \pi_V^* B_V - B$ is very ample, where, somewhat sloppily, π_U, π_V also denote the coordinate projections of $\operatorname{Alb}(U \times V)$.

Since X has rational singularities, $\operatorname{Alb}(X) = \operatorname{Alb}(U \times V)$. Replace H by

$$H^* := H + \operatorname{alb}_X^*(\pi_U^* B_U + \pi_V^* B_V - B).$$

Then

$$\begin{aligned} \phi_* H^* &= \phi_* H + \operatorname{alb}_{U \times V}^*(\pi_U^* B_U + \pi_V^* B_V - B) \\ &= \pi_U^*(H_U + \operatorname{alb}_U^* B_U) + \pi_V^*(H_V + \operatorname{alb}_V^* B_V). \end{aligned}$$

Since H^* is again very ample, we are done. □

18.1 (Quotients of trivial families). We consider families $X \to C$ over a smooth pointed curve germ $0 \in C$ such that after a finite base change and normalization we get a trivial family. This means that we start with a trivial family $F \times D$ over a disc D, an automorphism τ of F of order dividing m and take the quotient $X := (F \times D)/(\tau, e(1/m))$.

If the order of τ is less than m then there is a subgroup which acts trivially on F and the quotient is again a trivial family. Thus we may assume that the order of τ is precisely m.

Fix a top form ω on F. Pulling back by τ gives an isomorphism $\omega = \eta \tau^* \omega$ for some mth root of unity η. If $\eta \neq 1$ then on the quotient family the monordomy around $0 \in C$ has finite order $\neq 1$, and the boundary term B in Proposition 16 is nonzero.

Finally, if $\omega = \tau^* \omega$ then $F_0 := F/(\tau)$ also has trivial canonical class. Thus by the adjunction formula we see that $\omega_{X/C} \cong \mathcal{O}_X((m-1)F_0)$ is not trivial.

Next we apply Theorem 14 to study those quotients of Calabi-Yau varieties which are uniruled but not rationally connected. Let us see first some examples of how this can happen. Then we see that these trivial examples exhaust all possibilities.

Example 19. Let $\Pi : X' \to Z'$ be an étale locally trivial fiber bundle whose base Z' and typical fiber F' are both projective Calabi-Yau varieties. Then X' is also a projective Calabi-Yau variety. Let G' be a finite group acting on X' and assume that Π is G'-equivariant. Assume that

1. $\kappa(Z'/G') = 0$, and
2. for general $z \in Z'$, the quotient $\Pi^{-1}(z)/\operatorname{Stab}_z G'$ is rationally connected.

Then $\Pi/G' : X'/G' \to Z'/G'$ is the MRC-fibration of X'/G'.

More generally, let $H \subset G'$ be a normal subgroup such that $X := X'/H$ is a Calabi-Yau variety and set $G := G'/H$. Then $X/G \cong X'/G'$ and its MRC-fibration is given by $\Pi/G' : X'/G' \to Z'/G'$.

Theorem 20. *Let X be a projective Calabi–Yau variety and G a finite group acting on X. Assume that X/G is uniruled but not rationally connected. Let $\pi : X/G \dashrightarrow Z$ be the MRC fibration. Then there is*

1. *a finite, Calabi–Yau, Galois cover $X' \to X$,*
2. *a proper morphism $\Pi : X' \to Z'$ which is an étale locally trivial fiber bundle whose base Z' and typical fiber F' are both projective Calabi-Yau varieties, and*
3. *a group G' acting on X' where $G \supset \operatorname{Gal}(X'/X)$ and $G'/\operatorname{Gal}(X'/X) = G$,*

such that $\Pi/G' : X'/G' \to Z'/G'$ is birational to the MRC fibration $\pi : X/G \dashrightarrow Z$.

Proof. Let $X/G \dashrightarrow Z$ be the MRC fibration and let $\pi : X \to X/G \dashrightarrow Z$ be the composite. Z is not uniruled by [GHS03] thus Theorem 14 applies and we get a direct product $F \times Z$ mapping to X. Since both X and $F \times Z$ have trivial canonical class, $F \times Z \to X$ is étale in codimension 1.

In order to lift the G-action from X to a cover, we need to take the Galois closure of $F \times Z \to X/G$. Let G' be its Galois group. This replaces $F \times Z$ with a finite cover which is étale in codimension 1. The latter need not be globally a product, only étale locally a product. □

Corollary 21. *Let A be an Abelian variety and G a finite group acting on A. There is a unique maximal G-equivariant quotient $A \to B$ such that $A/G \to B/G$ is the MRC quotient.*

Proof. Let $\Pi : A^0 \to Z^0$ be the quotient constructed in Theorem 20. Its fibers F_z are smooth subvarieties of A with trivial canonical class. Thus each F_z is a translation of a fixed Abelian subvariety $C \subset A$ (cf. [GH79, 4.14]). Set $B = A/C$. □

Definition 22. *Let G be a finite group acting on a vector space V. Let $G^{RT} < G$ be the subgroup generated by all elements of age < 1 and V^{RT} the complement of the fixed space of G^{RT}.*

Definition 23. *Let A be an Abelian variety and G a finite group acting on A. For every $x \in A$, let $G_x := \operatorname{Stab}(x) < G$ denote the stabilizer and $i_x : A \to A$ the translation by x. Consider the action of G_x on T_xA, the tangent space of A at x. Let G_x^{RT} and $(T_xA)^{RT}$ be as above. Note that $(T_xA)^{RT}$ is the tangent space of a translate of an Abelian subvariety $A_x \subset A$ since it is the intersection of the kernels of the endomorphisms $g - \mathbf{1}_A$ for $g \in G_x^{RT}$. Set*

$$G^{RT} := \langle G_x^{RT} : x \in A \rangle \quad \textit{and} \quad (TA)^{RT} := \langle i_x^*(T_x A)^{RT} : x \in A \rangle.$$

Then $(TA)^{RT}$ is the tangent space of the Abelian subvariety generated by the A_x. Denote it by A_1^{RT}.

The group G/G^{RT} acts on the quotient Abelian variety $q_1 : A \to A/A_1^{RT}$. If $q_i : A \to A/A_i^{RT}$ is already defined, set

$$A_{i+1}^{RT} := q_i^{-1}\left((A/A_i^{RT})_1^{RT}\right)$$

and let $q_{i+1} : A \to A/A_{i+1}^{RT}$ be the quotient map. The increasing sequence of Abelian subvarieties $A_1^{RT} \subset A_2^{RT} \subset \cdots$ eventually stabilizes to $A_{stab}^{RT} \subset A$.

Corollary 24. *Let A be an Abelian variety and G a finite group acting on A. Then*

1. *$\kappa(A/G) = 0$ iff $G^{RT} = \{1\}$, and*
2. *A/G is rationally connected iff $A_{stab}^{RT} = A$.*

3 Basic non-Reid-Tai pairs

Our goal in this section is to classify basic non-RT pairs.

There is a basic dichotomy:

Proposition 25. *If (G, V) is a basic non-RT pair, then either (G, V) is primitive or G respects a decomposition of V as a direct sum of lines:*

$$V = L_1 \oplus \cdots \oplus L_n.$$

In the latter case, the homomorphism $\phi\colon G \to S_n$ given by the permutation action of G on $\{L_1, \ldots, L_n\}$ is surjective, and every exceptional element in G maps to a transposition.

Proof. Suppose that G respects the decomposition $V \cong V_1 \oplus \cdots \oplus V_m$ for some $m \geq 2$. If there is more than one such decomposition, we choose one so that $m \geq 2$ is minimal. By irreducibility, G acts transitively on the set of V_i. As the conjugacy class of any exceptional element g generates G, it follows that g permutes the V_i non-trivially. Suppose that for $2 \leq k \leq m$, we have

$$g(V_1) = V_2,\ g(V_2) = V_3,\ g(V_k) = V_1.$$

Then g and $e(1/k)g$ are isospectral on $V_1 \oplus \cdots \oplus V_k$. Thus the eigenvalues of g constitute a union of $\dim V_1$ cosets of the cyclic group $\langle e(1/k) \rangle$. Such a union of cosets can satisfy the Reid-Tai condition only if $k = 2$ and $\dim V_1 = 1$, and then g must stabilize V_i for every $i \geq 3$. Thus g induces a transposition on the V_i, each of which must be of dimension 1. A transitive subgroup of S_m which contains a transposition must be of the form $S_a^b \rtimes T$, where $ab = m$, T is a transitive subgroup of S_b, and $a \geq 2$. It corresponds to a decomposition of the set of factors V_i into b sets of cardinality a. If W_j denote the direct sums of the V_i within each of the a-element sets of this partition, it follows that G respects this coarser decomposition, contrary to the assumption that m is minimal.

To analyze the primitive case, it is useful to quantify how far a unitary operator is from the identity.

Definition 26. *Let H be a Hilbert space, T a unitary operator on H, and B an orthonormal basis of H. The* deviation *of T with respect to B is given by*

$$d(T,B) := \sum_{b\in B} \|T(b)-b\|.$$

The deviation *of T is*

$$d(T) := \inf_B d(T,B),$$

as B ranges over all orthonormal bases. If $d(T)<\infty$, we say T has finite deviation.

As the arc of a circle cut off by a chord is always longer than the chord, if H is a finite-dimensional Hilbert space and $g\colon H\to H$ a unitary operator of finite order not satisfying the Reid-Tai condition, we have $d(g)<2\pi$. This is the primary motivation for our definition of deviation.

For any space H, unitary operator T, basis B, and real number $x>0$, we define

$$S(T,B,x) = \{b\in B \mid \|T(b)-b\|\geq x\}.$$

If 1_I denotes the characteristic function of the interval I,

$$\int_0^\infty |S(T,B,x)|dx = \int_0^\infty \sum_{b\in B} 1_{[0,\|T(b)-b\|]}dx \tag{1}$$

$$= \sum_{b\in B}\int_0^\infty 1_{[0,\|T(b)-b\|]}dx = \sum_{b\in B}\|T(b)-b\| = d(T,B). \tag{2}$$

(3)

It is obvious that deviation is symmetric in the sense that $d(T)=d(T^{-1})$. Next we prove a lemma relating d to multiplication in the unitary group.

Proposition 27. *If $T_1, T_2, \ldots T_n$ are unitary operators of finite deviation on a Hilbert space H, then*

$$d(T_1T_2\cdots T_n) \leq n(d(T_1)+d(T_2)+\cdots d(T_n)).$$

Proof. Let $B_1, B_2, \dots, B_n$ denote orthonormal bases of H. We claim that there exists an orthonormal basis B such that for all $x > 0$,

$$|S(T_1T_2\cdots T_n, B, nx)| \le |S(T_1, B_1, x)| + |S(T_2, B_2, x)| + \cdots |S(T_n, B_n, x)|. \tag{4}$$

This claim implies the proposition, by integrating over x.

Given T_i, B_i and $x > 0$, we define

$$V_x = \mathrm{Span} \bigcup_{i=1}^{n} S(T_i, B_i, x).$$

As all T_i are of finite deviation, the set of "jumps" (x such that V_x is not constant in a neighborhood of x) is discrete in $(0,\infty)$. Arranging them in reverse order, we see that there exists a (possibly infinite) increasing chain of finite-dimensional subspaces W_i of H such that each V_x is equal to one of the W_i. We choose B to be any orthonormal basis adapted to $W_1 \subset W_2 \subset \cdots$ in the sense that $B \cap W_i$ is an orthonormal basis of W_i for all i.

For all $b \in B$, by the triangle inequality,

$$\|T_1T_2\cdots T_n(b) - b\| \le \sum_{i=1}^{n} \|T_1T_2\cdots T_{i-1}(T_i(b) - b)\| = \sum_{i=1}^{n} \|T_i(b) - b\|.$$

If $b \notin V_x$, then b is orthogonal to every element of $S(T_i, B_i, x)$ for $i = 1, \dots, n$, and therefore, $\|T_i(b) - b\| \le x$ for all i. It follows that

$$\|T_1T_2\cdots T_n(b) - b\| \le nx,$$

or $b \notin S(T_1T_2\cdots T_n, B, nx)$. This implies (4).

Proposition 28. *If T_1 and T_2 are operators on a Hilbert space H such that T_1 is of bounded deviation, then*

$$d(T_1^{-1}T_2^{-1}T_1T_2) \le 4d(T_1).$$

Proof. As $d(T_1^{-1}) = d(T_1) = d(T_2^{-1}T_1T_2)$, the proposition follows from Proposition 27.

Lemma 29. *Let G be a compact group and (ρ, V) a non-trivial representation of G such that $V^G = (0)$. Then there exists $g \in G$ with $d(g) \ge \dim V$.*

Proof. As V has no G-invariants,

$$\int_G \mathrm{tr}(\rho(g))\, dg = 0,$$

there exists $g \in G$ with $\Re(\rho(g)) \le 0$. If $d(g) < \dim V$, there exists an orthonormal basis B of V such that $\sum_{b\in B} |g(b) - b| < \dim V$. If a_{ij} is the matrix of $\rho(g)$ with respect to such a basis, $\Re(\sum_i a_{ii}) < 0$, so

$$\sum_{b\in B} \|b-g(b)\| > \sum_i |1-a_{ii}| \geq \sum_i (1-\Re(a_{ii})) > \dim V,$$

which gives a contradiction.

Lemma 30. *Let (G,V) be a finite group and a representation such that $V^G = (0)$. Suppose that for some integer k, every element of G can be written as a product of at most k conjugates of g. Then $d(g) \geq \frac{\dim V}{k^2}$.*

Proof. This is an immediate consequence of Proposition 27 and Lemma 29.

We recall that a *characteristically simple* group G is isomorphic to a group of the form K^r, where K is a (possibly abelian) finite simple group.

Proposition 31. *There exists a constant C such that if H is a perfect central extension of a characteristically simple group K^r and the conjugacy class of $h \in H$ generates the whole group, then every element in H is the product of no more than $C \log |H|$ conjugates of h.*

Proof. If H is perfect, then K^r is perfect, so K is a non-abelian finite simple group and therefore perfect. Let $\tilde{K}$ denote the universal central extension of K, so $\tilde{K}^r$ is the universal central extension of K^r as well as of H. As the only subgroup of $\tilde{K}^r$ mapping onto H is $\tilde{K}^r$, it suffices to prove that if $\tilde{h} = (x_1, \ldots, x_r) \in \tilde{K}^r$ has the property that its conjugacy class generates $\tilde{K}^r$, then every element of $\tilde{K}^r$ is the product of at most $C \log |H|$ conjugates of $\tilde{h}$. For any t from 1 to r, we can choose $(1, \ldots, 1, y, 1, \ldots, 1) \in \tilde{K}^r$ (with y in the tth coordinate) whose commutator with $\tilde{h}$ is an element $(1, \ldots, 1, z, 1 \ldots, 1)$ not in the center of $\tilde{K}^r$. This element is the product of two conjugates of $\tilde{h}$. If we can find an absolute constant A such that for every finite simple group K and every non-central element $z \in \tilde{K}$, every element of $\tilde{K}$ is the product of at most $A \log |K|$ conjugates of z, the proposition holds with $C = 2A$.

To prove the existence of A, we note that we may assume that K has order greater than any specified constant. In particular, we may assume that K is either an alternating group A_m, $m \geq 8$, or a group of Lie type, and in the latter case, if $K \neq \tilde{K}$, we can identify $\tilde{K}$ with the group of points of a simply connected, almost simple algebraic group over a finite field. It is known that the *covering number* of A_m is $\lfloor m/2 \rfloor$ (see, e.g., [ASH85]), so every element of the group can be written as a product of $\leq m/2$ elements belonging to any given non-trivial conjugacy class X. The universal central extension of A_m is of order $m!$, and at least half of those elements are products of $\leq m/2$ elements in any fixed conjugacy class X, so all of the elements are products of $\leq m < \log m!/2$ elements of X. For the groups of Lie type, we have an upper bound linear in the absolute rank ([EGH99],[LL98]) and therefore sublogarithmic in order.

Lemma 32. *For every integer $n > 0$, there are only finitely many classes of primitive finite subgroups $G \subset \mathrm{GL}_n(\mathbb{C})$ up to projective equivalence.*

Proof. As G is primitive, a normal abelian subgroup of G lies in the center of $\mathrm{GL}_n(\mathbb{C})$. By Jordan's theorem, G has a normal abelian subgroup whose index can be bounded in terms of n. Thus the image of G in $\mathrm{PGL}_n(\mathbb{C})$ is bounded in terms of n. For each isomorphism class of finite groups, there are only finitely many projective n-dimensional representations.

32.1 (Proof of Theorem 8).

First we assume G stabilizes a set $\{L_1, \ldots, L_n\}$ of lines which give a direct sum decomposition of V. We have already seen that the resulting homomorphism $\phi\colon G \to S_n$ is surjective. Let $t \in G$ lie in $\ker\phi$, so $t(v_i) = \lambda_i v_i$ for all i and all $v_i \in L_i$. The commutator of t with any preimage of the transposition $(i\,j) \in S_n$ gives an element of $\ker\phi$ that has eigenvalues λ_i/λ_j, λ_j/λ_i and 1 (of multiplicity $n-2$). The G-conjugacy class of this element consists of all diagonal matrices with this multiset of eigenvalues. Thus $\ker\phi$ contains $\ker\det_C\colon C^n \to C$, where C is the group generated by all ratios of eigenvalues of all elements of $\ker\phi$. It follows that $\ker\phi$ is the product of $\ker\det_C$ and a group of scalar matrices. If we pass to $\mathrm{PGL}(V)$, therefore, the image of G is an extension of C^{n-1} by S_n.

We claim that this extension is split if n is sufficiently large. To prove this, it suffices to prove $H^2(S_n, C^{n-1}) = 0$. This follows if we can show that $H^2(S_n, \mathbb{Z}^{n-1}) = H^3(S_n, \mathbb{Z}^{n-1}) = 0$, or equivalently, the sum-of-coordinate maps

$$H^i(S_n, \mathbb{Z}^n) \to H^i(S_n, \mathbb{Z})$$

are isomorphisms for $i = 1, 2, 3$, where S_n acts on $\mathbb{Z}^n$ by permutations. By Shapiro's lemma, the composition of restriction and sum-of-coordinates gives an isomorphism $H^i(S_n, \mathbb{Z}^n) \tilde{\to} H^i(S_{n-1}, \mathbb{Z})$, so we need to know that the restriction homomorphisms $H^i(S_n, \mathbb{Z}) \to H^i(S_{n-1}, \mathbb{Z})$ are isomorphisms when n is large compared to i, which follows from [Nak60].

Thus the image of G in $\mathrm{PGL}(V)$ is $C^{n-1} \rtimes S_n$, which is the same as the image in $\mathrm{PGL}(V)$ of the imprimitive unitary reflection group $G(|C|, k, n)$, where k is any divisor of G.

It remains to consider the primitive case. Let Z denote the center of G. As Z is abelian and has a faithful isotypic representation, it must be cyclic. If G is abelian, then $G = Z$, and we are done. (This can be regarded as a subcase of the case that G stabilizes a decomposition of V into lines.) Otherwise, let $\overline{H} \cong K^r$ denote a characteristically simple normal subgroup of $\overline{G} := G/Z$, where K is a (possibly abelian) finite simple group and $r \geq 1$. If K is abelian, we let $H \subset G$ denote the inverse image of $\overline{H}$ in G. In the non-abelian case, we let H denote the derived group of the inverse image of $\overline{H}$ in G, which is perfect and again maps onto $\overline{H}$. We know that H is not contained in the center of G, so some inner automorphism of G acts non-trivially on H. It follows that conjugation by g acts non-trivially on H. By Proposition 28, there exists a non-trivial element $h \in H$ with $d(h) < 8\pi$.

We consider five cases:

1. H is abelian.

2. $\overline{H}$ is abelian but H is not.
3. K is a group of Lie type.
4. K is an alternating group A_m, where m is greater than a sufficiently large constant.
5. K is non-abelian but not of type (3) or (4).

We prove that case (4) leads to reflection groups, and all of the other cases contribute only finitely many solutions.

If H is abelian, then the restriction of V to H is isotypical and H is central, contrary to the definition of H.

If $\overline{H}$ is abelian and H is not, then H is a central extension of a vector group and is therefore the product of its center Z by an extraspecial p-group H_p for some prime p. The kernel $Z[p]$ of multiplication by p on Z is isomorphic to $\mathbb{Z}/p\mathbb{Z}$, and the commutator map $\overline{H}\times\overline{H} \to Z[p]$ gives a non-degenerate pairing. Therefore, conjugation by any element in $G \setminus H$ induces a non-trivial map on $\overline{H}$, Thus we can take h with $d(h) < 8\pi$ to lie outside the center of H.

The image of every element of H in $\mathrm{Aut}(V)$ is the product of a scalar matrix and the image of an element of H_p in a direct sum of $m \geq 1$ copies of one of its faithful irreducible representations. By the Stone-von Neumann theorem, a faithful representation of an extraspecial p-group is determined by a central character; its dimension is p^n, where $|H_p| = p^{2n+1}$, and every non-central element has eigenvalues $\omega, \omega e(1/p), \omega e(2/p), \ldots, \omega e(-1/p)$, each occurring with multiplicity p^{n-1}, where $\omega^{p^2} = 1$. As h is a scalar multiple of an element with these eigenvalues,

$$8\pi > d(h) \geq 2\pi m p^{n-1} \frac{p(p-1)}{2p} \geq \frac{2\pi m p^n}{4},$$

so $\dim V = mp^n < 16$. By Lemma 32, there are only finitely many possibilities for (G, V) up to projective equivalence.

In cases (3)–(5), H is perfect. If the conjugacy class of h in G does not generate H, it generates a proper normal subgroup of H, which is a central extension of a subgroup of K^r which is again normal in $\overline{G}$. Such a subgroup is of the form K^s for $s < r$. Replacing H if necessary by a smaller group, we may assume that the G-conjugacy class of h generates H. By Proposition 31, every element of H is the product of at most $C \log |H|$ conjugates of h, and by Lemma 30, this implies

$$\dim V < 8\pi C^2 \log^2 |H|. \tag{5}$$

For case (3), we note that by [SZ93, Table 1], a faithful irreducible projective representation of a finite simple group K which is not an alternating group always has dimension at least $e^{c_1\sqrt{\log |K|}}$ for some positive absolute constant c_1. A faithful irreducible representation of K^r is the tensor power of r faithful irreducible representations of K, so its dimension is at least $e^{c_1\sqrt{\log |\overline{H}|}} > e^{c_1\sqrt{\log |H|/2}}$. For $|H| \gg 0$, this is in contradiction with (5).

Thus there are only finitely many possibilities for H up to isomorphism, and this gives an upper bound for $\dim V$. By Lemma 32, there are only finitely many possibilities for (G,V) up to projective equivalence.

For case (4), we need to consider both ordinary representations of A_m and spin representations (i.e., projective representations which do not lift to linear representations). By [KT04], the minimal degree of a spin representation of A_m grows faster than any polynomial, in particular, faster than $m^{5/2}$. To every irreducible linear representation of A_m one can associate a partition λ of m for which the first part λ_1 is greater than or equal to the number of parts. There may be one or two representations associated to λ, and in the latter case their degrees are equal and their direct sum is irreducible as an S_m-representation. By [LS04, 2.1,2.4], if $\lambda_1 \leq m-3$, the degree of any S_m-representation associated to λ is at least $\binom{m-3}{3}$, so the degree of any A_m-representation associated to λ is at least half of that. Thus, for $m \gg 0$, the only faithful representations of A_m which have degree less than $m^{5/2}$ are $V_{m-1,1}$, $V_{m-2,2}$, and $V_{m-2,1,1}$ of degrees $m-1$, $\frac{(m-1)(m-2)}{2}$ and $\frac{m(m-3)}{2}$ respectively. Now, $\log|H| \leq \log(m!)^r < rm^{1.1}$ for $m \gg 0$, and the minimal degree of any faithful representation of H is at least $(m-1)^r \gg r^2m^{2.2}$ for $r \geq 3$. By (5), there are only four possibilities which need be considered. If $r=2$, then $H = \overline{H} = A_m^2$, and V must be the tensor product of two copies of $V_{m-1,1}$. Otherwise $r=1$, $H = \overline{H} = A_m$, and V is $V_{m-1,1}$, $V_{m-2,2}$, or $V_{m-2,1,1}$. In the first case, the normalizer of H in $\mathrm{GL}(V)$ is S_n^2; in the remaining cases, it is S_n. As representations of S_n^2 or S_n respectively are all self-dual, if $g \in G$ or any scalar multiple thereof satisfies the Reid-Tai condition, all eigenvalues of g must be 1 except for a single -1. By the classification of reflection groups, the only one of these possibilities which can actually occur is the case $V = V_{m-1,1}$ which corresponds to the Weyl group of type A_{m-1}.

For case (5), there are only finitely many possibilities for K, and for each K, we have $\log|H| \leq r\log|\tilde{K}|$, while the minimal dimension of a faithful irreducible representation of H grows exponentially. Thus, (5) gives an upper bound on $\dim V$. The theorem follows from Lemma 32.

4 Quotients of Abelian varieties

Let us now specialize to the case when $X = A$ is an Abelian variety and G a finite group acting on A. For any $x \in A$, the dual of the tangent space T_xA can be canonically identified with $H^0(A, \Omega_A)$. By Hodge theory the representation of $\mathrm{Aut}_x(A)$ on $H^1(A,\mathbb{Q}) \otimes \mathbb{C}$ is isomorphic to the direct sum of the dual representations on $H^0(A,\Omega_A)$ and on $H^1(A,\mathcal{O}_A)$.

Definition 33. *A pair (G,V) is of* AV-type *if $V \oplus V^*$ is isomorphic to the complexification of a rational representation of G.*

We have the following elementary proposition.

Proposition 34. *Let (G, V) denote a non-RT pair of AV-type. Let $G_1 \subset G$ be a subgroup and $V_1 \subset V$ a G_1-subrepresentation such that (G_1, V_1) is a basic non-RT pair. Let $g_1 \in G_1$ be exceptional for V, and let S_1 denote the set of eigenvalues of g_1 acting on V_1, excluding 1. Then every element of S_1 is a root of unity whose order lies in*

$$\{2, 3, 4, 5, 6, 7, 8, 10, 12, 14, 18\}. \tag{6}$$

If $|S_1| > 1$, then S_1 is one of the following:

1. $\{e(1/6), e(1/3)\}$.
2. $\{e(1/6), e(1/2)\}$.
3. $\{e(1/6), e(2/3)\}$.
4. $\{e(1/3), e(1/2)\}$.
5. $\{e(1/8), e(3/8)\}$.
6. $\{e(1/8), e(5/8)\}$.
7. *A subset of* $\{e(1/12), e(1/4), e(5/12)\}$.

Proof. Let $\Sigma \subset \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ be the set of automorphisms σ such that V_1^σ is a G_1-subrepresentation of V. As $V \oplus V^*$ is $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable, $\Sigma \cup c\Sigma = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, where c denotes complex conjugation. Let $e(r_1)$ be an element of S_1, set

$$S_0 = \{\sigma(e(r_1)) \mid \sigma \in \Sigma\},$$

and let $r_1, \ldots, r_k$ denote pairwise distinct rational numbers in $(0, 1)$ such that $S_0 = \{e(r_1), \ldots, e(r_k)\}$. The r_i have a common denominator d, and we write $a_i = dr_i$. As $\mathrm{age}(g_1) < 1$,

$$d > a_1 + \cdots + a_k \geq 1 + 2 + \cdots + k \geq \binom{k+1}{2} \geq \binom{\phi(d)/2 + 1}{2} \geq \frac{\phi(d)^2}{8}.$$

On the other hand,

$$\phi(d) = d \prod_{p|d} \frac{p-1}{p} \geq \frac{d}{3} \prod_{p|d, p\geq 5} p^{\frac{\log 4}{\log 5} - 1} \geq \frac{d^{\frac{\log 4}{\log 5}}}{3}.$$

Thus, $d < 372$, and an examination of cases by machine leads to the conclusion d belongs to the set (6).

If α and β are two distinct elements of S_1, then there exists Σ for which $\Sigma(\alpha) \cup \Sigma(\beta)$ satisfies the Reid-Tai condition. On the other hand, $\alpha^f = \beta^f = 1$ for some $f \leq 126$. We seek to classify triples of integers (a, b, f), $0 < a < b < f \leq 126$, for which there exists a subset $\Sigma \subset \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ with $\Sigma \cup c\Sigma = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ for which $\Sigma(e(a/f)) \cup \Sigma(e(b/f))$ satisfies the Reid-Tai condition. A machine search for such triples is not difficult and reveals that the only possibilities are given by the first six cases of the proposition together with the three pairs obtained by omitting a single element from $\{e(1/12), e(1/4), e(5/12)\}$. The proposition follows.

34.1 (Proof of Theorem 9). If A is a simple Abelian variety of dimension ≥ 4, $V = T_0A$, and G is a finite automorphism group of A which constitutes an exception to the statement of the theorem, then (G, V) is a non-RT pair of AV-type. For every $g \in G$ and every integer k, the identity component of the kernel of $g^k - 1$, regarded as an endomorphism of A, is an Abelian subvariety of A and therefore either trivial or equal to the whole of A. It follows that all eigenvalues of g are roots of unity of the same order.

Let g denote an exceptional element. Let S_g be the multiset of eigenvalues of g on V and $S_g \cup \overline{S}_g$ the multiset of eigenvalues of g on $V \oplus V^*$. Then $S_g \cup \overline{S}_g$ can be partitioned into a union (in the sense of multisets) of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-orbits of roots of unity. Thus S_g can be partitioned into subsets $S_{g,i}$ such that for each $X_{g,i}$ either $S_{g,i} \cup \overline{S}_{g,i}$ or $S_{g,i}$ itself is a single $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-orbit. If each $S_{g,i}$ is written as a set $\{e(r_{i,1}), \ldots, e(r_{i,j_i})\}$ where the $r_{i,j}$ lie in $(0,1)$, then for some i, the mean of the values $r_{i,j}$ is less than $\frac{1}{4}$. As every root of unity is Galois-conjugate to its inverse and $\mathrm{age}(g) < 1$, if $S_{g,i}$ consists of a single $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-orbit, then $S_{g,i}$ is $\{1\}$ or $\{-1\}$, in which case all eigenvalues of S_g are equal, so the mean of the values $r_{i,j}$ is always $\geq \frac{1}{2}$. For each $n > 2$ in the set (6), we present the set $S = \{e(r_1), e(r_2), \ldots, e(r_{\phi(n)/2})\}$ such that $S \cup \overline{S}$ contains all primitive nth roots of unity and $\sum_j r_j$ is minimal, $r_j \geq 0$:

n	$\phi(n)/2$	*Values of* r_j	*Mean of* r_j
3	1	$\frac{1}{3}$	$\frac{1}{3}$
4	1	$\frac{1}{4}$	$\frac{1}{4}$
5	2	$\frac{1}{5}, \frac{2}{5}$	$\frac{3}{10}$
6	1	$\frac{1}{6}$	$\frac{1}{6}$
7	3	$\frac{1}{7}, \frac{2}{7}, \frac{3}{7}$	$\frac{2}{7}$
8	2	$\frac{1}{8}, \frac{3}{8}$	$\frac{1}{4}$
9	3	$\frac{1}{9}, \frac{2}{9}, \frac{4}{9}$	$\frac{7}{27}$
10	2	$\frac{1}{10}, \frac{3}{10}$	$\frac{1}{5}$
12	2	$\frac{1}{12}, \frac{5}{12}$	$\frac{1}{4}$
14	3	$\frac{1}{14}, \frac{3}{14}, \frac{5}{14}$	$\frac{3}{14}$
18	3	$\frac{1}{18}, \frac{5}{18}, \frac{7}{18}$	$\frac{13}{54}$

Table 1

Inspection of this table reveals that the mean of the values r_j is less than $\frac{1}{4}$, only if $n = 18$, $n = 14$, $n = 10$, or $n = 6$. In the first two cases, the condition $\dim A \geq 4$ implies there must be at least two subsets $S_{g,i}$ in the partition, which implies $\dim A \geq 6$. As the mean of the r_j exceeds $\frac{1}{6}$ for $n = 14$ and $n = 18$, this is impossible. If $n = 6$, all the eigenvalues of g must be $\frac{1}{6}$, and there could be as many as five. However, an abelian variety A with an automorphism which acts as the scalar $e(1/6)$ on T_0A is of the form $\mathbb{C}^g/\Lambda$, where Λ is a torsion-free $\mathbb{Z}[e(1/6)] = \mathbb{Z}[e(1/3)]$-module with the inclusion $\Lambda \to$

$\mathbb{C}^g$ equivariant with respect to $\mathbb{Z}[e(1/3)]$. Every finitely generated torsion-free module over $\mathbb{Z}[e(1/3)]$ is free (as $\mathbb{Z}[e(1/3)]$ is a PID), so A decomposes as a product of elliptic curves with CM by $\mathbb{Z}[e(1/3)]$, contrary to hypothesis. If $n = 10$, the only possibility is that $\dim A = 4$, and the eigenvalues of g are $e(1/10), e(1/10), e(3/10), e(3/10)$. Again, $\mathbb{Z}[e(1/5)]$ is a PID, so $A = \mathbb{C}^4/\Lambda$, where $\Lambda \cong \mathbb{Z}[e(1/5)] \oplus \mathbb{Z}[e(1/5)]$. Let $\Lambda_1 \subset \Lambda$ denote the first summand. As A is simple, the $\mathbb{C}$-span of Λ_1 must have dimension > 2. However, letting $\lambda \in \Lambda_1$ be a generator, we can write λ as a sum of two eigenvectors for $e(1/10) \in \mathbb{Z}[e(1/5)]$. Every element of $\mathbb{Z}[e(1/5)]\lambda$ is then a complex linear combination of these 2 eigenvectors.

Lemma 35. *Let (G, V) denote a non-RT pair of AV-type. Let $G_1 \subset G$ be a subgroup and $V_1 \subset V$ a G_1-subrepresentation such that (G_1, V_1) is an imprimitive basic non-RT pair. Then (G_1, V_1) is a complex reflection group.*

Proof. By Proposition 25, V_1 decomposes as a direct sum of lines which G_1 permutes. Let g_1 be an element of G_1 which is exceptional for V_1. By Proposition 25, after renumbering the L_i, g_1 interchanges L_1 and L_2 and stabilizes all the other L_i. The eigenvalues of g_1 acting on $L_1 \oplus L_2$ are therefore of the form $e(r)$ and $e(r + 1/2)$ for some $r \in [0, 1/2)$. By Proposition 34, this means $r = 0$, $r = 1/6$, or $r = 1/8$. In the first case, g might have an additional eigenvalue $e(1/6)$ or $e(1/3)$ on one of the lines L_i, $i \geq 3$ and fix all the remaining lines pointwise. In all other cases, g_1 must fix L_i pointwise for $i \geq 3$. If g_1 has eigenvalues $-1, 1, e(1/6), 1, \ldots, 1$, eigenvalues $-1, 1, e(1/3), 1 \ldots, 1$, eigenvalues $e(1/6), e(2/3), 1, \ldots, 1$, or eigenvalues $e(1/8), e(5/8), 1, \ldots, 1$, then g_1^2 is again exceptional but stabilizes all of the lines L_i, which is impossible by Proposition 25. In the remaining case, g_1 has eigenvalues $-1, 1 \ldots, 1$, so g_1 is a reflection, and G_1 is a complex reflection group.

Theorem 36. *Let (G, V) be a non-RT pair of AV-type, $G_1 \subset G$ a subgroup, and $V_1 \subset V$ a G_1-subrepresentation such that (G_1, V_1) is a basic non-RT pair. If (G_1, V_1) is not of reflection type, then $\dim V_1 = 4$, and G_1 is contained in the reflection group G_{31} in the Shephard-Todd classification.*

Proof. Let $g_1 \in G_1$ be an exceptional element for V. If g_1 or any of its powers is a pseudoreflection on V_1, then G_1 is generated by pseudoreflections and is therefore a reflection group on V_1. We may therefore assume that every power of g_1 which is non-trivial on V_1 has at least two non-trivial eigenvalues in its action on V_1. Also, by Lemma 35, (G_1, V_1) may be assumed primitive.

We consider first the case that S_1 consists of a single element of order n. By a well-known theorem of Blichfeldt (see, e.g., [Coh76, 5.1]) a non-scalar element in a primitive group cannot have all of its eigenvalues contained in an arc of length $\pi/3$. If follows that $n \leq 5$. If $n = 5$, V contains at least two different fifth roots of unity, and as g_1 is exceptional on V, it follows that the multiplicity of the non-trivial eigenvalue of g_1 on V_1 is 1, contrary to hypothesis. If $n = 4$, then by [Wal01], the eigenvalues 1 and i have the same multiplicity (which must be at least 2), and by [Kor86], $\dim V_1$ is a power of 2. As the multiplicity of i is at most 3, the only possibility is that $\dim V_1 = 4$ and the eigenvalues of g_1 are $1, 1, i, i$. The classification of primitive 4-dimensional groups [Bli17] shows that the only such groups containing such an element are contained in the group G_{31} in the Shephard-Todd classification. If $n = 3$, the multiplicity of the eigenvalue $e(1/3)$ must be 2, and [Wal01] show that there are only two possible examples, one in dimension 3 (which is projectively equivalent to the Hessian reflection group G_{25}) and one in dimension 5 (which is projectively equivalent to the reflection group G_{33}). The case $n = 2$ does not arise, since the non-trivial eigenvalue multiplicity is ≥ 2.

Thus, we need only consider the cases that $|S_1| \geq 2$. The possibilities for the multiset of non-trivial eigenvalues of g_1 acting on V_1 which are consistent with g_1 being exceptional for an AV-pair (G_1, V) are as follows:

a. $e(1/6), e(1/3)$
b. $e(1/6), e(1/6), e(1/3)$
c. $e(1/6), e(1/6), e(1/6), e(1/3)$
d. $e(1/6), e(1/3), e(1/3)$
e. $e(1/6), e(1/2)$
f. $e(1/6), e(1/6), e(1/2)$
g. $e(1/6), e(2/3)$
h. $e(1/3), e(1/2)$
i. $e(1/8), e(3/8)$
j. $e(1/8), e(5/8)$
k. $e(1/12), e(1/4)$
l. $e(1/12), e(5/12)$
m. $e(1/4), e(5/12)$
n. $e(1/12), e(1/4), e(5/12)$

Cases (a), (d), (e), (g), (h), (k), and (m) are ruled out because no power of g_1 may be a pseudoreflection. In case (f), g_1^2 has two non-trivial eigenvalues, both equal to $e(1/3)$, and we have already treated this case. Likewise, cases (j) and (l) are subsumed in our analysis of the case that there are two non-trivial eigenvalues, both equal to i.

For the four remaining cases, we observe that the conjugacy class of g_1 generates the non-abelian group G_1, so g_1 fails to commute with some conjugate h_1. The group generated by g_1 and h_1 fixes a subspace W_1 of V_1 of codimension at most 6, 8, 4, and 6 in cases (b), (c), (i), and (n) respectively which g_1 and h_1 fix pointwise. Let $U_1 \subset V_1/W_1$ denote a space on which $\langle g_1, h_1 \rangle$ acts irreducibly and on which g_1 and h_1 do not commute. The non-trivial eigenvalues of g_1 and h_1 on U_1 form subsets of the non-trivial eigenvalues of g_1 and h_1 on V_1, and the action of $\langle g_1, h_1 \rangle$ on U_1 is primitive because the eigenvalues of g_1 do not include a coset of any non-trivial subgroup of $\mathbb{C}^\times$.

We claim that if $\dim U_1 > 1$, all the non-trivial eigenvalues of g_1 on V_1 occur already in U_1. In cases (i) and (n), we have already seen that no proper subset of indicated sets of eigenvalues can appear, together with the eigenvalue 1 with some multiplicity, in any primitive irreducible representation. In cases (b) and (c), Blichfeldt's $\pi/3$ theorem implies that if the eigenvalues of some element in a primitive representation of a finite group are 1 with some multiplicity, $e(1/6)$ with some multiplicity, and possibly $e(1/3)$, then $e(1/3)$ must actually appear. Therefore, the factor U_1 must have $e(1/3)$ as eigenvalue, and every other irreducible factor of V_1 must be 1-dimensional. If no eigenvalue $e(1/6)$ appears in g_1 acting on U_1, then g_1^3 and h_1^3 commute. If all conjugates of g_1^3 commute, then G has a normal abelian subgroup. Such a subgroup must consist of scalar elements of $\mathrm{End}(V_1)$, but this is not possible given that at least one eigenvalue of g_1^3 on V_1 is 1 and at least one eigenvalue is -1. Without loss of generality, therefore, we may assume that g_1^3 and h_1^3 fail to commute. It follows that both $e(1/3)$ and $e(1/6)$ are eigenvalues of g_1 on U_1. As case (a) has already been disposed of, the multiplicity of $e(1/6)$ as an eigenvalue of g_1 on U_1 is at least 2. In the case (b), this proves the claim. Once it has been shown that there are no solutions of type (b), it will follow that the eigenvalue $e(1/6)$ must appear with multiplicity 3, which proves the claim for (c).

Finally, we show that for each of the cases (b), (c), (i), and (n) there is no finite group G_1 with a primitive representation U_1 and an element g_1 whose multiset of non-trivial eigenvalues is as specified. First we consider whether G_1 can stabilize a non-trivial tensor decomposition of U_1. The only possibilities for g_1 respecting such a decomposition are case (b) with eigenvalues $1, e(1/6), e(1/6), e(1/3)$ decomposing as a tensor product of two representations with eigenvalues $1, e(1/6)$ and case (c) with eigenvalues $1, 1, e(1/6), e(1/6), e(1/6), e(1/3)$ decomposing as a tensor product of representations with eigenvalues $1, e(1/6)$ and $1, 1, e(1/6)$. As U_1 is a primitive representation of G_1, Blichfeldt's theorem rules out both possibilities.

Next we rule out the possibility that G_1 normalizes a tensor decomposition with g_1 permuting tensor factors non-trivially. Given that $\dim U_1 \leq 8$, this could only happen if there are two or three tensor factors, each of dimension 2. It is easy to see that if $T_1, \ldots, T_n$ are linear transformations on a vector space V, the transformation on $V^{\otimes n}$ defined by $v_1 \otimes \cdots \otimes v_n \mapsto T_n(v_n) \otimes T_1(v_1) \otimes \cdots \otimes T_{n-1}(v_{n-1})$ has the same trace as $T_1 T_2 \cdots T_n$. It follows that any unitary transformation T on $V^{\otimes n}$ which normalizes the tensor decomposition but permutes the factors nontrivially satisfies

$$|\mathrm{tr}(T)| \leq (\dim V)^{n-1},$$

with equality only if the permutation is a transposition (ij), $T_i T_j$ is scalar, and all other factors T_i are scalar; in particular, equality implies that T^2 is scalar. The following table gives for each case the absolute value of the trace of g_1 acting on U_1 in terms of the dimension of U_1:

Case	3	4	5	6	7	8
(b)	$\sqrt{3}$	2	$\sqrt{7}$	$2\sqrt{3}$		
(c)		$\sqrt{7}$	3	$\sqrt{13}$	$\sqrt{19}$	$3\sqrt{3}$
(i)	$\sqrt{3}$	$\sqrt{6}$				
(n)	2	$\sqrt{5}$	$2\sqrt{2}$	$\sqrt{13}$		

Table 2

In each case, except (b) and $\dim U_1 = 4$, $\mathrm{tr}(T)$ violates the inequality, and in this case, T^2 is not scalar.

Let H_1 denote a characteristically simple normal subgroup of G_1. As G_1 does not normalize a tensor decomposition, U_1 is an irreducible representation of H_1. Either H_1 is the product of an extraspecial p-group H_p and a group Z of scalars or H_1 is a central extension of a product of mutually isomorphic finite simple groups by a scalar group Z. As $\dim U_1 \leq 8$, in the former case, $|H_p| \in \{2^3, 2^5, 2^7, 3^3, 5^3, 7^3\}$. In the latter case, either $\overline{H}_1 = H_1/Z$ is isomorphic to K^r for some finite simple group K, and $r = 1$ since G_1 does not normalize a tensor decomposition. For a list of possibilities for H_1, we use the tables of Hiss and Malle [HM01], which are based on the classification of finite simple groups. Note that primitive groups were classified up through dimension 10 before the classification of finite simple groups was available (see, e.g., [Fei71], [Fei76] and the references therein). The following table enumerates the possibilities for $\overline{H}_1$, where representation numbering is that of [CCN$^+$85] and asterisks indicate a Stone-von Neumann representation:

	Representation Degree						
Group	2	3	4	5	6	7	8
$(\mathbb{Z}/2\mathbb{Z})^2$	*						
$(\mathbb{Z}/3\mathbb{Z})^2$		*					
$(\mathbb{Z}/2\mathbb{Z})^4$			*				
$(\mathbb{Z}/5\mathbb{Z})^2$				*			
$(\mathbb{Z}/7\mathbb{Z})^2$						*	
A_5	6	3	$4, 8$	5	9		
$(\mathbb{Z}/2\mathbb{Z})^6$							*
$L_2(7)$		2	7		$4, 9$	5	$6, 11$
A_6		14	8	2	$16, 19$		$4, 10$
$L_2(8)$						$2, 3$	6
$L_2(11)$				2	9		
$L_2(13)$					10	2	
$L_2(17)$							12
A_7			10		$2, 17, 24$		
$U_3(3)$					2	$3, 4$	
A_8						2	15
$L_3(4)$					41		19
$U_4(2)$			21	2	4		
A_9							$2, 19$
J_2					22		
$S_6(2)$						2	31
$U_4(3)$					72		
$O_8^+(2)$							54

Table 3

For the finite simple groups $\overline{H}_1$, we consult character tables [CCN+85]. This is easy to do so, since only a few of the characters in Table 3 take values whose absolute values are large enough to appear in Table 2. There are no cases where an element of order 6 has a character absolute value as given in row (b) or (c) of Table 2; an element of order 8 has an absolute value as given by row (i); or an element of order 12 has an absolute value as given by row (n).

For the case that H_1 is an extraspecial p-group, every non-zero character value is an integral power of $\sqrt{p}$. This was proved for $p > 2$ by Howe [How73, Prop. 2(ii)]. For lack of a reference for $p = 2$, we sketch a proof which works in general. The embedding $H_1 \to \mathrm{GL}(U_1)$ is a Stone-von Neumann representation with central character χ. Let GG_1 denote the group of pairs $(g_1, g_2) \in G_1^2$ such that $g_1H_1 = g_2H_1$. There is a natural action of GG_1 on $\mathbb{C}[H_1]$ given by

$$(g_1, g_2)([h_1]) = [g_1h_1g_2^{-1}].$$

The restriction of this representation to $H_1^2 \subset GG_1$ is of the form $\bigoplus V_i \boxtimes V_i^*$, where the sum is taken over all irreducible representations V_i of H_1. The factor $U_1 \boxtimes U_1^*$ is the $\chi \boxtimes \chi^*$ eigenspace of the center Z^2 of H_1^2 acting on $\mathbb{C}[H_1]$, where

χ is the central character of Z on U_1. As the action of GG_1 on this eigenspace of $\mathbb{C}[H_1]$ extends the irreducible representation of H_1^2 on $U_1 \boxtimes U_1^*$, any other extension of $(H_1^2, U_1 \boxtimes U_1^*)$ to GG_1 is projectively equivalent to this one. The particular extension we have in mind is obtained by letting $(g_1, g_2) \in GG_1$ act on $U_1 \boxtimes U_1^*$ according to the action of g_1 on U_1 and the action of g_2 on U_1^* coming from the inclusion $G_1 \subset \mathrm{GL}(U_1)$. From this it is easy to see that the character value of (g_1, g_1) on each Z^2-eigenspace of $\mathbb{C}[H_1]$ is either 0 or $|\overline{H}_1^{g_1}|$. As $\overline{H}_1^{g_1}$ is a vector space over $\mathbb{Z}/p\mathbb{Z}$, $\mathrm{tr}(g_1|U_1)\mathrm{tr}(g_1|U_1^*)$ is either 0 or an integer power of p. Consulting Table 2, we see that this rules out every possibility except a character value 2 and $\dim U_1 = 4$. This can actually occur, but not with the eigenvalues of case (b).

5 Examples

We conclude with some examples to illustrate various aspects of the classification given above. We begin with some examples from group theory.

In principle all non-RT pairs can be built up from basic pairs, by reversing the operations which led to constructing basic pairs in the first place, i.e. by replacing G by an extension $\tilde{G}$ of G whose image in $\mathrm{Aut}(V)$ is the same as that of G; by combining (G, V_1) and (G, V_2) to give the pair $(G, V_1 \oplus V_2)$ (which may or may not be non-RT); and by replacing (G, V) by (G', V), where G' lies between G and its normalizer in $\mathrm{Aut}(V)$. To illustrate that, we observe that all non-RT pairs of the form $((\mathbb{Z}/2\mathbb{Z})^n \rtimes H, \mathbb{C}^n)$, where $H \subset S_n$ is a transitive group, arise from the basic non-RT pair $(\mathbb{Z}/2\mathbb{Z}, \mathbb{C})$. This accounts for the series of Weyl groups of type B_n/C_n, but not for the Weyl groups of type D_n, which are primitive. This construction can be used more generally to build non-RT pairs of the form $(G^n \rtimes H, V^n)$ starting with a non-RT pair (G, V) and a transitive permutation group $H \subset S_n$.

It may happen that a basic non-RT pair (G, V) of reflection type nevertheless fails to have an exceptional element which is a scalar multiple of a pseudoreflection. Consider the case $G = U_4(2) \times \mathbb{Z}/3\mathbb{Z}$ and V is a faithful irreducible 5-dimensional representation of G. Then G has an exceptional element g whose eigenvalues are $1, 1, 1, e(1/3), e(1/3)$ and whose conjugacy class generates G. It has another element h with eigenvalues $1, -1, -1, -1, -1$ which is not exceptional. As $-h$ is a reflection, it is easy to see that $U_4(2) \times \mathbb{Z}/2\mathbb{Z}$ is a 5-dimensional reflection group (in fact, it is G_{33} in the Shephard-Todd classification), and of course this reflection group is projectively equivalent to (G, V).

There really does exist a primitive 4-dimensional non-RT pair (G, V) which is not of reflection type. By [Bli17], there is a short exact sequence

$$0 \to I_4 \to G_{31} \to S_6 \to 0$$

where I_4 is the central product of $\mathbb{Z}/4\mathbb{Z}$ and any extraspecial 2-group of order 32. The group S_6 contains two non-conjugate subgroups isomorphic to S_5, whose inverse images in G_{31} are primitive. One is the reflection group G_{29}, and one contains elements with eigenvalues $1, 1, i, i$. The question arises as to whether these two groups are conjugate. The character table of G_{29}, provided by the software package CHEVIE [GHL$^+$96], reveals that this group has two faithful 4-dimensional representations. One has reflections and the other has elements with spectrum $1, 1, i, i$. It follows that G_{29} with respect to this non-reflection representation, or equivalently, the non-reflection index-6 subgroup of G_{31}, gives the desired example. This example (in fact all of G_{31}) can actually be realized inside $\mathrm{GL}_4(\mathbb{Z}[i])$, as shown in [Bli17].

The set of projective equivalence classes of basic non-RT pairs which are of AV-type is infinite, as is the set of basic non-RT pairs which are not. We have already mentioned the Weyl groups of type D_n as examples of the first kind; the reflection groups $(\mathbb{Z}/k\mathbb{Z})^{n-1} \times S_n$ are never of AV-type if $k > 4$.

We conclude with some geometric examples.

If (G, V) is a non-RT pair and $V = V_0 \otimes_{\mathbb{Q}} \mathbb{C}$ for some rational representation V_0 of G, then there exists an Abelian variety A, and a homomorphism $G \to \mathrm{Aut}(A)$ such that A/G is uniruled and the Lie algebra of A is isomorphic to V as G-module. Indeed, we may choose any integral lattice $\Lambda_0 \subset V_0$ which is G-stable and define $A = \mathrm{Hom}(\Lambda_0, E)$ for any elliptic curve E. If V is irreducible, then A/G is rationally connected. This includes all examples where G is the Weyl group of a root system and V_0 is the $\mathbb{Q}$-span of the root system. When Λ_0 is taken to be the root system, the quotients A/G are in fact weighted projective spaces by a theorem of E. Looijenga [Loo77] which was one of the motivations for this paper.

Let Λ_0 denote the (12-dimensional) Coxeter-Todd lattice, which we regard as a free module of rank 6 over $R := \mathbb{Z}[e(1/3)]$. Let $G = G_{34}$ denote the group of R-linear isometries of this lattice. If E denotes the elliptic curve over $\mathbb{C}$ with complex multiplication by R, then $\mathrm{Hom}_R(\Lambda_0, E)/G$ is rationally connected. The group G is a reflection group but not a Weyl group, and we do not know whether this variety is rational or even unirational.

We have already observed that there is a 4-dimension basic non-RT pair (G, V) which is not of reflection type and such that $G \subset \mathrm{GL}_4(\mathbb{Z}[i]) \subset \mathrm{GL}(V)$. If E denotes the elliptic curve with CM by $\mathbb{Z}[i]$, E^4/G is rationally connected. Again, we do not know about rationality or unirationality.

Let A be an abelian variety with complex multiplication by $\mathbb{Z}[e(1/7)]$ with CM type chosen so that some automorphism g of order 7 has eigenvalues $e(1/7), e(2/7), e(3/7)$ acting on T_0A. Then $A/\langle g \rangle$ is rationally connected, but once again we do not know whether it is rational or unirational.

Let E be any elliptic curve, $A = E^3$, and $G = S_3 \times \{\pm 1\}$. Let the factors S_3 and $\{\pm 1\}$ of G act on A by permuting factors and by multiplication respectively. If $V = T_0A = \mathbb{C}^3$ and W denotes the plane in which coordinates sum to zero, then the images of G in W and in V/W are reflection groups. Thus $A^{RT}_{stab} = A$, so A/G is rationally connected by Corollary 24. Yet again, we do not know about the rationality or unirationality of the quotient.

There are imprimitive basic non-RT triples of arbitrarily large degree which can be realized by automorphism groups of Calabi-Yau varieties but not as automorphism groups of Abelian varieties. For example,

$$G_{n+2,1,n} = (\mathbb{Z}/(n+2)\mathbb{Z})^n \rtimes S_n$$

acts on the n-dimensional Fermat hypersurface $x_0^{n+2} + \cdots + x_{n+1}^{n+2} = 0$ fixing the coordinates x_0 and x_1 and therefore the point

$$P = (1 : e^{\frac{\pi i}{n+2}} : 0 : \cdots : 0).$$

The action of G on the tangent space to P gives the reflection representation of $G_{n+2,1,n}$.

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