Finite subgroups of the plane Cremona group

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Summary. This paper completes the classic and modern results on classification of conjugacy classes of finite subgroups of the group of birational automorphisms of the complex projective plane.

CONTENTS

1 Introduction

The Cremona group $\mathrm{Cr}_k(n)$ over a field k is the group of birational automorphisms of the projective space $\mathbb{P}^n_k$, or, equivalently, the group of k-automorphisms of the field $k(x_1, x_2, \ldots, x_n)$ of rational functions in n independent variables. The group $\mathrm{Cr}_k(1)$ is the group of automorphisms of the projective line, and hence it is isomorphic to the projective linear group $\mathrm{PGL}_k(2)$. Already in the case $n = 2$ the group $\mathrm{Cr}_k(2)$ is not well understood in spite

of extensive classical literature (e.g. [20], [34]) on the subject as well as some modern research and expositions of classical results (e.g. [2]). Very little is known about the Cremona groups in higher-dimensional spaces.

In this paper we consider the plane Cremona group over the field of complex numbers, denoted by $\mathrm{Cr}(2)$. We return to the classical problem of classification of finite subgroups of $\mathrm{Cr}(2)$. The classification of finite subgroups of $\mathrm{PGL}(2)$ is well-known and goes back to F. Klein. It consists of cyclic, dihedral, tetrahedral, octahedral and icosahedral groups. Groups of the same type and order constitute a unique conjugacy class in $\mathrm{PGL}(2)$. Our goal is to find a similar classification in the two-dimensional case.

The history of this problem begins with the work of E. Bertini [9] who classified conjugacy classes of subgroups of order 2 in $\mathrm{Cr}(2)$. Already in this case the answer is drastically different. The set of conjugacy classes is parametrized by a disconnected algebraic variety whose connected components are respectively isomorphic to either the moduli spaces of hyperelliptic curves of genus g (de Jonquiéres involutions), or the moduli space of canonical curves of genus 3 (Geiser involutions), or the moduli space of canonical curves of genus 4 with vanishing theta characteristic (Bertini involutions). Bertini's proof was considered to be incomplete even according to the standards of rigor of the 19th century algebraic geometry. A complete and short proof was published only a few years ago by L. Bayle and A. Beauville [5].

In 1894 G. Castelnuovo [15], as an application of his theory of adjoint linear systems, proved that any element of finite order in $\mathrm{Cr}(2)$ leaves invariant either a net of lines, or a pencil of lines, or a linear system of cubic curves with $n \leq 8$ base points. A similar result was claimed earlier by S. Kantor in his memoir which was awarded a prize by the Accademia delle Scienze di Napoli in 1883. However Kantor's arguments, as was pointed out by Castelnuovo, required justifications. Kantor went much further and announced a similar theorem for arbitrary finite subgroups of $\mathrm{Cr}(2)$. He proceeded to classify possible groups in each case (projective linear groups, groups of de Jonquiéres type, and groups of type M_n). A much clearer exposition of his results can be found in a paper of A. Wiman [49]. Unfortunately, Kantor's classification, even with some correction made by Wiman, is incomplete for the following two reasons. First, only maximal groups were considered and even some of them were missed. The most notorious example is a cyclic group of order 8 of automorphisms of a cubic surface, also missed by B. Segre [47] (see [33]). Second, although Kantor was aware of the problem of conjugacy of subgroups, he did not attempt to fully investigate this problem.

The goal of our work is to complete Kantor's classification. We use a modern approach to the problem initiated in the works of Yuri Manin and the second author (see a survey of their results in [38]). Their work gives a clear understanding of the conjugacy problem via the concept of a rational G-surface. It is a pair (S, G) consisting of a nonsingular rational projective surface and a subgroup G of its automorphism group. A birational map $S - \to \mathbb{P}^2_k$ realizes G as a finite subgroup of $\mathrm{Cr}(2)$. Two birational isomorphic G-

surfaces define conjugate subgroups of Cr(2), and conversely a conjugacy class of a finite subgroup G of Cr(2) can be realized as a birational isomorphism class of G-surfaces. In this way classification of conjugacy classes of subgroups of Cr(2) becomes equivalent to the birational classification of G-surfaces. A G-equivariant analog of a minimal surface allows one to concentrate on the study of minimal G-surfaces, i.e. surfaces which cannot be G-equivariantly birationally and regularly mapped to another G-surface. Minimal G-surfaces turn out to be G-isomorphic either to the projective plane, or a conic bundle, or a Del Pezzo surface of degree $d = 9 - n \leq 6$. This leads to groups of projective transformations, or groups of de Jonquiéres type, or groups of type M_n, respectively. To complete the classification one requires

- to classify all finite groups G which may occur in a minimal G-pair (S, G);
- to determine when two minimal G-surfaces are birationally isomorphic.

To solve the first part of the problem one has to compute the full automorphism group of a conic bundle surface or a Del Pezzo surface (in the latter case this was essentially accomplished by Kantor and Wiman), then make a list of all finite subgroups which act minimally on the surface (this did not come up in the works of Kantor and Wiman). The second part is less straightforward. For this we use the ideas from Mori's theory to decompose a birational map of rational G-surfaces into elementary links. This theory was successfully applied in the arithmetic case, where the analog of the group G is the Galois group of the base field (see [38]). We borrow these results with obvious modifications adjusted to the geometric case. Here we use the analogy between k-rational points in the arithmetic case (fixed points of the Galois action) and fixed points of the G-action. As an important implication of the classification of elementary G-links is the rigidity property of groups of type M_n with $n \geq 6$: any minimal Del Pezzo surface (S, G) of degree $d \leq 3$ is not isomorphic to a minimal G-surface of different type. This allows us to avoid much of the painful analysis of possible conjugacy for a lot of groups.

The large amount of group-theoretical computations needed for the classification of finite subgroups of groups of automorphisms of conic bundles and Del Pezzo surfaces makes us expect some possible gaps in our classification. This seems to be a destiny of enormous classification problems. We hope that our hard work will be useful for the future faultless classification of conjugacy classes of finite subgroups of Cr(2).

It is appropriate to mention some recent work on classification of conjugacy classes of subgroups of $Cr(2)$. We have already mentioned the work of L. Bayle and A. Beauville on groups of order 2. The papers [7],[22], [51] study groups of prime orders, Beauville's paper [8] classifies p-elementary groups, and a thesis of J. Blanc [6] contains a classification of all finite abelian groups. The second author studies two non-conjugate classes of subgroups isomorphic to $S_3 \times \mathbb{Z}/2\mathbb{Z}$. In the work of S. Bannai and H. Tokunaga examples are given of non-conjugate subgroups isomorphic to S_4 and A_5.

This paper is partly based on the lectures by the first author in workshops on Cremona transformations in Torino in September 2005 and Lisbon in May 2006. He takes the opportunity to thank the organizers for the invitation and for providing a good audience. We both like to thank A. Beauville, Chenyang Xu and, especially, J. Blanc for pointing out some errors in previous versions of our paper.

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2 First examples

2.1 Homaloidal linear systems

We will be working over the field of complex numbers. Recall that a dominant rational map $\chi : \mathbb{P}^2 - \to \mathbb{P}^2$ is given by a 2-dimensional linear system $\mathcal{H}$ equal to the proper transform of the linear system of lines $\mathcal{H}' = |\ell|$ in the target plane. A choice of a basis in $\mathcal{H}$ gives an explicit formula for the map in terms of homogeneous coordinates

$$(x_0', x_1', x_2') = (P_0(x_0, x_1, x_2), P_1(x_0, x_1, x_2), P_2(x_0, x_1, x_2)),$$

where P_0, P_1, P_2 are linear independent homogeneous polynomials of degree d, called the (algebraic) *degree* of the map. This is the smallest number d such that $\mathcal{H}$ is contained in the complete linear system $|\mathcal{O}_{\mathbb{P}^2}(d)|$ of curves of degree d in the plane. By definition of the proper transform, the linear system $\mathcal{H}$ has no fixed components, or, equivalently, the polynomials P_i's are mutually coprime. The birational map χ is not a projective transformation if and only if the degree is larger than 1, or, equivalently, when χ has *base points*, the common zeros of the members of the linear system. A linear system defining a birational map is called a *homaloidal* linear system. Being proper transform of a general line under a birational map, its general member is an irreducible rational curve. Also two general curves from the linear system intersect outside the base points at one point. These two conditions characterize homaloidal linear systems (more about this later).

2.2 Quadratic transformations

A quadratic Cremona transformation is a birational map $\chi : \mathbb{P}^2 - \to \mathbb{P}^2$ of degree 2. The simplest example is the *standard quadratic transformation* defined by the formula

$$\tau_1 : (x_0, x_1, x_2) \mapsto (x_1x_2, x_0x_2, x_0x_1). \tag{1}$$

In affine coordinates this is given by $\tau_1 : (x, y) \mapsto (\frac{1}{x}, \frac{1}{y})$. It follows from the definition that $\tau_1^{-1} = \tau_1$, i.e., τ_1 is a birational involution of $\mathbb{P}^2$. The base

points of T are the points $p_0 = (1,0,0), p_1 = (0,1,0), p_2 = (0,0,1)$. The transformation maps an open subset of the coordinate line $x_i = 0$ to the point p_i. The homaloidal linear system defining τ_1 is the linear system of conics passing through the points p_0, p_1, p_2.

The Moebius transformation $x \mapsto x^{-1}$ of $\mathbb{P}^1$ is conjugate to the transformation $x \mapsto -x$ (by means of the map $x \mapsto \frac{x-1}{x+1}$). This shows that the standard Cremona transformation τ_1 is conjugate in $\mathrm{Cr}(2)$ to a projective transformation given by

$$(x_0, x_1, x_2) \mapsto (x_0, -x_1, -x_2).$$

When we change the homaloidal linear system defining τ_1 to the homaloidal linear system of conics passing through the point p_1, p_2 and tangent at p_2 to the line $x_0 = 0$ we obtain the transformation

$$\tau_2 : (x_0, x_1, x_2) \mapsto (x_1^2, x_0x_1, x_0x_2). \tag{2}$$

In affine coordinates it is given by $(x, y) \mapsto (\frac{1}{x}, \frac{y}{x^2})$. The transformation τ_2 is also a birational involution conjugate to a projective involution. To see this we define a rational map $\chi : \mathbb{P}^2 - \to \mathbb{P}^3$ by the formula $(x_0, x_1, x_2) \mapsto (x_1^2, x_0x_1, x_0x_2, x_1x_2)$. The Cremona transformation τ_2 acts on $\mathbb{P}^3$ via this transformation by $(u_0, u_1, u_2, u_3) \mapsto (u_1, u_0, u_3, u_2)$. Composing with the projection of the image from the fixed point $(1,1,1,1)$ we get a birational map $(x_0, x_1, x_2) \mapsto (y_0, y_1, y_2) = (x_1(x_0 - x_1), x_0x_2 - x_1^2, x_1(x_2 - x_1))$. It defines the conjugation of τ_2 with the projective transformation $(y_0, y_1, y_2) \mapsto (-y_0, y_2 - y_0, y_1 - y_0)$.

Finally, we could further "degenerate" τ_1 by considering the linear system of conics passing through the point p_3 and intersecting each other at this point with multiplicity 3. This linear system defines a birational involution

$$\tau_3 : (x_0, x_1, x_2) \mapsto (x_0^2, x_0x_1, x_1^2 - x_0x_2). \tag{3}$$

Again it can be shown that τ_3 is conjugate to a projective involution.

Recall that a birational transformation is not determined by the choice of a homaloidal linear system, one has to choose additionally a basis of the linear system. In the above examples, the basis is chosen to make the transformation an involution.

2.3 De Jonquiéres involutions

Here we exhibit a series of birational involutions which are not conjugate to each other and not conjugate to a projective involution. In affine coordinates they are given by the formula

$$\mathrm{dj}_P : (x, y) \mapsto (x, \frac{P(x)}{y}), \tag{4}$$

where $P(x)$ a polynomial of degree $2g+1$ or $2g+2$ without multiple roots. The conjugation by the transformation $(x,y) \mapsto (\frac{ax+b}{cx+d}, y)$ shows that the conjugacy class of dj_P depends only on the orbit of the set of roots of P with respect to the group $\mathrm{PGL}(2)$, or, in other words, on the birational class of the hyperelliptic curve

$$y^2 + P(x) = 0. \tag{5}$$

The transformation dj_P has the following beautiful geometric interpretation. Consider the projective model H_{g+2} of the hyperelliptic curve (5) given by the homogeneous equation of degree $g+2$

$$T_2^2 F_g(T_0,T_1) + 2T_2 F_{g+1}(T_0,T_1) + F_{g+2}(T_0,T_1) = 0, \tag{6}$$

where

$$D = F_{g+1}^2 - F_g F_{g+2} = T_0^{2g+2} P(T_1/T_0)$$

is the homogenization of the polynomial $P(x)$. The curve H_{g+2} has an ordinary singular point of multiplicity g at $q = (0,0,1)$ and the projection from this point to $\mathbb{P}^1$ exhibits the curve as a double cover of $\mathbb{P}^1$ branched over the $2g+2$ zeroes of D.

Consider the affine set $T_2 = 1$ with affine coordinates $(x,y) = (T_0/T_2, T_1/T_2)$. A general line $y = kx$ intersects the curve H_{g+2} at the point $q = (0,0)$ with multiplicity g and at two other points $(\alpha, k\alpha)$ and $(\alpha', k\alpha')$, where α, α' are the roots of the quadratic equation

$$t^2 F_{g+2}(1,k) + 2tF_{g+1}(1,k) + F_g(1,k) = 0.$$

Take a general point $p = (x, kx)$ on the line and define the point $p' = (x', kx')$ such that the pairs $(\alpha, k\alpha), (\alpha', k\alpha')$ and $(x, kx), (x', kx')$ are harmonic conjugate. This means that x, x' are the roots of the equation $at^2+2bt+c = 0$, where $aF_g(1,k) + cF_{g+2}(1,k) - 2bF_{g+1}(1,k) = 0$. Since $x + x' = -2b/c, xx' = c/a$ we get $F_g(1,k) + xx'F_{g+2}(1,k) + (x+x')F_{g+1}(1,k) = 0$. We express x' as $(ax+b)/(cx+d)$ and solve for (a,b,c,d) to obtain

$$x' = \frac{-F_{g+1}(1,k)x - F_g(1,k)}{xF_{g+2}(1,k) + F_{g+1}(1,k)}.$$

Since $k = y/x$, after changing the affine coordinates $(x,y) = (x_0/x_2, x_1/x_2)$ to $(X,Y) = (x_1/x_0, x_2/x_0) = (y/x, 1/x)$, we get

$$IH_{g+2} : (X,Y) \mapsto (X',Y') := \big(X, \frac{-YP_{g+1}(X) - P_{g+2}(X)}{P_g(X)Y + P_{g+1}(X)}\big), \tag{7}$$

where $P_i(X) = F_i(1,X)$. Let $T : (x,y) \mapsto (x, yP_g + P_{g+1})$. Taking $P(x) = P_{g+1}^2 - P_g P_{g+2}$, we check that $T^{-1} \circ \mathrm{dj}_P \circ T = IH_{g+2}$. This shows that our geometric de Jonquiéres involution IH_{g+2} given by (7) is conjugate to the de Jonquiéres involution dj_P defined by (4).

Let us rewrite (7) in homogeneous coordinates:

$$\begin{aligned} x_0' &= x_0(x_2F_g(x_0,x_1)+F_{g+1}(x_0,x_1)),\\ x_1' &= x_1(x_2F_g(x_0,x_1)+F_{g+1}(x_0,x_1)),\\ x_2' &= -x_2F_{g+1}(x_0,x_1)-F_{g+2}(x_0,x_1). \end{aligned} \tag{8}$$

Now it is clear that the homaloidal linear system defining IH_{g+2} consists of curves of degree $g+2$ which pass through the singular point q of the hyperelliptic curve (6) with multiplicity g. Other base points satisfy

$$x_2F_g(x_0,x_1)+F_{g+1}(x_0,x_1) = -x_2F_{g+1}(x_0,x_1)-F_{g+2}(x_0,x_1) = 0.$$

Eliminating x_2, we get the equation $F_{g+1}^2 - F_gF_{g+2} = 0$ which defines the set of the $2g+2$ ramification points $p_1,\dots,p_{2g+2}$ of the projection $H_{g+2}\setminus\{q\}\to\mathbb{P}^1$.

Let

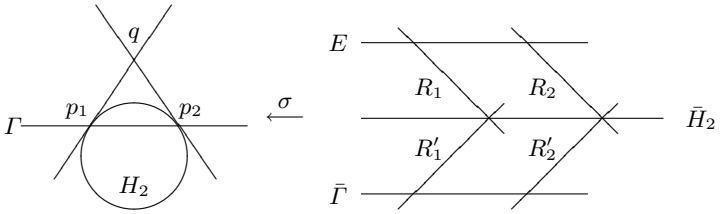
$$\Gamma : T_2F_g(T_0,T_1)+F_{g+1}(T_0,T_1) = 0$$

be the first polar Γ of H_{g+2} with respect to the point q. The transformation IH_{g+2} blows down Γ and the lines $\langle q,p_i\rangle$ to points. It follows immediately from (8) that the set of fixed points of the involution IH_{g+2} outside the base locus is equal to the hyperelliptic curve (6). Also we see that the pencil of lines through q is invariant with respect to IH_{g+2}.

Let $\sigma : S\to\mathbb{P}^2$ be the blow-up of the point q and the points $p_1,\dots,p_{2g+2}$. The full pre-image of the line $\ell_i = \langle q,p_i\rangle$ consists of two irreducible components, each isomorphic to $\mathbb{P}^1$. They intersect transversally at one point. We will call such a reducible curve, a *bouquet* of two $\mathbb{P}^1$'s. One component is the exceptional curve $R_i = \sigma^{-1}(p_i)$ and another one is the proper transform R_i' of the line ℓ_i. The proper transform of H_{g+2} intersects $\sigma^{-1}(\ell_i)$ at its singular point. Thus the proper transform $\bar{H}_{g+2}$ of the hyperelliptic curve H_{g+2} intersects the exceptional curve $E=\sigma^{-1}(q)$ at the same points where the proper transform of lines ℓ_i intersect E. The proper transform $\bar{\Gamma}$ of Γ intersects R_i at one nonsingular point, and intersects E at g points, the same points where the proper inverse transform $\bar{H}_{g+2}$ of H_{g+2} intersects E. The involution IH_{g+2} lifts to a biregular automorphism τ of S. It switches the components R_i and R_i' of $\sigma^{-1}(\ell_i)$, switches E with $\bar{\Gamma}$ and fixes the curve $\bar{H}_{g+2}$ pointwiseley. The pencil of lines through q defines a morphism $\phi : S\to\mathbb{P}^1$ whose fibres over the points corresponding to the lines ℓ_i are isomorphic to a bouquet of two $\mathbb{P}^1$'s. All other fibres are isomorphic to $\mathbb{P}^1$. This is an example of a *conic bundle* or a Mori fibration (or in the archaic terminology of [37], a minimal rational surface with a pencil of rational curves).

To show that the birational involutions $IH_{g+2}, g>0$, are not conjugate to each other or to a projective involution we use the following.

Lemma 2.1. *Let G be a finite subgroup of* $\mathrm{Cr}(2)$ *and let $C_1,\dots,C_k$ be non-rational irreducible curves on $\mathbb{P}^2$ such that each of them contains an open subset C_i^0 whose points are fixed under all $g\in G$. Then the set of birational isomorphism classes of the curves C_i is an invariant of the conjugacy class of G in* $\mathrm{Cr}(2)$.



Proof. Suppose $G = T \circ G' \circ T^{-1}$ for some subgroup G' of $\mathrm{Cr}(2)$ and some $T \in \mathrm{Cr}(2)$. Then, replacing C_i^0 by a smaller open subset we may assume that $T^{-1}(C_i^0)$ is defined and consists of fixed points of G'. Since C_i is not rational, $T^{-1}(C_i^0)$ is not a point, and hence its Zariski closure is a rational irreducible curve C_i' birationally isomorphic to C_i which contains an open subset of fixed points of G'.

Since a connected component of the fixed locus of a finite group of projective transformations is a line or a point, we see that IH_{g+2} is not conjugate to a subgroup of projective transformations for any $g > 0$. Since IH_{g+2} is conjugate to some involution (4), where $P(x)$ is determined by the birational isomorphism class of H_{g+2}, we see from the previous lemma that IH_{g+2} is conjugate to $IH'_{g'+2}$ if and only if $g = g'$ and the curves H_{g+2} and H'_{g+2} are birationally isomorphic. Finally, let us look at the involution IH_2. It is a quadratic transformation which is conjugate to the quadratic transformation $\tau_2 : (x, y) \mapsto (x, x/y)$.

A Jonquiéres involution (4) is a special case of a Cremona transformation of the form

$$(x, y) \mapsto \left(\frac{ax + b}{cx + d}, \frac{r_1(x)y + r_2(x)}{r_3(x)y + r_4(x)}\right),$$

where $a, b, c, d \in \mathbb{C}, ad - bc \neq 0$ and $r_i(x) \in \mathbb{C}(x)$ with $r_1(x)r_4(x) - r_2(x)r_3(x) \neq 0$. These transformations form a subgroup of $\mathrm{Cr}(2)$ called a *de Jonquiéres subgroup* and denoted by $\mathrm{dJ}(2)$. Of course, its definition requires a choice of a transcendence basis of the field $\mathbb{C}(\mathbb{P}^2)$. If we identify $\mathrm{Cr}(2)$ with the group $\mathrm{Aut}_{\mathbb{C}}(\mathbb{C}(x, y))$, and consider the field $\mathbb{C}(x, y)$ as a field $K(y)$, where $K = \mathbb{C}(x)$, then

$$\mathrm{dJ}(2) \cong \mathrm{PGL}_{\mathbb{C}(x)}(2) \rtimes \mathrm{PGL}(2)$$

where $\mathrm{PGL}(2)$ acts on $\mathrm{PGL}_{\mathbb{C}(x)}(2)$ via Moebius transformations of the variable x.

It is clear that all elements from $\mathrm{dJ}(2)$ leave the pencil of lines parallel to the y-axis invariant. One can show that a subgroup of $\mathrm{Cr}(2)$ which leaves a pencil of rational curves invariant is conjugate to $\mathrm{dJ}(2)$.

2.4 Geiser and Bertini involutions

The classical definition of a *Geiser involution* is as follows [29]. Fix 7 points $p_1, \ldots, p_7$ in $\mathbb{P}^2$ in general position (we will make this more precise later). The

linear system L of cubic curves through the seven points is two-dimensional. Take a general point p and consider the pencil of curves from L passing through p. Since a general pencil of cubic curves has 9 base points, we can define $\gamma(p)$ as the ninth base point of the pencil. One can show that the algebraic degree of a Geiser involution is equal to 8. Another way to see a Geiser involution is as follows. The linear system L defines a rational map of degree 2

$$f : \mathbb{P}^2 - \to |L|^* \cong \mathbb{P}^2.$$

The points p and $\gamma(p)$ lie in the same fibre. Thus γ is a birational deck transformation of this cover. Blowing up the seven points, we obtain a Del Pezzo surface S of degree 2 (more about this later), and a regular map of degree 2 from S to $\mathbb{P}^2$. The Geiser involution γ becomes an automorphism of the surface S.

It is easy to see that the fixed points of a Geiser involution lie on the ramification curve of f. This curve is a curve of degree 6 with double points at the points $p_1, \ldots, p_7$. It is birationally isomorphic to a canonical curve of genus 3. Applying Lemma 2.1, we obtain that a Geiser involution is not conjugate to any de Jonquiéres involution IH_{g+2}. Also, as we will see later, the conjugacy classes of Geiser involutions are in a bijective correspondence with the moduli space of canonical curves of genus 3.

To define a *Bertini involution* we fix 8 points in $\mathbb{P}^2$ in general position and consider the pencil of cubic curves through these points. It has the ninth base point p_9. For any general point p there will be a unique cubic curve $C(p)$ from the pencil which passes through p. Take p_9 for the zero in the group law of the cubic $C(p)$ and define $\beta(p)$ as the negative $-p$ with respect to the group law. This defines a birational involution on $\mathbb{P}^2$, a Bertini involution [9]. One can show that the algebraic degree of a Bertini involution is equal to 17. We will see later that the fixed points of a Bertini involution lie on a canonical curve of genus 4 with vanishing theta characteristic (isomorphic to a nonsingular intersection of a cubic surface and a quadric cone in $\mathbb{P}^3$). So, a Bertini involution is not conjugate to a Geiser involution or a de Jonquiéres involution. It can be realized as an automorphism of the blow-up of the eight points (a Del Pezzo surface of degree 1), and the quotient by this involution is isomorphic to a quadratic cone in $\mathbb{P}^3$.

3 Rational G-surfaces

3.1 Resolution of indeterminacy points

Let $\chi : S - \to S'$ be a birational map of nonsingular projective surfaces. It is well-known (see [32]) that there exist birational morphisms of nonsingular surfaces $\sigma : X \to S$ and $\phi : X \to S'$ such that the following diagram is commutative

$$\begin{array}{ccc} & X & \\ {\scriptstyle\sigma}\swarrow & & \searrow{\scriptstyle\phi} \\ S & \dashrightarrow & S'. \end{array} \tag{9}$$

It is called a *resolution of indeterminacy points* of χ. Recall also that any birational morphism can be factored into a finite sequence of blow-ups of points. Let

$$\sigma : X = X_N \xrightarrow{\sigma_N} X_{N-1} \xrightarrow{\sigma_{N-1}} \cdots \xrightarrow{\sigma_2} X_1 \xrightarrow{\sigma_1} X_0 = S \tag{10}$$

be such a factorization. Here $\sigma_i : X_i \to X_{i-1}$ is the blow-up of a point $x_i \in X_{i-1}$. Let

$$E_i = \sigma_i^{-1}(x_i), \quad \mathcal{E}_i = (\sigma_{i+1} \circ \dots \sigma_N)^{-1}(E_i). \tag{11}$$

Let H' be a very ample divisor class on S' and $\mathcal{H}'$ be the corresponding complete linear system $|H'|$. Let $\mathcal{H}_N = \phi^*(\mathcal{H}')$. Define $m(x_N)$ as the smallest positive number such that $\mathcal{H}_N + m(x_N)E_N = \sigma_N^*(\mathcal{H}_{N-1})$ for some linear system $\mathcal{H}_{N-1}$ on X_{N-1}. Then proceed inductively to define linear systems $\mathcal{H}_k$ on each X_k such that $\mathcal{H}_{k+1} + m(x_{k+1})E_{k+1} = \sigma_{k+1}^*(\mathcal{H}_k)$, and finally a linear system $\mathcal{H} = \mathcal{H}_0$ on S such that $\mathcal{H}_1 + m(x_1)E_1 = \sigma_1^*(\mathcal{H})$. It follows from the definition that

$$\phi^*(\mathcal{H}') = \sigma^*(\mathcal{H}) - \sum_{i=1}^{N} m(x_i)\mathcal{E}_i. \tag{12}$$

The proper transform of $\mathcal{H}'$ on S under χ is contained in the linear system $\mathcal{H}$. It consists of curves which pass through the points x_i with multiplicity $\geq m_i$. We denote it by

$$\chi^{-1}(\mathcal{H}') = |H - m(x_1)x_1 - \dots - m(x_N)x_N|,$$

where $\mathcal{H} \subset |H|$. Here for a curve on S to pass through a point $x_i \in X_{i-1}$ with multiplicity $\geq m(x_i)$ means that the proper transform of the curve on X_{i-1} has x_i as a point of multiplicity $\geq m(x_i)$. The divisors $\mathcal{E}_i$ are called the *exceptional curves* of the resolution $\sigma : X \to S$ of the birational map χ. Note that $\mathcal{E}_i$ is an irreducible curve if and only if $\sigma_{i+1} \circ \dots \circ \sigma_N : X \to X_i$ is an isomorphism over $E_i = \sigma^{-1}(x_i)$.

The set of points $x_i \in X_i, i = 1, \dots, N$, is called the set of *indeterminacy points*, or *base points*, or *fundamental points* of χ. Note that, strictly speaking, only one of them, x_1, lies in S. However, if $\sigma_1 \circ \dots \circ \sigma_i : X_i \to S$ is an isomorphism in a neighborhood of x_{i+1} we can identify this point with a point in S. Let $\{x_i, i \in I\}$ be the set of such points. Points $x_j, j \notin I$, are *infinitely near points*. A precise meaning of this classical notion is as follows.

Let S be a nonsingular projective surface and $\mathcal{B}(S)$ be the category of birational morphisms $\pi : S' \to S$ of nonsingular projective surfaces. Recall that a morphism from $(S' \xrightarrow{\pi'} S)$ to $(S'' \xrightarrow{\pi''} S)$ in this category is a regular map $\phi : S' \to S''$ such that $\pi'' \circ \phi = \pi'$.

Definition 3.1. *The* bubble space S^{bb} *of a nonsingular surface* S *is the factor set*

$$S^{\mathrm{bb}} = \Big(\bigcup_{(S' \xrightarrow{\pi'} S) \in \mathcal{B}(S)} S' \Big) / R,$$

where R *is the following equivalence relation:* $x' \in S'$ *is equivalent to* $x'' \in S''$ *if the rational map* $\pi''^{-1} \circ \pi' : S' - \to S''$ *maps isomorphically an open neighborhood of* x' *to an open neighborhood of* x''.

It is clear that for any $\pi : S' \to S$ from $\mathcal{B}(S)$ we have an injective map $i_{S'} : S' \to S^{\mathrm{bb}}$. We will identify points of S' with their images. If $\phi : S'' \to S'$ is a morphism in $\mathcal{B}(S)$ which is isomorphic in $\mathcal{B}(S')$ to the blow-up of a point $x' \in S'$, any point $x'' \in \phi^{-1}(x')$ is called *infinitely near point* to x' of the first order. This is denoted by $x'' \succ x'$. By induction, one defines an infinitely near point of order k, denoted by $x'' \succ_k x'$. This defines a partial order on S^{bb}.

We say that a point $x \in S^{\mathrm{bb}}$ is of height k, if $x \succ_k x_0$ for some $x_0 \in S$. This defines the *height function* on the bubble space

$$\mathrm{ht}_S : S^{\mathrm{bb}} \to \mathbb{N}.$$

Clearly, $S = \mathrm{ht}^{-1}(0)$.

It follows from the known behavior of the canonical class under a blow-up that

$$K_X = \sigma^*(K_S) + \sum_{i=1}^{N} \mathcal{E}_i. \tag{13}$$

The intersection theory on a nonsingular surface gives

$$\mathcal{H}'^2 = (\phi^*(\mathcal{H}'))^2 = (\sigma^*(\mathcal{H}) - \sum_{i=1}^{N} m(x_i)\mathcal{E}_i)^2 = \mathcal{H}^2 - \sum_{i=1}^{N} m(x_i)^2, \tag{14}$$

$$K_{S'} \cdot \mathcal{H}' = K_S \cdot \mathcal{H} - \sum_{i=1}^{N} m(x_i).$$

Example 3.2. Let $\chi : \mathbb{P}^2 - \to \mathbb{P}^2$ be a Cremona transformation, $\mathcal{H}' = |\ell|$ be the linear system of lines in $\mathbb{P}^2$, and $\mathcal{H} \subset |n\ell|$. The formulas (14) give

$$n^2 - \sum_{i=1}^{N} m(x_i)^2 = 1, \tag{15}$$

$$3n - \sum_{i=1}^{N} m(x_i) = 3.$$

The linear system $\mathcal{H}$ is written in this situation as $\mathcal{H} = |n\ell - \sum_{i=1}^{N} m_i x_i|$. For example, a quadratic transformation with 3 base points p_1, p_2, p_3 is given by the linear system $|2\ell - p_1 - p_2 - p_3|$. In the case of the standard quadratic

transformation τ_1 the curves $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$ are irreducible, the map $\sigma_1 : X_1 \to X_0 = \mathbb{P}^2$ is an isomorphism in a neighborhood of p_2, p_3 and the map $\sigma_2 : X_2 \to X_1$ is an isomorphism in a neighborhood of $\sigma^{-1}(p_3)$. This shows that we can identify p_1, p_2, p_3 with points on $\mathbb{P}^2$. In the case of the transformation (2), we have $\sigma_1(p_2) = p_1$ and p_3 can be identified with a point on $\mathbb{P}^2$. So in this case $p_2 \succ p_1$. For the transformation (3) we have $p_3 \succ p_2 \succ p_1$.

For a Geiser involution (resp. a Bertini involution) we have $\mathcal{H} = |8\ell - 3p_1 - \cdots - 3p_7|$ (resp. $\mathcal{H} = |17\ell - 6p_1 - \cdots - 6p_8|$).

3.2 *G*-surfaces

Let G be a finite group. A G-surface is a pair (S, ρ), where S is a nonsingular projective surface and ρ is an isomorphism from G to a group of automorphisms of S. A morphism of the pairs $(S, \rho) \to (S', \rho')$ is defined to be a morphism of surfaces $f : S \to S'$ such that $\rho'(G') = f \circ \rho(G) \circ f^{-1}$. In particular, two subgroups of $\mathrm{Aut}(S)$ define isomorphic G-surfaces if and only if they are conjugate inside of $\mathrm{Aut}(S)$. Often, if no confusion arises, we will denote a G-surface by (S, G).

Let $\chi : S \to S'$ be a birational map of G-surfaces. Then one can G-equivariantly resolve χ, in the sense that one can find the diagram (9) where all maps are morphisms of G-surfaces. The group G acts on the surface X permuting the exceptional configurations $\mathcal{E}_i$ in such a way that $\mathcal{E}_i \subset \mathcal{E}_j$ implies $g(\mathcal{E}_i) \subset g(\mathcal{E}_j)$. This defines an action of G on the set of indeterminacy points of χ ($g(x_i) = x_j$ if $g(\mathcal{E}_i) = g(\mathcal{E}_j)$). The action preserves the order, i.e. $x_i \succ x_j$ implies $g(x_i) \succ g(x_j)$, so the function $\mathrm{ht} : \{x_1, \ldots, x_N\} \to \mathbb{N}$ is constant on each orbit Gx_i.

Let $\mathcal{H}' = |H'|$ be an ample linear system on S' and

$$\phi^*(\mathcal{H}') = \sigma^*(\mathcal{H}) - \sum_{i=1}^{N} m(x_i)\mathcal{E}_i$$

be its inverse transform on X as above. Everything here is G-invariant, so $\mathcal{H}$ is a G-invariant linear system on S and the multiplicities $m(x_i)$ are constant on the G-orbits. So we can rewrite the system in the form

$$\phi^*(\mathcal{H}') = \sigma^*(\mathcal{H}) - \sum_{\kappa \in \mathcal{I}} m(\kappa)\mathcal{E}_\kappa,$$

where $\mathcal{I}$ is the set of G-orbits of indeterminacy points. For any $\kappa \in \mathcal{I}$ we set $m(\kappa) = m(x_i)$, where $x_i \in \kappa$ and $\mathcal{E}_\kappa, = \sum_{x_i \in \kappa} \mathcal{E}_i$. Similarly one can rewrite the proper transform of $\mathcal{H}'$ on S

$$|H - \sum_{\kappa \in \mathcal{I}} m(\kappa)\kappa|. \tag{16}$$

Now we can rewrite the intersection formula (14) in the form

$$H'^2 = H^2 - \sum_{\kappa\in\mathcal{I}} m(\kappa)^2 d(\kappa) \tag{17}$$
$$K_{S'} \circ H' = K_S \cdot H - \sum_{\kappa\in\mathcal{I}} m(\kappa) d(\kappa),$$

where $d(\kappa) = \#\{i : x_i \in \kappa\}$.

Remark 3.3. In the arithmetical analog of the previous theory all the notations become more natural. Our maps are maps over a perfect ground field k. A blow-up is the blow-up of a closed point in the scheme-theoretical sense, not necessary k-rational. An exceptional curve is defined over k but when we replace k with its algebraic closure $\bar{k}$, it splits into the union of conjugate exceptional curves over $\bar{k}$. So, in the above notation, κ means a closed point on S or an infinitely near point. The analog of $d(\kappa)$ is of course the degree of a point, i.e. the extension degree $[k(x) : k]$, where $k(x)$ is the residue field of x.

3.3 The G-equivariant bubble space

Here we recall Manin's formalism of the theory of linear systems with basis condition in its G-equivariant form (see [45]).

First we define the *G-equivariant bubble space* of a G-surface S as a G-equivariant version $(S, G)^{\mathrm{bb}}$ of Definition 3.1. One replaces the category $\mathcal{B}(S)$ of birational morphisms $S' \to S$ with the category $\mathcal{B}(S, G)$ of birational morphisms of G-surfaces. In this way the group G acts on the bubble space $(S, G)^{\mathrm{bb}}$ and the height function becomes constant on G-orbits. Let

$$Z^*(S, G) = \varinjlim \mathrm{Pic}(S'), \tag{18}$$

where the inductive limit is taken with respect to the functor Pic from the category $\mathcal{B}(S, G)$ with values in the category of abelian groups defined by $S' \to \mathrm{Pic}(S')$. The group $Z^*(S, G)$ is equipped with a natural structure of G-module. Also it is equipped with the following natural structures.

(a) A symmetric G-invariant pairing

$$Z^*(S, G) \times Z^*(S, G) \to \mathbb{Z}$$

induced by the intersection pairing on each $\mathrm{Pic}(S')$.

(b) A distinguished cone of effective divisors classes in $Z^*(S, G)$

$$Z^*_+(S, G) = \varinjlim \mathrm{Pic}_+(S'),$$

where $\mathrm{Pic}_+(S')$ is the cone of effective divisor classes on each S' from $\mathcal{B}(S, G)$.

(c) A distinguished G-equivariant homomorphism

$$K : Z^*(S, G) \to \mathbb{Z}, \quad K(z) = K_{S'} \cdot z, \text{ for any } S' \to S \text{ from } \mathcal{B}(S, G).$$

Let $f : S' \to S$ be a morphism from $\mathcal{B}(S, G)$ and $\mathcal{E}_1, \dots, \mathcal{E}_n$ be its exceptional curves. We have a natural splitting

$$\mathrm{Pic}(S') = f^*(\mathrm{Pic}(S)) \oplus \mathbb{Z}[\mathcal{E}_1] \oplus \cdots \oplus \mathbb{Z}[\mathcal{E}_n].$$

Now let $Z_0(S, G) = \mathbb{Z}^{(S,G)^{\mathrm{bb}}}$ be the free abelian group generated by the set $(S, G)^{\mathrm{bb}}$. Identifying exceptional curves with points in the bubble space, and passing to the limit we obtain a natural splitting

$$Z^*(S, G) = Z_0(S, G) \oplus \mathrm{Pic}(S). \tag{19}$$

Passing to invariants we get the splitting

$$Z^*(S, G)^G = Z_0(S, G)^G \oplus \mathrm{Pic}(S)^G. \tag{20}$$

Write an element of $Z^*(S, G)^G$ in the form

$$z = D + \sum_{\kappa \in O} m(\kappa)\kappa,$$

where O is the set of G-orbits in $Z_0(S, G)^G$ and D is a G-invariant divisor class on S. Then

(a) $z \cdot z' = D \cdot D' - \sum_{\kappa \in O} m(\kappa)m'(\kappa)d(\kappa)$;
(b) $z \in Z^*_+(S, G)$ if and only if $D \in \mathrm{Pic}_+(S)^G, m(\kappa) \geq 0$ and $m(\kappa') \leq m(\kappa)$ if $\kappa' \succ \kappa$;
(c) $K(z) = D \cdot K_S - \sum_{\kappa \in O} m(\kappa)d(\kappa)$.

Let $\phi : S' \to S$ be an object of $\mathcal{B}(S, G)$. Then we have a natural map $\phi_{\mathrm{bb}} : (S', G)^{\mathrm{bb}} \to (S, G)^{\mathrm{bb}}$ which induces an isomorphism $\phi^*_{\mathrm{bb}} : Z(S, G) \to Z(S', G)$. We also have a natural isomorphism $\phi^{\mathrm{bb}}_* : Z(S', G) \to Z(S, G)$. None of these maps preserves the splitting (19). Resolving indeterminacy points of any birational map $\chi : (S, G) - \to (S', G')$ we can define

- *proper direct transform map* $\chi_* : Z^*(S, G) \xrightarrow{\sim} Z^*(S', G)$;
- *proper inverse transform map* $\chi^* : Z^*(S', G) \xrightarrow{\sim} Z^*(S, G)$.

The group $Z^*(S, G)$ equipped with all above structures is one of the main G-birational invariants of S. It can be viewed as the Picard group of the bubble space $(S, G)^{\mathrm{bb}}$.

The previous machinery gives a convenient way to consider the linear systems defining rational maps of surfaces. Thus we can rewrite (12) in the form $|z|$, where $z = H - \sum m_i x_i$ is considered as an element of $Z^*_+(S, G)$. The condition that $|z|$ is homaloidal becomes equivalent to the conditions

$$\begin{aligned} z^2 = H^2 - \sum m_i^2 = H'^2, \\ K(z) = H \cdot K_S - \sum m_i = H' \cdot K_{S'}. \end{aligned} \tag{21}$$

When $S = S' = \mathbb{P}^2$ we get the equalities (15).

3.4 Minimal rational G-surfaces

Let (S, ρ) be a rational G-surface. Choose a birational map $\phi : S - \to \mathbb{P}^2$. For any $g \in G$, the map $\phi \circ g \circ \phi^{-1}$ belongs to $\mathrm{Cr}(2)$. This defines an injective homomorphism

$$\iota_\phi : G \to \mathrm{Cr}(2). \tag{22}$$

Suppose (S', ρ') is another rational G-surface and $\phi' : S' - \to \mathbb{P}^2$ is a birational map.

The following lemma is obvious.

Lemma 3.4. *The subgroups $\iota_\phi(G)$ and $\iota_{\phi'}(G)$ of $\mathrm{Cr}(2)$ are conjugate if and only if there exists a birational map of G-surfaces $\chi : S' - \to S$.*

The lemma shows that a birational isomorphism class of G-surfaces defines a conjugacy class of subgroups of $\mathrm{Cr}(2)$ isomorphic to G. The next lemma shows that any conjugacy class is obtained in this way.

Lemma 3.5. *Suppose G is a finite subgroup of $\mathrm{Cr}(2)$, then there exists a rational G-surface (S, ρ) and a birational map $\phi : S \to \mathbb{P}^2$ such that, for all $g \in G$,*

$$G = \phi \circ \rho(G) \circ \phi^{-1}.$$

Proof. We give two proofs. The fist one is after A. Verra. Let $D = \cap_{g \in G} \mathrm{dom}(g)$, where $\mathrm{dom}(g)$ is an open subset on which g is defined. Then $U = \cap_{g \in G} g(D)$ is an open invariant subset of $\mathbb{P}^2$ on which $g \in G$ acts biregularly. Order G in some way and consider a copy of $\mathbb{P}^2_g$ indexed by $g \in G$. For any $u \in U$ let $g(u) \in \mathbb{P}^2_g$. We define a morphism

$$\phi : U \to \prod_{g \in G} \mathbb{P}^2_g, \ u \mapsto (g(u))_{g \in G}.$$

Define an action of G on $\phi(U)$ by $g'((x_g)_{g \in G}) = (x_{gg'})_{g \in G}$. Then ϕ is obviously G-equivariant. Now define V as the Zariski closure of $\phi(U)$ in the product. It is obviously a G-invariant surface which contains an open G-invariant subset G-isomorphic to U. It remains to replace V by its G-equivariant resolution of singularities (which always exists).

The second proof is standard. Let U be as above and $U' = U/G$ be the orbit space. It is a normal algebraic surface. Choose any normal projective completion X' of U'. Let S' be the normalization of X' in the field of rational functions of U. This is a normal projective surface on which G acts by biregular transformations. It remains to define S to be a G-invariant resolution of singularities (see also [23]).

Summing up, we obtain the following result.

Theorem 3.6. *There is a natural bijective correspondence between birational isomorphism classes of rational G-surfaces and conjugate classes of subgroups of $\mathrm{Cr}(2)$ isomorphic to G.*

So our goal is to classify G-surfaces (S, ρ) up to birational isomorphism of G-surfaces.

Definition 3.7. *A* minimal *G-surface is a G-surface (S, ρ) such that any birational morphism of G-surfaces $(S, \rho) \to (S', \rho')$ is an isomorphism. A group G of automorphisms of a rational surface S is called a minimal group of automorphisms if the pair (S, ρ) is minimal.*

Obviously, it is enough to classify minimal rational G-surfaces up to birational isomorphism of G-surfaces.

Before we state the next fundamental result, let us recall some terminology.

A *conic bundle structure* on a rational G-surface (S, G) is a G-equivariant morphism $\phi : S \to \mathbb{P}^1$ such that the fibres are isomorphic to a reduced conic in $\mathbb{P}^2$. A *Del Pezzo surface* is a surface with ample anti-canonical divisor $-K_S$.

Theorem 3.8. *Let S be a minimal rational G-surface. Then either S admits a structure of a conic bundle with* $\mathrm{Pic}(S)^G \cong \mathbb{Z}^2$*, or S is isomorphic to a Del Pezzo surface with* $\mathrm{Pic}(S)^G \cong \mathbb{Z}$.

An analogous result from the classical literature is proven by using the method of the termination of adjoints, first introduced for linear system of plane curves in the work of G. Castelnuovo. It consists in replacing a linear system $|D|$ with the linear system $|D+K_S|$ and repeat doing this to stop only if the next step leads to the empty linear system. The application of this method to finding a G-invariant linear system of curves on the plane was initiated in the works of S. Kantor [41], who essentially stated the theorem above but without the concept of minimality. In arithmetical situation this method was first applied by Enriques [28]. A first modern proof of the theorem was given by Manin [44] and by the second author [37] (an earlier proof of Manin used the assumption that G is an abelian group). Nowadays the theorem follows easily from a G-equivariant version of Mori's theory (see [42], Example 2.18) and the proof can be found in literature ([7], [22]). For this reason we omit the proof.

Recall the classification of Del Pezzo surfaces (see [24], [45]). The number $d = K_S^2$ is called the *degree*. By Noether's formula, $1 \leq d \leq 9$. For $d \geq 3$, the anti-canonical linear system $|-K_S|$ maps S onto a nonsingular surface of degree d in $\mathbb{P}^d$. If $d = 9$, $S \cong \mathbb{P}^2$. If $d = 8$, then $S \cong \mathbb{P}^1 \times \mathbb{P}^1$, or $S \cong \mathbf{F}_1$, where as always we denote by $\mathbf{F}_n$ the minimal ruled surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$. For $d \leq 7$, a Del Pezzo surface S is isomorphic to the blow-up of $n = 9 - d$ points in $\mathbb{P}^2$ satisfying the following conditions

- no three are on a line;
- no six are on a conic;
- if $n = 8$, then then the points are not on a plane cubic which has one of them as its singular point.

For $d = 2$, the linear system $|-K_S|$ defines a finite morphism of degree 2 from S to $\mathbb{P}^2$ with a nonsingular quartic as the branch curve. Finally, for $d = 1$, the linear system $|-2K_S|$ defines a finite morphism of degree 2 onto a quadric cone $Q \subset \mathbb{P}^3$ with the branch curve cut out by a cubic.

For a minimal Del Pezzo G-surface the group $\mathrm{Pic}(S)^G$ is generated by K_S if S is not isomorphic to $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$. In the latter cases it is generated by $\frac{1}{3}K_S$ or $\frac{1}{2}K_S$, respectively.

A conic bundle surface is either isomorphic to $\mathbf{F}_n$ or to a surface obtained from $\mathbf{F}_n$ by blowing up a finite set of points, no two lying in a fibre of a ruling. The number of blow-ups is equal to the number of singular fibres of the conic bundle fibration. We will exclude the surfaces $\mathbf{F}_0$ and $\mathbf{F}_1$, considering them as Del Pezzo surfaces.

There are minimal conic bundles with ample $-K_S$ (see Proposition 5.2).

4 Automorphisms of minimal ruled surfaces

4.1 Some of group theory

We employ the standard notations for groups used by group-theorists (see [18]):

C_n, a cyclic group of order n,

$n = C_n$ if no confusion arises;

$n^r = C_n^r$, the direct sum of r copies of C_n (not to be confused with cyclic group of order n^r);

S_n, the *permutation group* of degree n,

A_n, the *alternating group* of degree n,

D_{2n}, the *dihedral group* of order $2n$,

$Q_{4n} = \langle a, b \mid a^{2n} = 1, b^2 = a^n, b^{-1}ab = a^{-1} \rangle$, *dicyclic group* of order $4n$, a *generalized quaternion group* if $n = 2^k$;

$H_n(p)$, the *Heisenberg group* of unipotent $n \times n$-matrices with entries in $\mathbb{F}_p$.

$\mathrm{GL}(n) = \mathrm{GL}(n, \mathbb{C})$, general linear group over $\mathbb{C}$,

$\mathrm{SL}(n) = \mathrm{SL}(n, \mathbb{C})$, special linear group over $\mathbb{C}$,

$\mathrm{PGL}(n) = \mathrm{GL}(n, \mathbb{C})/\mathbb{C}^*$, general projective linear group over $\mathbb{C}$,

$\mathrm{O}(n)$, the orthogonal linear group over $\mathbb{C}$,

$\mathrm{PO}(n)$, the projective orthogonal linear group over $\mathbb{C}$,

$L_n(q) = \mathrm{PSL}(n, \mathbb{F}_q)$, where $q = p^r$ is a power of a prime number p;

$T \cong A_4$, $O \cong S_4 \cong \mathrm{PGL}(2, \mathbb{F}_3)$, $I \cong A_5 \cong L_2(5) \cong L_2(2^2)$, tetrahedral, octahedron, icosahedron subgroups of PGL(2),

$\overline{T} \cong \mathrm{SL}(2, \mathbb{F}_3)$, $\bar{O} \cong \mathrm{GL}(2, \mathbb{F}_3)$, $\bar{I} \cong \mathrm{SL}(2, \mathbb{F}_5), \overline{D}_{2n} \cong Q_{4n}$, binary tetrahedral, binary octahedron,binary icosahedron, binary dihedral subgroups of SL(2),

$A \bullet B$ is an upward extension of B with help of a normal subgroup A;

$A : B$ is a split extension $A\bullet B$, i.e. a semi-direct product $A \rtimes B$ ((it is defined by a homomorphism $\phi : B \to \mathrm{Aut}(A)$);

$A^\bullet B$ is a non-split extension $A\bullet B$,

$nA = n^\bullet A$, where the normal group n is equal to the center.

$p^{a+b} = C_p^a \bullet C_p^b$, where p is prime;

$A \triangle B$ (or $A \triangle_D B$), the diagonal product of A and B over their common homomorphic image D (i.e. the subgroup of $A \times B$ of pairs (a, b) such that $\alpha(a) = \beta(b)$ for some surjections $\alpha : A \to D, \beta : B \to D$). When D is omitted it means that D is the largest possible;

$\frac{1}{m}[A \times B] = A \triangle_D B$, where $\#D = m$;

$A \wr S_n$, the wreath product, i.e. $A^n : S_n$, where S_n is the symmetric group acting on A^n by permuting the factors;

μ_n, the group of nth roots of unity with generator $\epsilon_n = e^{2\pi i/n}$.

We will often use the following simple result from group theory which is known as Goursat's Lemma.

Lemma 4.1. *Let G be a finite subgroup of the product $A \times B$ of two groups A and B. Let $p_1 : A \times B \to A, p_2 : A \times B \to B$ be the projection homomorphisms. Let $G_i = p_i(G), H_i = \mathrm{Ker}(p_j|G), i \neq j = 1, 2$. Then H_i is a normal subgroup in G_i. The map $\phi : G_1/H_1 \to G_2/H_2$ defined by $\phi(aH_1) = p_2(a)H_2$ is an isomorphism, and*

$$G = G_1 \triangle_D G_2,$$

where $D = G_1/H_1, \alpha : G_1 \to D$ is the projection map to the quotient, and β is the composition of the projection $G_2 \to G_2/H_2$ and ϕ^{-1}.

Note some special cases:

$$G \triangle_1 G' \cong G \times G', \quad G \triangle_{G'} G' = \{(g, \alpha(g)) \in G \times G', g \in G_1\},$$

where $\alpha : G \to G'$ is a surjection and $G' \to G'$ is the identity.

We will be dealing with various group extensions. The following lemma is known in group theory as the Schur-Zassenhaus Theorem. Its proof can be found in [30], 6.2.

Lemma 4.2. *Let $A\bullet B$ be an extension of groups. Suppose that the orders of A and B are coprime. Then the extension splits. If moreover A or B is solvable, then all subgroups of $A : B$ defining splittings are conjugate.*

We will often use the following simple facts, their proofs are left to the reader (or can be found in www.planetmath.org).

Lemma 4.3. *A subgroup of $D_{2n} = \langle a, b \mid a^n = b^2 = b^{-1}aba = 1\rangle$ is either cyclic or dihedral. A normal subgroup H is either cyclic subgroup $\langle a\rangle$, or $n = 2k$ and H is one of the following two subgroups $\langle a^2, b\rangle, \langle a^2, ab\rangle$ of index 2. These two subgroups are interchanged under the outer automorphism $a \mapsto$*

$a, b \mapsto ab$. If H is cyclic of order n/k, the quotient group is isomorphic to $D_{2n/k}$.

The group of $\mathrm{Aut}(D_{2n})$ *is isomorphic to* $(\mathbb{Z}/n)^* : n$ *and it is generated by the transformations* $a \mapsto a^s, b \mapsto a^t b$. *The subgroup of interior automorphisms is generated by transformations* $a \mapsto a^{-1}, b \mapsto b$ *and* $a \mapsto a, b \mapsto a^2 b$.

It will be convenient to list all isomorphism classes of non-abelian groups of order 16.

Notation	Center	LCS	Extensions	Presentation
$2 \times D_8$	2^2	16, 2, 1	$2^{1+3}, 2^{2+2}, 2^{3+1}$, $(2 \times 4) : 2$	$a^4 = b^2 = c^2 = 1$, $[a,b]a^2 = [a,c] = [b,c] = 1$
$2 \times Q_8$	2^2	16, 2, 1	2^{2+2}, $(2 \times 4)^\bullet 2$	$a^4 = a^2 b^{-2} = a^2[a,b] = 1$
D_{16}	2	16, 4, 2, 1	$8 : 2, 2D_8$, $(2^2)^\bullet 4, D_8^\bullet 2$	$a^8 = b^2 = a^2[a,b] = 1$
SD_{16}	2	16, 4, 2, 1	$8 : 2, D_8^\bullet 2$, $2D_8, (2^2)^\bullet 4$	$a^8 = b^2 = [a,b]a^{-2} = 1$
Q_{16}	2	16, 4, 2, 1	$8^\bullet 2, 2D_8$, $4^\bullet(2^2)$	$a^8 = a^4 b^{-2} = [a,b]a^2 = 1$
AS_{16}	4	16, 2, 1	$2^{1+3}, D_8 : 2$ $4(2^2), (2 \times 4) : 2$	$a^4 = b^2 = c^2 = [a,b] = 1$ $[c,b]a^{-2} = [c,a] = 1$
K_{16}	2^2	16, 2, 1	$2^{2+2}, (2 \times 4)^\bullet 2$ $2^\bullet(2 \times 4), 4 : 4$	$a^4 = b^4 = [a,b]a^2 = 1$
L_{16}	2^2	16, 2, 1	$2^2 : 4, 2^\bullet(2 \times 4)$ $(2 \times 4) : 2$	$a^4 = b^2 = c^2 = 1$, $[c,a]b = [a,b] = [c,b] = 1$
M_{16}	4	16, 2, 1	$8 : 2,\ 4(2^2)$ $2^\bullet(2 \times 4)$	$a^8 = b^2 = 1, [a,b]a^4 = 1$

Table 1. Non-abelian groups of order 16

Recall that there are two non-isomorphic non-abelian groups of order 8, D_8 and Q_8.

Finally we describe central extension of polyhedral and binary polyhedral groups. Recall that the isomorphism classes of central extensions $A \bullet G$, where A is an abelian group, are parametrized by the 2-cohomology group $H^2(G, A)$. We will assume that $A \cong p$, where p is prime. We will use the following facts about the cohomology groups of polyhedral and binary polyhedral groups which can be found in text-books on group cohomology (see, for example, [1]).

Lemma 4.4. *Let G be a polyhedral group or a binary polyhedral group. If $G \cong n$ is cyclic, then $H^2(G, p) \cong p$ if $p \mid n$ and zero otherwise. If G is not cyclic, then $H^2(G, p) = 0$ if $p \neq 2, 3$. Moreover*

(i) If G is a polyhedral group, then

$$H^2(G,2) \cong \begin{cases} 2 & \text{if } G \cong D_{2n}, n \text{ odd}, \\ 2^3 & \text{if } G \cong D_{2n}, n \text{ even}, \\ 2 & \text{if } G \cong T, \\ 2^2 & \text{if } G \cong O, \\ 2 & \text{if } G \cong I. \end{cases}$$

$$H^2(G,3) \cong \begin{cases} 3 & \text{if } G \cong T, \\ 1 & \text{otherwise.} \end{cases}$$

(ii)If G is a binary polyhedral group, then

$$H^2(G,2) \cong \begin{cases} 2 & \text{if } G \cong \overline{D}_{2n}, n \text{ odd}, \\ 2^2 & \text{if } G \cong \overline{D}_{2n}, n \text{ even}, \\ 2 & \text{if } G \cong \overline{O}, \\ 1 & \text{otherwise.} \end{cases}$$

$$H^2(G,3) \cong \begin{cases} 3 & \text{if } G \cong \overline{T}, \\ 1 & \text{otherwise.} \end{cases}$$

4.2 Finite groups of projective automorphisms

We start with the case $S = \mathbb{P}^2$, where $\mathrm{Aut}(S) \cong \mathrm{PGL}(3)$. To save space we will often denote a projective transformation

$$(x_0, x_1, x_2) \mapsto (L_0(x_0, x_1, x_2), L_1(x_0, x_1, x_2), L_2(x_0, x_1, x_2))$$

by $[L_0(x_0, x_1, x_2), L_1(x_0, x_1, x_2), L_2(x_0, x_1, x_2)]$.

Recall some standard terminology from the theory of linear groups. Let G be a subgroup of the general linear group $\mathrm{GL}(V)$ of a complex vector space V. The group G is called *intransitive* if the representation of G in V is reducible. Otherwise it is called *transitive*. A transitive group G is called *imprimitive* if it contains an intransitive normal subgroup G'. In this case V decomposes into a direct sum of G'-invariant proper subspaces, and elements from G permute them. A group is *primitive* if it is neither intransitive, nor imprimitive. We reserve this terminology for subgroups of $\mathrm{PGL}(V)$ keeping in mind that each such group can be represented by a subgroup of $\mathrm{GL}(V)$.

Let G' be a finite intransitive subgroup of GL(3) and G be its image in PGL(3). Then G' is conjugate to a subgroup $\mathbb{C}^* \times \mathrm{GL}(2)$ of block matrices.

To classify such subgroups we have to classify subgroups of GL(2). We will use the well-known classification of finite subgroups of PGL(2). They are isomorphic to one of the following *polyhedral groups*

- a cyclic group C_n;
- a dihedral group D_{2n} of order $2n \geq 2$;

- the tetrahedral group $T \cong A_4$ of order 12;
- the octahedron group $O \cong S_4$ of order 24;
- the icosahedron group $I \cong A_5$ of order 60.

Two isomorphic subgroups are conjugate subgroups of PGL(2).

The pre-image of such group in $\mathrm{SL}(2,\mathbb{C})$ under the natural map

$$\mathrm{SL}(2) \to \mathrm{PSL}(2) = \mathrm{SL}(2)/(\pm 1) \cong \mathrm{PGL}(2)$$

is a double extension $\overline{G} = 2\bullet G$. The group $\overline{G} = 2\bullet G$ is called a *binary polyhedral group*. A cyclic group of odd order is isomorphic to a subgroup SL(2) intersecting trivially the center.

Consider the natural surjective homomorphism of groups

$$\beta : \mathbb{C}^* \times \mathrm{SL}(2) \to \mathrm{GL}(2), \quad (c, A) \mapsto cA.$$

Its kernel is the subgroup $\{(1, I_2), (-1, -I_2)\}$.

Let G be a finite subgroup of GL(2) with center $Z(G)$. Since $cA = (-c)(-A)$ and $\det(cA) = c^2 \det A$, we see that $\widetilde{G} = \beta^{-1}(G)$ is a subgroup of $\mu_{2m} \times \bar{G}'$, where $\overline{G'}$ is a binary polyhedral group with nontrivial center whose image G' in PGL(2) is isomorphic to $G/Z(G)$. The homomorphism β defines an isomorphism from the kernel H_2 of the first projection $\widetilde{G} \to \mu_{2m}$ onto the subgroup $G_0 = \mathrm{Ker}(\det : G \to \mathbb{C}^*)$. Also it defines an isomorphism from the kernel H_1 of the second projection $\widetilde{G} \to \bar{G}'$ onto $Z(G)$. Applying Lemma 4.1, we obtain

$$\widetilde{G} \cong \mu_{2m} \triangle_D \overline{G}', \ D = \bar{G}'/G_0.$$

Lemma 4.5. *Let G be a finite non-abelian subgroup of* GL(2). *Then $G = \beta(\widetilde{G})$, where $\widetilde{G} \subset \mathbb{C}^* \times \mathrm{SL}(2,\mathbb{C})$ is conjugate to one of the following groups*

(i) $\widetilde{G} = \mu_{2m} \times \bar{I}, \ G \cong m \times \bar{I}$;
(ii) $\widetilde{G} = \mu_{2m} \times \overline{O}, \ G \cong m \times \overline{O}$;
(iii) $\widetilde{G} = \mu_{2m} \times \overline{T}, \ G \cong m \times \overline{T}$;
(iv) $\widetilde{G} = \mu_{2m} \times Q_{4n}, \ G \cong m \times Q_{4n}$;
(v) $\widetilde{G} = \frac{1}{2}[\mu_{4m} \times \overline{O}], \ G \cong 2m\bullet O \cong (m \times \overline{T})\bullet 2$ *(split if $m = 1, 2$)*;
(vi) $\widetilde{G} = \frac{1}{3}[\mu_{6m} \times \overline{T}], \ G \cong 2m\bullet T \cong (m \times 2^2)\bullet 3$ *(split if $m = 1, 3$)*;
(vii) $\widetilde{G} \cong \frac{1}{2}[\mu_{4m} \times Q_{8n}], G \cong 2m\bullet D_{4n} \cong (m \times Q_{4n})\bullet 2$ *(split if $m = 1, 2$)*;
(viii) $\widetilde{G} = \frac{1}{2}[\mu_{4m} \times Q_{4n}], \ G \cong 2m\bullet D_{2n} \cong (m \times 2n)\bullet 2$ *(split if $m = 1, 2$)*;
(ix) $\widetilde{G} = \frac{1}{4}[\mu_{4m} \times Q_{4n}], n$ *is odd,* $G \cong m\bullet D_{2n} \cong (m \times n)\bullet 2$ *(split if $m = 1, 2$)*.

Note that, although Q_{8n} has two different non-cyclic subgroups of index 2, they are conjugate under an element of SL(2), so they lead to conjugate subgroups in GL(2).

Lemma 4.4 gives us some information when some of these extensions split.

An abelian subgroup $G \subset \mathrm{GL}(2)$ is conjugate to a subgroup of diagonal matrices of the form $(\epsilon_m^a, \epsilon_n^b)$, where ϵ_m, ϵ_n are primitive roots of unity and $a, b \in \mathbb{Z}$. Let $d = (m, n), m = du, n = dv, d = kq$ for some fixed positive integer k. Let $H_1 = \langle \epsilon_m^k \rangle \subset \langle \epsilon_m \rangle, H_2 = \langle \epsilon_n^k \rangle \subset \langle \epsilon_n \rangle$ be cyclic subgroups of index k. Applying Lemma 4.1 we obtain

$$G \cong \langle \epsilon_m \rangle \triangle_k \langle \epsilon_n \rangle$$

where the homomorphisms $\langle \epsilon_m \rangle \to k, \langle \epsilon_n \rangle \to k$ differ by an automorphism of the cyclic group $\langle \epsilon_k \rangle \cong k$ defined by a choice of a new generator $\epsilon_m^s, (s, k) = 1$. In this case

$$G = (\langle \epsilon_m^k \rangle \times \langle \epsilon_n^k \rangle_\bullet \langle \epsilon_k \rangle \tag{23}$$

is of order $mn/k = uvkq^2$. In other words, G consists of diagonal matrices of the form $(\epsilon_m^a, \epsilon_n^b)$, where $a \equiv sb \mod k$.

Corollary 4.6. *Let G be an intransitive finite subgroup of* $\mathrm{GL}(3)$. *Then its image in* $\mathrm{PGL}(3)$ *consists of transformations* $[ax_0 + bx_1, cx_0 + dx_1, x_2]$, *where the matrices* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ *form a non-abelian finite subgroup H of* $\mathrm{GL}(2)$ *from Lemma 4.5 or an abelian group of the form* (23).

Now suppose G is transitive but imprimitive subgroup of PGL(3). Let G' be its largest intransitive normal subgroup. Then G/G' permutes transitively the invariant subspaces of G', hence we may assume that all of them are one-dimensional. Replacing G by a conjugate group we may assume that G' is a subgroup of diagonal matrices. We can represent its elements by diagonal matrices $g = (\epsilon_m^a, \epsilon_n^b, 1)$, where $a \equiv sb \mod k$ as in (23). The group G contains a cyclic permutation τ of coordinates. Since G' is a normal subgroup of G, we get $\tau^{-1} g \tau = (\epsilon_n^{-b}, \epsilon_n^{-b} \epsilon_m^a, 1) \in G'$. This implies that $n | bm, m | an$, hence $u | b, v | a$. Since $(\epsilon_m, \epsilon_n^s, 1)$ or $(\epsilon_m^{s'}, \epsilon_n, 1), ss' \equiv 1 \mod k$, belongs to G we must have $u = v = 1$, i.e. $m = n = d$. Therefore G' consists of diagonal matrices $g = (\epsilon_d^a, \epsilon_d^{sa}, 1)$. Since $\tau^{-1} g \tau = (\epsilon_d^{-sa}, \epsilon_d^{a-sa}, 1) \in G'$, we get $a - sa \equiv -s^2 a \mod k$ for all $a \in \mathbb{Z}/m\mathbb{Z}$. Hence the integers s satisfy the congruence $s^2 - s + 1 \equiv 0 \mod k$. If, moreover, $G/G' \cong S_3$, then we have an additional condition $s^2 \equiv 1 \mod k$, and hence either $k = 1$ and $G' = \mu_n \times \mu_n$ or $k = 3$, $s = 2$ and $G' = n \times n/k$.

This gives the following.

Theorem 4.7. *Let G be a transitive imprimitive finite subgroup of* $\mathrm{PGL}(3)$. *Then G is conjugate to one of the following groups*

- $G \cong n^2 : 3$ *generated by transformations*

$$[\epsilon_n x_0, x_1, x_2], \ [x_0, \epsilon_n x_1, x_2], \ [x_2, x_0, x_1];$$

- $G \cong n^2 : S_3$ *generated by transformations*

$$[\epsilon_n x_0, x_1, x_2], \ [x_0, \epsilon_n x_1, x_2], \ [x_0, x_2, x_1], \ [x_2, x_0, x_1];$$

- $G = G_{n,k,s} \cong (n \times \frac{n}{k}) : 3$, *where* $k > 1, k|n$ *and* $s^2 - s + 1 = 0 \mod k$. *It is generated by transformations*

$$[\epsilon_{n/k}x_0, x_1, x_2],\ [\epsilon_n^s x_0, \epsilon_n x_1, x_2],\ [x_2, x_0, x_1].$$

- $G \simeq (n \times \frac{n}{3}) : S_3$ *generated by transformations*

$$[\epsilon_{n/3}x_0, x_1, x_2],\ [\epsilon_n^2 x_0, \epsilon_n x_1, x_2],\ [x_0, x_2, x_1],\ [x_1, x_0, x_2].$$

The next theorem is a well-known result of Blichfeldt [10].

Theorem 4.8. *Any primitive finite subgroup* G *of* PGL(3) *is conjugate to one of the following groups.*

1. *The icosahedron group* A_5 *isomorphic to* $L_2(5)$. *It leaves invariant a nonsingular conic.*
2. *The* Hessian group *of order 216 isomorphic to* $3^2 : L_2(3)$. *It is realized as the group of automorphisms of the Hesse pencil of cubics*

$$x^3 + y^3 + z^3 + txyz = 0.$$

The subgroup $L_2(3)$ *is isomorphic to* $2\bullet A_4 \cong \overline{T}$ *and permutes the four reducible members of the pencil.*
3. *The* Klein group *of order 168 isomorphic to* $L_2(7)$ *(realized as the full group of automorphisms of the Klein quartic* $x^3y + y^3z + z^3x = 0$*).*
4. *The* Valentiner group *of order 360 isomorphic to* A_6. *It can be realized as the full group of automorphisms of the nonsingular plane sextic*

$$10x^3y^3 + 9zx^5 + y^5 - 45x^2y^2z^2 - 135xyz^4 + 27z^6 = 0.$$

5. *Subgroups of the Hessian group:*
   - $3^2 : 4$;
   - $3^2 : Q_8$.

4.3 Finite groups of automorphisms of $\mathbf{F_0}$

Since $\mathbf{F}_0$ is isomorphic to a nonsingular quadric in $\mathbb{P}^3$, the group $\mathrm{Aut}(\mathbf{F}_0)$ is isomorphic to the projective orthogonal group PO(4). The classification of finite subgroups of $O(4)$ is due to É. Goursat [31] (in the real case see a modern account in [19]). Goursat' Lemma 4.1 plays an important role in this classification.

Obviously,

$$\mathrm{Aut}(\mathbf{F}_0) \cong \mathrm{PGL}(2) \wr S_2,$$

First we classify subgroups of $\mathrm{PGL}(2) \times \mathrm{PGL}(2)$ by applying Goursat's Lemma.

Observe the following special subgroups of $\mathrm{PGL}(2) \times \mathrm{PGL}(2)$.

1. $G = G_1 \times G_2$ is the *product subgroup.*
2. $G \triangle_1 G = \{(g_1, g_2) \in G \times G : \alpha(g_1) = g_2\} \cong G$ is a *α-twisted diagonal subgroup.* If $\alpha = \mathrm{id}_G$, we get the *diagonal subgroup.*

Note that α-twisted diagonal groups are conjugate in $\mathrm{Aut}(\mathbf{F}_0)$ if $\alpha(g) = xgx^{-1}$for some x in the normalizer of G inside $\mathrm{Aut}(\mathbb{P}^1)$. In particular, we may always assume that α is an exterior automorphism of G.

We will use the notation $[p_1, \dots, p_r]$ for the Coxeter group defined by the Coxeter diagram

Following [19] we write $[p_1, \dots, p_r]^+$ to denote the index 2 subgroup of even length words in standard generators of the Coxeter group. If exactly one of the numbers $p_1, \dots, p_r$ is even, say p_k, there are two other subgroups of index 2 denoted by $[p_1, \dots, p_r^+]$ (resp. $[^+p_1, \dots, p_r]$). They consist of words which contain each generator $R_1, \dots, R_{k-1}$ (resp. $R_{k+1}, \dots, R_r$) even number of times. The intersection of these two subgroups is denoted by $[^+p_1, \dots, p_r^+]$. For example,

$$D_{2n} = [n],\ T = [3,3]^+,\ O = [3,4]^+,\ I = [3,5]^+.$$

Recall that each group $[p_1, \dots, p_r]$ has a natural linear representation in $\mathbb{R}^r$ as a reflection group. If $r = 3$, the corresponding representation defines a subgroup of PO(4). If $r = 2$, it defines a subgroup of PO(3) which acts diagonally on $\mathbb{P}^2 \times \mathbb{P}^2$ and on $\mathbb{P}^1 \times \mathbb{P}^1$ embedded in $\mathbb{P}^2 \times \mathbb{P}^2$ by the product of the Veronese maps. We denote $\overline{[p_1, \dots, p_r]}$ the quotient of $[p_1, \dots, p_r]$ by its center. Similar notation is used for the even subgroups of $[p_1, \dots, p_r]$.

Theorem 4.9. *Let G be a finite subgroup of* $\mathrm{PGL}(2) \times \mathrm{PGL}(2)$ *not conjugate to the product $A \times B$ of subgroups of* $\mathrm{PGL}(2)$*. Then G is conjugate to one of the following groups or its image under the switching of the factors.*

- $\frac{1}{60}[I \times I] \cong I \cong [3,5]^+$*;*
- $\frac{1}{60}[I \times I] \cong I \cong [3,3,3]^+$*;*
- $\frac{1}{24}[O \times O] \cong O \cong [3,4]^+$*;*
- $\frac{1}{24}[O \times O] \cong O \cong [2,3,3]^+$*;*
- $\frac{1}{12}[T \times T] \cong T \cong [3,3]^+$*;*
- $\frac{1}{2}[O \times O] \cong (T \times T) : 2 \cong \overline{[3,4,3]^+}$*;*
- $\frac{1}{6}[O \times O] \cong 2^4 : S_3 \cong \overline{[3,3,4]^+}$*;*
- $\frac{1}{3}[T \times T] \cong 2^4 : 3 \cong \overline{[^+3,3,4^+]}$;
- $\frac{1}{2}[D_{2m} \times D_{4n}] \cong (m \times D_{2n})^\bullet 2$ *$(m, n \geq 2)$;*
- $\frac{1}{4}[D_{4m} \times D_{4n}] \cong (m \times n) : 4$ *$(m, n$ odd$)$;*
- $\frac{1}{2k}[D_{2mk} \times D_{2nk}]_s \cong (m \times n) : D_{2k},\ (s,k) = 1$*);*
- $\frac{1}{2k}[D_{2mk} \times D_{2nk}]_s \cong (m \times n) : D_{2k},\ (s,2k) = 1, m, n$ *odd);*

- $\frac{1}{k}[C_{mk} \times C_{nk}]_s \cong (m \times n)\bullet k$ $((s,k)=1)$;
- $\frac{1}{k}[C_{mk} \times C_{nk}]_s \cong (m \times n)\bullet k$ $((s,2k)=1, m, n$ *odd*;);
- $\frac{1}{2}[D_{2m} \times O] \cong (m \times T) : 2$;
- $\frac{1}{2}[D_{4m} \times O] \cong (D_{2m} \times T) : 2 (m \geq 2)$;
- $\frac{1}{6}[D_{6n} \times O] \cong (m \times 2^2) : S_3 (m \geq 2)$;
- $\frac{1}{2}[C_{2m} \times O] \cong (m \times T)\bullet 2$ (*split if* $m = 1$);
- $\frac{1}{3}[C_{3m} \times T] \cong (m \times 2^2)\bullet 3$ (*split if* $m = 1$);
- $\frac{1}{2}[D_{4m} \times D_{4n}] \cong (D_{2m} \times D_{2n})^\bullet 2$ $(m, n \geq 2)$;
- $\frac{1}{2}[C_{2m} \times D_{4n}] \cong (m \times D_{2n})^\bullet 2$ $(n \geq 2)$;
- $\frac{1}{2}[C_{2m} \times D_{2n}] \cong (m \times n) : 2 \cong m\bullet D_{2n}$.

All other finite subgroups of $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ are conjugate to a group $G^0\bullet 2$, where the quotient 2 is represented by an automorphism which interchanges the two rulings of $\mathbf{F}_0$. It is equal to $\tau \circ g$, where τ is the switch $(x, y) \mapsto (y, x)$ and $g \in \mathrm{PGL}(2) \times \mathrm{PGL}(2)$.

4.4 Finite groups of automorphisms of $\mathbf{F}_n, n \neq 0$

Let S be a minimal ruled surface $\mathbf{F}_n, n \neq 0$. If $n = 1$, the group $\mathrm{Aut}(\mathbf{F}_1)$ is isomorphic to a subgroup of $\mathrm{Aut}(\mathbb{P}^2)$ leaving one point fixed. We will not be interested in such subgroups so we assume that $n \geq 2$.

Theorem 4.10. *Let* $S = \mathbf{F}_n, n \neq 2$. *We have*

$$\mathrm{Aut}(\mathbf{F}_n) \cong \mathbb{C}^{n+1} : (\mathrm{GL}(2)/\mu_n),$$

where $\mathrm{GL}(2)/\mu_n$ *acts on* $\mathbb{C}^n$ *by means of its natural linear representation in the space of binary forms of degree* n. *Moreover,*

$$\mathrm{GL}(2)/\mu_n \cong \begin{cases} \mathbb{C}^* : \mathrm{PSL}(2), & \text{if } n \text{ is even}, \\ \mathbb{C}^* : \mathrm{SL}(2), & \text{if } n \text{ is odd}. \end{cases}$$

Proof. This is of course well-known. We identify $\mathbb{F}_n$ with the weighted projective plane $\mathbb{P}(1, 1, n)$. An automorphism is given by the formula

$$(t_0, t_1, t_2) \mapsto (at_0 + bt_1, ct_0 + dt_1, et_2 + f_n(t_0, t_1)),$$

where f_n is a homogeneous polynomial of degree n. The vector space $\mathbb{C}^{n+1}$ is identified with the normal subgroup of transformations $[t_0, t_1, t_2 + f_n(t_0, t_1)]$. The quotient by this subgroup is isomorphic to the subgroup of transformations $[at_0 + bt_1, ct_0 + dt_1, et_2]$ modulo transformations of the form $[ct_0, ct_1, c^n t_2]$. This group is obviously isomorphic to $\mathrm{GL}(2)/\mu_n$. Consider the natural projection $\mathrm{GL}(2)/\mu_n \to \mathrm{PGL}(2) \cong \mathrm{PSL}(2)$. Define a homomorphism $\mathrm{SL}(2) \to \mathrm{GL}(2)/\mu_n$ by assigning to a matrix A the coset of A modulo μ_n. If n is even, the kernel of this homomorphism is $\langle -I_2 \rangle$, so we have a splitting $\mathrm{GL}(2) \cong \mathbb{C}^* : \mathrm{PGL}(2)$. If n is odd, the homomorphism is injective and defines a splitting $\mathrm{GL}(2)/\mu_n \cong \mathbb{C}^* : \mathrm{SL}(2)$.

Suppose G is a finite subgroup of $\mathrm{Aut}(\mathbb{F}_n)$. Obviously, G is contained in the subgroup $\mathrm{GL}(2)/\mu_n$.

Suppose $G \cap \mathbb{C}^* = \{1\}$. Then G is isomorphic to a subgroup of $\mathrm{PGL}(2)$ (resp. $\mathrm{SL}(2)$) over which the extension splits. Note that the kernel $\mathbb{C}^*$ of the projections $\mathrm{GL}(2)/\mu_n \to \mathrm{PGL}(2)$ or $\mathrm{GL}(2)/\mu_n \to \mathrm{SL}(2)$ is the center. Thus each finite subgroup H of $\mathrm{PGL}(2)$ (resp. $\mathrm{SL}(2)$) defines k conjugacy classes of subgroups isomorphic to H, where $k = \#\mathrm{Hom}(H, \mathbb{C}^*) = \#G/[G, G]$.

If $G \cong \mathbb{C}^* \cong \mu_m$ is non-trivial, the group is a central extension $m\bullet H$, where H is a polyhedral group, if n is even, and a binary polyhedral group otherwise. We can apply Lemma 4.4 to find some cases when the extension must split. In other cases the structure of the group is determined by using Theorem 4.9. We leave this to the reader.

5 Automorphisms of conic bundles

5.1 Geometry of conic bundles

Let $\pi : S \to \mathbb{P}^1$ be a conic bundle with singular fibres over points in a finite set $\Sigma \subset \mathbb{P}^1$. We assume that $k = \#\Sigma > 0$. Recall that each singular fibre $F_x, x \in \Sigma$, is the bouquet of two $\mathbb{P}^1$'s.

Let E be a section of the conic bundle fibration π. The Picard group of S is freely generated by the divisor classes of E, the class F of a fibre, and the classes of k components of singular fibres, no two in the same fibre. The next lemma follows easily from the intersection theory on S.

Lemma 5.1. *Let E and E' be two sections with negative self-intersection $-n$. Let r be the number of components of singular fibres which intersect both E and E'. Then $k - r$ is even and*

$$2E \cdot E' = k - 2n - r.$$

In particular,

$$k \geq 2n + r.$$

Since a conic bundle S is isomorphic to a blow-up of a minimal ruled surface, it always contains a section E with negative self-intersection $-n$. If $n \geq 2$, we obviously get $k \geq 4$. If $n = -1$, since (S, g) is minimal, there exists $g \in G$ such that $g(E) \neq E$ and $E \cap g(E) \neq \emptyset$. Applying the previous lemma we get

$$k \geq 4.$$

5.2 Exceptional conic bundles

We give three different constructions of the same conic bundle, which we will call an *exceptional conic bundle.*

First construction.

Choose a ruling $p : \mathbf{F}_0 \to \mathbb{P}^1$ on $\mathbf{F}_0$ and fix two points on the base, say 0 and ∞. Let F_0 and F_∞ be the corresponding fibres. Take $g+1$ points $a_1, \dots, a_{g+1}$ on F_0 and $g+1$ points $a_{g+2}, \dots, a_{2g+2}$ on F_∞ such that no two lie in the same fibre of the second ruling $q : \mathbf{F}_0 \to \mathbb{P}^1$. Let $\sigma : S \to \mathbf{F}_0$ be the blow-up of the points $a_1, \dots, a_{2g+2}$. The composition $\pi = q \circ \sigma : S \to \mathbb{P}^1$ is a conic bundle with $2g+2$ singular fibres $R_i + R_i'$ over the points $x_i = q(a_i), i = 1, \dots, 2g+2$. For $i = 1, \dots, g+1$, $R_i = \sigma^{-1}(a_i)$ and R_{n+i} is the proper transform of the fibre $q^{-1}(a_i)$. Similarly, for $i = 1, \dots, n$, R_i' is the proper transform of the fibre $q^{-1}(a_i)$ and $R'_{g+1+i} = \sigma^{-1}(a_{g+1+i})$.

Let E_0, E_∞ be the proper transforms of F_0, F_∞ on S. Each is a section of the conic bundle π. The section E_0 intersects $R_1, \dots, R_{2g+2}$, and the section E_∞ intersects $R_1', \dots, R_{2g+2}'$.

Let

$$D_0 = 2E_0 + \sum_{i=1}^{2g+2} R_i, \quad D_\infty = 2E_\infty + \sum_{i=1}^{2g+2} R_i'.$$

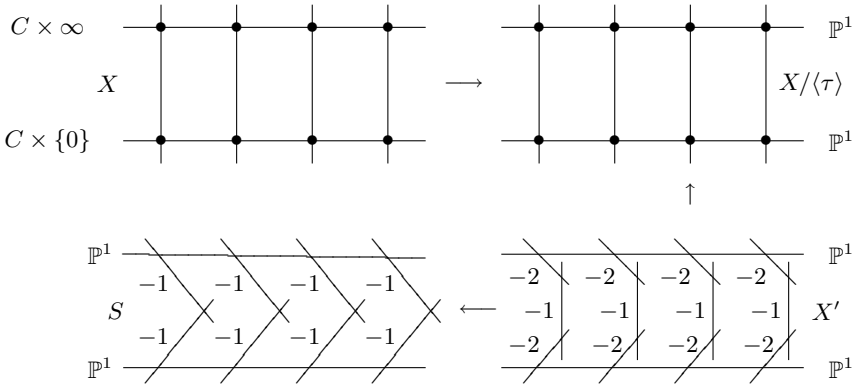
It is easy to check that $D_0 \sim D_\infty$. Consider the pencil $\mathcal{P}$ spanned by the curves D_0 and D_∞. It has $2g+2$ simple base points $p_i = R_i \cap R_i'$. Its general member is a nonsingular curve C. In fact, a standard formula for computing the Euler characteristic of a fibred surface in terms of the Euler characteristics of fibres shows that all members except D_0 and D_∞ are nonsingular curves. Let F be a fibre of the conic bundle. Since $C \cdot F = 2$, the linear system $|F|$ cuts out a g_2^1 on C, so it is a hyperelliptic curve or the genus g of C is 0 or 1. The points p_i are obviously the ramification points of the g_2^1. Computing the genus of C we find that it is equal to g, thus $p_1, \dots, p_{2g+2}$ is the set of ramification points. Obviously all nonsingular members are isomorphic curves. Let $\sigma : S' \to S$ be the blow-up the base points $p_1, \dots, p_{2g+2}$ and let $\overline{D}$ denote the proper transform of a curve on S. We have

$$2\bar{E}_0 + 2\bar{E}_\infty + \sum_{i=1}^{2g+2} (\bar{R}_i + \bar{R}_i' + 2\sigma^{-1}(p_i)) \sim 2\sigma^*(C).$$

This shows that there exists a double over $X' \to S'$ branched along the divisor $\sum_{i=1}^{2g+2}(\bar{R}_i + \bar{R}_i')$. Since $\bar{R}_i^{\,2} = \bar{R}_i'^{\,2} = -2$, the ramification divisor on X' consists of $4g+4$ (-1)-curves. Blowing them down we obtain a surface X isomorphic to the product $C \times \mathbb{P}^1$. This gives us the following.

Second construction. A pair (C, h) consisting of a nonsingular curve and an involution $h \in \mathrm{Aut}(C)$ with quotient $\mathbb{P}^1$ will be called a *hyperelliptic curve*. If C is of genus $g \geq 2$, then C is a hyperelliptic curve and h is its involution defined by the unique g_2^1 on C. Let δ be an involution of $\mathbb{P}^1$ defined by $(t_0, t_1) \mapsto (t_0, -t_1)$. Consider the involution $\tau = h \times \delta$ of the product $X = C \times \mathbb{P}^1$. Its fixed points are $4g+4$ points $c_i \times \{0\}$ and $c_i \times \{\infty\}$, where $X^{\langle h \rangle} = \{c_1, \dots, c_{2g+2}\}$. Let X' be a minimal resolution of $X/(\tau)$. It is easy to

see that the images of the curves $\{c_i\} \times \mathbb{P}^1$ are (-1)-curves on X'. Blowing them down we obtain our exceptional conic bundle.



Third construction.

Let us consider a quasi-smooth hypersurface Y of degree $2g+2$ in weighted projective space $\mathbb{P} = \mathbb{P}(1, 1, g+1, g+1)$ given by an equation

$$F_{2g+2}(T_0, T_1) + T_2T_3 = 0, \tag{24}$$

where $F_{2g+2}(T_0, T_1)$ is a homogeneous polynomial of degree $2g + 2$ without multiple roots. The surface is a double cover of $\mathbb{P}(1, 1, g + 1)$ (the cone over a Veronese curve of degree $g+1$) branched over the curve $F_{2g+2}(T_0, T_1)+T_2^2 = 0$. The pre-images of the singular point of $\mathbb{P}(1, 1, g + 1)$ with coordinate $(0, 0, 1)$ is a pair of singular points of Y with coordinates $(0, 0, 1, 0)$ and $(0, 0, 0, 1)$. The singularities are locally isomorphic to the singular points of a cone of the Veronese surface of degree $g + 1$. Let S be a minimal resolution of Y. The pre-images of the singular points are disjoint smooth rational curves E and E' with self-intersection $-(g + 1)$. The projection $\mathbb{P}(1, 1, g + 1, g + 1) \to \mathbb{P}^1, (t_0, t_1, t_2, t_3) \mapsto (t_0, t_1)$ lifts to a conic bundle on S with sections E, E'. The pencil $\lambda T_2 + \mu T_3 = 0$ cuts out a pencil of curves on Y which lifts to a pencil of bisections of the conic bundle S with $2g + 2$ base points $(t_0, t_1, 0, 0)$, where $F_{2g+2}(t_0, t_1) = 0$.

It is easy to see that this is a general example of an exceptional conic bundle. In Construction 2, we blow down the sections E_0, E_∞ to singular points. Then consider an involution g_0 of the surface which is a descent of the automorphism of the product $C \times \mathbb{P}^1$ given by $\mathrm{id}_C \times \psi$, where $\psi : (t_0, t_1) \mapsto (t_1, t_0)$. The quotient by (g_0) gives $\mathbb{P}(1, 1, g+1)$ and the ramification divisor is the image on S of the curve $C \times (1, 1)$ or $C \times (1, -1)$. On one of these curves g_0 acts identically, on the other one it acts as the involution defined by the g_2^1.

Proposition 5.2. *Let $\phi : S \to \mathbb{P}^1$ be a minimal conic G-bundle with $k \leq 5$ singular fibres. Then S is a Del Pezzo surface, unless $k = 4$ and S is an exceptional conic bundle.*

Proof. Since $k \leq 5$ we have $K_S^2 = 8 - k \geq 3$. By Riemann-Roch, $|-K_S| \neq \emptyset$. Suppose S is not a Del Pezzo surface. Then there exists an irreducible curve C such that $-K_S \cdot C \leq 0$. Suppose, the equality takes place. By Hodge's Index Theorem, $C^2 < 0$, and by the adjunction formula, $C^2 = -2$ and $C \cong \mathbb{P}^1$. If the strict inequality takes place, then C is a component of a divisor $D \in |-K_S|$, hence $|-K_S - C| \neq \emptyset$ and $|K_S + C| = \emptyset$. Moreover, since $K_S^2 > 0$, we have $C \notin |-K_S|$. Applying Riemann-Roch to the divisor $K_S + C$ we easily obtain that C is of arithmetic genus 0, and hence $C \cong \mathbb{P}^1$. By adjunction, $C^2 \leq -2$. In both cases we have a smooth rational curve with $C^2 \leq -2$.

If $k = 4$ and S is an exceptional conic bundle, then S is not a Del Pezzo surface since it has sections with self-intersection -2. Assume this is not the case. Let C be the union of smooth rational curves with self-intersection < -2. It is obviously a G-invariant curve, so we can write $C \sim -aK_S - bf$, where f is the divisor class of a fibre of ϕ. Intersecting with f we get $a > 0$. Intersecting with K_S, we get $2b > ad$, where $d = 8 - k \geq 3$. It follows from Lemma 5.1, that S contains a section E with self-intersection -2 or -1. Intersecting C with E we get $0 \leq C \cdot E = a(-K_S \cdot E) - b \leq a - b$. This contradicts the previous inequality. Now let us take C to be the union of (-2)-curves. Similarly, we get $2b = ad$ and $C^2 = -aK_S \cdot C - bC \cdot f = -bC \cdot f = -2ab$. Let r be the number of irreducible components of C. We have $2a = C \cdot f \geq r$ and $-2r \leq C^2 = -2ab \leq -br$. If $b = 2$, we have the equality everywhere, hence C consists of $r = 2a$ disjoint sections, and $8 = rd$. Since $d \geq 3$, the only solution is $d = 4, r = 2$, and this leads to the exceptional conic bundle. Assume $b = 1$. Since $C^2 = -2a$ is even, a is a positive integer, and we get $2 = ad$. Since $d \geq 3$, this is impossible.

5.3 Automorphisms of an exceptional conic bundle

Let us describe the automorphism group of an exceptional conic bundle. The easiest way to do it to use Construction 3. We denote by Y_g an exceptional conic bundle given by equation (24). Since we are interested only in minimal groups we assume that $g \geq 1$.

Since $K_{Y_g} = \mathcal{O}_{\mathbb{P}}(-2)$, any automorphism σ of Y_g is a restriction of an automorphism of $\mathbb{P}$. Let G_1 be the subgroup of $\mathrm{SL}(2)$ of transformations preserving the zero divisor of $F_{2g+2}(T_0, T_1)$ and $\chi_1 : G_1 \to \mathbb{C}^*$ be the multiplicative character of G_1 defined by $\sigma_1^*(F_{2g+2}) = \chi_1(\sigma_1)F_{2g+2}$. Similarly, let G_2 be the subgroup of $\mathrm{GL}(2)$ of matrices preserving the zeroes of T_2T_3 and let $\chi_2 : G_2 \to \mathbb{C}^*$ be the character defined by $\sigma^*(T_2T_3) = \chi_2(\sigma_2)T_2T_3$. Let

$$(G_1 \times G_2)^0 = \{(\sigma_1, \sigma_2) \in G_1 \times G_2 : \chi_1(\sigma_1) = \chi_2(\sigma_2)\}.$$

In the notation of the diagonal products,

$$(G_1 \times G_2)^0 = \frac{1}{m}[G_1 \times G_2],$$

where $\chi_1(G_1) = \mu_m \subset \mathbb{C}^*$. The subgroup

$$K = \langle (-I_2, (-1)^{g+1} I_2) \rangle$$

acts identically on Y_g and the quotient group is isomorphic to $\mathrm{Aut}(Y_g)$.

Let $\mathrm{Aut}(Y_g) \cong (G_1 \times G_2)^0/K \to \mathrm{PGL}(2)$ be the homomorphism induced by the projection of G_1 to $\mathrm{PGL}(2)$. Its image is a finite subgroup P of $\mathrm{PGL}(2)$. Its kernel H consists of cosets modulo K of pairs $(\pm I_2, \sigma_2)$, where $\chi_2(\sigma_2) = 1$. Clearly, $H \cong \mathrm{Ker}(\chi_2)$.

It is easy to see that $\mathrm{Ker}(\chi_2) \cong \mathbb{C}^* : 2$ is generated by diagonal matrices with determinant 1 and the matrix which switches the coordinates. Inside of $\mathrm{GL}(2)$ it is conjugate to the normalizer N of the maximal torus in $\mathrm{SL}(2)$. So we obtain an isomorphism

$$\mathrm{Aut}(Y_g) \cong N \bullet P. \tag{25}$$

Suppose there exists a homomorphism $\eta : G_1 \to \mathbb{C}^*$ such that $\eta(-1) = (-1)^{g+1}$. Then the homomorphism

$$G_1 \to (G_1 \times G_2)^0/K, \ \sigma_1 \mapsto (\sigma_1, \eta(\sigma_1) I_2) \mod K$$

factors through a homomorphism $P \to (G_1 \times G_2)^0/K$ and defines a splitting of the extension (25). Since elements of the form $(\sigma_1, \eta(\sigma_1) I_2)$ commute with elements of N, we see that the extension is trivial when it splits. It is easy to see that the converse is also true. Since the trivial η works when g is odd, we obtain that the extension always splits in this case. Assume g is even and G_1 admits a 1-dimensional representation η with $\eta(-I_2) = -1$. If its kernel is trivial, G_1 is isomorphic to a subgroup of $\mathbb{C}^*$, hence cyclic. Otherwise, the kernel is a subgroup of $\mathrm{SL}(2)$ not containing the center. It must be a cyclic subgroup of odd order. The image is a cyclic group. Thus G_1 is either cyclic, or a binary dihedral group D_{2n} with n odd.

To summarize we have proved the following.

Proposition 5.3. *The group of automorphisms of an exceptional conic bundle* (24) *is isomorphic to an extension $N \bullet P$, where P is the subgroup of* $\mathrm{PGL}(2)$ *leaving the set of zeroes of $F_{2g+2}(T_0, T_1)$ invariant and $N \cong \mathbb{C}^* : 2$ is a group of matrices with determinant ± 1 leaving $T_2 T_3$ invariant. Moreover, the extension splits and defines an isomorphism*

$$\mathrm{Aut}(Y_g) \cong N \times P$$

if and only if g is odd, or g is even and P is either a cyclic group or a dihedral group D_{4k+2}.

Now let G be a finite minimal subgroup of $\mathrm{Aut}(Y_g)$. Assume first that $\mathrm{Aut}(Y_g) \cong N \times P$. Let N' be the projection of G to N and P' be the projection to P. Since G is minimal, N' contains an element which switches $V(T_2)$ and $V(T_3)$. Thus N' is isomorphic to a dihedral group D_{2n}. Applying Goursat's Lemma we obtain that

$$G \cong N' \triangle_D Q,$$

where D is a common quotient of N' and Q. If N' is a dihedral group, then D is either dihedral group or a cyclic of order 2. Using Goursat's Lemma it is easy to list all possible subgroups. We leave it as an exercise to the reader.

If $\mathrm{Aut}(Y_g)$ is not isomorphic to the direct product $N \times P$, we can only say that

$$G \cong H \bullet Q,$$

where H is a subgroup of D_{2n} or Q_{4n}, and Q is a polyhedral group. Note that we can write these extensions in the form $n\bullet(2\bullet Q)$ or $n\bullet(2^2\bullet Q)$.

Example 5.4. Let $\pi : S \to \mathbb{P}^1$ be an exceptional conic bundle with $g = 1$. It has 4 singular fibres. According to the first construction, the blow up S' of S at the four singular points of the singular fibres admits an elliptic fibration $f : S' \to \mathbb{P}^1$ with two singular fibres of type I_0^* in Kodaira's notation. The j-invariant of the fibration is zero, and after the degree 2 base change $\mathbb{P}^1 \to \mathbb{P}^1$ ramified at 2 points, the surface becomes isomorphic to the product $E \times \mathbb{P}^1$, where E is an elliptic curve. Its j-invariant corresponds to the cross ratio of the 4 points, where a section of π with self-intersection -2 intersects the singular fibres. Conversely, starting from the product, we can divide it by an elliptic involution, to get the conic bundle. This is our second construction.

According to the third construction, the surface can be given by an equation

$$F_4(T_0, T_1) + T_2T_3 = 0$$

in the weighted projective space $\mathbb{P}(1, 1, 2, 2)$. The projection to (T_0, T_1) is a rational map undefined at the four points $P_i = (a, b, 0, 0)$, where $F_4(a, b) = 0$. After we blow them up we get the conic bundle. The projection to $(T_2, T_3$ is a rational map undefined at the two singular points $(0, 0, 1, 0)$ and $(0, 0, 0, 1)$. After we blow them up, we get the elliptic fibration. We have two obvious commuting involutions $\sigma_1 = [t_0, t_1, -t_3, -t_2]$ and $\sigma_2 = [t_0, t_1, t_3, t_2]$. The locus of fixed points of each of them is an elliptic curve with equation $T_2 = 0$ or $T_3 = 0$. The group $\langle\sigma_1, \sigma_2\rangle \cong 2^2$ permutes these two curves.

The groups of automorphisms of S is easy to describe. It follows from Proposition 5.3 that G is a finite subgroup of $P \times K$, where P is a subgroup of $\mathrm{PGL}(2)$ leaving the zeros of F_4 invariant and K is a subgroup of $\mathrm{GL}(2)$ leaving the zeroes of t_2t_3 invariant. First we choose coordinates T_0, T_1 to write F_4 in the form $T_0^4 + T_1^4 + aT_0^2T_1^2, a^2 \neq 4$. It is always possible if F_4 has 4 distinct roots (true in our case). Let P be the subgroup of $\mathrm{PGL}(2)$ leaving the set of zeroes of F_4 invariant. It is one of the following groups $1, 2, 4, 2^2, D_8, A_4$. If

$a^2 \neq 0, -12, 36$, then P is a subgroup of 2^2. If $a^2 = 0, 36$, then P is a subgroup of D_8. If $a^2 = -12$, then P is a subgroup of A_4.

Suppose a is not exceptional. Let $\bar{P}$ be the corresponding binary group. Then it leaves F_4 invariant, so G is a subgroup of $K' \times P$, where K' consists of matrices leaving t_2t_3 invariant. It is generated by diagonal matrices with determinant 1 and the transformations $[t_1, t_0]$. Thus

$$G \subset D_{2n} \times 2^2,$$

where we use that a a finite subgroup of K' is either cyclic or binary dihedral.

Suppose $a^2 = 0, 36$ and G contains an element of order 4. The form can be transformed to the form $T_0^4 + T_1^4$. The value of the character at an element τ of order 4 is equal to -1. We obtain

$$G \subset \tfrac{1}{2}[Q \times P] \cong (Q' \times P')^{\bullet}2 \cong Q'_{\bullet}P$$

where Q is a finite subgroup of K and the diagonal product is taken with respect to the subgroup $Q' = Q \cap K'$ of Q and the subgroup $P \cap 2^2$ of index 2 of P. The group Q' is cyclic or dihedral.

Suppose $a = -2\sqrt{3}i$ and G contains an element g of order 3 given by $[\frac{1-i}{2}t_0 + \frac{1-i}{2}t_1, -\frac{1+i}{2}t_0 - \frac{1+i}{2}t_1]$. Its character is defined by $\chi(g) = \epsilon_3$. We obtain

$$G \subset \frac{1}{3}[Q \times P],$$

where the diagonal product is taken with respect to the subgroup $Q = \chi_2^{-1}(\mu_3)$ of K and the subgroup $P' = P \cap 2^2$ of index 3 of P. Again G is a subgroup of $D_{2n}{\bullet}P$.

5.4 Minimal conic bundles G-surfaces

Now assume (S, G) is a minimal G-surface such that S admits a conic bundle map $\phi : S \to \mathbb{P}^1$. As we had noticed before the number of singular fibres k is greater or equal to 4. Thus

$$K_S^2 = 8 - k \leq 4. \tag{26}$$

Let (S, G) be a rational G-surface and

$$a : G \to \mathrm{O}(\mathrm{Pic}(X)), \quad g \mapsto (g^*)^{-1} \tag{27}$$

be the natural representation of G in the orthogonal group of the Picard group. We denote by G_0 the kernel of this representation. Since $k > 2$ and G_0 fixes any component of a singular fibre, it acts identically on the base of the conic bundle fibration. Since G_0 fixes the divisor class of a section, and sections with negative self-intersection do not move in a linear system, we see that G_0 fixes pointwisely any section with negative self-intersection. If we consider a section as a point of degree 1 on the generic fibre, we see that G_0 must be is a cyclic group.

Proposition 5.5. *Assume $G_0 \neq \{1\}$. Then S is an exceptional conic bundle.*

Proof. Let g_0 be a non-trivial element from G_0. Let E be a section with $E^2 = -n < 0$. Take an element $g \in G$ such that $E' = g(E) \neq E$. Since g_0 has two fixed points on each component of a singular fibre we obtain that E and E' do not intersect the same component. By Lemma 5.1, we obtain that $k = 2n$. Now we blow down n components in n fibres intersecting E and n components in the remaining n fibres intersecting E' to get a minimal ruled surface with two disjoint sections with self-intersection 0. It must be isomorphic to $\mathbf{F}_0$. So, we see that S is an exceptional conic bundle (Construction 1) with $n = g+1$.

From now on in this section, we assume that $G_0 = \{1\}$.

Let S_η be the general fibre of ϕ. By Tsen's theorem it is isomorphic to $\mathbb{P}^1_K$, where $K = \mathbb{C}(t)$ is the field of rational functions of the base. Consider S_η as a scheme over $\mathbb{C}$. Then

$$\mathrm{Aut}_{\mathbb{C}}(S_\eta) \cong \mathrm{Aut}_K(S_\eta) : \mathrm{PGL}(2) \cong \mathrm{dJ}(2),$$

where $\mathrm{dJ}(2)$ is a de Jonquiéres subgroup of $\mathrm{Cr}(2)$ and $\mathrm{Aut}_K(S_\eta) \cong \mathrm{PGL}(2, K)$. A finite minimal group G of automorphisms of a conic bundle is isomorphic to a subgroup of $\mathrm{Aut}_{\mathbb{C}}(S_\eta)$. Let $G_K = G \cap \mathrm{Aut}_K(S_\eta)$ and $G_B \cong G/G_K$ be the image of G in $\mathrm{PGL}(2)$. We have an extension of groups

$$1 \to G_K \to G \to G_B \to 1 \tag{28}$$

Let $\mathcal{R}$ be the subgroup of $\mathrm{Pic}(S)$ spanned by the divisor classes of $R_i - R'_i, i = 1, \ldots, k$. It is obviously G-invariant and $\mathcal{R}_{\mathbb{Q}}$ is equal to the orthogonal complement of $\mathrm{Pic}(S)^G_{\mathbb{Q}}$ in $\mathrm{Pic}(S)_{\mathbb{Q}}$. The orthogonal group of the quadratic lattice $\mathcal{R}$ is isomorphic to the wreath product $2 \wr S_k$. The normal subgroup 2^k consists of transformations which switch some of the R_i's with R'_i. A subgroup isomorphic to S_k permutes the classes $R_i - R'_i$.

Lemma 5.6. *Let G be a minimal group of automorphisms of S. There exists an element $g \in G_K$ of order 2 which switches the components of some singular fibre.*

Proof. Since G is minimal, the G-orbit of any R_i cannot consist of disjoint components of fibres. Thus it contains a pair R_j, R'_j and hence there exists an element $g \in G$ such that $g(R_j) = R'_j$. If g is of odd order $2k+1$, then g^{2k} and g^{2k+1} fix R_j, hence g fixes R_j. This contradiction shows that g is of even order $2m$. Replacing g by an odd power, we may assume that g is of order $m = 2^a$.

Assume $a = 1$. Obviously the singular point $p = R_j \cap R'_j$ of the fibre belongs to the fixed locus S^g of g. Suppose p is an isolated fixed point. Then we can choose local coordinates at p such that g acts by $(z_1, z_2) \mapsto (-z_1, -z_2)$, and hence acts identically on the tangent directions. So it cannot switch the

components. Thus S^g contains a curve not contained in fibres which passes through p. This implies that $g \in G_K$.

Suppose $a > 1$. Replacing g by $g' = g^{m/2}$ we get an automorphism of order 2 which fixes the point x_j and the components R_j, R'_j. Suppose $S^{g'}$ contains one of the components, say R_j. Take a general point $y \in R_j$. We have $g'(g(y)) = g(g'(y)) = g(y)$. This shows that g' fixes R'_j pointwisely. Since $S^{g'}$ is smooth, this is impossible. Thus g' has 3 fixed points y, y', p on F_j, two on each component. Suppose y is an isolated fixed point lying on R_j. Let $\pi : S \to S'$ be the blowing down of R_j. The element g' descends to an automorphism of order 2 of S' which has an isolated fixed point at $q = \pi(R_j)$. Then it acts identically on the tangent directions at q, hence on R_j. This contradiction shows that $S^{g'}$ contains a curve intersecting F_j at y or at p, and hence $g' \in G_K$. Since g' is an even power, it cannot switch any components of fibres. Hence g' acts identically on $\mathcal{R}$ and hence on all $\mathrm{Pic}(S)$. Thus $g' = g^{m/2} = 1$ (recall that we assume that G_0 is trivial), contradicting the definition of order of g.

The restriction of the homomorphism $G \to \mathrm{O}(\mathcal{R}) \cong 2^k : S_k$ to G_K defines a surjective homomorphism

$$\rho : G_K \to 2^s, \ s \leq k.$$

An element from $\mathrm{Ker}(\rho)$ acts identically on $\mathcal{R}$ and hence on $\mathrm{Pic}(S)$. By Lemma 5.6, G_K is not trivial and $s > 0$. A finite subgroup of $\mathrm{PGL}(2, K)$ does not admit a surjective homomorphism to 2^s for $s > 2$. Thus $s = 1$ or 2.

Case 1: $s = 1$.

Let Σ' be the non-empty subset of Σ such that G_K switches the components of fibres over Σ'. Since G_K is a normal subgroup of G, the set Σ' is a G-invariant set. If $\Sigma \neq \Sigma'$, we repeat the proof of Lemma 5.6 starting from some component R_i of a fibre over a point $x \notin \Sigma'$, and find an element in G_K of even order which switches the components of maybe another fibre F_x, where $x \notin \Sigma'$. Since $G_K = 2$, we get a contradiction.

Let $G_K = \langle h \rangle$. The element h fixes two points on each nonsingular fibre. The closure of these points is a one-dimensional component C of S^h. It is a smooth bi-section of the fibration. Since we know that h switches all components, its trace on the subgroup $\mathcal{R}$ generated by the divisor classes $R_i - R'_i$ is equal to $-k$. Thus its trace on $H^2(S, \mathbb{Q})$ is equal to $2 - k$. Applying the Lefschetz fixed-point-formula, we get $e(S^h) = 4 - k$. If C is the disjoint union of two components, then S^h consists of k isolated fixed points (the singular points of fibres) and C. We get $e(S^h) = 4 + k$. This contradiction shows that C is irreducible and $e(C) = 4 - k$. Since h fixes pointwisely C and switches the components R_i and R'_i, the intersection point $R_i \cap R'_i$ must be on C. Thus the projection $C \to \mathbb{P}^1$ has $\geq k$ ramification points. Hence $4 - k = e(C) = 4 - (2 + 2g(C)) \leq 4 - k$. This shows that $k = 2g(C) + 2$, i.e. the singular points of fibres are the ramification points of the g^1_2.

Case 2: $s = 2$.

Let g_1, g_2 be two elements from G_K which are mapped to generators of the image of G_K in 2^k. Let C_1 and C_2 be one-dimensional components of the sets S^{g_1} and S^{g_2}. As in the previous case we show that C_1 and C_2 are smooth hyperelliptic curves of genera $g(C_1)$ and $g(C_2)$. Let Σ_1 and Σ_2 be the sets of branch points of the corresponding double covers. As in the previous case we show that $\Sigma = \Sigma_1 \cup \Sigma_2$. For any point $x \in \Sigma_1 \cap \Sigma_2$ the transformation $g_3 = g_1 g_2$ fixes the components of the fibre F_x. For any point $x \in \Sigma_1 \setminus \Sigma_2$, g_3 switches the components of F_x. Let C_3 be the one-dimensional component of S^{g_3} and Σ_3 be the set of branch points of g_3. We see that $\Sigma_i = \Sigma_j + \Sigma_k$ for distinct i, j, k, where $\Sigma_j + \Sigma_k = (\Sigma_j \cup \Sigma_k) \setminus (\Sigma_j \cap \Sigma_k)$. This implies that there exist three binary forms $p_1(t_0, t_1), p_2(t_0, t_1), p_3(t_0, t_1)$, no two of which have a common root, such that $\Sigma_1 = V(p_2 p_3), \Sigma_2 = V(p_1 p_3), \Sigma_3 = V(p_1 p_2)$. Setting $d_i = \deg p_i$, we get

$$2g(C_i) + 2 = d_j + d_k.$$

Let us summarize what we have learnt.

Theorem 5.7. *Let G be a minimal finite group of automorphisms of a conic bundle $f : S \to \mathbb{P}^1$ with a set Σ of singular fibres. Assume $G_0 = \{1\}$. Then $k = \#\Sigma > 2$ and one of the following cases occurs.*

(1) $G = 2P$, where the central involution h fixes pointwisely an irreducible smooth bisection C of π and switches the components in all fibres. The curve C is a curve of genus $g = (k-2)/2$. The conic bundle projection defines a g^1_2 on C with ramification points equal to singular points of fibres. The group P is isomorphic to the group of automorphisms of C modulo the involution defined by the g^1_2.

(2) $G \cong 2^2 \bullet P$, each nontrivial element g_i of the subgroup 2^2 fixes pointwisely an irreducible smooth bisection C_i. The set Σ is partitioned in 3 subsets $\Sigma_1, \Sigma_2, \Sigma_3$ such that the projection $f : C_i \to \mathbb{P}^1$ ramifies over $\Sigma_j + \Sigma_k, i \neq j \neq k$. The group P is subgroup of $\mathrm{Aut}(\mathbb{P}^1)$ leaving the set Σ and its partition into 3 subsets Σ_i invariant.

It follows from Lemma 4.4 that in Case 1 the non-split extension is isomorphic to a binary polyhedral group, unless $G = O$ or D_{2n}, where n is even.

Remark 5.8. It follows from the previous description that any abelian group G of automorphisms of a conic bundle must be a subgroup of some extension $Q \bullet P$, where Q is a dihedral, binary dihedral or cyclic group, and P is a polyhedral group. This implies that G is either a cyclic group, or a group $2 \times m$, or $2^2 \times m$, or 2^4. All these groups occur (see Example 5.12 and [6]).

5.5 Automorphisms of hyperelliptic curves

We consider a curve of genus g equipped with a linear series g^1_2 as a curve C of degree $2g+2$ in $\mathbb{P}(1,1,g+1)$ given by an equation

$$T_2^2 + F_{2g+2}(T_0, T_1) = 0.$$

An automorphism σ of C is defined by a transformation

$$(t_0, t_1, t_2) \mapsto (at_1 + bt_0, ct_1 + dt_0, \alpha t_2),$$

where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2)$ and $F(aT_0+bT_1, cT_0+dT_1) = \alpha^2 F(T_0, T_1)$. So to find the group of automorphisms of C we need to know relative invariants $\Phi(T_0, T_1)$ for finite subgroups $\bar{P}$ of $\mathrm{SL}(2, \mathbb{C})$ (see [48]). The set of relative invariants is a finitely generated $\mathbb{C}$-algebra. Its generators are called Gründformen. We will list the Gründformen (see [48]). We will use them later for the description of automorphism groups of Del Pezzo surfaces of degree 1.

- *$\bar{P}$ is a cyclic group of order n.*

A generator is given by the matrix

$$g = \begin{pmatrix} \epsilon_n & 0 \\ 0 & \epsilon_n^{-1} \end{pmatrix}.$$

The Gründformen are t_0 and t_1 with characters determined by

$$\chi_1(g) = \epsilon_n, \quad \chi_2(g) = \epsilon_n^{-1}.$$

- *$\bar{P} \cong Q_{4n}$ is a binary dihedral group of order $4n$.*

  Its generators are given by the matrices

$$g_1 = \begin{pmatrix} \epsilon_{2n} & 0 \\ 0 & \epsilon_{2n}^{-1} \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

The Gründformen are

$$\Phi_1 = t_0^n + t_1^n, \quad \Phi_2 = t_0^n - t_1^n, \quad \Phi_3 = t_0 t_1. \tag{29}$$

The generators g_1 and g_2 act on the Gründformen with characters

$$\chi_1(g_1) = \chi_2(g_1) = -1, \chi_1(g_2) = \chi_2(g_2) = i^n,$$

$$\chi_3(g_1) = 1, \chi_3(g_2) = -1.$$

- *$\bar{P}$ is a binary tetrahedral group of order 24.*

Its generators are given by the matrices

$$g_1 = \begin{pmatrix} \epsilon_4 & 0 \\ 0 & \epsilon_4^{-1} \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad g_3 = \frac{1}{i-1}\begin{pmatrix} i & i \\ 1 & -1 \end{pmatrix}.$$

The Gründformen are

$$\Phi_1 = t_0 t_1 (t_0^4 - t_1^4),\ \Phi_2,\ \Phi_3 = t_0^4 \pm 2\sqrt{-3} t_0^2 t_1^2 + t_1^4.$$

The generators g_1, g_2, g_3 act on the Gründformen with characters

$$\chi_1(g_1) = \chi_1(g_2) = \chi_1(g_3) = 1,$$
$$\chi_2(g_1) = \chi_2(g_2) = 1,\ \chi_2(g_3) = \epsilon_3,$$
$$\chi_3(g_1) = \chi_3(g_2) = 1,\ \chi_3(g_3) = \epsilon_3^2.$$

- *$\bar{P}$ is a binary octahedral group of order 48.*

  Its generators are

$$g_1 = \begin{pmatrix} \epsilon_8 & 0 \\ 0 & \epsilon_8^{-1} \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad g_3 = \frac{1}{i-1}\begin{pmatrix} i & i \\ 1 & -1 \end{pmatrix}.$$

The Gründformen are

$$\Phi_1 = t_0 t_1 (t_0^4 - t_1^4),\ \Phi_2 = t_0^8 + 14 t_0^4 t_1^4 + t_1^8,\ \Phi_3 = (t_0^4 + t_1^4)((t_0^4 + t_1^4)^2 - 36 t_0^4 t_1^4).$$

The generators g_1, g_2, g_3 act on the Gründformen with characters

$$\chi_1(g_1) = -1, \chi_1(g_2) = \chi_1(g_3) = 1,$$
$$\chi_2(g_1) = \chi_2(g_2) = \chi_2(g_3) = 1,$$
$$\chi_3(g_1) = -1, \quad \chi_3(g_2) = \chi(g_3) = 1.$$

- *$\bar{P}$ is a binary icosahedral group of order 120.*

  Its generators are

$$g_1 = \begin{pmatrix} \epsilon_{10} & 0 \\ 0 & \epsilon_{10}^{-1} \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad g_3 = \frac{1}{\sqrt{5}}\begin{pmatrix} \epsilon_5 - \epsilon_5^4 & \epsilon_5^2 - \epsilon_5^3 \\ \epsilon_5^2 - \epsilon_5^3 & -\epsilon_5 + \epsilon_5^4 \end{pmatrix}.$$

The Gründformen are

$$\Phi_1 = t_0^{30} + t_1^{30} + 522(t_0^{25} t_1^5 - t_0^5 t_1^{25}) - 10005(t_0^{20} t_1^{10} + t_0^{10} t_1^{20}),$$
$$\Phi_2 = -(t_0^{20} + t_1^{20}) + 228(t_0^{15} t_1^5 - t_0^5 t_1^{15}) - 494 t_0^{10} t_1^{10},$$
$$\Phi_3 = t_0 t_1 (t_0^{10} + 11 t_0^5 t_1^5 - t_1^{10}).$$

Since $P/(\pm 1) \cong A_5$ is a simple group and all Gründformen are of even degree, we easily see that the characters are trivial.

5.6 Commuting de Jonquiéres involutions

Recall that a de Jonquiéres involution IH_{g+2} is regularized by an automorphism of the surface S which is obtained from $\mathbf{F}_1$ by blowing up $2g+2$ points. Their images on $\mathbb{P}^2$ are the $2g+2$ fixed points of the involution of H_{g+2}. Let $\pi : S \to X = S/IH_{g+2}$. Since the fixed locus of the involution is a smooth hyperelliptic curve of genus g, the quotient surface X is a nonsingular surface. Since the components of singular fibres of the conic bundle on S are switched by IH_{g+2}, their images on X are isomorphic to $\mathbb{P}^1$. Thus X is a minimal ruled surface $\mathbf{F}_e$. What is e?

Let $\bar{C} = \pi(C)$ and $\bar{E} = \pi(E)$, where E is the exceptional section on S. The curve $\bar{E}$ is a section on X whose preimage in the cover splits. It is either tangent to $\bar{C}$ at any of its of g intersection points (since $IH_{g+2}(E) \cdot E = g$) or is disjoint from $\bar{C}$ if $g = 0$. Let s be the divisor class of a section on $\mathbf{F}_e$ with self-intersection $-e$ and f be the class of a fibre. It is easy to see that

$$\bar{C} = (g+1+e)f + 2s, \quad \bar{E} = \frac{g+e-1}{2}f + s.$$

Let $\bar{R}$ be a section with the divisor class s. Suppose $\bar{R} = \bar{E}$, then $\bar{R} \cdot \bar{C} = g+1-e = 2g$ implies $g = 1-e$, so $(g,e) = (1,0)$ or $(0,1)$. In the first case, we get an elliptic curve on $\mathbf{F}_0$ with divisor class $2f+2s$ and S is non-exceptional conic bundle with $k = 4$. In the second case S is the conic bundle (non-minimal) with $k = 2$.

Assume that $(g,e) \neq (1,0)$. Let $R = \pi^{-1}(\bar{R})$ be the pre-image of $\bar{R}$. We have $R^2 = -2e$. If it splits into two sections $R_1 + R_2$, then $R_1 \cdot R_2 = \bar{C} \cdot \bar{R} = g+1-e$, hence $-2e = 2(g+1-e)+2R_1^2$ gives $R_1^2 = -g-1$. Applying Lemma (5.1), we get $R_1 \cdot R_2 = g+1-e = g-1+(2g+2-a)/2 = -a/2$, where $a \geq 0$. This gives $e = g+1$, but intersecting $\bar{E}$ with $\bar{R}$ we get $e \leq g-1$. This contradiction shows that $\bar{R}$ does not split, and hence R is an irreducible bisection of the conic bundle with $R^2 = -2e$. We have $R \cdot E = (g-e-1)/2, R \cdot R_i = R \cdot R_i' = 1$, where $R_i + R_i'$ are reducible fibres of the conic fibration.

This shows that the image of R in the plane is a hyperelliptic curve $H'_{g'+2}$ of degree $d = (g-e+3)/2$ and genus $g' = d-2 = (g-e-1)/2$ with the point q of multiplicity g'. It also passes through the points $p_1, \ldots, p_{2g+2}$. Its Weierstrass points $p_1', \ldots, p_{2g'+2}'$ lie on H_{g+2}. Here we use the notation from 2.3. Also the curve $H'_{g'+2}$ is invariant with respect to the de Jonquiéres involution.

Write the equation of $H'_{g'+2}$ in the form

$$A_{g'}(T_0,T_1)T_2^2 + 2A_{g'+1}(T_0,T_1)T_2 + A_{g'+2}(T_0,T_1) = 0. \tag{30}$$

It follows from the geometric definition of the de Jonquiéres involution that we have the following relation between the equations of $H'_{g'+2}$ and H_{g+2} (cf. [17], p.126)

$$F_g A_{g'+2} - 2F_{g+1}A_{g'+1} + F_{g+2}A_{g'} = 0. \tag{31}$$

Consider this as a system of linear equations with coefficients of $A_{g'+2}, A_{g'+1}, A_{g'}$ considered as the unknowns. The number of the unknowns is equal to $(3g-3e+9)/2$. The number of the equations is $(3g-e+5)/2$. So, for a general H_{g+2} we can solve these equations only if $g=2k+1, e=0, d=k+2$ or $g=2k, e=1, d=k+1$. Moreover, in the first case we get a pencil of curves R satisfying these properties, and in the second case we have a unique such curve (as expected). Also the first case covers our exceptional case $(g,e)=(1,0)$.

For example, if we take $g=2$ we obtain that the six Weierstrass points $p_1,\dots,p_6$ of H_{g+2} must be on a conic. Or, if $g=3$, the eight Weierstrass points together with the point q must be the base points of a pencil of cubics. All these properties are of course not expected for a general set of 6 or 8 points in the plane.

To sum up, we have proved the following.

Theorem 5.9. *Let H_{g+2} be a hyperelliptic curve of degree $g+2$ and genus g defining a de Jonquiéres involution IH_{g+2}. View this involution as an automorphism τ of order 2 of the surface S obtained by blowing up the singular point q of H_{g+2} and its $2g+2$ Weierstrass points $p_1,\dots,p_{2g+2}$. Then*

(i) the quotient surface $X=S/(\tau)$ is isomorphic to $\mathbf{F}_e$ and the ramification curve is $C=S^\tau$;

(ii) if H_{g+2} is a general hyperelliptic curve then $e=0$ if g is odd and $e=1$ if g is even;

(iii) the branch curve $\bar{C}$ of the double cover $S\to\mathbf{F}_e$ is a curve from the divisor class $(g+1+e)f+2s$;

(iv) there exists a section from the divisor class $\frac{g+e-1}{2}f+s$ which is tangent to $\bar{C}$ at each g intersection points unless $g=0, e=1$ in which case it is disjoint from $\bar{C}$;

(v) the reducible fibres of the conic bundle on S are the pre-images of the $2g+2$ fibres from the pencil $|f|$ which are tangent to $\bar{C}$;

(vi) the pre-image of a section from the divisor class s either splits if $(g,e)=(1,0)$ or a curve of genus $g=0$, or a hyperelliptic curve C' of genus $g'=(g-e-1)/2\geq 1$ which is invariant with respect to τ. It intersects the hyperelliptic curve C at its $2g'+2$ Weierstrass points.

(vii) the curve C' is uniquely defined if $e>0$ and varies in a pencil if $e=0$.

Let $IH'_{g'+2}$ be the de Jonquiéres involution defined by the curve $H'_{g'+2}$ from equation (30). Then it can be given in affine coordinates by formulas (7), where F_i is replaced with A_i. Thus we have two involutions defined by the formulas

$$IH_{g+2}:\ (x',y')=\big(x,\frac{-yP_{g+1}(x)-P_{g+2}(x)}{P_g(x)y+P_{g+1}(x)}\big), \tag{32}$$
$$IH'_{g'+2}:(x',y')=\big(x,\frac{-yQ_{g'+1}(x)-Q_{g'+2}(x)}{Q_g(x)y+Q_{g'+1}(x)}\big),$$

where P_i are the dehomogenizations of the F_i's and Q_i are the dehomogenizations of the A_i's. Composing them in both ways we see that the relation (31) is satisfied if and only if the two involutions commute. Thus a de Jonquiéres involution can be always included in a group of de Jonquiéres transformations isomorphic to 2^2. In fact, for a general IH_{g+2} there exists a unique such group if g is even and there is a ∞^1 such groups when g is odd. It is easy to check that the involution $IH_{g+2} \circ H'_{g'+2}$ is the de Jonquiéres involution defined by the third hyperelliptic curve with equation

$$\det \begin{pmatrix} F_g & F_{g+1} & F_{g+2} \\ A_{g'} & A_{g'+1} & A_{g''+2} \\ 1 & -T_2 & T_2^2 \end{pmatrix} = B_{g''}T_2^2 + 2B_{g''-1}T_2 + B_{g''+2} = 0, \qquad (33)$$

(cf. [17], p.126).

If we blow up the Weierstrass point of the curve C' (the proper transform of $H'_{g'+2}$ in S), then we get a conic bundle surface S' from case (2) of Theorem 5.7.

5.7 A question on extensions

It still remains to decide which extensions

$$1 \to G_K \to G \to G_B \to 1 \qquad (34)$$

describe minimal groups of automorphisms of conic bundles. We do not have the full answer and only make a few remarks and examples. Lemma 4.4 helps to decide on splitting in the case when K is abelian and central.

Example 5.10. Consider a de Jonquiéres transformation

$$\mathrm{dj}_P : (x, y) \mapsto (x, P(x)/y),$$

where $P(T_1/T_0) = T_0^{-2g}F_{2g+2}(T_0, T_1)$ is a dehomogenization of ahomogeneous polynomial $F_{2g+2}(T_0, T_1)$ of degree $2g + 2$ defining a hyperelliptic curve of genus g. Choose F_{2g+2} to be a relative invariant of a binary polyhedral group $\bar{P}$ with character $\chi : \bar{P} \to \mathbb{C}^*$. We assume that $\chi = \alpha^2$ for some character $\alpha : \bar{P} \to \mathbb{C}^*$. For any $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \bar{P}$ define the transformation

$$g : (x, y) \mapsto (\frac{ax + b}{cx + d}, \alpha(g)(cx + d)^{-g-1}y).$$

We have

$$P(\frac{ax + b}{cx + d}) = \alpha^2(g)(cx + d)^{-2g-2}P(x).$$

It is immediate to check that g and dj_p commute. The matrix $-I_2$ defines the transformation $g_0 : (x, y) \mapsto (x, \alpha(-I_2)(-1)^{g+1}y)$. So, if

$$\alpha(-I_2) = (-1)^{g+1},$$

the action of $\bar{P}$ factors through P and together with dj_P generate the group $2 \times P$. On the other hand, if $\alpha(-I_2) = (-1)^g$, we get the group $G = 2 \times \bar{P}$. In this case the group G is regularized on an exceptional conic bundle with $G_0 \cong 2$. The generator corresponds to the transformation g_0.

Our first general observation is that the extension $G = 2P$ always splits if g is even, and of course, if P is a cyclic group of odd order. In fact, suppose G does not split. We can always find an element $g \in G$ which is mapped to an element $\bar{g}$ in P of even order $2d$ such that $g^{2d} = g_0 \in G_K$. Now $g_1 = g^d$ defines an automorphism of order 2 of the hyperelliptic curve $C = S^{g_0}$ with fixed points lying over two fixed points of $\bar{g}$ in $\mathbb{P}^1$. None of these points belong to Σ, since otherwise g_0, being a square of g_1, cannot switch the components of the corresponding fibre. Since g_1 has two fixed points on the invariant fibre and both of them must lie on C, we see that g_1 has 4 fixed points. However this contradicts the Hurwitz formula.

Recall that a double cover $f : X \to Y$ of nonsingular varieties with branch divisor $W \subset Y$ is given by an invertible sheaf $\mathcal{L}$ together with a section $s_W \in \Gamma(Y, \mathcal{L}^2)$ with zero divisor W. Suppose a group G acts on Y leaving invariant W. A *lift* of G is a group $\widetilde{G}$ of automorphisms of X such that it commutes with the covering involution τ of X and the corresponding homomorphism $\widetilde{G} \to \mathrm{Aut}(Y)$ is an isomorphism onto the group G.

The following lemma is well-known and is left to the reader.

Lemma 5.11. *A subgroup $G \subset \mathrm{Aut}(Y)$ admits a lift if and only if $\mathcal{L}$ admits a G-linearization and in the corresponding representation of G in $\Gamma(Y, \mathcal{L}^2)$ the section s_W is G-invariant.*

Example 5.12. Let $p_i(t_0, t_1), i = 0, 1, 2$, be binary forms of degree d. Consider a curve C in $\mathbf{F}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$ given by an equation

$$F = p_0(t_0, t_1)x_0^2 + 2p_1(t_0, t_1)x_0x_1 + p_2(t_0, t_1)x_1^2 = 0.$$

Assume that the binary form $D = p_1^2 - p_0p_2$ does not have multiple roots. Then C is a nonsingular hyperelliptic curve of genus $d - 1$. Suppose $d = 2a$ is even so that the genus of the curve is odd. Let P be a polyhedral group not isomorphic to a cyclic group of odd order. Let $V = \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})$ and $\rho : P \to \mathrm{GL}(S^{2a}V \otimes S^2V)$ be its natural representation, the tensor product of the two natural representations of P in the space of binary forms of even degree. Suppose that $F \in S^{2a}V \otimes S^2V$ is an invariant. Consider the double cover $S \to \mathbf{F}_0$ defined by the section F and the invertible sheaf $\mathcal{L} = \mathcal{O}_{\mathbf{F}_0}(a, 1)$. Now assume additionally that P does not have a linear representation in $S^aV \otimes V$ whose tensor square is equal to ρ. Thus $\mathcal{L}$ does not admit a P-linearization and we cannot lift P to a group of automorphisms of the double cover. However, the binary polyhedral group $\bar{P}$ lifts. Its central involution acts identically on $\mathbb{F}_0$, hence lifts to the covering involution of S. It follows from the discussion in the previous subsection that S is a non-exceptional conic bundle, and the group $\bar{P}$ is a minimal group of automorphisms of S with $G_K \cong 2$ and $G_B \cong P$.

Here is a concrete example. Take

$$p_0 = t_0t_1(t_0^2 + t_1^2), \quad p_1 = t_0^4 + t_1^4, \quad p_2 = t_0t_1(t_0^2 - t_1^2).$$

Let $\bar{P} \subset \mathrm{SL}(2)$ be a cyclic group of order 4 that acts on the variables t_0, t_1 via the transformation $[it_0, -it_1]$ and on the variables x_0, x_1 via the transformation $[ix_0, -ix_1]$. Then $\bar{P}$ acts on $S^2V \otimes V$ via $[-1, 1, -1] \otimes [i, -i]$. The matrix $-I_2$ acts as $1 \otimes -1$ and hence $P = \bar{P}/(\pm I_2)$ does not act on $S^2V \otimes V$. This realizes the cyclic group C_4 as a minimal group of automorphisms of a conic bundle with $k = 2g + 2 = 8$.

The previous example shows that for any $g \equiv 1 \mod 4$ one can realize a binary polyhedral group $\bar{P} = 2.P$ as a minimal group of automorphisms of a conic bundle with $2g + 2$ singular fibres. We do not know whether the same is true for $g \equiv 3 \mod 4$.

Example 5.13. Let $p_i(t_0, t_1), i = 1, 2, 3$, be three binary forms of even degree d with no multiple roots. Assume no two have common zeroes. Consider a surface S in $\mathbb{P}^1 \times \mathbb{P}^2$ given by a bihomogeneous form of degree $(d, 2)$

$$p_1(t_0, t_1)z_0^2 + p_2(t_0, t_1)z_1^2 + p_3(t_0, t_1)z_2^2 = 0, \tag{35}$$

The surface is nonsingular. The projection to $\mathbb{P}^1$ defines a conic bundle structure on S with singular fibres over the zeroes of the polynomials p_i. The curves C_i equal to the pre-images of the lines $z_i = 0$ under the second projection are hyperelliptic curves of genus $g = d - 1$. The automorphisms σ_1, σ_2 defined by the negation of one of the first two coordinates z_0, z_1, z_2 form a subgroup of $\mathrm{Aut}(S)$ isomorphic to 2^2. Let P be a finite subgroup of $\mathrm{SL}(2, \mathbb{C})$ and $g \mapsto g^*$ be its natural action on the space of binary forms. Assume that p_1, p_2, p_3 are relative invariants of P with characters χ_1, χ_2, χ_3 such that we can write them in the form η_i^2 for some characters η_1, η_2, η_3 of P. Then P acts on S by the formula

$$g((t_0, t_1), (z_0, z_1, z_2)) = ((g^*(t_0), g^*(t_1)), (\eta_1(g)^{-1}z_0, \eta_2(g)^{-1}z_1, \eta_3(g)^{-1}z_2)).$$

For example, let $P = \langle g \rangle$ be a cyclic group of order 4. We take $p_1 = t_0^2 + t_1^2, p_2 = t_0^2 - t_1^2, p_3 = t_0t_1$. It acts on S by the formula

$$g : ((t_0, t_1), (z_0, z_1, z_2)) \mapsto ((it_1, it_0), (iz_0, z_1, iz_2)).$$

Thus g^2 acts identically on t_0, t_1, z_1 and multiplies z_0, z_2 by -1. We see that $G_K = \langle g^2 \rangle$ and the extension $1 \to G_K \to G \to G_B \to 1$ does not split. If we add to the group the transformation $(t_0, t_1, z_0, z_1, z_2) \mapsto (t_0, t_1, z_0, -z_1, z_2)$ we get a non-split extension 2^{2+1}.

On the other hand, let us replace p_2 with $t_0^2 + t_1^2 + t_0t_1$. Define g_1 as acting only on t_0, t_1 by $[it_1, it_0]$, g_2 acts only on z_0 by $z_0 \mapsto -z_0$ and g_3 acts only on z_1 by $z_1 \mapsto -z_1$. We get the groups $\langle g_1, g_2 \rangle = 2^2$ and $\langle g_1, g_2, g_3 \rangle = 2^3$.

In another example we take P to be the dihedral group D_8. We take $p_1 = t_0^2 + t_1^2, p_2 = t_0^2 - t_1^2, p_3 = t_0t_1$. It acts on S by the formula

$$\left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right) : ((t_0, t_1), (z_0, z_1, z_2)) \mapsto ((it_0, -it_1), (iz_0, iz_1, z_2)),$$

$$\left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right) : ((t_0, t_1), (z_0, z_1, z_2)) \mapsto ((it_1, it_0), (z_0, iz_1, z_2)),$$

The scalar matrix $c = -I_2$ belongs to $G_K \cong 2^2$ and the quotient $P/(c) \cong 2^2$ acts faithfully on the base. This gives a non-split extension 2^{2+2}.

Finally, let us take

$$p_1 = t_0^4 + t_1^4,\ p_2 = t_0^4 + t_1^4 + t_0^2t_1^2,\ p_3 = t_0^4 + t_1^4 - t_0^2t_1^2.$$

These are invariants for the group D_4 acting via $g_1 : (t_0, t_1) \mapsto (t_0, -t_1),\ g_2 : (t_0, t_1) \mapsto (t_1, t_0)$. Together with transformations σ_1, σ_2 this generates the group 2^4 (see another realization of this group in [8]).

6 Automorphisms of Del Pezzo surfaces

6.1 The Weyl group

Let S be a Del Pezzo surface of degree d not isomorphic to $\mathbb{P}^2$ or $\mathbf{F}_0$. It is isomorphic to the blow-up of $N = 9 - d \leq 8$ points in $\mathbb{P}^2$ satisfying the conditions of generality from section 3.4. The blow-up of one or 2 points is obviously non-minimal (since the exceptional curve in the first case and the proper transform of the line through the two points is G-invariant). So we may assume that S is a Del Pezzo surface of degree $d \leq 6$.

Let $\pi : S \to \mathbb{P}^2$ be the blowing-up map. Consider the factorization (10) of π into a composition of blow-ups of $N = 9 - d$ points. Because of the generality condition, we may assume that none of the points $p_1, \ldots, p_N$ is infinitely near, or, equivalently, all exceptional curves $\mathcal{E}_i$ are irreducible curves. We identify them with curves $E_i = \pi^{-1}(p_i)$. The divisor classes $e_0 = [\pi^*(\text{line}], e_i = [E_i], i = 1, \ldots, N$, form a basis of $\mathrm{Pic}(S)$. It is called a *geometric basis*.

Let

$$\alpha_1 = e_0 - e_1 - e_2 - e_3, \alpha_2 = e_1 - e_2, \ldots, \alpha_N = e_{N-1} - e_N.$$

For any $i = 1, \ldots, N$ define a *reflection* isometry s_i of the abelian group $\mathrm{Pic}(S)$

$$s_i : x \mapsto x + (x \cdot \alpha_i)\alpha_i.$$

Obviously, $s_i^2 = 1$ and s_i acts identically on the orthogonal complement of α_i. Let W_S be the group of automorphisms of $\mathrm{Pic}(S)$ generated by the transformations $s_1, \ldots, s_N$. It is called the *Weyl group of* S. Using the basis $(e_0, \ldots, e_N)$ we identify W_S with a group of isometries of the odd unimodular quadratic form $q : \mathbb{Z}^{N+1} \to \mathbb{Z}$ of signature $(1, N)$ defined by

$$q_N(m_0, \dots, m_N) = m_0^2 - m_1^2 - \cdots - m_N^2.$$

Since $K_S = -3e_0 + e_1 + \cdots + e_N$ is orthogonal to all α_i's, the image of W_S in $\mathrm{O}(q_N)$ fixes the vector $k_N = (-3, 1, \dots, 1)$. The subgroup of $\mathrm{O}(q_N)$ fixing k_N is denoted by W_N and is called the *Weyl group of type* E_N. The orthogonal complement $\mathcal{R}_N$ of k_N equipped with the restricted inner-product, is called the *root lattice* of W_N.

We denote by $\mathcal{R}_S$ the sublattice of $\mathrm{Pic}(S)$ equal to the orthogonal complement of the vector K_S. The vectors $\alpha_1, \dots, \alpha_N$ form a $\mathbb{Z}$-basis of $\mathcal{R}_S$. By restriction the Weyl group W_S is isomorphic to a subgroup of $\mathrm{O}(\mathcal{R}_S)$. A choice of a geometric basis $\alpha_1, \dots, \alpha_N$ defines an isomorphism from $\mathcal{R}_S$ to the root lattice Q of a finite root system of type $E_N (N = 6, 7, 8), D_5 (N = 5), A_4 (N = 4)$ and $A_2 + A_1 (N = 3)$. The group W_S becomes isomorphic to the corresponding Weyl group $W(E_N)$.

The next lemma is well-known and its proof goes back to Kantor [41] and Du Val [27]. We refer for modern proofs to [2] or [25].

Lemma 6.1. *Let* $(e'_0, e'_1, \dots, e'_N)$ *be another geometric basis in* $\mathrm{Pic}(S)$ *defined by a birational morphism* $\pi' : S \to \mathbb{P}^2$ *and a choice of a factorization of* π' *into a composition of blow-ups of points. Then the transition matrix is an element of* W_N*. Conversely, any element of* W_N *is a transition matrix of two geometric bases in* $\mathrm{Pic}(S)$*.*

The next lemma is also well-known and is left to the reader.

Lemma 6.2. *If* $d \leq 5$*, then the natural homomorphism*

$$\rho : \mathrm{Aut}(S) \to W_S$$

is injective.

We will use the known classification of conjugacy classes in the Weyl groups. According to [14] they are indexed by certain graphs. We call them *Carter graphs*. One writes each element $w \in W$ as the product of two involutions $w_1 w_2$, where each involution is the product of reflections with respect to orthogonal roots. Let $\mathcal{R}_1, \mathcal{R}_2$ be the corresponding sets of such roots. Then the graph has vertices identified with elements of the set $\mathcal{R}_1 \cup \mathcal{R}_2$ and two vertices α, β are joined by an edge if and only if $(\alpha, \beta) \neq 0$. A Carter graph with no cycles is a Dynkin diagram. The subscript in the notation of a Carter graph indicates the number of vertices. It is also equal to the difference between the rank of the root lattice Q and the rank of its fixed sublattice $Q^{(w)}$.

Note that the same conjugacy classes may correspond to different graphs (e.g. D_3 and A_3, or $2A_3 + A_1$ and $D_4(a_1) + 3A_1$).

The Carter graph determines the characteristic polynomial of w. In particular, it gives the trace $\mathrm{Tr}_2(g)$ of g^* on the cohomology space $H^2(S, \mathbb{C}) \cong \mathrm{Pic}(S) \otimes \mathbb{C}$. The latter should be compared with the Euler-Poincarè characteristic of the fixed locus S^g of g by applying the Lefschetz fixed-point formula.

Graph	Order	Characteristic polynomial
A_k	$k+1$	$t^k + t^{k-1} + \cdots + 1$
D_k	$2k-2$	$(t^{k-1}+1)(t+1)$
$D_k(a_1)$	l.c.m$(2k-4, 4)$	$(t^{k-2}+1)(t^2+1)$
$D_k(a_2)$	l.c.m$(2k-6, 6)$	$(t^{k-3}+1)(t^3+1)$
$\vdots$	$\vdots$	$\vdots$
$D_k(a_{\frac{k}{2}-1})$	even k	$(t^{\frac{k}{2}}+1)^2$
E_6	12	$(t^4-t^2+1)(t^2+t+1)$
$E_6(a_1)$	9	t^6+t^3+1
$E_6(a_2)$	6	$(t^2-t+1)^2(t^2+t+1)$
E_7	18	$(t^6-t^3+1)^2(t+1)$
$E_7(a_1)$	14	t^7+1
$E_7(a_2)$	12	$(t^4-t^2+1)(t^3+1)$
$E_7(a_3)$	30	$(t^5+1)(t^2-t+1)$
$E_7(a_4)$	6	$(t^2-t+1)^2(t^3+1)$
E_8	30	$t^8+t^7-t^5-t^4-t^3+t+1$
$E_8(a_1)$	24	t^8-t^4+1
$E_8(a_2)$	20	$t^8-t^6+t^4-t^2+1$
$E_8(a_3)$	12	$(t^4-t^2+1)^2$
$E_8(a_4)$	18	$(t^6-t^3+1)(t^2-t+1)$
$E_8(a_5)$	15	$t^8-t^7+t^5-t^4+t^3-t+1$
$E_8(a_6)$	10	$(t^4-t^3+t^2-t+1)^2$
$E_8(a_7)$	12	$(t^4-t^2+1)(t^2-t+1)^2$
$E_8(a_8)$	6	$(t^2-t+1)^4$

Table 2. Carter graphs and characteristic polynomials

$$\mathrm{Tr}_2(g) = s - 2 + \sum_{i\in I}(2-2g_i), \tag{36}$$

where S^g the disjoint union of smooth curves $R_i, i \in I$, of genus g_i and s isolated fixed points.

To determine whether a finite subgroup G of $\mathrm{Aut}(S)$ is minimal, we use the well-known formula from the character theory of finite groups

$$\mathrm{rank}\ \mathrm{Pic}(S)^G = \frac{1}{\#G}\sum_{g\in G}\mathrm{Tr}_2(g).$$

The tables for conjugacy classes of elements from the Weyl group W_S give the values of the trace on the lattice $\mathcal{R}_S = K_S^{\perp}$. Thus the group is minimal if and only if the sum of the traces add up to 0.

6.2 Del Pezzo surfaces of degree 6

Let S be a Del Pezzo surface of degree 6. We fix a geometric basis e_0, e_1, e_2, e_3 which is defined by a birational morphism $\pi : S \to \mathbb{P}^2$ with indeterminacy points $p_1 = (1,0,0), p_2 = (0,1,0)$ and $p_3 = (0,0,1)$. The vectors

$$\alpha_1 = e_0 - e_1 - e_2 - e_3, \alpha_2 = e_1 - e_2, \alpha_3 = e_2 - e_3)$$

form a basis of the lattice $\mathcal{R}_S$ with Dynkin diagram of type $A_2 + A_1$. The Weyl group

$$W_S = \langle s_1 \rangle \times \langle s_2, s_3 \rangle \cong 2 \times S_3.$$

The representation $\rho : \mathrm{Aut}(S) \to W_S$ is surjective. The reflection s_1 is realized by the lift of the standard quadratic transformation τ_1. The reflection s_2(resp. s_3) is realized by the projective transformations $[x_1, x_0, x_2]$ (resp. $[x_0, x_2, x_1]$). The kernel of ρ is the maximal torus T of PGL(3), the quotient of $(\mathbb{C}^*)^3$ by the diagonal subgroup $\mathbb{C}^*$. Thus

$$\mathrm{Aut}(S) \cong T : (S_3 \times 2) \cong N(T) : 2,$$

where $N(T)$ is the normalizer of T in PGL(2). It is easy to check that s_1 acts on T as the inversion automorphism.

Let G be a minimal finite subgroup of $\mathrm{Aut}(S)$. Obviously, $\rho(G)$ contains s_1 and $s_2 s_2$ since otherwise G leaves invariant α_1 or one of the vectors $2\alpha_1 + \alpha_2$, or $\alpha_1 + 2\alpha_2$. This shows that $G \cap N(T)$ is an imprimitive subgroup of PGL(3). This gives

Theorem 6.3. *Let G be a minimal subgroup of a Del Pezzo surface of degree* 6. *Then*

$$G = H_\bullet \langle s_1 \rangle,$$

where H is an imprimitive finite subgroup of PGL(3).

Note that one the groups from the theorem is the group $2^2 : S_3 \cong S_4$. Its action on S given by the equation

$$x_0 y_0 z_0 - x_1 y_1 z_1 = 0$$

in $(\mathbb{P}^1)^3$ is given in [4].

6.3 Del Pezzo surfaces of degree $d = 5$.

In this case S is isomorphic to the blow-up of the reference points $p_1 = (1,0,0),\ p_2 = (0,1,0),\ p_3 = (0,0,1), p_4 = (1,1,1)$. The lattice $\mathcal{R}_S$ is of type A_4 and $W_S \cong S_5$. It is known that the homomorphism $\rho : \mathrm{Aut}(S) \to W_S$ is an isomorphism. We already know that it is injective. To see the surjectivity one can argue, for example, as follows.

Let τ be the standard quadratic transformation with base points p_1, p_2, p_3. It follows from its formula that the point p_4 is a fixed point. We know that τ can be regularized on the Del Pezzo surface S' of degree 6 obtained by the blow-up of the first three points. Since the pre-image of p_4 in S' is a fixed point, τ lifts to an automorphism of S. Now let ϕ be a projective transformation such that $\phi(p_1) = p_1, \phi(p_2) = p_2, \phi(p_4) = p_3$. For example, we take $A = [t_0 - t_2, t_1, -t_2, -t_2]$. Then the quadratic transformation $\phi^{-1}\tau\phi$ is not defined at the points p_1, p_2, p_4 and fixes the point p_3. As above, it can be lifted to an involution of S. Proceeding in this way we define 4 involutions $\tau = \tau_1, \dots, \tau_4$ of S each fixes one of the exceptional curves. One checks that their images in the Weyl group W_S generate the group.

Another way to see the isomorphism $\mathrm{Aut}(S) \cong S_5$ is to use a well-known isomorphism between S and the moduli space $\overline{\mathcal{M}_{0,5}} \cong (\mathbb{P}^1)^5//\mathrm{SL}(2)$. The group S_5 acts by permuting the factors.

Theorem 6.4. *Let (S, G) be a minimal Del Pezzo surface of degree $d = 5$. Then $G = S_5, A_5, 5:4, 5:2$, or C_5.*

Proof. As we have just shown $\mathrm{Aut}(S) \cong W_4 \cong S_5$. The group S_5 acts on $\mathcal{R}_S \cong \mathbb{Z}^4$ by means of its standard irreducible 4-dimensional representation (view $\mathbb{Z}^4$ as a subgroup of $\mathbb{Z}^5$ of vectors with coordinates added up to zero and consider the representation of S_5 by switching the coordinates). It is known that a maximal proper subgroup of S_5 is equal (up to a conjugation) to one of three subgroups $S_4, S_3 \times 2, A_5, 5:4$. A maximal subgroup of A_5 is either 5×2 or S_3 or $D_{10} = 5:2$. It is easy to see that the groups S_4 and $S_3 \times 2$ have invariant elements in the lattice Q_4. It is known that an element of order 5 in S_5 is a cyclic permutation, and hence has no invariant vectors. Thus any subgroup G of S_5 containing an element of order 5 defines a minimal surface (S, G). So, if (S, G) is minimal, G must be equal to one of the groups from the assertion of the theorem.

6.4 Automorphisms of a Del Pezzo surface of degree $d = 4$

In this case $\mathcal{R}$ is of type D_5 and $W_S \cong 2^4 : S_5$. We use the following well-known classical result.

Lemma 6.5. *Let S be a Del Pezzo surface of degree 4. Then S is isomorphic to a nonsingular surface of degree 4 in $\mathbb{P}^4$ given by equations*

$$F_1 = \sum_{i=0}^{4} t_i^2 = 0, \quad F_2 = \sum_{i=0}^{4} a_i t_i^2 = 0, \tag{37}$$

where all a_i's are distinct.

Proof. It is known that a Del Pezzo surface in its anti-canonical embedding is projectively normal. Using Riemann-Roch, one obtains that S is a complete intersection $Q_1 \cap Q_2$ of two quadrics. Let $\mathcal{P} = \lambda Q_1 + \mu Q_2$ be the pencil

spanned by these quadrics. The locus of singular quadrics in the pencil is a homogeneous equation of degree 5 in the coordinates λ, μ. Since S is nonsingular, it is not hard to see that the equation has no multiple roots (otherwise $\mathcal{P}$ contains a reducible quadric or there exists a quadric in the pencil with singular point at S, in both cases S is singular). Let $p_1, \ldots, p_5$ be the singular points of singular quadrics from the pencil. Suppose they are contained in a hyperplane H. Since no quadrics in the pencil contains H, the restriction $\mathcal{P}|H$ of the pencil of quadrics to H contains ≥ 5 singular members. This implies that all quadrics in $\mathcal{P}|H$ are singular. By Bertini's theorem, all quadrics are singular at some point $p \in H$. This implies that all quadrics in $\mathcal{P}$ are tangent to H at p. One of the quadrics must be singular at p, and hence S is singular at p. This contradiction shows that $p_1, \ldots, p_5$ span $\mathbb{P}^4$. Choose coordinates in $\mathbb{P}^4$ such that the singular points of singular quadrics from $\mathcal{P}$ are the points $(1,0,0,0,0), (0,1,0,0,0)$, and so on. Then each hyperplane $V(t_i) = (t_i = 0)$ is a common tangent hyperplane of quadrics from $\mathcal{P}$ at the point p_i. This easily implies that the equations of quadrics are given by (37).

Let $Q_i = V(a_i F_1 - F_2), i = 0, \ldots, 4$, be one of the singular quadrics in the pencil $\mathcal{P}$. It is a cone over a nonsingular quadric in $\mathbb{P}^3$, hence it contains two families of planes. The intersection of a plane with any other quadric in the pencil is a conic contained in S. Thus each Q_i defines a pair of pencils of conics $|C_i|$ and $|C_i'|$, and it is easy to see that $|C_i + C_i'| = |-K_S|$.

Proposition 6.6. *Let S be a Del Pezzo surface given by equations* (37). *The divisor classes $c_i = [C_i]$ together with K_S form a basis of* $\mathrm{Pic}(S) \otimes \mathbb{Q}$. *The Weyl group W_S acts on this basis by permuting the c_i's and sending c_i to $c_i' = [C_i'] = -K_S - c_i$.*

Proof. If we choose a geometric basis $(e_0, e_1, \ldots, e_5)$ in $\mathrm{Pic}(S)$, then the 5 pairs of pencils of conics are defined by the classes $e_0 - e_i, 2e_0 - e_1 - \cdots - e_5 + e_i$. It is easy to check that the classes $[C_i]$'s and K_S form a basis in $\mathrm{Pic}(S) \otimes \mathbb{Q}$. The group W_S contains a subgroup isomorphic to S_5 generated by the reflections in vectors $e_1 - e_2, \ldots, e_4 - e_5$., It acts by permuting $e_1, \ldots, e_5$, hence permuting the pencils $|C_i|$. It is equal to the semi-direct product of S_5 and the subgroup isomorphic to 2^4 which is generated by the conjugates of the product s of two commuting reflections with respect to the vectors $e_0 - e_1 - e_2 - e_3$ and $e_4 - e_5$. It is easy to see that $s([C_4]) = [C_4'], s([C_5]) = [C_5']$ and $s([C_i]) = [C_i]$ for $i \neq 4, 5$. This easily implies that W_S acts by permuting the classes $[C_i]$ and switching even number of them to $[C_i']$.

Corollary 6.7. *Let $W(D_5)$ act in $\mathbb{C}^5$ by permuting the coordinates and switching the signs of even number of coordinates. This linear representation of $W(D_5)$ is isomorphic to the representation of $W(D_5)$ on $\mathcal{R}_S \otimes \mathbb{C}$.*

The group of projective automorphisms generated by the transformations which switch x_i to $-x_i$ generates a subgroup H of $\mathrm{Aut}(S)$ isomorphic to 2^4.

We identify the group H with the linear space of subsets of even cardinality of the set $J = \{0, 1, 2, 3, 4\}$, or, equivalently, with the subspace of $\mathbb{F}_2^J$ of functions with support at a subset of even cardinality. We equip H with the symmetric bilinear form defined by the dot-product in $\mathbb{F}_2^J$, or, equivalently, by $(A, B) = \#A \cap B \mod 2$. We denote elements of H by i_A, where i_A is the characteristic function of $A \subset J$.

There are two kinds of involutions i_A. An involution of the first kind corresponds to a subset A of 4 elements. The set of fixed points of such an involution is a hyperplane section of S, an elliptic curve. The trace formula (36) gives that the the trace of i_A in $\mathrm{Pic}(S)$ is equal to -2. The corresponding conjugacy class in W_5 is of type $4A_1$. There are 5 involutions of the first kind. The quotient surface $S/\langle i_A \rangle = Q$ is isomorphic to a nonsingular quadric. The map $S \to Q$ coincides with the map $S \to \mathbb{P}^1 \times \mathbb{P}^1$ that is given by the pencils $|C_i|$ and $|C_i'|$.

Involutions of the second type correspond to subsets A of cardinality 2. The fixed-point set of such involution consists of 4 isolated points. This gives that the trace is equal to 2, and the conjugacy class is of type $2A_1$. The quotient $S/(i_A)$ is isomorphic to the double cover of $\mathbb{P}^2$ branched along the union of two conics.

The subgroup of the Weyl group $W(D_5)$ generated by involutions from the conjugacy class of type $2A_1$ is the normal subgroup 2^4 in the decomposition $W(D_5) \cong 2^4 : S_5$. The product of two commuting involutions from this conjugacy class is an involution of type $4A_1$. Thus the image of H in W_S is a normal subgroup isomorphic to 2^4.

Let $G \cong 2^a$ be a subgroup of 2^4. All cyclic groups G are not minimal.

There are three kinds of subgroups H of order 4 in 2^4. A subgroup of the first kind does not contain an involution of the first kind. An example is the group generated by i_{01}, i_{12}. The trace of its nonzero elements equal to 1. So this group is not minimal.

A subgroup of the second type contains only one involution of the first kind. An example is the group generated by i_{01}, i_{23}. The trace formula gives $\mathrm{rank}\ \mathrm{Pic}(S)^H = 2$. So it is also non-minimal.

A subgroup of the third kind contains two involutions of the first kind. For, example a group generated by i_{1234}, i_{0234}. It contains 2 elements with trace -3 and one element with trace 1. Adding up the traces we see the group is a minimal group. It is easy to see that S^H consists of 4 isolated points.

Now let us consider subgroups of order 8 of 2^4. They are parametrized by the same sets which parametrize involutions. A subgroup H_A corresponding to a subset A consists of involutions i_B such that $\#A \cap B$ is even. The subsets A correspond to linear functions on 2^4. If $\#A = 2$, say $A = \{0, 1\}$, we see that H_A contains the involutions $i_{01}, i_{01ab}, i_{cd}, c, d \neq 0, 1$. Adding up the traces we obtain that these subgroups are minimal.

If $\#A = 4$, say $A = \{1, 2, 3, 4\}$, the subgroup H_A consists of i_{1234} and i_{ab}, where $a, b \neq 0$. Adding up the traces we obtain that H_A is not minimal.

Since 2^4 contains a minimal subgroup, it is minimal itself.

Now suppose that the image G' of G in S_5 is non-trivial. The subgroup S_5 of $\mathrm{Aut}(S)$ can be realized as the stabilizer of a set of 5 skew lines on S (corresponding to the basis vectors $e_1, \dots, e_5$). Thus any subgroup H of S_5 realized as a group of automorphisms of S is isomorphic to a group of projective transformations of $\mathbb{P}^2$ leaving invariant a set of 5 points. Since there is a unique conic through these points, the group is isomorphic to a finite group of PGL(2) leaving invariant a set of 5 distinct points. In section 4, we listed all possible subgroups of GL(2) and in section 5 we described their relative invariants. It follows that a subgroup leaves invariant a set of 5 distinct points if and only if it is one of the following groups $C_2, C_3, C_4, C_5, S_3, D_{10}$. The corresponding binary forms of degree 5 are projectively equivalent to the following binary forms:

- $C_2 : t_0(t_0^2 - t_1^2)(t_0^2 + at_1^2),\ a \neq -1, 0, 1;$
- $C_4 : t_0(t_0^2 - t_1^2)(t_0^2 + t_1^2);$
- $C_3, S_3 : t_0 t_1 (t_0 - t_1)(t_0 - \epsilon_3 t_1)(t_0 - \epsilon_3^2 t_1);$
- $C_5, D_{10} : (t_0 - t_1)(t_0 - \epsilon_5 t_1)(t_0 - \epsilon_5^2 t_1)(t_0 - \epsilon_5^3 t_1)(t_0 - \epsilon_5^4 t_1).$

The corresponding surfaces are projectively equivalent to the following surfaces

$$C_2 : x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_0^2 + ax_1^2 - x_2^2 - ax_3^2 = 0,\ a \neq -1, 0, 1 \tag{38}$$

$$C_4 : x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_0^2 + ix_1^2 - x_2^2 - ix_3^2 = 0 \tag{39}$$

$$S_3 : x_0^2 + \epsilon_3 x_1^2 + \epsilon_3^2 x_2^2 + x_3^2 = x_0^2 + \epsilon_3^2 x_1^2 + \epsilon_3 x_2^2 + x_4^2 = 0 \tag{40}$$

$$D_{10} : \sum_{i=0}^{4} \epsilon_5^i x_i^2 = \sum_{i=0}^{4} \epsilon_5^{4-i} x_i^2 = 0 \tag{41}$$

Remark 6.8. Note that equations (40), (41), (41) are specializations of equation (39). It is obvious for equation (40) where we have to take $a = i$. Equation (39) specializes to equation (41) when we take $a = \pm\frac{1}{\sqrt{-3}}$ (use that the Moebius transformation of order 3 $x \mapsto \frac{\sqrt{a}x+1}{x+\sqrt{a}}$ permutes cyclically $\infty, \sqrt{a}, -\sqrt{-a}$ and fixes $1, -1$). Equation (39) specializes to equation (41) if we take $a = -2 \pm \sqrt{5}$ (use that the Moebius transformation $x \mapsto \frac{x+2a-1}{x+1}$ permutes cyclically $(\infty, 1, a, -a, -1)$). We thank J. Blanc for this observation.

Since the subgroup S_5 leaves the class e_0 invariant, it remains to consider subgroups G of $2^4 : S_5$ which are not contained in 2^4 and not conjugate to a subgroup of S_5. We use the following facts.

1) Suppose G contains a minimal subgroup of 2^4. Then G is minimal.

2) Let $\overline{G}$ be the image of G in S_5. Then it is a subgroup of one of the groups listed above.

3) The group $W(D_5)$ is isomorphic to the group of transformations of $\mathbb{R}^5$ which consists of permutations of coordinates and changing even number of signs of the coordinates. Each element $w \in W(D_5)$ defines a permutation of the coordinate lines which can be written as a composition of cycles $(i_1 \dots i_k)$.

If w changes signs of even number of the coordinates $x_{i_1}, \ldots, x_{i_k}$, the cycle is called positive. Otherwise it is called a negative cycle. The conjugacy class of w is determined by the lengths of positive and negative cycles, except when all cycles of even length and positive in which case there are two conjugacy classes. The latter case does not occur in the case when n is odd. Assign to a positive cycle of length k the Carter graph A_{k-1}. Assign to a pair of negative cycles of lengths $i \geq j$ the Carter graph of type D_{i+1} if $j = 1$ and $D_{i+j}(a_{j-1})$ if $j > 1$. Each conjugacy class is defined by the sum of the graphs. We identify D_2 with $2A_1$, and D_3 with A_3. In Table 2 below we give the conjugacy classes of elements in $W(D_5)$, their characteristic polynomials and the traces in the root lattice of type D_5.

Order	Notation	Characteristic polynomial	Trace	Representatives
2	A_1	$t+1$	3	(ab)
2	$2A_1$	$(t+1)^2$	1	$(ab)(cd), (ab)(cd)i_{abcd}$
2	$2A_1^*$	$(t+1)^2$	1	$(ab)(cd), (ab)(cd)i_{ab}$
2	$3A_1$	$(t+1)^3$	-1	$(ab)i_{cd}$
2	$4A_1$	$(t+1)^4$	-3	i_{abcd}
3	A_2	t^2+t+1	2	$(abc), (abc)i_{ab}$
4	A_3	t^3+t^2+t+1	1	$(abcd), (abcd)i_{ab}, (abcd)i_{abcd}$
4	A_1+A_3	$(t^3+t^2+t+1)(t+1)$	-1	$(ab)(cd)i_{ae}$
4	$D_4(a_1)$	$(t^2+1)^2$	1	$(ab)(cd)i_{ac}$
5	A_4	$(t^4+t^3+t^2+t+1)$	0	$(abcde), (abcde)i_A$
6	A_2+A_1	$(t^2+t+1)(t+1)$	0	$(ab)(cde)$
6	A_2+2A_1	$(t^2+t+1)(t+1)^2$	-2	$(abc)i_{abde}, (abc)i_{de}$
6	D_4	$(t^3+1)(t+1)$	0	$(abc)i_{abce}$
8	D_5	$(t^4+1)(t+1)$	-1	$(abcd)i_{abce}, (abcd)i_{de}$
12	$D_5(a_1)$	$(t^3+1)(t^2+1)$	0	$(abc)(de)i_{ac}$

Table 3. Conjugacy classes in $W(D_5)$

In the following $\overline{G}$ denotes the image of G in $K = W(D_5)/2^4 \cong S_5$. Case 1. $\overline{G} \cong C_2$.

It follows from the description of the image of $\mathrm{Aut}(S)$ in $W(D_5)$ given in Corollary 6.7, that $\overline{G}$ is generated by the permutation $s = (02)(13)$. Let $g \notin G \cap 2^4$. Then $g = s$ or $g = si_A$ for some A. It follows from Table 3 that g is either of type $2A_1$, or of type $A_1 + A_3$, or of type $D_4(a_1)$. Let $K = G \cap 2^4 \cong 2^a$. If $a = 0$ or 1, the group is not minimal.

$a = 2$.

Suppose first that s acts identically on K. Then the group is commutative isomorphic to 2^3 if it does not contain elements of order 4 and 2×4 otherwise. In the first case

$$K = F := \langle i_{02}, i_{13} \rangle. \tag{42}$$

is the subspace of fixed points of s in 2^4. Since $(si_A)^2 = i_{A+s(A)}$, we see that $G \setminus K = \{si_A, A \in K\}$. Consulting Table 3 we compute the traces of all elements from G to conclude that the total sum is equal to 8. Thus the group is not minimal.

In the second case G contains 4 elements of order 4 of the form si_A, where $A \notin K$. Suppose $A + s(A) = \{0, 1, 2, 3\}$. Since K is a subspace of the second type, it contributes 4 to the total sum of the traces. Thus the sum of the traces of the elements of order 4 must be equal to -4. In other words they have to be elements with trace -1 of the form si_A, where $\#A = 2, 4 \in A$. This gives the unique conjugacy class of a minimal group isomorphic to 2×4. It is represented by the group $G = \langle K, si_{04} \rangle$.

Assume now that G is non-abelian, obviously isomorphic to D_8. The subspace K contains one element from the set F. The nontrivial coset contains 2 elements of order 4 and two elements of order 2. Suppose K is of the second type with the sum of the traces of its elements equal to 4. Two elements of order 2 in $G \setminus K$ have the trace equal to 1. Elements of order 4 have the trace equal to 1 or -1. So the group cannot be minimal. Thus K must be of the third type, the minimal one. This gives us the minimal group conjugate to the subgroup $G = \langle i_{1234}, i_{02}, si_{04} \rangle$ isomorphic to D_8.

$a = 3$.

There are three s invariant subspaces of 2^4 of dimension 3. Their orthogonal complements are spanned by the one of the vectors in the set (42). As we saw earlier, if $K^\perp = \langle i_A \rangle$, where $\#A = 2$, the subspace K is a minimal group. Otherwise the total sum of the traces of elements from K is equal to 8. In the first case we may assume that $K = \langle i_{14}, i_{34}, i_{02} \rangle$. All elements of order 2 in the nontrivial coset have the trace equal to 1. Thus we must have elements of order 4 in the coset with trace -1. Let si_A be such an element, where we may assume that $A = \{0, 4\}$. Thus $G = \langle K, si_{04} \rangle$. Its nontrivial coset has 4 elements of order 4 with trace 1 and 4 elements of order 4 with trace -1. The group is minimal. It is a non-split extension $(2^3)^\bullet 2$. Its center is isomorphic to 2^2. The classification of groups of order 16 from Table 1 shows that this is group is isomorphic to L_{16}.

$a = 4$.

In this case $G = 2^4 : 2$, where the extension is defined by the action of s in 2^4. The group has 2-dimensional center with the quotient isomorphic to 2^3.

Case 2. $\overline{G} \cong C_3$.

We may assume that $\overline{G} = \langle s \rangle$, where $s = (012)$. Applying Lemma 4.2, we obtain that G is a split extension $K : 3$, where $K = G \cap 2^4 \cong 2^a$. Since there are no minimal elements of order 3, we must have $a > 0$. If $a = 1$, the group is $2 : 3 \cong 6$. There are no minimal elements of order 6, so we may assume that $a > 1$.

Assume $a = 2$. The group is abelian $2^2 \times 3$ or non-abelian $2^2 : 3 \cong A_4$. In the first case, $K = \langle i_{0123}, i_{0124} \rangle$ is the subspace of the third type, the minimal

one. Thus the total number of elements in the nontrivial cosets is equal to 0. An element of order 3 has the trace equal to 1. An element of order 6 has trace equal to -2 of 0. So we must have an element of order 6 with trace -2. It must be equal to si_{34}. Its cube is $i_{34} \in K$. So we get one conjugacy class of a minimal group isomorphic to $2^2 \times 3$. It is equal to $\langle K, si_{34}\rangle$.

If $G \cong A_4$, the subspace K is not minimal. The group does not contain elements of order 6. So the traces of all elements not from K are positive. This shows that the group cannot be minimal.

Assume $a = 3$. The subspace K is minimal if and only if its orthogonal complement is generated by i_{34}. Again the group must contain si_{34} with trace -2 and hence equal to $\langle K, si_{34}\rangle = \langle K, s\rangle$. The group G is minimal and is isomorphic to $2 \times (2^2 : 3) \cong 2 \times A_4$.

Finally assume that $K = \langle i_{0123}\rangle^{\perp}$ is not minimal. We have computed earlier the sum of the traces of its elements. It is equal to 8. Again it must contain an element of order 6 equals si_{0123}. Since K contains i_{0123}, the group contains s. Now we can add all the traces and conclude that the group is not minimal.

Of course we should not forget the minimal group $2^4 : 3$.

Case 3. $\overline{G} \cong S_3$.

The group $\overline{G}$ is generated by the permutations of coordinates (012) and (12)(34). It is immediately checked that $H = G \cap 2^4$ is not trivial for minimal G. The only subgroup of H invariant with respect to the conjugation action of $\overline{G}$ on H is H itself. This gives a minimal group isomorphic to $2^4 : S_3$. The extension is defined by the restriction to S_3 of the natural action of $S_4 = W(A_4)$ on its root lattice modulo 2.

Case 4. $\overline{G} \cong C_4$.

The group $2^4 : 4$ contains $2^4 : 2$, so all minimal groups of the latter group are minimal subgroups of $2^4 : 4$. Without loss of generality, we may assume that the group $\overline{G}$ is generated by the permutations of coordinates $s = (0123)$. The only proper subgroup of 2^4 invariant with respect to the conjugation action of $\overline{G}$ on $K = G \cap 2^4$ is either $\langle i_{0123}\rangle$ or its orthogonal complement. In the first case $G \cong 2\bullet 4 \cong 2 \times 4$ or 8. In the first case the group is not minimal. In the second case $G = \langle (0123)i_{0123}\rangle$ is minimal.

Assume $K = \langle i_{0123}\rangle^{\perp}$. If $s \notin G$, then $si_A \in G$, where $A \notin K$. The Table 3 shows that all such elements are minimal of order 8. This gives a minimal group $G \cong 2^3 : 4 = 2^2 : 8$.

Next we have to consider the case when $s \in G$ so that $G = \langle K, s\rangle$. The total number of traces of elements from K is equal to 8. Consulting the Table 3 we obtain that the elements in the cosets sK, s^2K, s^3K have the trace equal to 1. So the group is not minimal.

Our last minimal group in this case is $2^4 : 4$.

Case 5. $\overline{G} = C_5$ or D_{10}.

Again, we check using the table of conjugacy classes that no group isomorphic to D_{10} is minimal. Also no proper subgroup of H is invariant with

respect to conjugation by a permutation of order 5, or by a subgroup of S_5 generated by (012) and (12)(34). Thus we get two minimal groups isomorphic to $2^4 : 5$ or $2^4 : D_{10}$.

The following theorem summarizes what we have found.

Theorem 6.9. *Let (S, G) be a minimal Del Pezzo surface of degree $d = 4$. Then G is isomorphic to one of the following groups*

1. $\mathrm{Aut}(S) \cong 2^4$.
$$2^4,\ 2^3,\ 2^2.$$
2. $\mathrm{Aut}(S) \cong 2^4 : 2$
$$2 \times 4,\ D_8,\ L_{16},\ 2^4 : 2,$$
and from the previous case.
3. $\mathrm{Aut}(S) \cong 2^4 : 4$
$$8,\ 2^2 : 8,\ 2^4 : 4,$$
and from the previous two cases.
4. $\mathrm{Aut}(S) \cong 2^4 : S_3$.
$$2^2 \times 3,\ 2 \times A_4,\quad 2^4 : 3,\ 2^4 : S_3,$$
and from Cases 1) and 2).
5. $\mathrm{Aut}(S) \cong 2^4 : D_{10}$
$$2^4 : D_{10},\ 2^4 : 5,$$
and from Cases 1) and 2).

6.5 Cubic surfaces

The following theorem gives the classification of cyclic subgroups of $\mathrm{Aut}(S)$ and identifies the conjugacy classes of their generators.

Theorem 6.10. *Let S be a nonsingular cubic surface admitting a non-trivial automorphism g of order n. Then S is equivariantly isomorphic to one of the following surfaces $V(F)$ with*

$$g = [x_0, \epsilon_n^a x_1, \epsilon_n^b x_2, \epsilon_n^c x_3]. \tag{43}$$

- $4A_1$ $(n = 2)$, $(a, b, c) = (0, 0, 1)$,
$$F = T_3^2 L_1(T_0, T_1, T_2) + T_0^3 + T_1^3 + T_2^3 + \alpha T_0 T_1 T_2.$$
- $2A_1$ $(n = 2)$, $(a, b, c) = (0, 1, 1)$,
$$F = T_0 T_2 (T_2 + \alpha T_3) + T_1 T_3 (T_2 + \beta T_3) + T_0^3 + T_1^3.$$
- $3A_2$ $(n = 3)$, $(a, b, c) = (0, 0, 1)$,
$$F = T_0^3 + T_1^3 + T_2^3 + T_3^3 + \alpha T_0 T_1 T_2.$$

- A_2 $(n=3)$, $(a,b,c)=(0,1,1)$,
$$F = T_0^3 + T_1^3 + T_2^3 + T_3^3.$$
- $2A_2$ $(n=3)$, $(a,b,c)=(0,1,2)$,
$$F = T_0^3 + T_1^3 + T_2T_3(T_0 + aT_1) + T_2^3 + T_3^3.$$
- $D_4(a_1)$ $(n=4)$, $(a,b,c)=(0,2,1)$,
$$F = T_3^2T_2 + L_3(T_0,T_1) + T_2^2(T_0 + \alpha T_1).$$
- $A_3 + A_1$ $(n=4)$, $(a,b,c)=(2,1,3)$,
$$F = T_0^3 + T_0T_1^2 + T_1T_3^2 + T_1T_2^2.$$
- A_4 $(n=5)$, $(a,b,c)=(4,1,2)$,
$$F = T_0^2T_1 + T_1^2T_2 + T_2^2T_3 + T_3^2T_0.$$
- $E_6(a_2)$ $(n=6)$, $(a,b,c)=(0,3,2)$,
$$F = T_0^3 + T_1^3 + T_3^3 + T_2^2(\alpha T_0 + T_1).$$
- D_4 $(n=6)$, $(a,b,c)=(0,2,5)$,
$$F = L_3(T_0,T_1) + T_3^2T_2 + T_2^3.$$
- $A_5 + A_1$ $(n=6)$, $(a,b,c)=(4,2,1)$,
$$F = T_3^2T_1 + T_0^3 + T_1^3 + T_2^3 + \lambda T_0T_1T_2.$$
- $2A_1 + A_2$ $(n=6)$, $(a,b,c)=(4,1,3)$,
$$F = T_0^3 + \beta T_0T_3^2 + T_2^2T_1 + T_1^3.$$
- D_5 $(n=8)$, $(a,b,c)=(4,3,2)$,
$$F = T_3^2T_1 + T_2^2T_3 + T_0T_1^2 + T_0^3.$$
- $E_6(a_1)$ $(n=9)$, $(a,b,c)=(4,1,7)$,
$$F = T_3^2T_1 + T_1^2T_2 + T_2^2T_3 + T_0^3.$$
- E_6 $(n=12)$, $(a,b,c)=(4,1,10)$,
$$F = T_3^2T_1 + T_2^2T_3 + T_0^3 + T_1^3.$$

We only sketch a proof, referring for the details to [26]. Let g be a nontrivial projective automorphism of $S = V(F)$ of order n. All possible values of n can be obtained from the classification of conjugacy classes of $W(E_6)$. Choose coordinates to assume that g acts as in (43). Then F is a sum of monomials which belong to the same eigensubspace of g in its action in the space of cubic polynomials. We list all possible eigensubspaces. Since $V(F)$ is nonsingular, the square or the cube of each variable divides some monomial entering in F. This allows one to list all possible nonsingular $V(F)$ admitting an automorphism g. Some obvious linear change of variables allows one to find normal forms. Finally, we determine the conjugacy class by using the trace formula (36) applied to the locus of fixed points of g and its powers.

The conjugacy class labeled by the Carter graph with 6 vertices defines a minimal cyclic group.

Corollary 6.11. *The following conjugacy classes define minimal cyclic groups of automorphisms of a cubic surface S.*

- $3A_2$ *of order 3,*
- $E_6(a_2)$ *of order 6,*
- $A_5 + A_1$ *of order 6,*
- $E_6(a_1)$ *of order 9,*
- E_6 *of order 12.*

Next we find all possible automorphism groups of nonsingular cubic surfaces. Using a normal form of a cubic admitting a cyclic group of automorphisms from given conjugacy class, we determine all other possible symmetries of the equation. We refer for the details to [26]. The list of possible automorphism groups of cubic surfaces is given in Table 4.

Remark 6.12. Note that there are various ways to write the equation of cubic surfaces from the table. For example, using the identity

$$(x+y+z)^3 + \epsilon_3(x+\epsilon_3 y+\epsilon_3^2 z)^3 + \epsilon_3^2(x+\epsilon_3^2 y+\epsilon_3 z)^3 = 9(x^2z+y^2x+z^2y)$$

we see that the Fermat cubic can be given by the equation

$$T_0^3 + T_1^2T_3 + T_3^2T_2 + T_2^2T_1 = 0.$$

Using Theorem 6.10 this exhibits a symmetry of order 9 of the surface, whose existence is not obvious in the original equation.

Using the Hesse form of an equation of a nonsingular plane cubic curve we see that a surface with equation

$$T_0^3 + F_3(T_1, T_2, T_3) = 0$$

is projectively equivalent to a surface with the equation

$$T_0^3 + T_1^3 + T_2^3 + T_3^3 + 6aT_0T_1T_2 = 0.$$

Type	Order	Structure	$F(T_0, T_1, T_2, T_3)$	Parameters
I	648	$3^3 : S_4$	$T_0^3 + T_1^3 + T_2^3 + T_3^3$	
II	120	S_5	$T_0^2T_1 + T_0T_2^2 + T_2T_3^2 + T_3T_1^2$	
III	108	$H_3(3) : 4$	$T_0^3 + T_1^3 + T_2^3 + T_3^3 + 6aT_1T_2T_3$	$20a^3 + 8a^6 = 1$
IV	54	$H_3(3) : 2$	$T_0^3 + T_1^3 + T_2^3 + T_3^3 + 6aT_1T_2T_3$	$a - a^4 \neq 0$, $8a^3 \neq -1$, $20a^3 + 8a^6 \neq 1$
V	24	S_4	$T_0^3 + T_0(T_1^2 + T_2^2 + T_3^2) + aT_1T_2T_3$	$9a^3 \neq 8a$ $8a^3 \neq -1$,
VI	12	$S_3 \times 2$	$T_2^3 + T_3^3 + aT_2T_3(T_0 + T_1) + T_0^3 + T_1^3$	$a \neq 0$
VII	8	8	$T_3^2T_2 + T_2^2T_1 + T_0^3 + T_0T_1^2$	
VIII	6	S_3	$T_2^3 + T_3^3 + aT_2T_3(T_0 + bT_1) + T_0^3 + T_1^3$	$a^3 \neq -1$
IX	4	4	$T_3^2T_2 + T_2^2T_1 + T_0^3 + T_0T_1^2 + aT_1^3$	$a \neq 0$
X	4	2^2	$T_0^2(T_1 + T_2 + aT_3) + T_1^3 + T_2^3 + T_3^3 + 6bT_1T_2T_3$	$8b^3 \neq -1$
XI	2	2	$T_1^3 + T_2^3 + T_3^3 + 6aT_1T_2T_3 + T_0^2(T_1 + bT_2 + cT_3)$	$b^3, c^3 \neq 1$ $b^3 \neq c^3$ $8a^3 \neq -1$,

Table 4. Groups of automorphisms of cubic surfaces

The special values of the parameters $a = 0, 1, \epsilon_3, \epsilon_3^2$ give the Fermat cubic. The values a satisfying $20a^3 + 8a^6 = 1$ give a plane cubic with an automorphism of order 4 (a harmonic cubic). Since a harmonic cubic is isomorphic to the cubic with equation $T_1^3 + T_1T_2^2 + T_3^3 = 0$, using Theorem 6.10 we see symmetries of order 6 from the conjugacy class $E_6(a_2)$ for surfaces of type III, IV and of order 12 for the surface

$$T_3^2T_1 + T_2^2T_3 + T_0^3 + T_1^3 = 0$$

of type III.

It remains to classify minimal groups G. Note that if G is realized as a group of projective (or weighted projective) automorphisms of a family of surfaces (S_t), then G is a subgroup of the group of projective automorphisms of any surface S_{t_0} corresponding to a special value t_0 of the parameters. We indicate this by writing $S' \to S$. The types of S' when it happens are

$$IV \to III,\ IV \to I, \quad VI, VIII, IX \to I, \quad XI \to X.$$

So it suffices to consider the surfaces of types I, II, III, V, VII, X.

We will be using the following lemma, kindly communicated to us by R. Griess. For completeness sake, we provide its proof.

Lemma 6.13. *Let S_{n+1} act naturally on its root lattice $\mathcal{R}_n = \{(a_1, \dots, a_{n+1}) \in \mathbb{Z}^{n+1} : a_1 + \cdots + a_{n+1} = 0\}$. Let $\mathcal{R}_n(p) \cong \mathbb{F}_p^n$ be the reduction of $\mathcal{R}_n$ modulo a prime number $p > 2$ not dividing $n + 1$. Then the set of conjugacy classes of*

subgroups dividing a splitting of $\mathbb{F}_p^n : S_{n+1}$ *is bijective to the set* $\mathbb{F}_p$ *if* $p \mid n+1$ *and consists of one element otherwise.*

Proof. It is easy to see that, fixing a splitting, there is a natural bijection between conjugacy classes of splitting subgroups in $A : B$ and the cohomology set $H^1(B, A)$, where B acts on A via the homomorphism $B \to \mathrm{Aut}(A)$ defining the semi-direct product. So, it suffices to prove that $H^1(S_{n+1}, \mathcal{R}_n(p)) \cong \mathbb{F}_p$ if $p \mid n+1$ and zero otherwise. Consider the permutation representation of S_{n+1} on $M = \mathbb{F}_p^{n+1}$. We have an exact sequence of S_{n+1}-modules

$$0 \to \mathcal{R}_n(p) \to M \to \mathbb{F}_p \to 0$$

defined by the map $(a_1, \dots, a_{n+1}) \to a_1 + \cdots + a_{n+1}$. The module M is the induced module of the trivial representation of the subgroup S_n of S_{n+1}. By Eckmann-Shapiro's Lemma,

$$H^1(S_{n+1}, M) = H^1(S_n, \mathbb{F}_p) = \mathrm{Hom}(S_n, \mathbb{F}_p).$$

Suppose $p \nmid n+1$, then the exact sequence splits, and we get

$$0 = H^1(S_{n+1}, M) = H^1(S_{n+1}, \mathcal{R}_n(p)) \oplus H^1(S_{n+1}, \mathbb{F}_p).$$

Since $H^1(S_{n+1}, \mathbb{F}_p) = \mathrm{Hom}(S_{n+1}, \mathbb{F}_p) = 0$, we get the result. If $p \mid n+1$, then $H^0(S_{n+1}, M) = 0, H^0(S_{n+1}, \mathbb{F}_p) = \mathbb{F}_p$ and the exact sequence of cohomology gives the desired result.

Type I.

Let us first classify $\mathbb{F}_3$-subspaces of the group $K = 3^3$. We view 3^3 as the S_4-module $\mathcal{R}_3(3) \cong \mathbb{F}_3^3$ from the previous lemma. We denote the image of a vector (a, b, c, d) in K by $[a, b, c, d]$. In our old notations

$$[a, b, c, d] = [\epsilon^a x_0, \epsilon^b x_1, \epsilon^c x_2, \epsilon^d x_3].$$

There are 13 $(= \#\mathbb{P}^2(\mathbb{F}_3))$ one-dimensional subspaces in 3^3. The group S_4 acts on this set with 3 orbits. They are represented by vectors $[1, 2, 0, 0], [1, 1, 1, 0], [1, 1, 2, 2]$ with respective stabilizer subgroups $2^2, S_3$ and D_8. We call them lines of type I,II,III, respectively. As subgroups they are cyclic groups of order 3 of the following types.

$$\begin{cases} 2A_2 & \text{Type I,} \\ 3A_2 & \text{Type II,} \\ A_2 & \text{Type III.} \end{cases}$$

The conjugacy class of a 2-dimensional subspace K is determined by its orthogonal complement in 3^3 with respect to the dot-product pairing on $\mathbb{F}_3^4$. Thus we have 3 types of 2-dimensional subspaces of types determined by the type of its orthogonal complement.

An easy computation gives the following table.

Type	$3A_2$	$2A_2$	A_2	Trace
I	4	2	2	0
II	2	6	0	0
III	0	4	4	2

(44)

Here we list the types of the nontrivial elements in the subspace and the last column is the sum $\frac{1}{9}\sum_{g\in L}\mathrm{Tr}(g|\mathcal{R}_S)$. This gives us two conjugacy classes of minimal subgroups isomorphic to 3^2.

Let G be a subgroup of $\mathrm{Aut}(S)$, $\overline{G}$ be its image in S_4 and $K = G\cap 3^3$. Let $k = \dim_{\mathbb{F}_3} K$.

Case 1: $k = 0$.

In this case G defines a splitting of the projection $3^3 : S_4 \to S_4$. Assume $G \cong S_4$. It follows from Lemma 6.13 that there are 3 conjugacy classes of subgroups isomorphic to S_4 which define a splitting. Let us show each of them is minimal.

We start with the standard S_4 generated by permutations of the coordinates. It contains 6 elements of type $4A_1$, 8 elements of type $2A_2$, 3 elements of type $2A_1$ and 6 elements of type $A_3 + A_1$. Adding up the traces we obtain that the group is minimal.

Suppose G is another subgroup isomorphic but not conjugate to the previous S_4. It corresponds to a 1-cocylce $\phi : S_4 \to \mathbb{F}_3^3$ defined by a vector $v = (a_1, a_2, a_3, a_4)$ with $a_1 + a_2 + a_3 + a_4$

$$\phi_v(\sigma) = (a_1, a_2, a_3, a_4) - (a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}, a_{\sigma(4)}) = (a, b, c, d).$$

The cohomology class of this cocycle depends only on the sum of the coordinates of the vector v. Without loss of generality we may choose $v = (1, 0, 0, 0)$ and drop the subscript in ϕ. We identify the vector $\phi(\sigma)$ with the automorphism $[\epsilon_3^a x_0, \epsilon_3^b x_1, \epsilon_3^c x_2, \epsilon_3^d x_3]$. Thus a new S_4 consists of transformation $\sigma\circ\phi(\sigma)$. We check that the type of an element $\sigma\phi(\sigma)$ is equal to the type of σ for each $\sigma \in S_4$. This shows that the representation of a new S_4 in $\mathcal{R}_S$ has the same character as that of the old S_4. This shows that all three S_4's are minimal.

Let G be mapped isomorphically to a subgroup G' of S_4. If G' is a 2-group, then it is contained in a 2-Sylow subgroup of one of the S_4's. By the above its representation in $\mathcal{R}_S$ is the same as the restriction of the representation of S_4. A 2-Sylow subgroup of S_4 contains 3 elements of type $2A_1$, two element of type $4A_1$ and 2 elements of type $A_3 + A_1$. Adding up the traces we get 8. Thus the group is not minimal.

If G' is not a 2-group, then G' is either S_3 or 3. A lift of a permutation $(i23)$ is given by a matrix

$$\begin{pmatrix} 0 & 0 & \epsilon_3^a \\ \epsilon_3^b & 0 & 0 \\ 0 & \epsilon_3^c & 0 \end{pmatrix},$$

where $a+b+c \equiv 0 \mod 3$. It is immediately checked that all such matrices define an element of type $2A_2$. Adding up the traces we see that a lift of S_3 is minimal.

Case 2: $k = 1$.

Let G' be the image of G in S_4 and $K = G \cap 3^3$. Clearly, the subspace K must be invariant with respect to the restriction of the homomorphism $S_4 \to \mathrm{Aut}(3^3)$ to G'.

Assume K is of type I, say generated by $\mathbf{n}_K = [1,2,0,0]$. Then its stabilizer in S_4 is generated by $(12),(34),(123)$ isomorphic to 2^2. The conjugation by (12) sends $\mathbf{n}_K$ to $-\mathbf{n}_K$, and the conjugation by (34) fixes $\mathbf{n}_K$. Thus the group $K:2^2$ is isomorphic $S_3 \times 2$. It is easy to check that the product $g = (34)\mathbf{n}_K$ is of order 6, of minimal type $A_5 + A_1$. Thus the groups $K:2^2$ and its cyclic subgroup of order 6 are minimal. Also its subgroup of order 6 is minimal. Its subgroup $\langle \mathbf{n}_K, (12)\rangle \cong S_3$ contains 3 elements of type $4A_1$ and 2 elements of type $2A_2$. Adding up the traces we see that S_3 is also minimal. It is obviously not conjugate from the S_3 from the previous case.

Assume K is of type II, say generated by $\mathbf{n}_K = [1,1,1,0]$. Since $\mathbf{n}_K$ is minimal, any subgroup in this case is minimal. The stabilizer of K in S_4 is generated by $(123),(12)$ and is isomorphic to S_3. Our group G is a subgroup of $3\bullet S_3$ with K contained in the center. There are three non-abelian groups of order 18: $D_{18}, 3 \times S_3, 3^2:2$. The extension in the last group is defined by the automorphism of 3^2 equal to the minus identity. In our case the image of G in S_3 acts identically on K. Since the center of D_{18} or $3^2:2$ is trivial, this implies that either G is a cyclic subgroup of D_{18} of order 9 or 3, or a subgroup of $S_3 \times 3$ in which case $G \cong 3, 3^2, 6, 3 \times S_3$. Note that the group 3^2 is not conjugate to a subgroup of 3^3. To realize a cyclic subgroup of order 9 is enough to take $g = \mathbf{n}_K(234)$. Note that the Sylow subgroup of $3\bullet S_3$ is of order 9, so all 3-subgroups of order 9 are conjugate.

Assume K is of type III, say generated by $\mathbf{n}_K = [1,1,2,2]$. The stabilizer group is generated by $(12),(34),(13)(24)$ and is isomorphic to D_8. Our group G is a subgroup of $3:D_8$. The split extension is defined by the homomorphism $D_8 \to 2$ with kernel $\langle (12),(34)\rangle \cong 2^2$. The subgroup isomorphic to D_8 is contained in a non-minimal S_4, hence is not minimal. Let $H = \langle \mathbf{n}_K, (12), (34)\rangle \cong 6 \times 2$ so that $3:D_8 \cong H:2$. The subgroup H contains 4 elements of type D_4, 4 elements of type A_2, 2 elements of type A_1 and one element of type $2A_1$. Adding up the traces of elements from H we get the sum equal to 24. The nontrivial coset contains 8 elements of type $A_3 + A_1$, 4 elements of type $4A_1$ and one element of type $2A_1$. Adding up the traces we get 0. Thus the group is not minimal. So this case does not reveal any minimal groups.

Case 3: $k = 2$.

The image of G in S_4 is contained in the stabilizer of the orthogonal vector $\mathbf{n}_K$. Thus G is a subgroup of one of the following groups

$$G = \begin{cases} K : 2^2 & \text{if } K \text{ is of type I,} \\ K \bullet S_3 & \text{if } K \text{ is of type II,} \\ K : D_8 & \text{if } K \text{ is of type III.} \end{cases}$$

Since K of type I or II contains a minimal element of order 3, the subgroups of $K : 2^2$ and $K \bullet S_3$ are minimal. Recall that they all contain K.

Assume K is of type I. Recall that the S_4-module 3^3 is isomorphic to the root lattice $\mathcal{R}_3(3)$ of A_3 modulo 3. A permutation σ of order 2 represented by the transposition (12) decomposes the module into the sum of one-dimensional subspaces with eigenvalues $-1, 1, 1$. So, if σ fixes $\mathbf{n}_K$ it acts on K with eigenvalues $-1, 1$. Otherwise it acts identically on K. The product (12)(34) acts with eigenvalues $(-1, -1, 1)$, so if it fixes $\mathbf{n}_K$ then it acts as the minus identity on K. In our case $\mathbf{n}_K = [1, 2, 0, 0]$ and $(12), (12)(34) \in 2^2$ fix $\mathbf{n}_K$. Accordingly, (12) acts as $(-1, 1)$ giving a subgroup $K : 2 \cong 3 \times S_3$, (12)(34) gives the subgroup $K : 2 \cong 3^2 : 2 \ncong 3^2 : 2$. Finally, (34) acts identically on K giving the subgroup $3^2 \times 2$. So we obtain 3 subgroups of index 2 of $K : 2^2$ isomorphic to $3^2 : 2, 3^2 \times 2, S_3 \times 3$. The remaining subgroups are $K \cong 3^2$ and $K : 2^2 \cong 3^2 : 2^2$.

Assume K is of type II. Again we have to find all subgroups H of $G = K \bullet S_3$ containing K. Elements of order 2 are transpositions in S_3. They fix $\mathbf{n}_K$. Arguing as above we see that $K : 2 \cong 3 \times S_3$. An element of order 3 in S_3 fixes $\mathbf{n}_K$. Hence it acts in the orthogonal complement as an element of type A_2 in the root lattice of type A_2 modulo 3. This defines a unique non-abelian group of order 27 isomorphic to the Heisenberg group $H_3(3)$. The third group is $K \bullet S_3 \cong H_3(2) : 2 \cong 3(3^2 : 2)$.

Assume K is of type III. This time K is not minimal. Each subgroup of order 2 of type $4A_1$ of D_8 defines a subgroup $K : 2 \cong 3 \times S_3$ of $K : D_8$. It contains 3 elements of order 2, of type $4A_1$, 6 elements of order 6, of type D_4, and the elements from K. Adding up the traces we get the sum equal to 18. So the subgroup is not minimal. An element of order 2 of type $2A_1$ defines a subgroup $3^2 : 2$ not isomorphic to $3 \times S_3$. It contains 9 elements of type $2A_1$, and the elements from K. Adding up the traces we get the sum equal to 36. So the group is not minimal.

Assume $G \cong K : 4$. It contains the previous group $3^2 : 2$ as a subgroup of index 2. It has 9 2-Sylow subgroups of order 4. Thus the nontrivial coset consists of 18 elements of order 4. Each element of order 4 has the trace equal tom 0. This shows that the group is not minimal.

Finally it remains to investigate the group $K : D_8$. It contains the previous group $K : 4$ as a subgroup of index 2. The sum of the traces of its elements is equal to 36. The nontrivial coset consists of the union of 4 subsets, each is equal to the set of nontrivial elements in the group of type $K : 2 \cong 3 \times S_3$. The sum of traces of elements in each subset is equal to 12. So the total sum is 72 and the group is not minimal.

Case 3: $k = 3$.

This gives the groups $3^3 : H$, where H is a subgroup of S_4 which acts on 3^3 via the restriction of the homomorphism $S_4 \to \mathrm{Aut}(3^3)$ describing the action

of $W(A_3)$ on its root lattice modulo 3. We have the groups

$$3^3 : S_4,\ 3^3 : D_8,\ 3^3 : S_3,\ 3^3 : 2^2(2),\ 3^3 : 3,\ 3^3 : 4,\ 3^3 : 2(2).$$

Type II.

The surface is isomorphic to the Segre cubic surface in $\mathbb{P}^4$ given by the equations

$$\sum_{i=0}^{4} x_i^3 = \sum_{i=0}^{4} x_i = 0.$$

The group S_5 acts by permuting the coordinates. The orbit of the line $x_0 = x_1 + x_2 = x_3 + x_4 = 0$ consists of 15 lines. It is easy to see that the remaining 12 lines form a double-six. The lines in the double-six are described as follows.

Let ω be a primitive 5th root of unity. Let $\sigma = (a_1, \dots, a_5)$ be a permutation of $\{0, 1, 2, 3, 4\}$. Each line ℓ_σ is the span by a pair of points $(\omega^{a_1}, \dots, \omega^{a_5})$ and $(\omega^{-a_1}, \dots, \omega^{-a_5})$. This gives 12 different lines. One immediately checks that $\ell_\sigma \cap \ell_{\sigma'} \neq \emptyset$ if and only if $\sigma' = \sigma \circ \tau$ for some odd permutation τ. Thus the orbit of the alternating subgroup A_5 of any line defines a set of 6 skew lines (a sixer) and therefore A_5 is not minimal. Let $\ell_1, \dots, \ell_6$ be a sixer. It is known that the divisor classes ℓ_i, K_S span $\mathrm{Pic}(S) \otimes \mathbb{Q}$. This immediately implies that $\mathrm{Pic}(S)^{A_5}$ is spanned (over $\mathbb{Q}$) by K_S and the sum $\sum \ell_i$. Since S_5 does not leave this sum invariant, we see that S_5 is a minimal group.

A maximal subgroup of S_5 not contained in A_5 is isomorphic to S_4, or $5 : 4$, or $2 \times S_3$. The subgroups isomorphic to S_4 are conjugate so we may assume that it consists of permutations of $1, 2, 3, 4$. The group has 6 elements of type $4A_1$ conjugate to (12), 3 elements of type $2A_1$ conjugate to (12)(34), 8 elements of type $2A_2$ conjugate to (123) and 6 elements of type $A_3 + A_1$ conjugate to (1234). The total sum of the traces is equal to 0. So the group is minimal. This gives another, non-geometric proof of the minimality of S_5.

Consider a 2-Sylow subgroup G of S_4 isomorphic to D_8. It consists of 5 elements of order 2, two of type $4A_1$ (from the conjugacy class of (12)) and 3 of type $2A_1$ (from the conjugacy class of (12)(34)). Its cyclic subgroup of order 4 is generated by an element of type $A_3 + A_1$. Adding up the traces we see that the subgroup is not minimal. Thus S_4 has no minimal proper subgroups.

A subgroup isomorphic to $5 : 4$ is conjugate to a subgroup generated by two cycles (01234) and (0123). Computing the traces, we find that the group is not minimal. The subgroup isomorphic to $2 \times S_3$ is conjugate to a subgroup generated by $(012), (01), (34)$. Its element of order 6 belongs to the conjugacy class D_4. So this group is different from the isomorphic group in the previous case. Computing the traces we find that it is not minimal.

Type III.

The surface is a specialization of a surface of type IV. Recall that each nonsingular plane cubic curve is isomorphic to a member of the Hesse pencil

$$T_1^3 + T_2^3 + T_3^3 + 6aT_1T_2T_3 = 0.$$

The group of projective automorphisms leaving the pencil invariant is the Hesse group G_{216} of order 216. It is isomorphic to $3^2 : \overline{T}$. The stabilizer of a general member of the pencil is isomorphic to a non-abelian extension $3^2 : 2$. It is generated by

$$g_1 = [x_1, \epsilon_3 x_2, \epsilon_3^2 x_3],\ g_2 = [x_2, x_3, x_1],\ g_3 = [x_2, x_1, x_3].$$

The pencil contains 6 members isomorphic to a harmonic cubic. They correspond to the values of the parameters satisfying the equation $8a^6 + 20a^3 - 1 = 0$. The stabilizer of a harmonic member is the group $3^2 : 4$. The additional generator is given by the matrix

$$g_4 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \epsilon_3 & \epsilon_3^2 \\ 1 & \epsilon_3^2 & \epsilon_3 \end{pmatrix}$$

The pencil also contains 4 anharmonic cubics isomorphic to the Fermat cubic. They correspond to the parameters a satisfying the equation $a^4 - a = 0$. The stabilizer of an anharmonic member is isomorphic to $3^2 : 6$. The additional generator is given by $g_5 = [x_1, \epsilon_3 x_2, \epsilon_3 x_3]$.

All curves from the Hesse pencil have 9 common inflection points. If we fix one of them, say $(1, -1, 0)$, all nonsingular members acquire a group law. The group of automorphisms generated by g_1, g_2 correspond to translations by 3-torsion points. The automorphism g_3 is the negation automorphism. The automorphism g_4 is the complex multiplication by $\sqrt{-1}$. The automorphism g_5 is the complex multiplication by $e^{2\pi i/3}$.

The Hesse group admits a central extension $3G_{216} \cong H_3(3) : \overline{T}$ realized as the complex reflection group in $\mathbb{C}^3$. It acts linearly on the variables T_1, T_2, T_3 leaving the polynomial $T_1^3 + T_2^3 + T_3^3 + 6aT_1T_2T_3$ unchanged. We denote by $\tilde{g}_i$ the automorphism of the cubic surface obtained from the automorphism g_i of the Hesse pencil by acting identically on the variable T_0. The center of the group $3G_{216}$ is generated by $c = [\tilde{g}_1, \tilde{g}_2] = [1, \epsilon_3, \epsilon_3, \epsilon_3]$. This is an element of order 3, of minimal type $3A_2$.

Now we have a complete description of the automorphism group of a surface of type IV. Any minimal subgroup of $H_3(3) : 2$ can be found among minimal subgroups of surfaces of type I. However, we have 2 non-conjugate subgroups of type S_3 equal to $\langle \tilde{g}_1, \tilde{g}_3 \rangle$ and $\langle \tilde{g}_2, \tilde{g}_3 \rangle$, and two non-conjugate subgroups in $S_3 \times 3$ obtained from the previous groups by adding the central element c.

Surfaces of type III acquire additional minimal subgroups of the form $A : 4$, where A is a subgroup of $H_3(3)$. The element $\tilde{g}_4$ acts by conjugation on the subgroup $H_3(3)$ via $(\tilde{g}_1, \tilde{g}_2) \mapsto (\tilde{g}_2^2, \tilde{g}_1)$. Using $\tilde{g}_4$, we can conjugate the subgroups isomorphic to $S_3, 3 \times S_3, 3^2$. Also we get two new minimal groups $H_3(3) : 4$ and 12.

Type V.

The group $S_4 \cong 2^2 : S_3$ acts by permuting the coordinates T_1, T_2, T_3 and multiplying them by -1 leaving the monomial $T_1T_2T_3$ unchanged. To make the action explicit, we identify 2^2 with the subspace of $\mathbb{F}_2^3$ of vectors whose coordinates add up to 0. The semi-direct product corresponds to the natural action of S_3 by permuting the coordinates. Thus $g = ((a,b,c),\sigma) \in 2^2 : S_3$ acts as the transformation $[x_0, (-1)^a x_{\sigma(1)}, (-1)^b x_{\sigma(2)}, (-1)^c x_{\sigma(3)}]$. It is easy to compute the types of elements of S_4 in their action on S. The group contains 3 elements of type $2A_1$, 6 elements of type $4A_1$, 8 elements of type $2A_2$ and 6 elements of type $A_3 + A_1$. Adding up the traces we see that the group is minimal. The subgroup S_3 is minimal. No other subgroup is minimal.

Type VII.

The automorphism group of the surface of type VII is a non-minimal cyclic group of order 8.

Type X.

The automorphism group of the surface of type X consists of the identity, two involutions of type $4A_1$ and one involution of type $2A_1$. Adding up the traces, we get that the group is not minimal.

Let us summarize our result in the following.

Theorem 6.14. *Let G be a minimal subgroup of automorphisms of a nonsingular cubic surface. Then G is isomorphic to one of the following groups.*

1. *G is a subgroup of automorphisms of a surface of type I.*

$$S_4(3),\ S_3\ (2),\ S_3 \times 2,\ S_3 \times 3\ (2),\ 3^2 : 2\ (2),\ 3^2 : 2^2,$$
$$H_3(3) : 2,\ H_3(3),\ 3^3 : 2\ (2),\ 3^3 : 2^2\ (2),\ 3^3 : 3,\ 3^3 : S_3,\ 3^3 : D_8,\ 3^3 : S_4,\ 3^3 : 4,$$
$$3^3,\ 3^2\ (3),\ 3^2 \times 2,\ 9,\ 6\ (2),\ 3.$$

2. *G is a subgroup of automorphisms of a surface of type II.*

$$S_5, \quad S_4.$$

3. *G is a subgroup of automorphisms of a surface of type III.*

$$H_3(3) : 4,\ H_3(3) : 2,\ H_3(3),\ S_3 \times 3,\ S_3,\ 3^2,\ 12,\ 6,\ 3.$$

4. *G is a subgroup of automorphisms of a surface of type IV.*

$$H_3(3) : 2,\ H_3(3),\ S_3\ (2),\ 3 \times S_3\ (2),\ 3^2\ (2),\ 6,\ 3.$$

5. *G is a subgroup of automorphisms of a surface of type V.*

$$S_4,\ S_3.$$

6. *G is a subgroup of automorphisms of a surface of type VI.*

$$6, \quad S_3 \times 2, \quad S_3.$$

7. *G is a subgroup of automorphisms of a surface of type VIII.*

$$S_3.$$

6.6 Automorphisms of Del Pezzo surfaces of degree 2

Recall that the center of the Weyl group $W(E(7))$ is generated by an element w_0 which acts on the root lattice as the negative of the identity. Its conjugacy class is of type A_1^7. The quotient group $W(E_7)' = W(E_7)/(\langle w_0\rangle$ is isomorphic to the simple group $\mathrm{Sp}(6, \mathbb{F}_2)$. The extension $2.\mathrm{Sp}(6, \mathbb{F}_2)$ splits by the subgroup $W(E_7)^+$ equal to the kernel of the determinant homomorphism $\det : W(E_7) \to \{\pm 1\}$. Thus we have

$$W(E_7) = W(E_7)^+ \times \langle w_0\rangle.$$

Let H be a subgroup of $W(E_7)'$. Denote by H^+ its lift to an isomorphic subgroup of W^+. Any other isomorphic lift of H is defined by a nontrivial homomorphism $\alpha : H \to \langle w_0\rangle \cong 2$. Its elements are the products $h\alpha(h), h \in H^+$. We denote such a lift by H_α. Thus all lifts are parametrized by the group $\mathrm{Hom}(H, \langle w_0\rangle)$and H^+ corresponds to the trivial homomorphism. Note that $wH_\alpha w^{-1} = (w'Hw'^{-1})_\alpha$, where w' is the image of w in $W(E_7)'$. In particular, two lifts of the same group are never conjugate.

Now we apply this to our geometric situation. Let S be a Del Pezzo surface of degree 2. Recall that the map $S \to \mathbb{P}^2$ defined by $|-K_S|$ is a degree 2 cover. Its branch curve is a nonsingular curve of degree 4. It is convenient to view a Del Pezzo surface of degree 2 as a hypersurface in the weighted projective space $\mathbb{P}(1, 1, 1, 2)$ given by an equation of degree 4

$$T_3^2 + F_4(T_0, T_1, T_2) = 0. \tag{45}$$

The automorphism of the cover $\gamma = [t_0, t_1, t_2, -t_3]$ defines the conjugacy class of a Geiser involution of $\mathbb{P}^2$. For any divisor class D on S we have $D+\gamma_0^*(D) \in |-mK_S|$ for some integer m. This easily implies that γ^* acts as the minus identity in $\mathcal{R}_S$. Its image in the Weyl group $W(E_7)$ is the generator w_0 of its center. Thus the Geiser involution is the geometric realization of w_0.

Let $\rho : \mathrm{Aut}(S) \to W(E_7)$ be the natural injective homomorphism corresponding to a choice of a geometric basis in $\mathrm{Pic}(S)$. Denote by $\mathrm{Aut}(S)^+$ the full preimage of $W(E_7)^+$. Since $W(E_7)^+$ is a normal subgroup, this definition is independent of a choice of a geometric basis. Under the restriction homomorphism $\mathrm{Aut}(S) \to \mathrm{Aut}(B)$ the group $\mathrm{Aut}(S)^+$ is mapped isomorphically to $\mathrm{Aut}(B)$ and we obtain

$$\mathrm{Aut}(S)^+ \cong \mathrm{Aut}(S)/\langle\gamma\rangle \cong \mathrm{Aut}(B).$$

¿From now on we will identify any subgroup G of $\mathrm{Aut}(B)$ with a subgroup of $\mathrm{Aut}(S)$ which we call the *even lift* of G. Under the homomorphism $\rho : \mathrm{Aut}(S) \to W(E_7)$ all elements of G define *even conjugacy classes*, i.e. the conjugacy classes of elements from $W(E_7)^+$. It is immediate to see that a conjugacy class is even if and only if the sum of the subscripts in its Carter graph is even. An isomorphic lift of a subgroup G to a subgroup of $\mathrm{Aut}(S)$

corresponding to some nontrivial homomorphism $G \to \langle\gamma\rangle$ (or, equivalently to a subgroup of index 2 of G) will be called an *odd lift* of G.

The odd and even lifts of the same group are never conjugate, two minimal lifts are conjugate in $\mathrm{Aut}(S)$ if and only if the groups are conjugate in $\mathrm{Aut}(B)$. Two odd lifts of G are conjugate if and only if they correspond to conjugate subgroups of index 2 (inside of the normalizer of G in $\mathrm{Aut}(B)$).

The following simple lemma will be heavily used.

Lemma 6.15. *Let G be a subgroup of $\mathrm{Aut}(B)$ and H be its subgroup of index 2. Assume H is a minimal subgroup of $\mathrm{Aut}(S)$ (i.e. its even lift is such a subgroup). Then G is minimal in its even lift and its odd lift corresponding to H. Conversely, if G is minimal in both lifts, then H is a minimal subgroup.*

Proof. Let $\mathrm{Tr}(g)_\pm$ be the trace of $g \in G$ in the representation of G in $\mathcal{R}_S$ corresponding to the minimal (resp. odd) lift. Suppose G is minimal in both lifts. Then

$$\sum_{g\in G} \mathrm{Tr}_+(g) = \sum_{g\in H} \mathrm{Tr}_+(g) + \sum_{g\notin H} \mathrm{Tr}_+(g) = 0,$$

$$\sum_{g\in G} \mathrm{Tr}_-(g) = \sum_{g\in H} \mathrm{Tr}_-(g) + \sum_{g\notin H} \mathrm{Tr}_-(g) =$$

$$= \sum_{g\in H} \mathrm{Tr}_+(g) - \sum_{g\notin H} \mathrm{Tr}_+(g) = 0.$$

This implies that $\sum_{g\in H} \mathrm{Tr}_+(g) = 0$, i.e. H is minimal. The converse is obviously true.

Since γ generates a minimal subgroup of automorphisms of S, any group containing γ is minimal. So, we classify first subgroups of $\mathrm{Aut}(B)$ which admit minimal lifts. These will be all minimal subgroups of $\mathrm{Aut}(S)$ which do not contain the Geiser involution γ. The remaining minimal groups will be of the form $\langle\gamma\rangle \times \widetilde{G}$, where $\widetilde{G}$ is any lift of a subgroup G of $\mathrm{Aut}(B)$. Obviously, the product does not depend on the parity of the lift.

As in the case of cubic surfaces we first classify cyclic subgroups.

Lemma 6.16. *Let g be an automorphism of order $n > 1$ of a nonsingular plane quartic $C = V(F)$. Then one can choose coordinates in such a way that $g = [t_0, \epsilon_n^a t_1, \epsilon_n^b t_2]$ and F is given in the following list.*

(i) $(n = 2), (a, b) = (0, 1)$,

$$F = T_2^4 + T_2^2 L_2(T_0, T_1) + L_4(T_0, T_1).$$

(ii) $(n = 3), (a, b) = (0, 1)$,

$$F = T_2^3 L_1(T_0, T_1) + L_4(T_0, T_1).$$

(iii) $(n = 3), (a, b) = (1, 2)$,

$$F = T_0^4 + \alpha T_0^2 T_1 T_2 + T_0 T_1^3 + T_0 T_2^3 + \beta T_1^2 T_2^2.$$

(iv) $(n = 4), (a, b) = (0, 1)$,

$$F = T_2^4 + L_4(T_0, T_1).$$

(v) $(n = 4), (a, b) = (1, 2)$,

$$F = T_0^4 + T_1^4 + T_2^4 + \alpha T_0^2 T_2^2 + \beta T_0 T_1^2 T_2.$$

(vi) $(n = 6), (a, b) = (3, 2)$,

$$F = T_0^4 + T_1^4 + \alpha T_0^2 T_1^2 + T_0 T_2^3.$$

(vii) $(n = 7), (a, b) = (3, 1)$,

$$F = T_0^3 T_1 + T_1^3 T_2 + T_2^3 T_0 + \alpha T_0 T_1^2 T_2.$$

(viii) $(n = 8), (a, b) = (3, 7)$,

$$F = T_0^4 + T_1^3 T_2 + T_1 T_2^3.$$

(ix) $(n = 9), (a, b) = (3, 2)$,

$$F = T_0^4 + T_0 T_1^3 + T_2^3 T_1.$$

(x) $(n = 12), (a, b) = (3, 4)$,

$$F = T_0^4 + T_1^4 + T_0 T_2^3.$$

Here the subscript in polynomial L_i indicates its degree.

Also observe that the diagonal matrix (t, t, t, t^2) acts identically on S.

Let $g \in \mathrm{Aut}(B)$ be an element of order n of type $(*)$ from the previous Lemma. The following Table identifies the conjugacy class of two lifts $\tilde{g}$ of g in the Weyl group $W(E_7)$. If n is even, then g admits two lifts in $\mathrm{Aut}(S)$ of order n. If n is odd, then one of the lifts is of order n and another is of order $2n$. We denote by $(*)_+$ the conjugacy class of the lift which is represented by an element from $W(E_7)^+$ (of order n if n is odd). The conjugacy class of another lift is denoted by $(*)_-$. The last column of the Table gives the trace of g on $\mathcal{R}_S$.

The following is the list of elements of finite order which generate a minimal cyclic group of automorphisms. To identify the conjugacy class of a minimal lift we use the trace formula (36). If both lifts have the same trace, we distinguish them by computing the traces of their powers.

Type	Order	Notation	Trace
$(0)_-$	2	$7A_1$	-7
$(i)_+$	2	$4A_1$	-1
$(i)_-$	2	$3A_1$	1
$(ii)_+$	3	$3A_2$	-2
$(ii)_-$	6	$E_7(a_4)$	2
$(iii)_+$	3	$2A_2$	1
$(iii)_-$	6	$D_6(a_2)+A_1$	-1
$(iv)_+$	4	$D_4(a_1)$	3
$(iv)_-$	4	$2A_3+A_1$	-3
$(v)_+$	4	$2A_3$	-1
$(v)_-$	4	$D_4(a_1)+A_1$	1
$(vi)_+$	6	$E_6(a_2)$	2
$(vi)_-$	6	A_2+A_5	-2
$(vii)_+$	7	A_6	0
$(vii)_-$	14	$E_7(a_1)$	0
$(viii)_+$	8	D_5+A_1	-1
$(viii)_-$	8	D_5	1
$(ix)_+$	9	$E_6(a_1)$	1
$(ix)_-$	18	E_7	-1
$(x)_+$	12	E_6	0
$(x)_-$	12	$E_7(a_2)$	0

Table 5. Conjugacy classes of automorphisms of a Del Pezzo surface of degree 2

1. Order 2 (A_1^7) (The Geiser involution) $g=[t_0,t_1,t_2,-t_3]$

$$F=T_3^2+F_4(T_0,T_1,T_2).$$

2. Order 4 ($2A_3+A_1$) $g=[t_0,t_1,it_2,t_3]$

$$F=T_3^2+T_2^4+L_4(T_0,T_1).$$

3. Order 6 ($E_7(a_4)$) $g=[t_0,t_1,\epsilon_3 t_2,-t_3]$

$$F=T_3^2+T_2^3L_1(T_0,T_1)+L_4(T_0,T_1).$$

4. Order 6 (A_5+A_2) $g=[t_0,-t_1,\epsilon_3 t_2,-t_3]$

$$F=T_3^2+T_0^4+T_1^4+T_0T_2^3+aT_0^2T_1^2.$$

5. Order 6 ($D_6(a_2)+A_1$) $g=[t_0,\epsilon_3 x_1,\epsilon_3^2 x_2,-x_3]$

$$F=T_3^2+T_0(T_0^3+T_1^3+T_2^3)+T_1T_2(\alpha T_0^2+\beta T_1T_2).$$

6. Order 12 ($E_7(a_2)$) $g=[t_0,\epsilon_4 t_1,\epsilon_3 t_2,t_3]$

$$F=T_3^2+T_0^4+T_1^4+T_0T_2^3,\ (t_0,t_1,t_2,t_3).$$

7. Order 14 ($E_7(a_1)$) $g = [t_0, \epsilon_4 t_1, \epsilon_3 t_2, t_3]$

$$F = T_3^2 + T_0^3 T_1 + T_1^3 T_2 + T_2^3 T_0.$$

8. Order 18 (E_7) $g = [t_0, \epsilon_3 t_1, \epsilon_9^2 t_2, -t_3]$

$$F = T_3^2 + T_0^4 + T_0 T_1^3 + T_2^3 T_1.$$

Using the information about cyclic groups of automorphisms of plane quartics, it is not hard to get the classification of possible automorphism groups (see [26]). It is given in Table 5.

Type	Order	Structure	Equation	Parameters
I	336	$2 \times L_2(7)$	$T_3^2 + T_0^3 T_1 + T_1^3 T_2 + T_2^3 T_0$	
II	192	$2 \times (4^2 : S_3)$	$T_3^2 + T_0^4 + T_1^4 + T_2^4$	
III	96	$2 \times 4A_4$	$T_3^2 + T_2^4 + T_0^4 + aT_0^2 T_1^2 + T_1^4$	$a^2 = -12$
IV	48	$2 \times S_4$	$T_3^2 + T_2^4 + T_1^4 + T_0^4 +$ $+a(T_0^2 T_1^2 + T_0^2 T_2^2 + T_1^2 T_2^2)$	$a \neq \frac{-1 \pm \sqrt{-7}}{2}$
V	32	$2 \times AS_{16}$	$T_3^2 + T_2^4 + T_0^4 + aT_0^2 T_1^2 + T_1^4$	$a^2 \neq 0, -12, 4, 36$
VI	18	18	$T_3^2 + T_0^4 + T_0 T_1^3 + T_1 T_2^3$	
VII	16	$2 \times D_8$	$T_3^2 + T_2^4 + T_0^4 + T_1^4 + aT_0^2 T_1^2 + bT_2^2 T_0 T_1$	$a, b \neq 0$
VIII	12	2×6	$T_3^2 + T_2^3 T_0 + T_0^4 + T_1^4 + aT_0^2 T_1^2$	
IX	12	$2 \times S_3$	$T_3^2 + T_2^4 + aT_2^2 T_0 T_1 + T_2(T_0^3 + T_1^3) + bT_0^2 T_1^2$	
X	8	2^3	$T_3^2 + T_2^4 + T_1^4 + T_0^4$ $+aT_2^2 T_0^2 + bT_1^2 T_2^2 + cT_0^2 T_1^2$	distinct $a, b, c \neq 0$
XI	6	6	$T_3^2 + T_2^3 T_0 + L_4(T_0, T_1)$	
XII	4	2^2	$T_3^2 + T_2^4 + T_2^2 L_2(T_0, T_1) + L_4(T_0, T_1)$	
XIII	2	2	$T_3^2 + F_4(T_0, T_1, T_2)$	

Table 6. Groups of automorphisms of Del Pezzo surfaces of degree 2

Next we find minimal subgroups of automorphisms of a Del Pezzo surface of degree 2.

As in the previous case it is enough to consider surfaces S' which are not specialized to surfaces S of other types. When this happens we write $S' \to S$. We have

$$IX \to IV \to I, II,$$

$$XII \to X \to VII \to V \to II, III,$$

$$XI \to VIII \to III.$$

All of this is immediate to see, except the degeneration $VIII \to III$. This is achieved by some linear change of variables transforming the form $x^3 y + y^4$ to the form $u^4 + 2\sqrt{3} i u^2 v^2 + v^4$. So it suffices to consider surfaces of types I, II, III, VI.

Before we start the classification we advice the reader to go back to the beginning of the section and recall the concepts of odd and even lifts of subgroups of $\mathrm{Aut}(B)$.

Type I.

Since $L_2(7)$ has no subgroups of index 2 (in fact, it is a simple group), it admits a unique lift to a subgroup of $\mathrm{Aut}(S)$. It is known that the group $L_2(7)$ is generated by elements of order 2, 3 and 7. Consulting Table 4, we find that an element of order 2 must be of type $4A_1$, an element of order 3 must be of types $3A_2$ or $2A_2$, and element of order 7 is of type A_6. To decide the type of a generator g of order 3, we use that it acts as a cyclic permutation of the coordinates in the plane, hence has 3 fixed points $(1,1,1), (1,\eta_3,\eta_3^2), (1,\eta_3^2,\eta_3)$. The last two of them lie on the quartic. This easily implies that g has 4 fixed points on S, hence its trace in $\mathrm{Pic}(S)$ is equal to 2. This implies that g is of type $2A_2$. Comparing the traces with the character table of the group $L_2(7)$ we find that the representation of $L_2(7)$ in $(\mathcal{R}_S)\otimes\mathbb{C}$ is an irreducible 7-dimensional representation of $L_2(7)$. Thus the group is minimal.

Assume G is a proper subgroup of $L_2(7)$. It is known that maximal subgroups of $L_2(7)$ are isomorphic to S_4 or $7:3$. There are two conjugacy classes of subgroups isomorphic to S_4 (in the realization $L_2(7)\cong L_3(2)$ they occur as the stabilizer subgroups of a point or a line in $\mathbb{P}^2(\mathbb{F}_2)$). Since S_4 contains a unique subgroup of index 2, each subgroup can be lifted in two ways. Consider the even lift of S_4 lying in $L_2(7)$. To find the restriction of the 7-dimensional representation $V_7=(\mathcal{R}_S)_{\mathbb{C}}$ to G we apply the Frobenius Reciprocity formula. Let χ_k denote a k-dimensional irreducible representation of $L_2(7)$ and $\bar{\chi}_k$ be its restriction to S_4. It is known that the induced character of the trivial representation of S_4 is equal to $\chi_1+\chi_6$ (see [18]). Applying the Frobenius Reciprocity formula we get $\langle\bar{\chi}_1,\bar{\chi}_7\rangle=\langle\chi_1+\chi_6,\chi_7\rangle=0$. This computation shows that the even lifts of the two conjugacy classes of S_4 in $L_2(7)$ are minimal subgroups. It follows from Lemma 6.15 the the odd lifts are minimal only if the lift of the subgroup A_4 of S_4 is minimal. One checks that the induced character of the trivial representation of A_4 is equal to $\chi_1+\chi_6+\chi_7$. By the Frobenius Reciprocity formula, the restriction of V_7 to A_4 contains the trivial summand. Thus A_4 is not minimal and we conclude that there are only 2 non-conjugate lifts of S_4 to a minimal subgroup of $\mathrm{Aut}(S)$.

Next consider the subgroup $7:3$. It admits a unique lift. The induced representation of its trivial representation has the character equal to $\chi_1+\chi_7$. Applying the Frobenius Reciprocity formula, we see that this group is not minimal.

Let H be any subgroup of $L_2(7)$ which admits a minimal lift. Since $\mathrm{Aut}(S)$ does not contain minimal elements of order 3 or 7, H must be a subgroup of S_4. Since A_4 does not admit a minimal lift, H is either a cyclic group or isomorphic to either 2^2 or D_8. The only cyclic group which may admit a minimal lift is a cyclic group of order 4. However, the character table for $L_2(7)$ shows that the value of the character χ_7 at an element of order 4 is equal to -1, hence

it is of type $2A_3$. It follows from the Table that this element does not admit minimal lifts.

Suppose $G \cong 2^2$. In the even lift, it contains 3 nontrivial elements of type $4A_1$. Adding up the traces we see that this group is not minimal. In the odd lift, it contains one element of type $4A_1$ and two of type $3A_1$. Again, we see that the group is not minimal.

Assume $G \cong D_8$. The group S_4 is the normalizer of D_8. This shows that there are two conjugacy classes of subgroups isomorphic to D_8. The group G admits 2 lifts. In the even lift it contains 2 elements of type $2A_3$ and 5 elements of type $4A_1$. Adding up the traces, we obtain that the lift is minimal. Since the lift of 4 is not minimal, the odd lift of D_8 is not minimal.

Type II.

The group $\mathrm{Aut}(B)$ is generated by the transformations

$$g_1 = [t_0, it_1, t_2, -t_3],\ \tau = [t_1, t_0, t_2, t_3],\ \sigma = [t_0, t_2, t_1, t_3]$$

of types $D_4(a_1), 4A_1, 4A_1$, respectively. Let $g_2 = \sigma g_1 \sigma^{-1} = [t_0, t_1, it_2, -t_3]$.

We have

$$\tau g_1 \tau^{-1} = g_1^{-1} g_2^{-1},\ \tau g_2 \tau^{-1} = g_2.$$

The elements g_1, g_2, γ generate a normal subgroup isomorphic to 4^2. The quotient group is isomorphic to S_3. Its generators of order 2 can be represented by τ and σ. The elements $g_1^2, g_2^2, \tau, \sigma$ generate a subgroup (not normal) isomorphic to S_4. Thus

$$\mathrm{Aut}(B) \cong 4^2 : S_3 \tag{46}$$

and

$$\mathrm{Aut}(S) \cong 2 \times (4^2 : S_3). \tag{47}$$

Consider the natural homomorphism $f : \mathrm{Aut}(B) \to S_3$ with kernel 4^2. We will consider different cases corresponding to a possible image of a subgroup $G \subset \mathrm{Aut}(B)$ in S_3. For the future use we observe that $\mathrm{Aut}(B)$ does not contain elements of order 6 because its square is an element of type $3A_2$ but all our elements of order 3 are of type $2A_2$. Also it does not contain 2^3 (this follow from the presentation of the group). We will also use that $\mathrm{Aut}(B)$ contains 2 conjugacy classes of elements of order 4 of types $D_4(a_1)$ (represented by g_1) and $2A_3$ (represented by $g_1 g_2$).

Case 3: $f(G) = \{1\}$.

In this case G is a subgroup of 4^2. The group itself contains 3 elements of type $4A_1$, 6 elements of type $D_4(a_1)$ and 6 elements of type $2A_3$. The sum of the traces is equal to 16. Thus the group is not minimal. So no subgroup is minimal in the even lift. An odd lift corresponding to the homomorphism $4^2 \to \langle\gamma\rangle$ sending an element of type $D_4(a_1)$ to γ defines an odd lift. There is only one conjugacy class of subgroups of index 2 in 4^2. It defines an odd lift of 4^2. We may assume that the subgroup of index 2 is generated by g_1, g_2^2. It admits two odd lifts corresponding to the subgroups $\langle g_1^2, g_2^2\rangle$ and $\langle g_1 g_2^2\rangle$. Finally a cyclic

subgroup 4 of type $D_4(a_1)$ admits an odd lift. No other subgroup admits a minimal lift.

Case 2: $|f(G)| = 2$.

Replacing the group by a conjugate group, we may assume that $f(G) = \langle\tau\rangle$. We have

$$G_1 = f^{-1}(\langle\tau\rangle) = \langle\tau, g_1, g_2\rangle \cong 4^2 : 2 \cong 4D_8,$$

where the center is generated by g_2.

Let $H = \langle\tau, g_1^2 g_2\rangle$. One immediately checks that H is normal in G_1 and isomorphic to D_8. We have $G_1 \cong D_8 : 4$. The subgroup H consists of 5 elements of type $4A_1$ and 2 elements of type $2A_3$. Adding up the traces we obtain that H is minimal in its even lift. Thus G_1 is minimal in its even lift. The subgroup $G_2 = \langle\tau, g_1^2, g_2\rangle$ is of order 16. It contains H defining a split extension $D_8 : 2$ with center generated by g_2. It is isomorphic to the group AS_{16} (see Table 1) is of index 2 in G_1. Since it is minimal, the odd lift of G_1 corresponding to this subgroup is minimal.

We check that τg_1 is of order 8 and the normalizer of the cyclic group $\langle\tau g_1\rangle$ is generated by this subgroup and g_1^2. This gives us another subgroup G_3 of index 2 of G_1. It is a group of order 16 isomorphic to M_{16}. An element of order 8 is of type $D_5(a_1) + A_1$. Thus the sum of the traces is equal to 8. Adding up the traces of elements in the nontrivial coset of $\langle\tau g_1\rangle$ we get that the sum is equal to -8 (all elements have the trace equal to -1). This shows that G_3 is minimal. Thus the corresponding odd lift of G_1 is minimal.

Let G be a subgroup of index 2 of G_1 and $g = \tau g_1^a g_2^b \in G$ be the element of largest possible order in H. We verify that $g^2 = g_2^{2b-a}$. If g is of order 8, we check that it generates either $\langle\tau g_1\rangle$ considered earlier or its conjugate subgroup. Its normalizer is conjugate to the subgroup G_3 considered earlier. If g is of order 4, then $2b - a \equiv 2 \mod 4$. We list all possible cases and find that all elements of order 4 are conjugate. Thus we may assume that G contains $g = \tau g_1^2$. Now we check that the normalizer of this group is our group G_2.

So, all subgroups of index 2 are accounted for. They are two of them isomorphic to AS_{16} and M_{16}. They are all minimal in their even lift, and hence define odd lifts of G_1.

Let G be a subgroup of index 4 of $f^{-1}(\tau)$. It follows from above argument that G is conjugate to a subgroup of index 2 of G_2 or G_3. It could be $D_8, 8$, or 2×4. The first group is minimal, hence $D_8 : 2$ admits an odd minimal lift. Other two groups are not minimal. The last group admits an odd minimal lift. Note that it is not conjugate to odd 2×4 from Case 1. Finally a cyclic group of type $D_4(a_1)$ admits an odd minimal lift. It is not conjugate to a group from Case 1.

Case 3: $|f(G)| = 3$.

Without loss of generality we may assume that $f(G) = \langle\sigma\tau\rangle$. By Lemma 4.2, G is a split extension $H : 3$, where H is a subgroup of $\langle g_1, g_2\rangle$. Let $G_1 = f^{-1}(\langle\sigma\tau\rangle)$. It is a split extension $4^2 : 3$. By Sylow's Theorem, all subgroups

of order 3 are conjugate. Thus we may assume that H contains $\sigma\tau$. The possibilities are G_1 or $G_2 = \langle g_1^2, g_2^2, \sigma\tau\rangle \cong 2^2 : 3 \cong A_4$. The group A_4 has 3 elements of type $4A_1$, 4 elements of type $2A_2$ and 4 elements of type $3A_2$. Adding up the traces we see that the group is minimal. Thus G_1 is minimal too. The group G_1 does not have subgroups of index 2, so it does not admit odd lifts. Other groups in this case are conjugate to the non-minimal group $\langle\sigma\tau\rangle$.

Case 4:$f(G) = S_3$.

In this case $G \cap f^{-1}(\langle\sigma\tau\rangle)$ is a subgroup of index 2 equal to one of the two groups considered in the previous case. We get $G = \mathrm{Aut}(B)$, or $G \cong 2^2 : S_3 \cong S_4$, or S_3. Considering the preimage of $\langle\tau\rangle$, we find that all groups isomorphic to S_4 are conjugate and their Sylow 2-subgroup is D_8 from the previous case. Thus both $\mathrm{Aut}(B)$ and S_4 admit two minimal lifts. A group isomorphic to S_3 contains 2 elements of type $2A_2$ and it is not minimal in any lift.

Type III.

We assume that $a = 2\sqrt{3}i$ in the equation of the surface. The group $\mathrm{Aut}(B)$ is isomorphic to $4A_4$. It is generated (as always in its even lift) by

$$g_1 = [t_1, t_0, t_2, -t_3],\ g_2 = [it_1, -it_0, t_2, -t_3],$$

$$g_3 = [\epsilon_8^7 t_0 + \epsilon_8^7 t_1, \epsilon_8^5 t_0 + \epsilon_8 t_1, \sqrt{2}\epsilon_{12} t_2, 2\epsilon_6 t_3], c = [t_0, t_1, it_2, -t_3]$$

The "complicated" transformation g_3 is of order 3 (see our list of Gründformen for binary polyhedral groups). The generators g_1, g_2 are of type $4A_1$, the generator g_3 is of type $2A_2$ and the generator c is of type $D_4(a_1)$.

The element c generates the center. We have $g_1 g_2 = g_2 g_1 c^2$. This shows that the quotient by $\langle c\rangle$ is isomorphic to A_4 and the subgroup $\langle c, g_1, g_2\rangle \cong 4D_4$ is a group of order 16 isomorphic to the group AS_{16} (see Table 1).

Let $f : \mathrm{Aut}(B) \to A_4$ be the natural surjection with kernel $\langle c\rangle$. Let G be a subgroup of $\mathrm{Aut}(B)$.

Case 1: $G \subset \mathrm{Ker}(f) \cong 4$.

There are no even minimal subgroups. The whole kernel admits a minimal odd lift.

Case 2: $\#f(G) = 2$.

Without loss of generality we may assume that $f(G) = \langle g_1\rangle$. The subgroup $f^{-1}(\langle g_1\rangle)$ is generated by c, g_1 and is isomorphic to 4×2. It is not minimal in the even lift and minimal in the unique odd lift. Its subgroup $\langle cg_1\rangle$ of order 4 is of type $2A_3$ and does not admit minimal lifts.

Case 3:$f(G) = \langle g_3\rangle$.

We have $G = f^{-1}(\langle g_3\rangle) = \langle c, g_3\rangle = \langle cg_3\rangle \cong 12$. The element cg_3 is of type E_6, hence not minimal. Its square is an element of type $E_6(a_2)$, also not minimal. The subgroup $\langle (cg_3)^2\rangle \cong 6$ defines an odd minimal lift of G. The subgroup $\langle (cg_3)^4\rangle$ is of order 3. It defines an odd minimal lift of $\langle (cg_3)^2\rangle$. The group $\langle g_3\rangle$ admits an odd minimal lift.

Case 4: $f(G) = \langle g_1, g_2\rangle \cong 2^2$.

The subgroup $H = f^{-1}(\langle g_1, g_2\rangle)$ is generated by c, g_1, g_2. As we observed earlier, it is isomorphic to the group AS_{16} from Table 1. A proper subgroup is conjugate in $\mathrm{Aut}(B)$ to either $\langle g_1, g_2\rangle \cong D_8$ or $\langle cg_1, g_2\rangle$. All of the subgroups are isomorphic to D_8 with center generated by c^2. The cyclic subgroup of order 4 is of type $2A_3$, thus the subgroups are minimal in the even lift (we have done this computation for surfaces of type II). Thus the group H is minimal in the even lift and also minimal in two odd lifts corresponding to its two subgroups of index 2.

Case 5: $f(G) = A_4$.

It is easy to see that G has non-trivial center (the center of its Sylow 2-subgroup). It is equal to $\langle c\rangle$ or $\langle c^2\rangle$. In the first case $G = \mathrm{Aut}(B)$. Since it contains minimal subgroups it is minimal.

A subgroup G of index 2 is isomorphic to $2A_4 \cong D_8 : 3$. Its Sylow 2-subgroup is equal to one of the two subgroups isomorphic to D_8 from Case 4. Thus $\mathrm{Aut}(B)$ admits two odd lifts. Since G has no subgroups of index 2, the odd lifts of G do not exist.

Type VI.

In this case $\mathrm{Aut}(B) \cong 9$ is not minimal so does not admit minimal lifts.

To summarize our investigation we give two lists. In the first we list all groups which do not contain the Geiser involution γ. We indicate by $+$ or $-$ the types of their lifts. Also we indicate the number of conjugacy classes.

All other minimal groups are of the form $\langle\gamma\rangle \times G$, where G is one of the lifts of a subgroup of $\mathrm{Aut}(B)$. In the second list we give only groups $2 \times G$, where G does not admit a minimal lift. All other groups are of the form $2 \times G$, where G is given in the previous table.

Theorem 6.17. *Let G be a minimal group of automorphisms of a Del Pezzo surface of degree 2. Then G is either equal to a minimal lift of a subgroup from Table 7 or equal to $\gamma \times G'$, where G' is either from the table of one of the following groups of automorphisms of the branch quartic curve B*

1. *Type I:* $7 : 3,\ A_4,\ S_3,\ 7,\ 4,\ 3,\ 2$.
2. *Type II:* $2^2,\ S_3,\ 8,\ 4,\ 3,\ 2$.
3. *Type III:* $2^2,\ 4,\ 2$.
4. *Type IV:* $S_3,\ 2^2,\ 3,\ 2$.
5. *Type V:* $2^2,\ 2$.
6. *Type VI:* $9,\ 3$.
7. *Type VII:* $2^2,\ 4,\ 2$
8. *Type VIII:* 3.
9. *Type IX:* $S_3,\ 3,\ 2$.
10. *Type X:* $2^2,\ 2$.
11. *Type XI:* 3.
12. *Type XII:* $\{1\}$.

Type of S	Group	Lift	Conjugacy classes
I	$L_2(7)$	+	1
	S_4	+	2
	D_8	+	2
II	$4^2 : S_3$	+,-	2
	S_4	+,-	2
	$4^2 : 3$	+	1
	A_4	+	1
	$4^2 : 2 \cong D_8 : 4$	+,-,-	3
	M_{16}	+	1
	AS_{16}	+,-	2
	D_8	+	1
	4^2	-	1
	2×4	-	2
	4	-	2
III	$4A_4$	+,-	2
	$2A_4 \cong D_8 : 3$	+	1
	AS_{16}	+,-	2
	D_8	+	1
	12	-	1
	6	-	1
	2×4	-	1
	4	-	1
IV	S_4	+	1
	D_8	+	1
V	AS_{16}	+,-	2
	D_8	+	1
	2×4	-	2
	4	-	1
VII	D_8	+	1
VIII	6	-	1

Table 7. Minimal groups of automorphisms not containing γ

6.7 Automorphisms of Del Pezzo surfaces of degree 1

Let S be a Del Pezzo surface of degree 1. The linear system $|-2K_S|$ defines a finite map of degree 2 onto a quadric cone Q in $\mathbb{P}^3$. Its branch locus is a nonsingular curve B of genus 4 cut out by a cubic surface. Recall that a singular quadric is isomorphic to the weighted projective space $\mathbb{P}(1,1,2)$. A curve of genus 4 of degree 6 cut out in Q by a cubic surface is given by equation $F(T_0,T_1,T_2)$ of degree 6. After change of coordinates it can be given by an equation $T_2^3 + F_4(T_0,T_1)T_2 + F_6(T_0,T_1) = 0$, where F_4 and F_6 are binary forms of degree 4 and 6. The double cover of Q branched along such curve is isomorphic to a hypersurface of degree 6 in $\mathbb{P}(1,1,2,3)$

$$T_3^2 + T_2^3 + F_4(T_0,T_1)T_2 + F_6(T_0,T_1) = 0. \tag{48}$$

The vertex of Q has coordinates $(0,0,1)$ and its pre-image in the cover consist of one point $(0,0,1,a)$, where $a^2+1=0$ (note that $(0,0,1,a)$ and $(0,0,1,-a)$ represent the same point on $\mathbb{P}(1,1,2,3)$). This is the base-point of $|-K_S|$. The members of $|-K_S|$ are isomorphic to genus 1 curves with equations $y^2+x^3+F_4(t_0,t_1)x+F_6(t_0,t_1)=0$. The locus of zeros of $\Delta = F_4^3+27F_6^2$ is the set of points in $\mathbb{P}^1$ such that the corresponding genus 1 curve is singular. It consists of a simple roots and b double roots. The zeros of F_4 are either common zeros with F_6 and Δ, or represent nonsingular elliptic curves isomorphic to an anharmonic plane cubic curve. The zeros of F_6 are either common zeros with F_4 and Δ, or represent nonsingular elliptic curves isomorphic to a harmonic plane cubic curve.

Observe that no common root of F_4 and F_6 is a multiple root of F_6 since otherwise the surface is singular.

Since the ramification curve of the cover $S \to Q$ (identified with the branch curve B) is obviously invariant with respect to $\mathrm{Aut}(S)$ we have a natural surjective homomorphism

$$Aut(S) \to \mathrm{Aut}(B). \tag{49}$$

Its kernel is generated by the deck involution β which we call the *Bertini involution*. It defines the Bertini involution in $\mathrm{Cr}(2)$. The Bertini involution is the analog of the Geiser involution for Del Pezzo surfaces of degree 2. The same argument as above shows that β acts in $\mathcal{R}_S$ as the minus of the identity map. Under the homomorphism $\mathrm{Aut}(S) \to W(E_8)$ defined by a choice of a geometric basis, the image of β is the elements w_0 generating the center of $W(E_8)$. This time w_0 is an even element, i.e. belongs to $W(E_8)^+$. The quotient group $W(E_8)^+/\langle w_0\rangle$ is isomorphic to the simple group $\mathrm{O}(8,\mathbb{F}_2)^+$.

Since Q is a unique quadric cone containing B, the group $\mathrm{Aut}(B)$ is a subgroup of $\mathrm{Aut}(Q)$. Consider the natural homomorphism

$$r : \mathrm{Aut}(B) \to \mathrm{Aut}(\mathbb{P}^1).$$

Let G be a subgroup of $\mathrm{Aut}(B)$ and P be its image in $\mathrm{Aut}(\mathbb{P}^1)$. We assume that elements from G act on the variables T_0, T_1 by linear transformations with determinant 1. The polynomials F_4 and F_6 are the relative invariants of the binary group $\bar{P}$. They are polynomials in Gründformen which were listed in section 5.5. Let χ_4, χ_6 be the corresponding characters of $\bar{P}$ defined by the binary forms F_4, F_6. Let χ_2, χ_3 be the characters of G defined by the action on the variables T_2, T_3. Assume that $F_4 \neq 0$. Then

$$\chi_4\chi_2 = \chi_6 = \chi_3^3 = \chi_3^2.$$

If $g \in G \cap \mathrm{Ker}(r) \setminus \{1\}$, then g acts on the variables T_0, T_1 by either the identity or the minus identity. Thus $\chi_4(g) = \chi_6(g) = 1$ and we must have $\chi_2(g) = \chi_3(g)^2 = 1$. This shows that $g = [t_0, t_1, t_3, -t_3] = [-t_0, -t_1, t_2, -t_3] = \beta$.

If $F_4 = 0$, then we must have only $\chi_2(g)^3 = \chi_3(g)^2 = 1$. Since $[-t_0, -t_1, t_2, -t_3]$ is the identity transformation, we may assume that $\chi_3(g) = 1$ and represent g by $g = [t_0, t_1, \epsilon_3 t_3, \pm t_3]$. Thus $G \cap \mathrm{Ker}(r) = \langle \beta, \alpha\rangle \cong 6$.

Conversely, start with a polyhedral group P such that its lift to a binary polyhedral group $\bar{P}$ acts on the variables T_0, T_1 leaving $V(F_4)$ and $V(F_6)$ invariant. Let χ_4, χ_6 be the corresponding characters. Assume that there exist character $\chi_2, \chi_3 : \bar{P} \to \mathbb{C}^*$ such that

$$\chi_0^2 = \chi_4\chi_1 = \chi_6 = \chi_1^3. \tag{50}$$

Then $g = [at_0+bt_1, ct_0+t_1] \in \bar{P}$ acts on S by $[at_0+bt_1, ct_0+t_1, \chi_2(g)t_2, \chi_3(t_3)]$. This transformation is the identity in $\mathrm{Aut}(S)$ if and only if $g = [-t_0, -t_1]$ and $\chi_2(-1) = 1, \chi_3(-1) = -1$. This shows that $\bar{P}$ can be identified with a subgroup of $\mathrm{Aut}(S)$ with $-I_2 = \beta$ if and only if $\chi_3(-1) = -1$. If $\chi_3(-1) = 1$, then P can be identified with a subgroup of $\mathrm{Aut}(S)$ not containing β. In the latter case,

$$r^{-1}(P) = \begin{cases} P \times \langle\beta\rangle & \text{if } F_4 \neq 0 \\ P \times \langle\beta, \alpha\rangle & \text{otherwise.} \end{cases}$$

In particular, if $F_4 = 0$, there are three subgroups of $\mathrm{Aut}(S)$ which are mapped surjectively to P.

In the former case

$$r^{-1}(P) = \begin{cases} \bar{P} & \text{if } F_4 \neq 0, \\ \bar{P} \times \langle\alpha\rangle & \text{otherwise.} \end{cases}$$

Of course it could happen that neither P nor $\bar{P}$ lifts to a subgroup of $\mathrm{Aut}(S)$. In this case $r^{-1}(P) \cong 2P \ncong \bar{P}$ or $r^{-1}(P) \cong 3 \times 2P$ (if $F_4 = 0$).

In the following list we give a nontrivial subgroup $P$of $\mathrm{Aut}(\mathbb{P}^1)$ as a group of automorphisms of B and a smallest lift $\tilde{P}$ of P to a subgroup of $r^{-1}(P)$. If $F_4 \neq 0$, then we will see that $\tilde{P} = r^{-1}(P)$ or $\tilde{P} \cong P$. In the latter case $r^{-1}(P) \cong 2 \times P$. If $F_4 = 0$, and $\tilde{P} \cong P$, then $r^{-1}(P) \cong 6 \times \tilde{P}$. Otherwise $r^{-1}(P) \cong 3 \times \tilde{P}$.

Also we give generators of $\tilde{P}$ to $\mathrm{Aut}(S)$ as a group acting on t_0, t_1 with determinant 1 and the Bertini involution as an element of the lift.

1. *Cyclic groups P*

   (i) $P = \{1\}, F_4 = 0, r^{-1}(P) = \langle\beta, \alpha\rangle \cong 6$.

   (ii) $P \cong 2, g = [it_0, -it_1, -t_2, it_3]$,

   $$F_4 = F_2(T_0^2, T_1^2) \neq 0, \quad F_6 = F_3(T_0^2, T_1^2);$$

   (iii)$P \cong 2, \widetilde{P} \cong 4, g = [it_0, -it_1, t_2, t_3], \beta = g^2$,

   $$F_4 = a(T_0^4 + T_1^4) + bT_0^2T_1^2, \quad F_6 = T_0T_1F_2(T_0^2, T_1^2);$$

   (iv)$P \cong 2, \widetilde{P} = 4, g = [-t_1, t_0, t_2, t_3], \beta = g^2$,

   $$F_4 \text{ as in (iii)}, \ F_6 = (T_0^2 + T_1^2)(a(T_0^4 + T_1^4) + T_0T_1(bT_0T_1 + c(T_0^2 - T_1^2));$$

(v) $P \cong 2, g = [-t_1, t_0, -t_2, it_3]$,

$$F_4 \text{ as in (iii)}, \ F_6 = a(T_0^6 - T_1^6) + bT_0T_1(T_0^4 + T_1^4);$$

(vi) $P \cong 3, g = [\epsilon_3 t_0, \epsilon_3^2 t_1, \epsilon_3^2 t_2, t_3]$,

$$F_4 = T_0(aT_0^3 + bT_1^3), \quad F_6 = F_2(T_0^3, T_1^3);$$

(vii) $P \cong 3, g = [\epsilon_3 t_0, \epsilon_3^2 t_1, t_2, t_3]$,

$$F_4 = aT_0^2T_1^2, \quad F_6 = F_2(T_0^3, T_1^3);$$

(viii) $P \cong 4, g = [\epsilon_8 t_0, \epsilon_8^{-1} t_1, it_2, \epsilon_8^3 t_3]$,

$$F_4 = aT_0^4 + bT_1^4, \quad F_6 = T_0^2(cT_0^4 + dT_1^4);$$

(ix) $P \cong 4, \widetilde{P} \cong 8, g = [\epsilon_8 t_0, \epsilon_8^{-1} t_1, -t_2, t_3], \beta = g^4$,

$$F_4 = aT_0^2T_1^2, \quad F_6 = T_0T_1(T_0^4 + T_1^4);$$

(x) $P \cong 5, g = [\epsilon_{10} t_0, \epsilon_{10}^{-1} t_1, \epsilon_5 t_2, \epsilon_{10}^3 t_3]$,

$$F_4 = aT_0^4, \quad F_6 = T_0(T_0^5 + T_1^5);$$

(xi) $P \cong 6, g = [\epsilon_{12} t_0, \epsilon_{12}^{-1} t_1, \epsilon_6 t_2, it_3]$,

$$F_4 = aT_0^4, \quad F_6 = bT_0^6 + T_1^6, b \neq 0;$$

(xii) $P \cong 6, g = [\epsilon_{12} t_0, \epsilon_{12}^{-1} t_1, -t_2, it_3]$,

$$F_4 = aT_0^2T_1^2, \quad F_6 = T_0^6 + T_1^6;$$

(xiii) $P \cong 10, \widetilde{P} \cong 20, g = [\epsilon_{20} t_0, \epsilon_{20}^{-1} t_1, \epsilon_{10}^8 t_2, \epsilon_{10}^{-1} t_3], g^{10} = \beta$,

$$F_4 = T_0^4, \quad F_6 = T_0T_1^5;$$

(xiv) $P \cong 12, g = [\epsilon_{24} t_0, \epsilon_{24}^{-1} t_1, \epsilon_{12} t_2, \epsilon_{24} t_3]$,

$$F_4 = T_0^4, \quad F_6 = T_1^6.$$

2. *Dihedral groups*

(i) $P \cong 2^2, \widetilde{P} \cong D_8, g_1 = [it_1, it_0, -t_2, it_3], \ g_2 = [-t_1, t_0, -t_2, it_3], \beta = (g_1g_2)^2, g_1^2 = g_2^2 = 1$,

$$F_4 = a(T_0^4 + T_1^4) + bT_0^2T_1^2, \quad F_6 = T_0T_1[c(T_0^4 + T_1^4) + dT_0^2T_1^2];$$

(ii) $P \cong 2^2, \widetilde{P} \cong Q_8, g_1 = [it_1, it_0, t_2, t_3], \ g_2 = [-t_1, t_0, t_2, t_3], \beta = g_1^2 = g_2^2$,

$$F_4 = a(T_0^4 + T_1^4) + bT_0^2T_1^2, \quad F_6 = T_0T_1(T_0^4 - T_1^4);$$

(iii)$P \cong D_6, g_1 = [\epsilon_6 t_0, \epsilon_6^{-1} t_1, t_2, -t_3], g_2 = [it_1, it_0, -t_2, it_3]$,

$$F_4 = aT_0^2T_1^2, \quad F_6 = T_0^6 + T_1^6 + bT_0^3T_1^3;$$

(iv)$P \cong D_8, \widetilde{P} \cong D_{16},\ g_1 = [\epsilon_8 t_0, \epsilon_8^{-1} t_1, -t_2, it_3], g_2 = [-t_1, t_0, -t_2, it_3]$, $g_1^4 = \beta, g_2^2 = 1$,

$$F_4 = aT_0^2T_1^2, \quad F_6 = T_0T_1(T_0^4 + T_1^4);$$

(v) $P \cong D_{12}, \widetilde{P} \cong 2D_{12} \cong (2 \times 6)^{\bullet}2, g_1 = [\epsilon_{12} t_0, \epsilon_{12}^{-1} t_1, -t_2, it_3], g_2 = [-t_1, t_0, t_2, t_3], g_1^6 = 1, g_2^2 = \beta$,

$$F_4 = aT_0^2T_1^2, \quad F_6 = T_0^6 + T_1^6.$$

3. *Other groups*
(i) $P \cong A_4, \widetilde{P} \cong \overline{T}, g_1 = [\epsilon_8^7 t_0 + \epsilon_8^7 t_1, \epsilon_8^5 t_0 + \epsilon_8 t_1, \sqrt{2}\epsilon_3 t_2, 2t_3],\ g_2 = [it_0, -it_1, t_2, t_3],\ g_1^3 = g_2^2 = \beta$,

$$F_4 = T_0^4 + 2\sqrt{-3}T_0^2T_1^2 + T_2^4, \quad F_6 = T_0T_1(T_0^4 - T_1^4),$$

(ii) $P \cong O, \widetilde{P} \cong \overline{T} : 2, g_1 = [\epsilon_8^7 t_0 + \epsilon_8^7 t_1, \epsilon_8^5 t_0 + \epsilon_8 t_1, \sqrt{2}\epsilon_3 t_2, 2t_3],\ g_2 = [\epsilon_8 t_0, \epsilon_8^{-1} t_1, -t_2, it_3],\ g_3 = [-\epsilon_8 t_1, \epsilon_8^7 t_0, -t_2, it_3], g_1^3 = g_2^4 = \beta, g_3^2 = 1, r^{-1}(P) = 3 \times \bar{O}$,

$$F_4 = 0, \quad F_6 = T_0T_1(T_0^4 - T_1^4),$$

Table 6 gives the list of the full automorphism groups of Del Pezzo surfaces of degree 1.

The following is the list of cyclic minimal groups $\langle g \rangle$ of automorphisms of Del Pezzo surfaces $V(F)$ of degree 1.

1. Order 2
   - A_1^8 (the Bertini involution) $g = [t_0, t_1, t_2, -t_3]$

$$F = T_3^2 + T_2^3 + F_4(T_0, T_1)T_2 + F_6(T_0, T_1),$$

2. Order 3
   - $4A_2\ \ g = [t_0, t_1, \epsilon_3 t_2, t_3]$

$$F = T_3^2 + T_2^3 + F_6(T_0, T_1),$$

3. Order 4
   - $2D_4(a_1)\ g = [t_0, -t_1, -t_2, \pm it_3]$

$$F = T_3^2 + T_2^3 + (aT_0^4 + bT_0^2T_1^2 + cT_1^4)T_2 + T_0T_1(dT_0^4 + eT_1^4),$$

4. Order 5

Type	Order	Structure	F_4	F_6	Parameters
I	144	$3 \times (\overline{T} : 2)$	0	$T_0T_1(T_0^4 - T_1^4)$	
II	72	$3 \times 2D_{12}$	0	$T_0^6 + T_1^6$	
III	36	$6 \times D_6$	0	$T_0^6 + aT_0^3T_1^3 + T_1^6$	$a \neq 0$
IV	30	30	0	$T_0(T_0^5 + T_1^5)$	
V	24	$\overline{T}$	$a(T_0^4 + \alpha T_0^2T_1^2 + T_1^4)$	$T_0T_1(T_0^4 - T_1^4)$	$\alpha = 2\sqrt{-3}$
VI	24	$2D_{12}$	$aT_0^2T_1^2$	$T_0^6 + T_1^6$	$a \neq 0$
VII	24	2×12	T_0^4	T_1^6	
VIII	20	20	T_0^4	$T_0T_1^5$	
IX	16	D_{16}	$aT_0^2T_1^2$	$T_0T_1(T_0^4 + T_1^4)$	$a \neq 0$
X	12	D_{12}	$T_0^2T_1^2$	$T_0^6 + aT_0^3T_1^3 + T_1^6$	$a \neq 0$
XI	12	2×6	0	$G_3(T_0^2, T_1^2)$	
XII	12	2×6	T_0^4	$aT_0^6 + T_1^6$	$a \neq 0$
XIII	10	10	T_0^4	$T_0(aT_0^5 + T_1^5)$	$a \neq 0$
XIV	8	Q_8	$T_0^4 + T_1^4 + aT_0^2T_1^2$	$bT_0T_1(T_0^4 - T_1^4)$	$a \neq 2\sqrt{-3}$
XV	8	2×4	$aT_0^4 + T_1^4$	$T_0^2(bT_0^4 + cT_1^4)$	
XVI	8	D_8	$T_0^4 + T_1^4 + aT_0^2T_1^2$	$T_0T_1(b(T_0^4 + T_1^4) + cT_0^2T_1^2)$	$b \neq 0$
XVII	6	6	0	$F_6(T_0, T_1)$	
XVIII	6	6	$T_0(aT_0^3 + bT_1^3)$	$cT_0^6 + dT_0^3T_1^3 + T_1^6$	
XIX	4	4	$G_2(T_0^2, T_1^2$	$T_0T_1F_2(T_0^2, T_1^2)$	
XX	4	2^2	$G_2(T_0^2, T_1^2$	$G_3(T_0^2, T_1^2)$	
XXI	2	2	$F_4(T_0, T_1)$	$F_6(T_0, T_1)$	

Table 8. Groups of automorphisms of Del Pezzo surfaces of degree 1

- $2A_4$ $g = [t_0, \epsilon_5 t_1, t_2, t_3]$

$$F = T_3^2 + T_2^3 + aT_0^4T_2 + T_0(bT_0^5 + T_1^5),$$

5. Order 6
   - $E_6(a_2) + A_2$ $g = [t_0, -t_1, \epsilon_3 t_2, t_3]$

$$F = T_3^2 + T_2^3 + G_3(T_0^2, T_1^2),$$

   - $E_7(a_4) + A_1$ $g = [t_0, \epsilon_3 t_1, t_2, -t_3]$

$$F = T_3^2 + T_2^3 + (T_0^4 + aT_0T_1^3)T_2 + bT_0^6 + cT_0^3T_1^3 + dT_1^6,$$

   - $2D_4$ $g = [\epsilon_6 t_0, \epsilon_6^{-1} t_1, t_2, t_3]$

$$F = T_3^2 + aT_0^2T_1^2T_2 + bT_0^6 + cT_0^3T_1^3 + eT_1^6,$$

   - $E_8(a_8)$ $g = [t_0, t_1, \epsilon_3 t_2, -t_3]$

$$F = T_3^2 + T_2^3 + F_6(T_0, T_1),$$

   - $A_5 + A_2 + A_1$ $g = [t_0, \epsilon_6 t_1, t_2, t_3]$

$$F = T_3^2 + T_2^3 + aT_0^4T_2 + T_0^6 + bT_1^6,$$

6. Order 8
   - $D_8(a_3)$ $g = [it_0, t_1, -it_2, \pm\epsilon_8 t_3]$

$$F = T_3^2 + T_2^3 + aT_0^2T_1^2T_2 + T_0T_1(T_0^4 + T_1^4),$$

7. Order 10
   - $E_8(a_6)$ $g = [t_0, \epsilon_5 t_1, t_2, -t_3]$

$$F = T_3^2 + T_2^3 + aT_0^4T_2 + T_0(bT_0^5 + T_1^5),$$

8. Order 12
   - $E_8(a_3)$ $g = [-t_0, t_1, \epsilon_6 t_2, it_3]$

$$F = T_3^2 + T_2^3 + T_0T_1(T_0^4 + aT_0^2T_1^2 + T_1^4),$$

9. Order 15
   - $E_8(a_5)$ $g = [t_0, \epsilon_5 t_1, \epsilon_3 t_2, t_3]$

$$F = T_3^2 + T_2^3 + T_0(T_0^5 + T_1^5),$$

10. Order 20
    - $E_8(a_2)$ $g = [t_0, \epsilon_{10} t_1, -t_2, it_3]$

$$F = T_3^2 + T_2^3 + aT_0^4T_2 + T_0T_1^5,$$

11. Order 24
    - $E_8(a_1)$ $g = [it_0, t_1, \epsilon_{12} t_2, \epsilon_8 t_3]$

$$F = T_3^2 + T_2^3 + T_0T_1(T_0^4 + T_1^4),$$

12. Order 30
    - E_8 $g = [t_0, \epsilon_5 t_1, \epsilon_3 t_2, -t_3]$

$$F = T_3^2 + T_2^3 + T_0(T_0^5 + T_1^5).$$

To list all minimal subgroups of $\mathrm{Aut}(S)$ is very easy. We know that any subgroup in $\mathrm{Ker}(r)$ contains contains one of the elements $\alpha, \beta, \alpha\beta$ which are all minimal of types $8A_1, 4A_2, E_8(a_8)$. So, a subgroup is not minimal only if its image P in $\mathrm{Aut}(B)$ can be lifted isomorphically to $\mathrm{Aut}(S)$.

We will use the following lemma.

Lemma 6.18. *Let $P \subset \mathrm{Aut}(\mathbb{P}^1)$ and $G \subset \mathrm{Aut}(S)$ be contained in $r^{-1}(P)$. Then G is a minimal group unless $G = \tilde{P} \cong P$ and G is a non-minimal cyclic group or non-minimal dihedral group D_6.*

Proof. It follows from above classification of possible subgroups of $\mathrm{Aut}(B)$ and its lifts to $\mathrm{Aut}(S)$ that any non-isomorphic lift contains β or α, or $\beta\alpha$ which generate minimal cyclic groups. If the lift is isomorphic to P then P is either a cyclic group or $P \cong D_6$. The group D_6 contains 3 elements of type $4A_1$ and 2 elements of type $2A_2$. Adding up the traces we see that the group is not minimal.

Let us classify minimal groups of automorphisms of a Del Pezzo surface of degree 1. As in the previous cases, to find a structure of such groups is enough to consider the types of surfaces which are not specialized to surfaces of other types. The following notation Type A→Type B indicates that a surface of type A specializes to a surface of type B.

$$V, IX, XIV, XVI, XVII, XIX, XXI \to I,$$

$$III, VI, X, XI, XII, XVI, XVII, XVIII, XX, XXI \to II$$

$$XIII, XXI \to IV, \quad XIII, XXI \to VIII,$$

$$XII, XX, XXI \to VII, \ XX, XI \to XV.$$

It remains to consider surfaces of types

$$I, \ II, \ IV, \ VII, \ VIII, \ XV.$$

Type I. $P \cong S_4$.

Possible conjugacy classes of subgroups H are $\{1\}, 2, 2, 3, 2^2, 4, D_8, D_6, A_4, S_4$. Groups of order 2 have two conjugacy classes in P represented by $[it_0, -it_1]$ and $[-t_1, t_0]$. The groups are realized in cases (iii) and (iv). None of them lifts isomorphically. An cyclic group of order 3 is generated by a non-minimal element realized in case (vii). Its isomorphic minimal lift is not minimal. A cyclic group of order 4 does not admit an isomorphic lift. The dihedral subgroup 2^2 is of type (ii). This information, together with Lemma 6.18 allows us to classify all minimal subgroups.

- $P = \{1\}$: $\langle \beta\alpha \rangle \cong 6, \langle \alpha \rangle \cong 3, \langle \alpha \rangle \cong 2$;
- $P = 2$: 4, 12;
- $P = 2$: 4, 12;
- $P = 3$: 3^2, 3×6;
- $P = 2^2$: Q_8, $Q_8 \times 3$;
- $P = 2^2$: D_8, $D_8 \times 3$;
- $P = 4$: 8, 8×3;
- $P = D_8$: D_{16}, $D_8 \times 3$;
- $P = D_6$: $D_6 \times 2$, $D_6 \times 3, D_6 \times 6$;
- $P = A_4$: $\overline{T}$, $\overline{T} \times 3$;
- $P = S_4$: $\overline{T} : 2$, $3 \times (\overline{T} : 2)$.

Surfaces specializing to a surface of type I have the following minimal subgroups.

V: 4, 6, Q_8, $\overline{T}$.

IX: 4(2), 8, D_{16}.

XIV: 4, Q_8.

XVI: D_8.

XVII:2, 3, 6.

XIX: 2, 4.
XXI: 2.

Type II: $P = D_{12}$.

Possible subgroups are $\{1\}, 2, 2, 3, 2^2, 6, D_6, D_{12}$. Cyclic subgroups of order 2, 3 and 6 admit isomorphic non-minimal lifts. All these groups are not minimal. There are two conjugacy classes of subgroups of order 2 in P represented by $[it_0, -it_1]$ and $[-t_1, t_0]$. One subgroups lifts isomorphically, other one does not. The cyclic group of order 6 admits an isomorphic lift and not minimal. The dihedral group D_6 admits a non-minimal isomorphic lift.

- $P = \{1\}$: $\langle\beta\alpha\rangle \cong 6, \langle\alpha\rangle \cong 3, \langle\alpha\rangle \cong 2$;
- $P = 2$: 4, 12;
- $P = 2$: 2^2, $2^2 \times 3$, 6;
- $P = 3$: 3^2, $3^2 \times 2$;
- $P = 2^2$: Q_8, $Q_8 \times 3$;
- $P = 6$: 2×6,
- $P = D_6$: $2 \times D_6$, $D_6 \times 3$, $D_6 \times 6$;
- $P = D_{12}$: $2D_{12}$, $3 \times 2D_{12}$.

Surfaces specializing to a surface of type II have the following subgroups:
III: 4, 12, 2^2, $2^2 \times 3$, 6, 3^2, $3^2 \times 2$, Q_8, $Q_8 \times 3$, $2 \times D_6$, $D_6 \times 3, D_6 \times 6$.
VI: 4, 2^2, 3^2, Q_8, 2×6, $2 \times D_6$, $2D_{12} \cong (2 \times 6)^\bullet 2$.
X: 2, $2 \times D_6$.
XI: 2, 3, 6, 2^2, 2×6.
XII: $6 \times 2, 6, 2^2, 2 \times 6$.
XVI: 2, 4, D_8.
XVII: 2, 3, 6.
XVIII: 2, 6.
XX: 2, 2^2.
XXI: 2.

Type IV: $P = 5$ This is easy. We have $P \cong 5$. It admits an isomorphic lift to a non-minimal subgroup.

- $P = \{1\}$: $\langle\beta\alpha\rangle \cong 6, \langle\alpha\rangle \cong 3, \langle\alpha\rangle \cong 2$;
- $P = 5$: 5, $10, 15, 30$;

Surfaces specializing to a surface of type IV have the following subgroups:
XIII: 5, 10.
XXI: 2.

Type VII: $P \cong 12$.

- $P = 2$: 2^2.
- $P = 3$: 6;
- $P = 4$: 2×4;
- $P = 6$: 2×6;
- $P = 12$: 2×12.

Surfaces specializing to a surface of type VII have the following subgroups:
XII: 2, 6, 2×6. XX: 2, 2^2.
XXI: 2.

Type VIII: $P \cong 10$.

- $P = \{1\}$: $\langle \beta\alpha \rangle \cong 6, \langle \alpha \rangle \cong 3, \langle \alpha \rangle \cong 2$;
- $P = 2$: 2^2.
- $P = 5$: 10;
- $P = 10$: 20.

Surfaces specializing to a surface of type VIII have the following subgroups:
XIII: 5, 10.
XXI: 2.

Type XV: $P \cong 4$.

- $P = \{1\}$: $\langle \beta\alpha \rangle \cong 6, \langle \alpha \rangle \cong 3, \langle \alpha \rangle \cong 2$;
- $P = 2$: 2^2.
- $P = 4$: 2×4.

Surfaces specializing to a surface of type VIII have the following subgroups:
XX: 2, 2^2.
XXI: 2.

7 Elementary links and factorization theorem

7.1 Noether-Fano inequality

Let $|d\ell - m_1 x_1 - \cdots - m_N x_N|$ be a homaloidal net in $\mathbb{P}^2$. The following is a well-known classical result.

Lemma 7.1. *(Noether's inequality) Assume* $d > 1, m_1 \geq \cdots \geq m_N \geq 0$. *Then*

$$m_1 + m_2 + m_3 \geq d + 1,$$

and the equality holds if and only if either $m_1 = \cdots = m_N$ *or* $m_1 = n-1, m_2 = \cdots = m_N$.

Proof. We have

$$m_1^2 + \cdots + m_N^2 = d^2 - 1, \quad m_1 + \cdots + m_N = 3d - 3.$$

Multiplying the second equality by m_3 and subtracting from the first one, we get

$$m_1(m_1 - m_3) + m_2(m_2 - m_3) - \sum_{i \geq 4} m_i(m_3 - m_i) = d^2 - 1 - 3m_3(d-1).$$

¿From this we obtain

$$(d-1)(m_1+m_2+m_3-d-1) = (m_1-m_3)(d-1-m_1)+(m_2-m_3)(d-1-m_2)+ \\ +\sum_{i\geq 4} m_i(m_3-m_i).$$

Since $d-1-m_i \geq 0$, this obviously proves the assertion.

Corollary 7.2.

$$m_1 > d/3.$$

Let us generalize Corollary 7.2 to birational maps of any rational surfaces. The same idea works even for higher-dimensional varieties. Let $\chi : S \to S'$ be a birational map of surfaces. Let $\sigma : X \to S, \phi : X \to S'$ be its resolution. Let $|H'|$ be a linear system on S' without base points. Let

$$\phi^*(H') \sim \sigma^*(H) - \sum_i m_i \mathcal{E}_i$$

for some divisor H on S and exceptional curves $\mathcal{E}_i$ of the map σ.

Theorem 7.3. *(Noether-Fano inequality) Assume that there exists some integer $m_0 \geq 0$ such that $|H' + mK_{S'}| = \emptyset$ for $m \geq m_0$. For any $m \geq m_0$ such that $|H + mK_S| \neq \emptyset$ there exists i such that*

$$m_i > m.$$

Proof. We know that $K_X = \sigma^*(K_S) + \sum_i \mathcal{E}_i$. Thus we have the equality in $\mathrm{Pic}(X)$

$$\phi^*(H') + mK_X = (\sigma^*(H + mK_S)) + \sum (m - m_i)\mathcal{E}_i.$$

Applying f_* to the left-hand side we get the divisor class $H' + mK_{S'}$ which, by assumption, cannot be effective. Since $|\sigma^*(H + mK_S)| \neq \emptyset$, applying ϕ_* to the right-hand side, we get the sum of an effective divisor and the image of the divisor $\sum_i (m - m_i)\mathcal{E}_i$. If all $m - m_i$ are nonnegative, it is also an effective divisor, and we get a contradiction. Thus there exists i such that $m - m_i < 0$.

Example 7.4. Assume $S = S' = \mathbb{P}^2$, $H = d\ell$ and $H' = \ell$. We have $|H + K_{S'}| = |-2\ell| = \emptyset$. Thus we can take $m_0 = 1$. If $d \geq 3$, we have for any $1 \leq a \leq d/3$, $|H' + aK_S| = |(d - 3a)\ell| \neq \emptyset$. This gives $m_i > d/3$ for some i. This is Corollary 7.2.

Example 7.5. Let $\chi : S- \to S'$ be a birational map of Del Pezzo surfaces. Assume that S' is not a quadric or the plane. Consider the complete linear system $H' = |-K_{S'}|$. Then $|H' + mK_{S'}| = \emptyset$ for $m \geq 2$. Let $\chi^{-1}(H') = |D - \eta|$ be its proper transform on S. Choose a standard basis $(e_0, \ldots, e_k)$ in $\mathrm{Pic}(S)$ corresponding to the blow-up $S \to \mathbb{P}^2$. Since $K_S = -3e_0 + e_1 + \cdots + e_k$, we can write $\chi^{-1}(H') = |-aK_S - \sum m_i x_i|$, where $a \in \frac{1}{3}\mathbb{Z}$. Assume that $\chi^*(H') = -aK_S$. Then there exists a point with multiplicity $\geq a$ if $a > 1$ that we assume.

Remark 7.6. The Noether inequality is of course well-known (see, for example, [2], [34]). We give it here to express our respect of this important and beautiful result of classical algebraic geometry. Its generalization from Theorem 7.3 is also well-known (see [38], 1.3). Note that the result can be also applied to G-equivariant maps χ provided that the linear system $|H'|$ is G-invariant. In this case the linear system $|H - \eta|$ is also G-invariant and the bubble cycle $\eta = \sum m_i x_i$ consists of the sum of G-orbits.

The existence of base points of high multiplicity in the linear system $|H - \eta| = \chi^{-1}(H')$ follows from the classical theory of termination of the adjoint system for rational surfaces which goes back to G. Castelnuovo. Nowadays this theory has an elegant interpretation in the Mori theory which we give in the next section.

7.2 Elementary links

We will be dealing with minimal Del Pezzo G-surfaces or minimal conic bundles G-surfaces. In the G-equivariant version of the Mori theory they are interpreted as extremal contractions $\phi : S \to C$, where $C = \mathrm{pt}$ is a point in the first case and $C \cong \mathbb{P}^1$ in the second case. They are also two-dimensional analogs of rational Mori G-fibrations.

A birational G-map between the Mori fibrations are diagrams

$$\begin{array}{ccc} S & \overset{\chi}{\dashrightarrow} & S' \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi'} \\ C & & C' \end{array} \tag{51}$$

which in general do not commute with the fibrations. These maps are decomposed into *elementary links*. These links are divided into the four following types.

- Links of type I:

They are commutative diagrams of the form

$$\begin{array}{ccc} S & \xleftarrow{\sigma} & Z = S' \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi'} \\ C = \mathrm{pt} & \xleftarrow{\alpha} & C' = \mathbb{P}^1 \end{array} \tag{52}$$

Here $\sigma : Z \to S$ is the blow-up of a G-orbit, S is a minimal Del Pezzo surface, $\phi' : S' \to \mathbb{P}^1$ is a minimal conic bundle G-fibration, α is the constant map. For example, the blow-up of a G-fixed point on $\mathbb{P}^2$ defines a minimal conic G-bundle $\phi' : \mathbf{F}_1 \to \mathbb{P}^1$ with a G-invariant exceptional section.

- Links of type II:

They are commutative diagrams of the form

$$\begin{array}{ccccc} S & \xleftarrow{\sigma} & Z & \xrightarrow{\tau} & S' \\ {\scriptstyle\phi}\downarrow & & & & \downarrow{\scriptstyle\phi'} \\ C & & = & & C' \end{array} \tag{53}$$

Here $\sigma : Z \to S, \tau : Z \to S'$ are the blow-ups of G-orbits such that rank $\mathrm{Pic}(Z)^G = \text{rank } \mathrm{Pic}(S)^G + 1 = \text{rank } \mathrm{Pic}(S')^G + 1$, $C = C'$ is either a point or $\mathbb{P}^1$. An example of a link of type II is the Geiser (or Bertini) involution of $\mathbb{P}^2$, where one blows up 7 (or 8) points in general position which form one G-orbit. Another frequently used link of type II is an elementary transformation of minimal ruled surfaces and conic bundles.

- Links of type III:

These are the birational maps which are the inverses of links of type I.

- Links of type IV:

They exist when S has two different structures of G-equivariant conic bundles. The link is the exchange of the two conic bundle structures

$$\begin{array}{ccc} S & = & S' \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi'} \\ C & & C' \end{array} \tag{54}$$

One uses these links to relate elementary links with respect to one conic fibration to elementary links with respect to another conic fibration. Often the change of the conic bundle structures is realized via an involution in $\mathrm{Aut}(S)$, for example, the switch of the factors of $S = \mathbb{P}^1 \times \mathbb{P}^1$ (see the following classification of elementary links).

7.3 The factorization theorem

Let $\chi : S - \to S'$ be a G-equivariant birational map of minimal G-surfaces. We would like to decompose it into a composition of elementary links. This is achieved with help of G-equivariant theory of log-pairs (S, D), where D is a G-invariant $\mathbb{Q}$-divisor on S. It is chosen as follows. Let us fix a G-invariant very ample linear system H' on S'. If S' is a minimal Del Pezzo surface we take $\mathcal{H}' = |-a'K_{S'}|, a' \in \mathbb{Z}_+$. If S' is a conic bundle we take $\mathcal{H}' = |-a'K_{S'} + b'f'|$, where f' is the class of a fibre of the conic bundle, a', b' are some appropriate positive integers.

Let $\mathcal{H} = \mathcal{H}_S = \chi^{-1}(\mathcal{H}')$ be the proper transform of $\mathcal{H}'$ on S. Then

$$\mathcal{H} = |-aK_S - \sum m_x x|,$$

if S is a Del Pezzo surface, $a \in \frac{1}{2}\mathbb{Z}_+ \cup \frac{1}{3}\mathbb{Z}_+$, and

$$\mathcal{H} = |-aK_S + bf - \sum m_x x|,$$

if S is a conic bundle, $a \in \frac{1}{2}\mathbb{Z}_+, b \in \frac{1}{2}\mathbb{Z}$. The linear system $\mathcal{H}$ is G-invariant, and the 0-cycle $\sum m_x x$ is a sum of G-orbits with integer multiplicities. One uses the theory of log-pairs (S, D), where D is a general divisor from the linear system $\mathcal{H}$, by applying some "untwisting links" to χ in order to decrease the number a, the algebraic degree of $\mathcal{H}$. Since a is a rational positive number with bounded denominator, this process terminates after finitely many steps (see [21],[38]).

Theorem 7.7. *Let $f : S- \to S'$ be a birational map of minimal G-surfaces. Then χ is equal to a composition of elementary links.*

The proof of this theorem is the same as in the arithmetic case ([38], Theorem 2.5). Each time one chooses a link to apply and the criterion used for termination of the process is based on the following version of Noether's inequality in the Mori theory.

Lemma 7.8. *In the notation from above, if $m_x \leq a$ for all base points x of $\mathcal{H}$ and $b \geq 0$ in the case of conic bundles, then χ is an isomorphism.*

The proof of this lemma is the same as in the arithmetic case ([38], Lemma 2.4).

We will call a base points x of $\mathcal{H}$ with $m_x > a$ a *maximal singularity* of $\mathcal{H}$. It follows from 3.2 that if $\mathcal{H}$ has a maximal singularity of height > 0, then it also has a maximal singularity of height 0. We will be applying the "untwisting links" of types I-III to these points. If $\phi : S \to \mathbb{P}^1$ is a conic bundle with all its maximal singularities untwisted with helps of links of type II, then either the algorithm terminates, or $b < 0$. In the latter case the linear system $|K_S + \frac{1}{a}\mathcal{H}| = |\frac{b}{a}f|$ is not nef and has *canonical singularities* (i.e. no maximal singularities). Applying the theory of log-pairs to the pair $(S, |\frac{b}{a}f|)$ we find an extremal contraction $\phi' : S \to \mathbb{P}^1$, i.e. another conic bundle structure on S. Rewriting $\mathcal{H}$ in a new basis $-K_S, f'$ we find the new coefficient $a' < a$. Applying the link of type IV relating ϕ and ϕ', we start over the algorithm with decreased a.

It follows from the proofs of Theorem 7.7 and Lemma 7.8 that all maximal singularities of H are in general position in the following sense.

(i) If S is a minimal Del Pezzo G-surface, then the blow-up of all maximal singularities of $\mathcal{H}$ is a Del Pezzo surface (of course this agrees with the description of points in general position at the end of section 3.8).
(ii) If $\phi : S \to \mathbb{P}^1$ is a conic bundle, then none of the maximal singularities lie on a singular fibre of ϕ and no two lie on one fibre.

The meaning of these assertions is that the linear system $|H|$ has no fixed components. In the case of Del Pezzo surfaces with an orbit of maximal singular points we can find a link by blowing up this orbit to obtain a surface Z with $\mathrm{Pic}(Z)^G \cong \mathbb{Z}\oplus\mathbb{Z}$ and two extremal rays. By applying Kleiman's criterion this implies that $-K_Z$ is ample. The similar situation occurs in the case of conic bundles (see [38], Comment 2).

Let S be a minimal Del Pezzo G-surface of degree d. Let us write $\mathcal{H}_S = |-aK_S - \sum m_\kappa \kappa|$ as in (16).

Lemma 7.9. *Let $\kappa_1, \dots, \kappa_n$ be the G-orbits of maximal multiplicity. Then*

$$\sum_{i=1}^{n} d(\kappa_i) < d.$$

Proof. Let $D_1, D_2 \in \mathcal{H}_S$ be two general divisors from $\mathcal{H}_S$. Since $\mathcal{H}_S$ has no fixed components, we have

$$0 \le D_1 \cdot D_2 = a^2 d - \sum m_\kappa^2 d(\kappa) \le a^2 d - \sum_{i=1}^{n} m_{\kappa_i}^2 d(\kappa_i) =$$

$$a^2(d - \sum_{i=1}^{n} d(\kappa_i)) - \sum_{i=1}^{n} (m_{\kappa_i}^2 - a^2) d(\kappa_i).$$

It follows from Example 7.5 that $m_{\kappa_i} > a$ for all $i = 1, \dots, n$. This implies that $d > \sum_{i=1}^{n} d(\kappa_i)$.

Definition 7.10. *A minimal Del Pezzo G-surface is called* superrigid *(resp.* rigid*) if any birational G-map $\chi : S- \to S'$ is a G-isomorphism (resp. there exists a birational G-automorphism $\alpha : S- \to S$ such that $\chi \circ \alpha$ is a G-isomorphism).*

A minimal conic bundle $\phi : S \to \mathbb{P}^1$ is called superrigid *(resp.* rigid*) if for any birational G-map $\chi : S- \to S'$, where $\phi' : S' \to \mathbb{P}^1$ is a minimal conic bundle, there exists an isomorphism $\delta : \mathbb{P}^1 \to \mathbb{P}^1$ such that the following diagram is commutative*

$$\begin{array}{ccc} S & \overset{\chi}{\dashrightarrow} & S' \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi'} \\ \mathbb{P}^1 & \xrightarrow{\delta} & \mathbb{P}^1 \end{array} \tag{55}$$

(resp. there exists a birational G-automorphism $\alpha : S- \to S'$ such that the diagram is commutative after we replace χ with $\chi \circ \alpha$).

Applying Lemma 7.8 and Lemma 7.9, we get the following.

Corollary 7.11. *Let S be a minimal Del Pezzo G-surface of degree $d = K_S^2$. If S has no G-orbits κ with $d(\kappa) < d$, then S is superrigid. In particular, a Del Pezzo surface of degree 1 is always superrigid and a Del Pezzo surface of degree 2 is superrigid unless G has a fixed point.*

A minimal conic G-bundle with $K_S^2 \le 0$ is superrigid.

The first assertion is clear. To prove the second one, we untwist all maximal base points of $\mathcal{H}_S$ with help of links of type II to get a conic bundle $\phi' : S' \to \mathbb{P}^1$ with $b' < 0$. Since $H_{S'}^2 = a^2 K_{S'}^2 + 4ab' - \sum {m'_x}^2 \geq 0$ and $K_{S'}^2 = K_S^2 \leq 0, 4ab' < 0$, we get a contradiction with Lemma 7.8. Thus χ after untwisting maximal base points terminates at an isomorphism (see [35], [36], [38], Theorem 1.6).

7.4 Classification of elementary links

Here we consider an elementary link $f : S- \to S'$ defined by a resolution $(S \xleftarrow{\sigma} Z \xrightarrow{\tau} S')$. We take $H_{S'}$ to be the linear system $|-aK_{S'}|$ if S' is a Del Pezzo surface and $|f|$ if S' is a conic bundle, where f is the divisor class of a fibre. It is assumed that the point which we blow up are in general position in sense of the previous subsection.

We denote by $\mathcal{D}P_k$ (resp. $\mathcal{C}_k$) the set of isomorphism classes of minimal Del Pezzo surfaces (resp. conic bundles) with $k = K_S^2$ (resp. $k = 8 - K_S^2$).

Proposition 7.12. *Let S, S' be as in Link I of type I. The map $\sigma : Z = S' \to S$ is the blowing up of a G-invariant bubble cycle η with $\mathrm{ht}(\eta) = 0$ of some degree d. The proper transform of the linear system $|f|$ on S' is equal to the linear system $\mathcal{H}_S = |-aK_S - m\eta|$. Here f is the class of a fibre of the conic bundle structure on S'. The following cases are possible:*

1. $K_S^2 = 9$
   - $S = \mathbb{P}^2, S' = \mathbf{F}_1, d = 1, m = 1, a = \frac{1}{3}$.
   - $S = \mathbb{P}^2, S' \in \mathcal{C}_3, d = 4, m = 1, a = \frac{2}{3}$.
2. $K_S^2 = 8$
   - $S = \mathbf{F}_0, \pi : S' \to \mathbb{P}^1$ *a conic bundle with* $k = 2$, $d = 2$, $m = 1$, $a = \frac{1}{2}$.
3. $K_S^2 = 4$
   - $S \in \mathcal{D}P_4, p : S' \to \mathbb{P}^1$ *a conic bundle with* $f = -K_{S'} - l$, *where* l *is a (-1)-curve,* $d = a = 1, m = 2$.

Proof. Let $\mathcal{H}_S = |-aK_S - b\eta|$, where η is a G-invariant bubble cycle of degree d. We have

$$(-aK_S - b\eta)^2 = a^2 K_S^2 - b^2 d = 0, \quad (-aK_S - b\eta, -K_S) = aK_S^2 - bd = 2.$$

Let $t = b/a$. We have

$$(td)^2 = dK_S^2, \quad K_S^2 - td = 2/a > 0.$$

The second inequality, gives $td < K_S^2$, hence $d < K_S^2$. Giving the possible values for K_S^2 and using that $a \in \frac{1}{3}\mathbb{Z}$, we check that the only possibilities are:

$$(K_S^2, d, t) = (9, 1, 3),\ (8, 2, 2),\ (4, 1, 2),\ (4, 2, 1).$$

This gives our cases and one extra case $(4, 2, 1)$. In this case $a = 2$ and $\mathcal{H}_S = |-2K_S - 2x_1|$ contradicting the primitiveness of f. Note that this case is realized in the case when the ground field is not algebraically closed (see [38]).

Proposition 7.13. *Let S, S' be as in Link of type II. Assume that S, S' are both minimal Del Pezzo surfaces. Then $(S \xleftarrow{\sigma} Z \xrightarrow{\tau} S')$, where σ is the blow-up of a G-invariant bubble cycle η with $\mathrm{ht}(\eta) = 0$ and some degree d. The proper transform of the linear system $|-K_{S'}|$ on S is equal to $|-aK_S - m\eta|$. And similarly defined d', m', a' for τ. The following cases are possible:*

1. $K_S^2 = 9$
   - $S' \cong S = \mathbb{P}^2, d = d' = 8, m = m' = 18, a = a' = 17$ *$(S \leftarrow Z \rightarrow S')$ is a minimal resolution of a Bertini involution).*
   - $S' \cong S = \mathbb{P}^2, d = d' = 7, m = m' = 9, a = a' = 8$ *$(S \leftarrow Z \rightarrow S')$ is a minimal resolution of a Geiser transformation).*
   - $S' \cong S = \mathbb{P}^2, d = d' = 6, m = m' = 6, a = a' = 5$ *$(S \leftarrow Z \rightarrow S')$ is a minimal resolution of a Cremona transformation given by the linear system $|5\ell - 2p_1 - 2p_2 - 2p_3 - 2p_4 - 2p_5|$,*
   - $S \cong \mathbb{P}^2, S' \in \mathcal{D}P_5, d = 5, m' = 6, a = \frac{5}{3}, d' = 1, m = 2, a' = 3.$
   - $S \cong S' = \mathbb{P}^2, d = d' = 3, m = m' = 1, a = a' = \frac{2}{3}$, *$(S \leftarrow Z \rightarrow S')$ is a minimal resolution of a standard quadratic transformation.*
   - $S = \mathbb{P}^2, S' = \mathbf{F}_0, d = 2, m = 3, a' = \frac{3}{2}, d' = 1, a = \frac{4}{3}.$
2. $K_S^2 = 8$
   - $S \cong S' \cong \mathbf{F}_0, d = d' = 7, a = a' = 15, m = m' = 16.$
   - $S \cong S' \cong \mathbf{F}_0, d = d' = 6, a = a' = 7, m = m' = 8.$
   - $S \cong \mathbf{F}_0, S' \in \mathcal{D}P_5, d = 5, d' = 2, a = \frac{5}{2}, m = 4, a' = 4, m' = 6.$
   - $S \cong \mathbf{F}_0, S' \cong \mathbf{F}_0, d = d' = 4, a = a' = 3, m = m' = 4.$
   - $S \cong \mathbf{F}_0, S' \in \mathcal{D}P_6, d = 3, d' = 1, a = \frac{3}{2}, m = 2, m' = 4, a' = 2.$
   - $S \cong \mathbf{F}_0, S' \cong \mathbb{P}^2, d = 1, d' = 2, a = \frac{3}{2}, m = 3, a' = \frac{4}{3}, m' = 2.$ *This link is the inverse of the last case from the preceding list.*
3. $K_S^2 = 6$
   - $S \cong S' \in \mathcal{D}P_6, d = d' = 5, a = 11, m = 12.$
   - $S \cong S' \in \mathcal{D}P_6, d = d' = 4, a = 5, m = 6.$
   - $S \cong S' \in \mathcal{D}P_6, d = d' = 3, a = 3, m = 4.$
   - $S \cong S' \in \mathcal{D}P_6, d = d' = 2, a = 2, m = 3.$
   - $S \in \mathcal{D}P_6, S' = \mathbf{F}_0, d = 1, d' = 3, a = \frac{3}{2}, m = 2.$ *This link is the inverse of the link from the preceding list with $S' \in \mathcal{D}P_6, d = 3$.*
4. $K_S^2 = 5$
   - $S \cong S' \in \mathcal{D}P_5, d = d' = 4, m = m' = 10, a = a' = 5..$
   - $S = S' \in \mathcal{D}P_5, d = d' = 3, m = m' = 5, a = a' = 4.$
   - $S \in \mathcal{D}P_5, S' = \mathbf{F}_0, d = 2, d' = 5.$ *This link is inverse of the link with $S = \mathbf{F}_0, S' \in \mathcal{D}P_5, d = 5$.*
   - $S \in \mathcal{D}P_5, S' = \mathbb{P}^2, d = 1, d' = 5.$ *This link is inverse of the link with $S = \mathbb{P}^2, S' \in \mathcal{D}P_5, d = 5$.*
5. $K_S^2 = 4$
   - $S \cong S' \in \mathcal{D}P_4, d = d' = 3.$ *This is an analog of the Bertini involution.*
   - $S \cong S' \in \mathcal{D}P_4, d = d' = 2.$ *This is an analog of the Geiser involution.*
6. $K_S^2 = 3$
   - $S \cong S' \in \mathcal{D}P_3, d = d' = 2.$ *This is an analog of the Bertini involution.*

- $S \cong S' \in \mathcal{D}P_3, d = d' = 1$. *This is an analog of the Geiser involution.*

7. $K_S^2 = 2$
   - $S = S' \in \mathcal{D}P_2, d = d' = 1$. *This is an analog of the Bertini involution.*

Proof. Similar to the proof of the previous proposition, we use that

$$\mathcal{H}_S^2 = a^2 K_S^2 - b^2 d = K_{S'}^2, \quad aK_S^2 - bd = K_{S'}^2,$$

$$H_{S'}^2 = a'^2 K_{S'}^2 - b'^2 d = K_S^2, \quad a'K_S^2 - b'd = K_S^2.$$

Since the link is not a biregular map, by Noether's inequality we have $a > 1, a' > 1, b > a, b' > a'$. This implies

$$d < K_S^2 - \frac{1}{a} K_{S'}^2, \ d' < K_{S'}^2 - \frac{1}{a'} K_S^2.$$

It is not difficult to list all solutions. For example, assume $K_S^2 = 1$. Since d is a positive integer, we see that there are no solutions. If $K_S^2 = 2$, we must have $d = d' = 1$.

Proposition 7.14. *Let S, S' be as in Link of type II. Assume that S, S' are both minimal conic bundles. Then $(S \leftarrow Z \rightarrow S')$ is a composition of elementary transformations $\mathrm{elm}_{x_1} \circ \ldots \circ \mathrm{elm}_{x_s}$, where $(x_1, \ldots, x_s)$ is a G-orbit of points not lying on a singular fibre with no two points lying on the same fibre.*

We skip the classification of links of type III. They are the inverses of links of type I.

Proposition 7.15. *Let S, S' be as in Link of type IV. Recall that they consist of changing the conic bundle structure. The following cases are possible:*

- $K_S^2 = 8, S' = S, f' = -K_{S'} - f$, *it is represented by a switch automorphism;*
- $K_S^2 = 4, S' = S, f' = -K_{S'} - f$;
- $K_S^2 = 2, S' = S, f' = -2K_{S'} - f$; *it is represented by a Geiser involution;*
- $K_S^2 = 1, S' = S, f' = -4K_{S'} - f$; *it is represented by a Bertini involution;*

Proof. In this case S admits two extremal rays and $\mathrm{rank}\, \mathrm{Pic}(S)^G = 2$ so that $-K_S$ is ample. Let $|f'|$ be the second conic bundle. Write $f' \sim -aK_S + bf$. Using that $f'^2 = 0, f \cdot K_S = f' \cdot K_S = -2$, we easily get $b = -1$ and $aK_S^2 = 4$. This gives all possible cases from the assertion.

8 Birational classes of minimal G-surfaces

8.1

Let S be a minimal G-surface S and $d = K_S^2$. We will classify all isomorphism classes of (S, G) according to the increasing parameter d. Since the number of singular fibres of a minimal conic bundle is at least 4, we have $d \leq 4$ for conic bundles.

- $d \leq 0$.

By Corollary 7.11, S is a superrigid conic bundle with $k = 8 - d$ singular fibres. The number k is a birational invariant. The group G is of de Jonquiéres type and its conjugacy class in $\mathrm{Cr}(2)$ is determined uniquely by Theorem 5.7 or Theorem 5.3.

Also observe that if $\phi : S \to \mathbb{P}^1$ is an exceptional conic bundle and $G_0 = \mathrm{Ker}(G \to \mathrm{O}(\mathrm{Pic}(S))$ is non-trivial, then no links of type II is possible. Thus the conjugacy class of G is uniquely determined by the isomorphism class of S.

- $d = 1$, S is a Del Pezzo surface.

By Corollary 7.11, the surface S is superrigid. Hence the conjugacy class of G is determined uniquely by its conjugacy class in $\mathrm{Aut}(S)$. All such conjugacy classes are listed in subsection 6.7.

- $d = 1$, S is a conic bundle.

Let $\phi : S \to \mathbb{P}^1$ be a minimal conic bundle on S. It has $k = 7$ singular fibres. If $-K_S$ is ample, i.e. S is a (non-minimal) Del Pezzo surface, then the center of $\mathrm{Aut}(S)$ contains the Bertini involution β. We know that β acts as -1 on $\mathcal{R}_S$, thus β does not act identically on $\mathrm{Pic}(S)^G$, hence $\beta \notin G$. Since k is odd, the conic bundle is not exceptional, so we cam apply Theorem 5.7, Case (2). It follows that G must contain a subgroup isomorphic to 2^2, adding β we get that S is a minimal Del Pezzo 2^3-surface of degree 1. However, the classification shows that there are no such surfaces.

Thus $-K_S$ is not ample. It follows from Proposition 7.13 that the structure of a conic bundle on S is unique. Any other conic bundle birationally G-isomorphic to S is obtained from S by elementary transformations with G-invariant set of centers.

- $d = 2$, S is a Del Pezzo surface.

By Corollary 7.11, S is superrigid unless G has a fixed point on S. If $\chi : S - \to S'$ is a birational G-map, then H_S has only one maximal base point and it does not lie on a (-1)-curve. We can apply an elementary link $Z \to S, Z \to S$ of type II which together with the projections $S \to \mathbb{P}^2$ resolves the Bertini involution. These links together with the G-automorphisms (including the Geiser involution) generate the group of birational G-automorphisms of S (see [38], Theorem 4.6). Thus the surface is rigid. The conjugacy class of G in $\mathrm{Cr}(2)$ is determined uniquely by the conjugacy class of G in $\mathrm{Aut}(S)$. All such conjugacy classes are listed in Table 7 and Theorem 6.17.

- $d = 2$, $\phi : S \to \mathbb{P}^1$ is a conic bundle.

If $-K_S$ is ample, then ϕ is not exceptional. The center of $\mathrm{Aut}(S)$ contains the Geiser involution γ. Since γ acts non-trivially on $\mathrm{Pic}(S)^G = \mathbb{Z}^2$, we see

that $\gamma \notin G$. Applying γ we obtain another conic bundle structure. In other words, γ defines an elementary link of type IV. Using the factorization theorem we show that the group of birational G-automorphisms of S is generated by links of type II, the Geiser involution, and G-automorphisms (see [36], [39], Theorem 4.9). Thus $\phi : S \to \mathbb{P}^1$ is a rigid conic bundle.

If S is not a Del Pezzo surface, ϕ could be an exceptional conic bundle with $g = 2$. In any case the group G is determined in Theorem 5.3. We do not know whether S can be mapped to a conic bundle with $-K_S$ ample (see [36]).

Applying Proposition 5.2, we obtain that any conic bundle with $d \geq 3$ is a non-minimal Del Pezzo surface, unless $d = 4$ and S is an exceptional conic bundle. In the latter case, the group G can be found in Theorem 5.3. It is not known whether it is birationally G-isomorphic to a Del Pezzo surface. It is true in the arithmetic case.

- $d = 3$, S is a minimal Del Pezzo surface.

The classification of elementary links shows that S is rigid. Birational G-automorphisms are generated by links of type (6) from Proposition 7.12. The conjugacy class of G in $\mathrm{Cr}(2)$ is determined by the conjugacy class of G in $\mathrm{Aut}(S)$.

- $d = 3$, S is a minimal conic bundle.

Since $k = 5$ is odd, G has 3 commuting involutions, the fixed-point locus of one of them must be a rational 2-section of the conic bundle. It is easy to see that it is a (-1)-curve C from the divisor class $-K_S - f$. The other two fixed-point curves are of genus 2. The group G leaves the curve C invariant. Thus blowing it down, we obtain a minimal Del Pezzo G-surface S' of degree 4. The group G contains a subgroup isomorphic to 2^2. Thus G can be found in the list of minimal groups of degree 4 Del Pezzo surfaces with a fixed point. For example, the group 2^2 has 4 fixed points.

- $d = 4$, S is a minimal Del Pezzo surface.

If $S^G = \emptyset$, then S admits only self-links of type II, so it is rigid or super-rigid. The conjugacy class of G in $\mathrm{Cr}(2)$ is determined by the conjugacy class of G in $\mathrm{Aut}(S)$ and we can apply Theorem 6.9. If x is a fixed point of G, then we can apply a link of type I, to get a minimal conic bundle with $d = 3$. So, all groups with $S^G \neq \emptyset$ are conjugate to groups of de Jonquiéres type realized on a conic bundle $S \in \mathcal{C}_5$.

- $d = 4$, S is a minimal conic bundle.

Since $k = 4$, it follows from Lemma 5.1 that either S is an exceptional conic bundle with $g = 1$, or S is a Del Pezzo surface with two sections with self-intersection -1 intersecting at one point. In the latter case, S is obtained by regularizing a de Jonquéres involution IH_3 (see section 2.3). In the case when

S is an exceptional conic bundle the groups of automorphisms are described in Example 5.4. They are minimal if and only if the kernel of the map $G \to \mathrm{PGL}(2)$ contains an involution not contained in $G_0 = \mathrm{Ker}(G \to \mathrm{O}(\mathrm{Pic}(S))$. If G_0 is not trivial, then no elementary transformation is possible. So, S is not birationally isomorphic to a Del Pezzo surface.

- $d = 5$, S is a Del Pezzo surface, $G \cong 5$.

Let us show that (S, G) is birationally isomorphic to $(\mathbb{P}^2, G)$. Since rational surfaces are simply-connected, G has a fixed point x on S. The anti-canonical model of S is a surface of degree 5 in $\mathbb{P}^5$. Let P be the tangent plane of S at x. The projection from P defines a birational G-equivariant map $S - \to \mathbb{P}^2$ given by the linear system of anti-canonical curves with double point at x. It is an elementary link of type II.

- $d = 5$, S is a Del Pezzo surface, $G \cong 5 : 2, 5 : 4$.

The cyclic subgroup of order 5 of G has two fixed points on S. This immediately follows from the Lefschetz fixed-point formula. Since it is normal in G, the groups G has an orbit κ with $d(\kappa) = 2$. Using an elementary link of type II with $S' = \mathbf{F}_0$, we obtain that G is conjugate to a group acting on $\mathbf{F}_0$.

- $d = 5$, S is a Del Pezzo surface, $G \cong A_5, S_5$.

It is clear that $S^G = \emptyset$ since otherwise G admits a faithful 2-dimensional linear representation. It is known that it does not exist. Since A_5 has no index 2 subgroups G does not admit orbits κ with $d(\kappa) = 2$. The same is obviously true for $G = S_5$. It follows from the classification of links that (S, G) is superrigid.

- $d = 6$.

One of the groups from this case, namely $G \cong 2 \times S_3$ was considered in [39], [40] (the papers also discuss the relation of this problem to some questions in the theory of algebraic groups raised in [43]). It is proved there that (S, G) is not birationally isomorphic to $(\mathbb{P}^2, G)$ but birationally isomorphic to minimal $(\mathbf{F}_0, G)$. The birational isomorphism is easy to describe. We know that G contains the lift of the standard Cremona involution. It has 4 fixed points in S, the lifts of the points given in affine coordinates by $(\pm 1, \pm 1)$. The group S_3 fixes $(1, 1)$ and permutes the remaining points p_1, p_2, p_3. The proper transforms of the lines $\langle p_i, p_j \rangle$ in S are disjoint (-1)-curves E_i. The anti-canonical model of S is a surface of degree 6 in $\mathbb{P}^6$. The projection from the tangent plane to S at the fixed point, is a link of type II with $S' = \mathbf{F}_0$. It blows-up the fixed point, and then blows down the pre-images of the curves E_i. Now the group G acts on $\mathbf{F}_0$ with $\mathbf{F}_0^G = \emptyset$.

If minimal G contains some non-trivial imprimitive projective transformations, then G has no fixed points. It follows from the classification of links that S is rigid. If $G \cong 6$ or S_3, then it acts on $\mathbf{F}_0$ with a fixed point. The projection

from this point defines a birational isomorphism (S, G) and $(\mathbb{P}^2, G)$. Thus the only groups which are not conjugate to a group of projective transformations are the groups which are mapped surjectively onto $W_S = S_3 \times 2$. Those of them which are mapped isomorphically are conjugate to subgroups of $\mathbf{F}_0$.

- $d = 8$.

In this case $S = \mathbf{F}_0$ or $\mathbf{F}_n, n > 1$. In the first case (S, G) is birationally isomorphic to $(\mathbb{P}^2, G)$ if $S^G \neq \emptyset$ (via the projection from the fixed point). This implies that the subgroup G' of G belonging to the connected component of the identity of $\mathrm{Aut}(\mathbf{F}_0)$ is an extension of cyclic groups. As we saw in Theorem 4.9 this implies that G' is an abelian group of transformations $(x, y) \mapsto (\epsilon_{nk}^a x, \epsilon_{mk}^b y)$, where $a = sb \mod k$ for some s coprime to k. If $G \neq G'$, then we must have $m = n = 1$ and $s = \pm 1 \mod k$.

If $\mathbf{F}_0^G = \emptyset$ and $\mathrm{Pic}(\mathbf{F}_0)^G \cong \mathbb{Z}$, then the classification of links shows that links of type II with $d = d' = 7, 6, 5, d = 3, d' = 1$ map $\mathbf{F}_0$ to $\mathbb{F}_0$ or to minimal Del Pezzo surfaces of degrees 5 or 6. These cases have been already considered. If $\mathrm{rank}\ \mathrm{Pic}(S)^G = 2$, then any birational G-map $S - \to S'$ is composed of elementary transformations with respect to one of the conic bundle fibrations. They do not change K_S^2 and do not give rise a fixed points. So, G is not conjugate to any subgroup of $\mathrm{Aut}(\mathbb{P}^2)$.

Assume $n > 1$. Then $G = A.B$, where $A \cong n$ acts identically on the base of the fibration and $B \subset \mathrm{PGL}(2)$. The subgroup B fixes pointwisely two disjoint sections, one of them is the exceptional one. Let us consider different cases corresponding to possible groups B.

$B \cong C_n$. In this case B has two fixed points on the base, hence G has 2 fixed points on the non-exceptional section. Performing an elementary transformation with center at one of these points we descend G to a subgroup of $\mathbf{F}_{n-1}$. Proceeding in this way, we arrive to the case $n = 1$, and then obtain that G is a group of automorphisms of $\mathbb{P}^2$.

$B \cong D_n$. In this case B has an orbit of cardinality 2 in $\mathbb{P}^1$. A similar argument shows that G has an orbit of cardinality 2 on the non-exceptional section. Applying the product of the elementary transformations at these points we descend G to a subgroup of automorphisms of $\mathbf{F}_{n-2}$. Proceeding in this way we obtain that G is a conjugate to a subgroup of $\mathrm{Aut}(\mathbb{P}^2)$ or of $\mathrm{Aut}(\mathbf{F}_0)$. In the latter case it does not have fixed points, and hence is not conjugate to a linear subgroup of $\mathrm{Cr}(2)$.

$B \cong T$. The group B has an orbit of cardinality 4 on the non-exceptional section. A similar argument shows that G is conjugate to a group of automorphisms of $\mathbf{F}_2, \mathbf{F}_0$, or $\mathbb{P}^2$.

$B \cong O$. The group B has an orbit of cardinality 6. As in the previous case we show that G is conjugate to a group of automorphisms of $\mathbb{P}^2$, or $\mathbf{F}_0$, or $\mathbf{F}_2$, or $\mathbf{F}_3$.

$B \cong I$. The group B has an orbit of cardinality 12. We obtain that G is conjugate to a group of automorphisms of $\mathbb{P}^2$ or of $\mathbf{F}_n$, where $n = 0, 2, 3, 4, 5, 6$.

- $d = 9$.

In this case $S = \mathbb{P}^2$ and G is a finite subgroup of PGL(3). The methods of the representation theory allows us to classify them up to conjugacy in the group PGL(3). However, some of non-conjugate groups can be still conjugate inside the Cremona group.

For example all cyclic subgroups of PGL(3) of the same order n are conjugate in Cr(2). Any element g of order n in PGL(3) is conjugate to a transformation g given in affine coordinates by the formula $(x, y) \mapsto (\epsilon_n x, \epsilon_n^a y)$. Let $T \in \mathrm{dJ}(2)$ be given by the formula $(x, y) \mapsto (x, x^a/y)$. Let $g' : (x, y) \mapsto (\epsilon_n^{-1} x, y)$. We have

$$g' \circ T \circ g : (x, y) \mapsto (\epsilon_n x, \epsilon_n^a y) \mapsto (\epsilon_n x, x^a/y) \mapsto (x, x^a/y) = T.$$

This shows that g' and g are conjugate.

We do not know whether any two isomorphic non-conjugate subgroups of PGL(3) are conjugate in Cr(2).

9 What is left?

Here we list some problems which have not been yet resolved.

- Find the conjugacy classes in Cr(2) of subgroups of PGL(3). For example, there are two non-conjugate subgroups of PGL(3) isomorphic to A_5 or A_6 which differ by an outer automorphism of the groups. Are they conjugate in Cr(2)?

- Find the finer classification of the conjugacy classes of de Jonquiéres groups.

We already know that the number of singular fibres in a minimal conic bundle G-surface is an invariant. Even more, the projective equivalence class of the corresponding k points on the base of the conic fibration is an invariant. Are there other invariants? In the case when $G_K \cong 2$, we know that the quotient of the conic bundle by the involution generating G_K is a minimal rule surface $\mathbf{F}_e$. Is the number e a new invariant?

- Give a finer geometric description of the algebraic variety parametrizing conjugacy classes.

Even in the case of Del Pezzo surfaces we give only normal forms. What is precisely the moduli space of Del Pezzo surfaces with a fixed isomorphism class of a minimal automorphism group?

We know that conic bundles (S, G) with $k \geq 8$ singular fibres are superrigid, so any finite subgroup G' of Cr(2) conjugate to G is realized as an automorphism group of a conic bundle obtained from S by a composition of

elementary transformations with G-invariant centers. If S is not exceptional and $G \cong 2.P$, then the invariant of the conjugacy class is the hyperelliptic curve of fixed points of the central involution. If $G \cong 2^2.P$, then we have three commuting involutions and their curves of fixed points are the invariants of the conjugacy class. Do they determine the conjugacy class?

When $k = 6, 7$ we do not know whether (S, G) is birationally isomorphic to (S', G), where S' is a Del Pezzo surface. This is true when $k \in \{0, 1, 2, 3, 5\}$ or $k = 4$ and S is not exceptional.

- Find more explicit description of groups G as subgroups of $\mathrm{Cr}(2)$.

This has been done in the case of abelian groups in [6]. For example one may ask to reprove and revise Autonne's classification of groups whose elements are quadratic transformations [3]. An example of such non-cyclic group is the group of automorphisms S_5 of a Del Pezzo surface of degree 5.

- Finish the classical work on the birational classification of rational cyclic planes $z^n = f(x, y)$.

More precisely, the quotient S/G of a rational surface S by a cyclic group of automorphisms defines a cyclic extension of the fields of rational functions. Thus there exists a rational function $R(x, y)$ such that there exists an isomorphism of fields $\mathbb{C}(x, y)(\sqrt[n]{R(x, y)}) \cong \mathbb{C}(x, y)$, where n is the order of G. Obviously we may assume that $R(x, y)$ is a polynomial $f(x, y)$, hence we obtain an affine model of S in the form $z^n = f(x, y)$. A birational isomorphism of G-surfaces replaces the branch curve $f(x, y) = 0$ by a Cremona equivalent curve $g(x, y)$. The problem is to describe the Cremona equivalence classes of the branch curves which define rational cyclic planes.

For example, when (S, G) is birationally equivalent to $(\mathbb{P}^2, G)$, we may take $f(x, y) = x$ since all cyclic groups of given order are conjugate in $\mathrm{Cr}(2)$. When $n = 2$, the problem was solved by M. Noether [46] and later G. Castelnuovo and F. Enriques [16] had realized that the classification follows from Bertini's classification of involutions in $\mathrm{Cr}(2)$. When n is prime the problem was studied by A. Bottari [11]. We refer for modern work on this problem to [12], [13].

- Extend the classification to the case of non-algebraically closed fields, e.g. $\mathbb{Q}$, and algebraically closed fields of positive characteristic.

Note that there are more automorphism groups in the latter case. For example, the Fermat cubic surface $T_0^3 + T_1^3 + T_2^3 + T_3^3 = 0$ over a field of characteristic 2 has the automorphism group isomorphic to $U(4, \mathbb{F}_4)$, which is a subgroup of index 2 of the Weyl group $W(E_6)$.

10 Tables

In the following tables we give the tables of conjugacy classes of subgroups in $\mathrm{Cr}(2)$ which are realized as minimal automorphism groups of Del Pezzo

surfaces of degree $d \leq 5$ and not conjugate to subgroups of automorphisms of minimal rational surfaces or conic bundles. The information about other groups can be found in the corresponding sections of the paper. The tables contain the order of a group G, its structure, the type of a surface on which the group is realized as a minimal group of automorphisms, the equation of the surface, and the number of conjugacy classes, if finite, or the dimension of the variety parametrizing the conjugacy classes.

Order	Type	Surface	Equation	Conjugacy
2	A_1^7	$\mathcal{D}P_2$	XIII	∞^6
2	A_1^8	$\mathcal{D}P_1$	XXII	
3	$3A_2$	$\mathcal{D}P_3$	I,III,IV	∞^1
3	$4A_2$	$\mathcal{D}P_1$	XVIII	∞^3
4	$2A_3 + A_1$	$\mathcal{D}P_2$	II,III,V	∞^1
4	$2D_4(a_1)$	$\mathcal{D}P_1$	I,VI, X,XVII,XX	∞^5
5	$2A_4$	$\mathcal{D}P_1$	XIV	∞^2
6	$E_6(a_2)$	$\mathcal{D}P_3$	I,VI	∞^1
6	$A_5 + A_1$	$\mathcal{D}P_3$	I, III, IV	∞^1
6	$E_7(a_4)$	$\mathcal{D}P_2$	XI	∞^1
6	$A_5 + A_2$	$\mathcal{D}P_2$	VIII	∞^1
6	$D_6(a_2) + A_1$	$\mathcal{D}P_2$	II,III,IV,IX	∞^1
6	$A_5 + A_2 + A_1$	$\mathcal{D}P_1$	II,VIII,XIII	∞^2
6	$E_6(a_2) + A_2$	$\mathcal{D}P_1$	II,XII	∞^2
6	$E_8(a_8)$	$\mathcal{D}P_1$	XVIII	∞^3
6	$2D_4$	$\mathcal{D}P_1$	VII,XI	∞^1
6	$E_7(a_4) + A_1$	$\mathcal{D}P_1$	II,VIII,XIX	∞^4
8	D_5	$\mathcal{D}P_4$	(40)	1
8	$D_8(a_3)$	$\mathcal{D}P_1$	X	1
9	$E_6(a_1)$	$\mathcal{D}P_3$	I	1
10	$E_8(a_6)$	$\mathcal{D}P_1$	IV,IX,XIV	∞^2
12	E_6	$\mathcal{D}P_3$	III	1
12	$E_7(a_2)$	$\mathcal{D}P_2$	III	1
12	$E_8(a_3)$	$\mathcal{D}P_1$	I,V	∞^2
14	$E_7(a_1)$	$\mathcal{D}P_2$	I	1
15	$E_8(a_5)$	$\mathcal{D}P_1$	IV	1
18	E_7	$\mathcal{D}P_2$	VI	1
20	$E_8(a_2)$	$\mathcal{D}P_1$	IX	1
24	$E_8(a_1)$	$\mathcal{D}P_1$	I	1
30	E_8	$\mathcal{D}P_1$	IV	1

Table 9. Cyclic subgroups

Order	Structure	Surface	Equation	Conjugacy classes
4	2^2	$\mathcal{D}P_4$		∞^2
4	2^2	$\mathcal{D}P_2$	XII	∞^5
4	2^2	$\mathcal{D}P_1$	XXI	∞^5
4	2^2	$\mathcal{D}P_1$	V, VI,X,XV,XVII	∞^3
8	2×4	$\mathcal{D}P_4$	(39)	∞^1
8	2×4	$\mathcal{D}P_2$	V	$2 \times \infty^1$
8	2×4	$\mathcal{D}P_2$	I-V	∞^1
8	2×4	$\mathcal{D}P_2$	VII	∞^2
8	2×4	$\mathcal{D}P_1$	VIII,XVI	∞^2
8	2^3	$\mathcal{D}P_4$		∞^2
8	2^3	$\mathcal{D}P_2$	I-V,X	∞^3
9	3^2	$\mathcal{D}P_3$	I	1
9	3^2	$\mathcal{D}P_3$	I,IV	$2 \times \infty^1$
9	3^2	$\mathcal{D}P_3$	III	1
9	3^2	$\mathcal{D}P_1$	I,II,III	∞^1
12	2×6	$\mathcal{D}P_4$	(40)	1
12	2×6	$\mathcal{D}P_2$	III, VIII	∞^1
12	2×6	$\mathcal{D}P_1$	II,VIII,XIII	∞^2
12	2×6	$\mathcal{D}P_1$	III,XII	∞^2
12	2×6	$\mathcal{D}P_1$	II,VII	∞^1
16	2^4	$\mathcal{D}P_4$		∞^2
16	$2^2 \times 4$	$\mathcal{D}P_2$	II,III,V	∞^1
16	4^2	$\mathcal{D}P_2$	II	1
16	2×8	$\mathcal{D}P_2$	II	1
18	3×6	$\mathcal{D}P_3$	I	1
18	3×6	$\mathcal{D}P_1$	III	∞^1
18	3×6	$\mathcal{D}P_1$	II	1
24	2×12	$\mathcal{D}P_1$	VIII	∞^1
24	2×12	$\mathcal{D}P_2$	III	1
27	3^3	$\mathcal{D}P_3$	I	1
32	2×4^2	$\mathcal{D}P_2$	II	1
36	6^2	$\mathcal{D}P_1$	II	1

Table 10. Abelian non-cyclic subgroups

Order	Structure	Surface	Equation	Conjugacy classes
6	D_6	$\mathcal{D}P_3$	III,IV,VIII,XI	∞^2
6	D_6	$\mathcal{D}P_3$	I	1
8	D_8	$\mathcal{D}P_4$	(39)	∞^1
8	D_8	$\mathcal{D}P_2$	II,III,V,VII	∞^2
8	D_8	$\mathcal{D}P_1$	I, XVI	∞^3
8	Q_8	$\mathcal{D}P_1$	II,V,VI,XIV	∞^2
8	$3 \times D_8$	$\mathcal{D}P_1$	I, XVI	∞^3
8	$3 \times Q_8$	$\mathcal{D}P_1$	II,III	∞^1
12	$2 \times D_6$	$\mathcal{D}P_3$	I,VI	∞^1
12	T	$\mathcal{D}P_2$	II	1
12	$2 \times D_6$	$\mathcal{D}P_2$	I,II,IV,IX	∞^2
12	$2 \times D_6$	$\mathcal{D}P_2$	II	1
12	$2 \times D_6$	$\mathcal{D}P_1$	I,II,III,VI	∞^1
16	$2 \times D_8$	$\mathcal{D}P_2$	II,III,V,VII	∞^2
16	D_{16}	$\mathcal{D}P_1$	I,IX	∞^1
18	$3 \times D_6$	$\mathcal{D}P_3$	III	1
18	$3 \times D_6$	$\mathcal{D}P_3$	I,IV	$2 \times \infty^1$
18	$3 \times D_6$	$\mathcal{D}P_1$	I,II,III	∞^1
24	$\overline{T}$	$\mathcal{D}P_1$	I,V	∞^1
24	$2 \times T$	$\mathcal{D}P_4$	(41)	1
24	S_4	$\mathcal{D}P_3$	I	3
24	S_4	$\mathcal{D}P_3$	II	1
24	S_4	$\mathcal{D}P_2$	II	1
24	S_4	$\mathcal{D}P_2$	I, II,IV	∞^1
24	$2 \times T$	$\mathcal{D}P_2$	II	1
24	$3 \times Q_8$	$\mathcal{D}P_1$	I	1
24	$3 \times Q_8$	$\mathcal{D}P_1$	II	1
24	$3 \times D_8$	$\mathcal{D}P_1$	I	∞^2
36	$6 \times D_6$	$\mathcal{D}P_1$	I,II,III	∞^1
48	$2 \times O$	$\mathcal{D}P_2$	I,II,III,V	∞^1
48	$2 \times \overline{T}$	$\mathcal{D}P_1$	I	1
72	$3 \times \overline{T}$	$\mathcal{D}P_1$	I	1

Table 11. Products of cyclic groups and polyhedral or binary polyhedral non-cyclic group

Order	Structure	Surface	Equation	Conjugacy classes
8	D_8	$\mathcal{D}P_4$	(39)	∞^1
8	D_8	$\mathcal{D}P_2$	II,III,IV,V,VII	∞^2
8	Q_8	$\mathcal{D}P_1$	I,V,XIV	∞^2
8	D_8	$\mathcal{D}P_1$	I,V,XVI	∞^2
16	L_{16}	$\mathcal{D}P_4$	(39)	$2 \times \infty^1$
16	$2 \times D_8$	$\mathcal{D}P_2$	II,III,IV,V,VII	1
16	$2 \times D_8$	$\mathcal{D}P_2$	I	1
16	AS_{16}	$\mathcal{D}P_2$	II,III,V	$2 \times \infty^1$
16	M_{16}	$\mathcal{D}P_2$	II	1
16	D_{16}	$\mathcal{D}P_1$	I,IX	∞^1
32	$2^2 : 8$	$\mathcal{D}P_4$	(40)	1
32	$2^4 : 2$	$\mathcal{D}P_4$	(39)	∞^1
32	$D_8 : 4$	$\mathcal{D}P_2$	II	1
32	$2 \times AS_{16}$	$\mathcal{D}P_2$	III,V	$3 \times \infty^1$
32	$2 \times M_{16}$	$\mathcal{D}P_2$	II	1
64	$2^4 : 4$	$\mathcal{D}P_4$	(40)	1
64	$2 \times (D_8 : 4)$	$\mathcal{D}P_2$	II	1
27	$H_3(3)$	$\mathcal{D}P_3$	I,III,IV	∞^1
81	$3^3 : 3$	$\mathcal{D}P_3$	I	1

Table 12. Non-abelian p-groups

Order	Structure	Surface	Equation	Conjugacy classes
18	$3^2:2$	$\mathcal{D}P_3$	I	2
24	$2D_{12}$	$\mathcal{D}P_1$	II,VI	∞^1
24	$D_8:3$	$\mathcal{D}P_2$	III	1
36	$3^2:2^2$	$\mathcal{D}P_3$	I	1
42	$2\times(7:3)$	$\mathcal{D}P_1$	II	1
48	$2^4:3$	$\mathcal{D}P_4$	(40)	1
48	$2\times D_8:3$	$\mathcal{D}P_2$	III	1
48	$\overline{T}:2$	$\mathcal{D}P_1$	I	1
48	$4^2:3$	$\mathcal{D}P_2$	II	1
54	$H_3(3):2$	$\mathcal{D}P_3$	IV	∞^1
54	$3^3:2$	$\mathcal{D}P_3$	I	2
60	A_5	$\mathcal{D}P_5$		1
72	$3\times 2D_{12}$	$\mathcal{D}P_1$	II	1
80	$2^4:5$	$\mathcal{D}P_4$	(41)	1
96	$2^4:S_3$	$\mathcal{D}P_4$	(40)	1
96	$4^2:S_3$	$\mathcal{D}P_2$	I	2
96	$2\times(4^2:3)$	$\mathcal{D}P_2$	II	1
108	$3^3:4$	$\mathcal{D}P_3$	I	2
108	$3^3:2^2$	$\mathcal{D}P_3$	I	2
120	S_5	$\mathcal{D}P_5$		1
120	S_5	$\mathcal{D}P_3$	II	1
144	$3\times(\overline{T}:2)$	$\mathcal{D}P_1$	I	1
160	$2^4:D_{10}$	$\mathcal{D}P_4$	(41)	1
162	$3^3:S_3$	$\mathcal{D}P_3$	I	2
168	$L_2(7)$	$\mathcal{D}P_2$	I	1
192	$2\times(4^2:S_3)$	$\mathcal{D}P_3$	I	1
216	$3^3:D_8$	$\mathcal{D}P_3$	I	2
336	$2\times L_2(7)$	$\mathcal{D}P_2$	I	1
648	$3^3:S_4$	$\mathcal{D}P_3$	I	2

Table 13. Other groups

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