A generalization of the Capelli identity

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Summary. We prove a generalization of the Capelli identity. As an application we obtain an isomorphism of the Bethe subalgebras actions under the $(\mathfrak{gl}_N, \mathfrak{gl}_M)$ duality.

To Yuri Manin on the occasion of 70-th birthday, with admiration.

1 Introduction

Let $\mathcal{A}$ be an associative algebra over complex numbers. Let $A = (a_{ij})_{i,j=1}^n$ be an $n \times n$ matrix with entries in $\mathcal{A}$. The *row determinant of* A is defined by the formula:

$$\mathrm{rdet}(A) := \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) a_{1\sigma_1} \dots a_{n\sigma_n}.$$

Let x_{ij}, $i, j = 1, \dots, M$, be commuting variables. Let $\partial_{ij} = \partial / \partial x_{ij}$,

$$E_{ij} = \sum_{a=1}^{M} x_{ia} \partial_{ja}. \tag{1}$$

Let $X = (x_{ij})_{i,j=1}^M$ and $D = (\partial_{ij})_{i,j=1}^M$ be $M \times M$ matrices.

The classical Capelli identity [C1] asserts the following equality of differential operators:

$$\operatorname{rdet}\Big(E_{ji} + (M-i)\delta_{ij}\Big)_{i,j=1}^{M} = \det(X)\det(D). \tag{2}$$

This identity is a "quantization" of the identity

$$\det(AB) = \det(A)\det(B)$$

for any matrices A, B with commuting entries.

The Capelli identity has the following meaning in the representation theory. Let $\mathbb{C}[X]$ be the algebra of complex polynomials in variables x_{ij}. There are two natural actions of the Lie algebra $\mathfrak{gl}_M$ on $\mathbb{C}[X]$. The first action is given by operators from (1) and the second action is given by operators $\widetilde{E}_{ij} = \sum_{a=1}^{M} x_{ai}\partial_{aj}$. The two actions commute and the corresponding $\mathfrak{gl}_M \oplus \mathfrak{gl}_M$ action is multiplicity free.

It is not difficult to see that the right hand side of (2), considered as a differential operator on $\mathbb{C}[X]$, commutes with both actions of $\mathfrak{gl}_M$ and therefore lies in the image of the center of the universal enveloping algebra $U\mathfrak{gl}_M$ with respect to the first action. Then the left hand side of the Capelli identity expresses the corresponding central element in terms of $U\mathfrak{gl}_M$ generators.

Many generalizations of the Capelli identity are known. One group of generalizations considers other elements of the center of $U\mathfrak{gl}_M$, called quantum immanants, and then expresses them in terms of $\mathfrak{gl}_M$ generators, see [C2], [N1],[O]. Another group of generalizations considers other pairs of Lie algebras in place of $(\mathfrak{gl}_M, \mathfrak{gl}_M)$, e.g. $(\mathfrak{gl}_M, \mathfrak{gl}_N)$, $(\mathfrak{sp}_{2M}, \mathfrak{gl}_2)$, $(\mathfrak{sp}_{2M}, \mathfrak{so}_N)$, etc, see [MN], [HU]. The third group of generalizations produces identities corresponding not to pairs of Lie algebras, but to pairs of quantum groups [NUW] or superalgebras [N2].

In this paper we prove a generalization of the Capelli identity which seemingly does not fit the above classification.

Let $\boldsymbol{z} = (z_1, \dots, z_N)$, $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_M)$ be sequences of complex numbers. Let $Z = (z_i\delta_{ij})_{ij=1}^{N}$, $\Lambda = (\lambda_i\delta_{ij})_{ij=1}^{M}$ be the corresponding diagonal matrices. Let X and D be the $M \times N$ matrices with entries x_{ij} and ∂_{ij}, $i = 1, \dots, M$, $j = 1, \dots, N$, respectively. Let $\mathbb{C}[X]$ be the algebra of complex polynomials in variables x_{ij}, $i = 1, \dots, M$, $j = 1, \dots, N$. Let $E_{ij}^{(a)} = x_{ia}\partial_{ja}$, where $i, j = 1, \dots, M$, $a = 1, \dots, N$.

In this paper we prove that

$$\prod_{a=1}^{N}(u - z_a)\ \operatorname{rdet}\Big((\partial_u - \lambda_i)\delta_{ij} - \sum_{a=1}^{N} \frac{E_{ji}^{(a)}}{u - z_a}\Big)_{i,j=1}^{M} = \operatorname{rdet}\begin{pmatrix} u - Z & X^t \\ D & \partial_u - \Lambda \end{pmatrix} \tag{3}$$

The left hand side of (3) is an $M \times M$ matrix while the right hand side is an $(M+N) \times (M+N)$ matrix.

Identity (3) is a "quantization" of the identity

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A)\ \det(D - CA^{-1}B)$$

which holds for any matrices A, B, C, D with commuting entries, for the case when A and D are diagonal matrices.

By setting all z_i, λ_j and u to zero, and $N = M$ in (3), we obtain the classical Capelli identity (2), see Section 2.4.

Our proof of (3) is combinatorial and reduces to the case of 2×2 matrices. In particular, it gives a proof of the classical Capelli identity, which may be new.

We invented identity (3) to prove Theorem 6 below, and Theorem 6 in its turn was motivated by results of [MTV2]. In Theorem 6 we compare actions of two Bethe subalgebras.

Namely, consider $\mathbb{C}[X]$ as a tensor product of evaluation modules over the current Lie algebras $\mathfrak{gl}_M[t]$ and $\mathfrak{gl}_N[t]$ with evaluation parameters $\boldsymbol{z}$ and $\boldsymbol{\lambda}$, respectively. The action of the algebra $\mathfrak{gl}_M[t]$ on $\mathbb{C}[X]$ is given by the formula

$$E_{ij} \otimes t^n = \sum_{a=1}^{N} x_{ia} \partial_{ja} z_i^n ,$$

and the action of the algebra $\mathfrak{gl}_N[t]$ on $\mathbb{C}[X]$ is given by the formula

$$E_{ij} \otimes t^n = \sum_{a=1}^{M} x_{ai} \partial_{aj} \lambda_i^n .$$

In contrast to the previous situation, these two actions do not commute.

The algebra $U\mathfrak{gl}_M[t]$ has a family of commutative subalgebras $\mathcal{G}(M, \boldsymbol{\lambda})$ depending on parameters $\boldsymbol{\lambda}$ and called the Bethe subalgebras. For a given $\boldsymbol{\lambda}$, the Bethe subalgebra $\mathcal{G}(M, \boldsymbol{\lambda})$ is generated by the coefficients of the expansion of the expression

$$\operatorname{rdet} \Big((\partial_u - \lambda_i)\delta_{ij} - \sum_{a=1}^{N} \sum_{s=1}^{\infty} (E_{ji}^{(a)} \otimes t^s) u^{-s-1} \Big)_{i,j=1}^{M} \tag{4}$$

with respect to powers of u and ∂_u, cf. Section 3. For different versions of definitions of Bethe subalgebras and relations between them, see [FFR], [T], [R], [MTV1].

Similarly, there is a family of Bethe subalgebras $\mathcal{G}(N, \boldsymbol{z})$ in $U\mathfrak{gl}_N[t]$ depending on parameters $\boldsymbol{z}$.

For fixed $\boldsymbol{\lambda}$ and $\boldsymbol{z}$, consider the action of the Bethe subalgebras $\mathcal{G}(M, \boldsymbol{\lambda})$ and $\mathcal{G}(N, \boldsymbol{z})$ on $\mathbb{C}[X]$ as defined above. In Theorem 6 we show that the actions of the Bethe subalgebras on $\mathbb{C}[X]$ induce the same subalgebras of endomorphisms of $\mathbb{C}[X]$.

The paper is organized as follows. In Section 2 we describe and prove formal Capelli-type identities and in Section 3 we discuss the relations of the identities to the Bethe subalgebras.

Acknowledgment. We thank the referee for bringing to our attention the papers [KS] and [GR] which relate the Capelli identity to Jordan algebras and "quasideterminants", respectively.

Research of E.M. is supported in part by NSF grant DMS-0601005. Research of A.V. is supported in part by NSF grant DMS-0555327.

2 Identities

2.1 The main identity

We work over the field of complex numbers, however all results of this paper hold over any field of characteristic zero.

Let $\mathcal{A}$ be an associative algebra. Let $A = (a_{ij})_{i,j=1}^n$ be an $n \times n$ matrix with entries in $\mathcal{A}$. Define the *row determinant of* A by the formula:

$$\mathrm{rdet}(A) := \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) a_{1\sigma_1} \dots a_{n\sigma_n},$$

where S_n is the symmetric group on n elements.

Fix two natural numbers M and N and a complex number $h \in \mathbb{C}$. Consider noncommuting variables u, p_u, x_{ij}, p_{ij}, where $i = 1, \dots, M$, $j = 1, \dots, N$, such that the commutator of two variables equals zero except

$$[p_u, u] = h, \qquad [p_{ij}, x_{ij}] = h,$$

$i = 1, \dots, M$, $j = 1, \dots, N$.

Let X, P be two $M \times N$ matrices given by

$$X \ := \ (x_{ij})_{i=1,\dots,M}^{j=1,\dots,N}\ , \qquad P \ := \ (p_{ij})_{i=1,\dots,M}^{j=1,\dots,N}\ .$$

Let $\mathcal{A}_h^{(MN)}$ be the associative algebra whose elements are polynomials in p_u, x_{ij}, p_{ij}, $i = 1, \dots, M$, $j = 1, \dots, N$, with coefficients that are rational functions in u.

Let $\mathcal{A}^{(MN)}$ be the associative algebra of linear differential operators in u, x_{ij}, $i = 1, \dots, M$, $j = 1, \dots, N$, with coefficients in $\mathbb{C}(u) \otimes \mathbb{C}[X]$.

We often drop the dependence on M, N and write $\mathcal{A}_h$, $\mathcal{A}$ for $\mathcal{A}_h^{(MN)}$ and $\mathcal{A}^{(MN)}$, respectively.

For $h \neq 0$, we have the isomorphism of algebras

$$\begin{aligned} \iota_h \ : \ \mathcal{A}_h &\to \mathcal{A}\ , \\ u,\, x_{ij} &\mapsto u,\, x_{ij}\ , \\ p_u,\, p_{ij} &\mapsto h\frac{\partial}{\partial u},\, h\frac{\partial}{\partial x_{ij}}\ . \end{aligned} \tag{5}$$

Fix two sequences of complex numbers $\boldsymbol{z} = (z_1, \ldots, z_N)$ and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_M)$. Define the $M \times M$ matrix $G_h = G_h(M, N, u, p_u, \boldsymbol{z}, \boldsymbol{\lambda}, X, P)$ by the formula

$$G_h := \Big((p_u - \lambda_i)\, \delta_{ij} - \sum_{a=1}^{N} \frac{x_{ja} p_{ia}}{u - z_a} \Big)_{i,j=1}^{M} . \qquad (6)$$

Theorem 1. *We have*

$$\prod_{a=1}^{N} (u - z_a) \ \mathrm{rdet}(G_h) = \sum_{A,B,|A|=|B|} (-1)^{|A|} \prod_{a \notin B} (u - z_a) \prod_{b \notin A} (p_u - \lambda_b) \ \det(x_{ab})_{a \in A}^{b \in B} \ \det(p_{ab})_{a \in A}^{b \in B},$$

where the sum is over all pairs of subsets $A \subset \{1, \ldots, M\}$, $B \subset \{1, \ldots, N\}$ such that A and B have the same cardinality, $|A| = |B|$. Here the sets A, B inherit the natural ordering from the sets $\{1, \ldots, M\}$, $\{1, \ldots, N\}$. This ordering determines the determinants in the formula.

Theorem 1 is proved in Section 2.5.

2.2 A presentation as a row determinant of size $M + N$

Theorem 1 implies that the row determinant of G can be written as the row determinant of a matrix of size $M + N$.

Namely, let Z be the diagonal $N \times N$ matrix with diagonal entries $z_1, \ldots, z_N$. Let Λ be the diagonal $M \times M$ matrix with diagonal entries $\lambda_1, \ldots, \lambda_M$:

$$Z := (z_i \delta_{ij})_{i,j=1}^{N} , \qquad \Lambda := (\lambda_i \delta_{ij})_{i,j=1}^{M} .$$

Corollary 2. *We have*

$$\prod_{a=1}^{N} (u - z_a) \ \mathrm{rdet}\ G = \mathrm{rdet} \begin{pmatrix} u - Z & X^t \\ P & p_u - \Lambda \end{pmatrix},$$

where X^t denotes the transpose of the matrix X.

Proof. Denote

$$W := \begin{pmatrix} u - Z & X^t \\ P & p_u - \Lambda \end{pmatrix},$$

The entries of the first N rows of W commute. The entries of the last M rows of W also commute. Write the Laplace decomposition of $\mathrm{rdet}(W)$ with respect to the first N rows. Each term in this decomposition corresponds to a choice of N columns in the $N \times (N + M)$ matrix $(u - Z, X^T)$. We label such a choice by a pair of subsets $A \subset \{1, \ldots, M\}$ and $B \subset \{1, \ldots, N\}$ of the same cardinality. Namely, the elements of A correspond to the chosen columns in X^T and the elements of the complement to B correspond to the chosen columns in $u - Z$. Then the term in the Laplace decomposition corresponding to A and B is exactly the term labeled by A and B in the right hand side of the formula in Theorem 1. Therefore, the corollary follows from Theorem 1. □

Let A, B, C, D be any matrices with commuting entries of sizes $N \times N, N \times M, M \times N$ and $M \times M$, respectively. Let A be invertible. Then we have the equality of matrices of sizes $(M+N) \times (M+N)$:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & 0 \\ C & D - CA^{-1}B \end{pmatrix} \begin{pmatrix} 1 & A^{-1}B \\ 0 & 1 \end{pmatrix}$$

and therefore

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A) \det(D - CA^{-1}B) \ . \tag{7}$$

The identity of Corollary 2 for $h = 0$ turns into identity (7) with diagonal matrices A and D. Therefore, the identity of Corollary 2 may be thought of as a "quantization" of identity (7) with diagonal A and D.

2.3 A relation between determinants of sizes M and N

Introduce new variables v, p_v such that $[p_v, v] = h$.

Let $\bar{\mathcal{A}}_h$ be the associative algebra whose elements are polynomials in p_u, p_v, x_{ij}, p_{ij}, $i = 1, \dots, M$, $j = 1, \dots, N$, with coefficients in $\mathbb{C}(u) \otimes \mathbb{C}(v)$.

Let $e : \bar{\mathcal{A}}_h \to \bar{\mathcal{A}}_h$ be the unique linear map which is the identity map on the subalgebra of $\bar{\mathcal{A}}_h$ generated by all monomials which do not contain p_u and p_v and which satisfy

$$e(ap_u) = e(a)v, \qquad e(ap_v) = e(a)u,$$

for any $a \in \bar{\mathcal{A}}_h$.

Let $\bar{\mathcal{A}}$ be the associative algebra of linear differential operators in u, v, x_{ij}, $i = 1, \dots, M$, $j = 1, \dots, N$, with coefficients in $\mathbb{C}(u) \otimes \mathbb{C}(v) \otimes \mathbb{C}[x_{ij}]$. Then for $h \neq 0$, we have the isomorphism of algebras extending the isomorphism (5):

$$\begin{aligned} \bar{\iota}_h : \ & \bar{\mathcal{A}}_h \to \bar{\mathcal{A}}, \\ & u, v, x_{ij} \mapsto u, v, x_{ij}, \\ & p_u, p_v, p_{ij} \mapsto h\frac{\partial}{\partial u}, h\frac{\partial}{\partial v}, h\frac{\partial}{\partial x_{ij}} \ . \end{aligned}$$

For $a \in \bar{\mathcal{A}}$ and a function $f(u, v)$ let $a \cdot f(u, v)$ denotes the function obtained by the action of a considered as a differential operator in u and v on the function $f(u, v)$.

We have

$$\bar{\iota}_h(e(a)) = \exp(-uv/h)\bar{\iota}_h(a) \cdot \exp(uv/h)$$

for any $a \in \bar{A}_h$ such that a does not depend on either p_u or p_v.

Define the $N \times N$ matrix $H_h = H_h(M, N, v, p_v, \boldsymbol{z}, \boldsymbol{\lambda}, X, P)$ by

$$H_h := \Big((p_v - z_i)\delta_{ij} - \sum_{b=1}^{M} \frac{x_{bj} p_{bi}}{v - \lambda_b} \Big)_{i,j=1}^{N}, \tag{8}$$

cf. formula (6).

Corollary 3. *We have*

$$e\Big(\prod_{a=1}^{N}(u-z_a)\,\mathrm{rdet}(G_h)\Big)=e\Big(\prod_{b=1}^{M}(v-\lambda_b)\,\mathrm{rdet}(H_h)\Big).$$

Proof. Write the dependence on parameters of the matrix G: $G_h=G_h(M,N,u,p_u,\boldsymbol{z},\boldsymbol{\lambda},X,P)$. Then

$$H_h=G_h(N,M,v,p_v,\boldsymbol{\lambda},\boldsymbol{z},X^T,P^T).$$

The corollary now follows from Theorem 1. □

2.4 A relation to the Capelli identity

In this section we show how to deduce the Capelli identity from Theorem 1.

Let s be a complex number. Let $\alpha_s:\mathcal{A}_h\to\mathcal{A}_h$ be the unique linear map which is the identity map on the subalgebra of $\mathcal{A}_h$ generated by all monomials which do not contain p_u, and which satisfies

$$\alpha_s(aup_u)=s\alpha_s(a)$$

for any $a\in\bar{\mathcal{A}}_h$.

We have

$$\bar{\iota}_h(\alpha_s(a))=u^{-s/h}\bar{\iota}_h(a)\cdot u^{s/h}$$

for any $a\in\bar{A}_h$.

Consider the case $z_1=\dots=z_N=0$ and $\lambda_1=\dots=\lambda_M=0$ in Theorem 1.

Then it is easy to see that the row determinant $\mathrm{rdet}(G)$ can be rewritten in the following form

$$u^M\,\mathrm{rdet}(G_h)=\mathrm{rdet}\left(h(up_u-M+i)\delta_{ij}-\sum_{a=1}^{N}x_{ja}p_{ia}\right)_{i,j=1}^{M}.$$

Applying the map α_s, we get

$$\alpha_s(u^M\,\mathrm{rdet}(G_h))=\mathrm{rdet}\left(h(s-M+i)\delta_{ij}-\sum_{a=1}^{N}x_{ja}p_{ia}\right)_{i,j=1}^{M}.$$

Therefore applying Theorem 1 we obtain the identity

$$\mathrm{rdet}\left(h(s-M+i)\delta_{ij}-\sum_{a=1}^{N}x_{ja}p_{ia}\right)_{i,j=1}^{M}=\sum_{A,B,|A|=|B|}(-1)^{|A|}\prod_{b=0}^{M-|A|-1}(s-bh)\ \det(x_{ab})_{a\in A}^{b\in B}\ \det(p_{ab})_{a\in A}^{b\in B}.$$

In particular, if $M=N$, and $s=0$, we obtain the famous Capelli identity:

$$\mathrm{rdet}\Big(\sum_{a=1}^{M} x_{ja}p_{ia} + h(M-i)\delta_{ij}\Big)_{i,j=1}^{M} = \det X \ \det P.$$

If $h = 0$ then all entries of X and P commute and the Capelli identity reads $\det(XP) = \det(X)\det(P)$. Therefore, the Capelli identity can be thought of as a "quantization" of the identity $\det(AB) = \det(A)\det(B)$ for square matrices A, B with commuting entries.

2.5 Proof of Theorem 1

We denote

$$E_{ij,a} := x_{ja}p_{ia}/(u - z_a).$$

We obviously have

$$[E_{ij,a}, E_{kl,b}] = \delta_{ab}(\delta_{kj}(E_{il,a})' - \delta_{il}(E_{kj,a})'),$$

where the prime denotes the formal differentiation with respect to u.

Denote also $F^1_{jk,a} = -E_{jk,a}$ and $F^0_{jj,0} = (p_u - \lambda_j)$.

Expand $\mathrm{rdet}(G)$. We get an alternating sum of terms,

$$\mathrm{rdet}(G_h) = \sum_{\sigma,a,c} (-1)^{\mathrm{sgn}(\sigma)} F^{c(1)}_{1\sigma(1),a(1)} F^{c(2)}_{2\sigma(2),a(2)} \dots F^{c(M)}_{M\sigma(M),a(M)}, \tag{9}$$

where the summation is over all triples σ, a, c such that σ is a permutation of $\{1, \dots, M\}$ and a, c are maps $a : \{1, \dots, M\} \to \{0, 1, \dots, N\}$, $c : \{1, \dots, M\} \to \{0, 1\}$ satisfying: $c(i) = 1$ if $\sigma(i) \neq i$; $a(i) = 0$ if and only if $c(i) = 0$.

Let m be a product whose factors are of the form $f(u)$, p_u, p_{ij}, x_{ij} where $f(u)$ are some rational functions in u. Then the product m will be called *normally ordered* if all factors of the form p_u, p_{ij} are on the right from all factors of the form $f(u), x_{ij}$. For example, $(u-1)^{-2}x_{11}p_u p_{11}$ is normally ordered and $p_u(u-1)^{-2}x_{11}p_{11}$ is not.

Given a product m as above, define a new normally ordered product $: m :$ as the product of all factors of m in which all factors of the form p_u, p_{ij} are placed on the right from all factors of the form $f(u), x_{ij}$. For example, $: p_u(u-1)^{-2}x_{11}p_{11} := (u-1)^{-2}x_{11}p_u p_{11}$.

If all variables p_u, p_{ij} are moved to the right in the expansion of $\mathrm{rdet}(G)$ then we get terms obtained by normal ordering from the terms in (9) plus new terms created by the non-trivial commutators. We show that in fact all new terms cancel in pairs.

Lemma 4. *For $i = 1, \dots, M$, we have*

$$\mathrm{rdet}(G_h) = \sum_{\sigma,a,c} (-1)^{\mathrm{sgn}(\sigma)} F^{c(1)}_{1\sigma(1),a(1)} \dots F^{c(i-1)}_{(i-1)\sigma(i-1),a(i-1)} \Big(: F^{c(i)}_{i\sigma(i),a(i)} \dots F^{c(M)}_{M\sigma(M),a(M)} : \Big) \tag{10}$$

where the sum is over the same triples σ, a, c as in (9).

Proof. We prove the lemma by induction on i. For $i = M$ the lemma is a tautology. Assume it is proved for $i = M, M-1, \dots, j$, let us prove it for $i = j-1$.

We have

$$F^1_{(j-1)r,a} : F^{c(j)}_{j\sigma(j),a(j)} \dots F^{c(M)}_{M,\sigma(M),a(M)} := \tag{11}$$
$$: F^1_{(j-1)r,a} F^{c(j)}_{j\sigma(j),a(j)} \dots F^{c(M)}_{M\sigma(M),a(M)} : + \sum_k : F^{c(j)}_{j\sigma(j),a(j)} \dots (-E_{kr,a})' \dots F^{c(M)}_{M\sigma(M),a(M)} : ,$$

where the sum is over $k \in \{j, \dots, M\}$ such that $a(k) = a$, $\sigma(k) = j-1$ and $c(k) = 1$.

We also have

$$F^0_{(j-1)(j-1),0} : F^{c(j)}_{j\sigma(j),a(j)} \dots F^{c(M)}_{M\sigma(M),a(M)} := \tag{12}$$
$$: F^0_{(j-1)(j-1),0} F^{c(j)}_{j\sigma(j),a(j)} \dots F^{c(M)}_{M\sigma(M),a(M)} : + \sum_k : F^{c(j)}_{j\sigma(j),a(j)} \dots (-E_{k\sigma(k),a(k)})' \dots F^{c(M)}_{M\sigma(M),a(M)} : ,$$

where the sum is over $k \in \{j, \dots, M\}$ such that $c(k) = 1$.

Using (11), (12), rewrite each term in (10) with $i = j$. Then the k-th term obtained by using (11) applied to the term labeled by σ, c, a with $c(j-1) = 0$ cancels with the k-th obtained by using (12) applied to the term labeled by $\tilde\sigma, \tilde c, \tilde a$ defined by the following rules.

$$\tilde\sigma(i) = \sigma(i) \quad (i \neq j-1, k), \qquad \tilde\sigma(j-1) = j-1, \qquad \tilde\sigma(k) = \sigma(j-1),$$
$$\tilde c(i) = c(i) \quad (i \neq j-1), \qquad \tilde c(j-1) = 0,$$
$$\tilde a(i) = a(i) \quad (i \neq j-1), \qquad \tilde a(j-1) = 0.$$

After this cancellation we obtain the statement of the lemma for $i = j-1$. □

Remark 5. The proof of Lemma 4 implies that if the matrix σG_h is obtained from G_h by permuting the rows of G_h by a permutation σ then $\mathrm{rdet}(\sigma G_h) = (-1)^{\mathrm{sgn}(\sigma)} \mathrm{rdet}(G_h)$.

Consider the linear isomorphism $\phi_h : A_h \to A_0$ which sends any normally ordered monomial m in A_h to the same monomial m in A_0.

By (10) with $i = 1$, the image $\phi_h(\mathrm{rdet}(G_h))$ does not depend on h and therefore can be computed at $h = 0$. Therefore Theorem 1 for all h follows from Theorem 1 for $h = 0$. Theorem 1 for $h = 0$ follows from formula (7).

3 The $(\mathfrak{gl}_M, \mathfrak{gl}_N)$ duality and the Bethe subalgebras

3.1 Bethe subalgebra

Let E_{ij}, $i, j = 1, \dots, M$, be the standard generators of $\mathfrak{gl}_M$. Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{gl}_M$,

$$\mathfrak{h} = \oplus_{i=1}^{M} \mathbb{C} \cdot E_{ii}.$$

We denote $U\mathfrak{gl}_M$ the universal enveloping algebra of $\mathfrak{gl}_M$.

For $\mu \in \mathfrak{h}^*$, and a $\mathfrak{gl}_M$ module L denote by $L[\mu]$ the vector subspace of L of vectors of weight μ,

$$L[\mu] = \{v \in L \mid hv = \langle \mu, h\rangle \, v \text{ for any } h \in \mathfrak{h}\}.$$

We always assume that $L = \oplus_\mu L[\mu]$.

For any integral dominant weight $\Lambda \in \mathfrak{h}^*$, denote by L_Λ the finite-dimensional irreducible $\mathfrak{gl}_M$-module with highest weight Λ.

Recall that we fixed sequences of complex numbers $\boldsymbol{z} = (z_1, \dots, z_N)$ and $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_M)$. From now on we will assume that $z_i \neq z_j$ and $\lambda_i \neq \lambda_j$ if $i \neq j$.

For $i, j = 1, \dots, M,$, $a = 1, \dots, N$, let $E_{ji}^{(a)} = 1^{\otimes (a-1)} \otimes E_{ji} \otimes 1^{\otimes (N-a)} \in (U\mathfrak{gl}_M)^{\otimes N}$.

Define the $M \times M$ matrix $\widetilde{G} = \widetilde{G}(M, N, \boldsymbol{z}, \boldsymbol{\lambda}, u)$ by

$$\widetilde{G}(M, N, \boldsymbol{z}, \boldsymbol{\lambda}, u) := \Big((\frac{\partial}{\partial u} - \lambda_i)\delta_{ij} - \sum_{a=1}^{N} \frac{E_{ji}^{(a)}}{u - z_a} \Big)_{i,j=1}^{M}.$$

The entries of $\widetilde{G}$ are differential operators in u whose coefficients are rational functions in u with values in $(U\mathfrak{gl}_M)^{\otimes N}$.

Write

$$\mathrm{rdet}(\widetilde{G}(M, N, \boldsymbol{z}, \boldsymbol{\lambda}, u)) = \frac{\partial^M}{\partial u^M} + \widetilde{G}_1(M, N, \boldsymbol{z}, \boldsymbol{\lambda}, u) \frac{\partial^{M-1}}{\partial u^{M-1}} + \dots + \widetilde{G}_M(M, N, \boldsymbol{z}, \boldsymbol{\lambda}, u).$$

The coefficients $\widetilde{G}_i(M, N, \boldsymbol{z}, \boldsymbol{\lambda}, u)$, $i = 1, \dots, M$, are called *the transfer matrices of the Gaudin model.* The transfer matrices are rational functions in u with values in $(U\mathfrak{gl}_M)^{\otimes N}$.

The transfer matrices commute:

$$[\widetilde{G}_i(M, N, \boldsymbol{z}, \boldsymbol{\lambda}, u), \widetilde{G}_j(M, N, \boldsymbol{z}, \boldsymbol{\lambda}, v)] = 0,$$

for all i, j, u, v, see [T] and Proposition 7.2 in [MTV1].

The transfer matrices clearly commute with the diagonal action of $\mathfrak{h}$ on $(U\mathfrak{gl}_M)^{\otimes N}$.

For $i = 1, \dots, M$, it is clear that $\widetilde{G}_i(M, N, \boldsymbol{z}, \boldsymbol{\lambda}, u) \prod_{a=1}^{N} (u - z_a)^i$ is a polynomial in u whose coefficients are pairwise commuting elements of $(U\mathfrak{gl}_M)^{\otimes N}$. Let $\mathcal{G}(M, N, \boldsymbol{z}, \boldsymbol{\lambda}) \subset (U\mathfrak{gl}_M)^{\otimes N}$ be the commutative subalgebra generated by the coefficients of polynomials $\widetilde{G}_i(M, N, \boldsymbol{z}, \boldsymbol{\lambda}, u) \prod_{a=1}^{N} (u - z_a)^i$, $i = 1, \dots, M$. We call the subalgebra $\mathcal{G}(M, N, \boldsymbol{z}, \boldsymbol{\lambda})$ the *Bethe subalgebra.*

Let $\mathcal{G}(M, \boldsymbol{\lambda}) \subset U\mathfrak{gl}_M[t]$ be the subalgebra considered in the introduction. Let $U\mathfrak{gl}_M[t] \to (U\mathfrak{gl}_M)^{\otimes N}$ be the algebra homomorphism defined by $E_{ij} \otimes$

$t^n \mapsto \sum_{a=1}^{N} E_{ij}^{(a)} z_a^n$. Then the subalgebra $\mathcal{G}(M,N,\boldsymbol{z},\boldsymbol{\lambda})$ is the image of the subalgebra $\mathcal{G}(M,\boldsymbol{\lambda})$ under that homomorphism.

The Bethe subalgebra clearly acts on any N-fold tensor products of $\mathfrak{gl}_M$ representations.

Define the *Gaudin Hamiltonians*, $H_a(M,N,\boldsymbol{z},\boldsymbol{\lambda}) \subset (U\mathfrak{gl}_M)^{\otimes N}$, $a = 1,\dots,N$, by the formula

$$H_a(M,N,\boldsymbol{z},\boldsymbol{\lambda}) = \sum_{b=1, b\neq a}^{N} \frac{\Omega^{(ab)}}{z_a - z_b} + \sum_{b=1}^{M} \lambda_b E_{bb}^{(a)},$$

where $\Omega^{(ab)} := \sum_{i,j=1}^{M} E_{ij}^{(a)} E_{ji}^{(b)}$.

Define the *dynamical Hamiltonians* $H_a^{\vee}(M,N,\boldsymbol{z},\boldsymbol{\lambda}) \subset (U\mathfrak{gl}_M)^{\otimes N}$, $a = 1,\dots,M$, by the formula

$$H_a^{\vee}(M,N,\boldsymbol{z},\boldsymbol{\lambda}) = \sum_{b=1,\ b\neq a}^{M} \frac{(\sum_{i=1}^{N} E_{ab}^{(i)})(\sum_{i=1}^{N} E_{ba}^{(i)}) - \sum_{i=1}^{N} E_{aa}^{(i)}}{\lambda_a - \lambda_b} + \sum_{b=1}^{N} z_b\, E_{aa}^{(b)} .$$

It is known that the Gaudin Hamiltonians and the dynamical Hamiltonians are in the Bethe subalgebra, see e.g. Appendix B in [MTV1]:

$$H_a(M,N,\boldsymbol{z},\boldsymbol{\lambda}) \in \mathcal{G}(M,N,\boldsymbol{z},\boldsymbol{\lambda}), \qquad H_b^{\vee}(M,N,\boldsymbol{z},\boldsymbol{\lambda}) \in \mathcal{G}(M,N,\boldsymbol{z},\boldsymbol{\lambda}),$$

$a = 1,\dots,N$, $b = 1,\dots,M$.

3.2 The $(\mathfrak{gl}_M, \mathfrak{gl}_N)$ duality

Let $L_\bullet^{(M)} = \mathbb{C}[x_1,\dots,x_M]$ be the space of polynomials of M variables. We define the $\mathfrak{gl}_M$-action on $L_\bullet^{(M)}$ by the formula

$$E_{ij} \mapsto x_i \frac{\partial}{\partial x_j}.$$

Then we have an isomorphism of $\mathfrak{gl}_M$ modules

$$L_\bullet^{(M)} = \bigoplus_{m=0}^{\infty} L_m^{(M)}$$

the submodule $L_m^{(M)}$ being spanned by homogeneous polynomials of degree m. The submodule $L_m^{(M)}$ is the irreducible $\mathfrak{gl}_M$ module with highest weight $(m,0,\dots,0)$ and highest weight vector x_1^m.

Let $L_\bullet^{(M,N)} = \mathbb{C}[x_{11},\dots,x_{1N},\dots,x_{M1},\dots,x_{MN}]$ be the space of polynomials of MN commuting variables.

Let $\pi^{(M)} : (U\mathfrak{gl}_M)^{\otimes N} \to \mathrm{End}(L_\bullet^{(M,N)})$ be the algebra homomorphism defined by

$$E_{ij}^{(a)} \mapsto x_{ia} \frac{\partial}{\partial x_{ja}}.$$

In particular, we define the $\mathfrak{gl}_M$ action on $L_\bullet^{(M,N)}$ by the formula

$$E_{ij} \mapsto \sum_{a=1}^{N} x_{ia} \frac{\partial}{\partial x_{ja}}.$$

Let $\pi^{(N)} : (U\mathfrak{gl}_N)^{\otimes M} \to \mathrm{End}(L_\bullet^{(M,N)})$ be the algebra homomorphism defined by

$$E_{ij}^{(a)} \mapsto x_{ai} \frac{\partial}{\partial x_{aj}}.$$

In particular, we define the $\mathfrak{gl}_N$ action on $L_\bullet^{(M,N)}$ by the formula

$$E_{ij} \mapsto \sum_{a=1}^{M} x_{ai} \frac{\partial}{\partial x_{aj}}.$$

We have isomorphisms of algebras,

$$\begin{aligned} &\big(\mathbb{C}[x_1,\dots,x_M]\big)^{\otimes N} \to L_\bullet^{(M,N)}, \; 1^{\otimes(j-1)} \otimes x_i \otimes 1^{\otimes(N-j)} \mapsto x_{ij}, \\ &\big(\mathbb{C}[x_1,\dots,x_N]\big)^{\otimes M} \to L_\bullet^{(M,N)}, \; 1^{\otimes(i-1)} \otimes x_j \otimes 1^{\otimes(M-i)} \mapsto x_{ij}. \end{aligned} \tag{13}$$

Under these isomorphisms the space $L_\bullet^{(M,N)}$ is isomorphic to $(L_\bullet^{(M)})^{\otimes N}$ as a $\mathfrak{gl}_M$ module and to $(L_\bullet{}^{(N)})^{\otimes M}$ as a $\mathfrak{gl}_N$ module.

Fix $\boldsymbol{n} = (n_1,\dots,n_N) \in \mathbb{Z}_{\geq 0}^N$ and $\boldsymbol{m} = (m_1,\dots,m_M) \in \mathbb{Z}_{\geq 0}^M$ with $\sum_{i=1}^N n_i = \sum_{a=1}^M m_a$. The sequences $\boldsymbol{n}$ and $\boldsymbol{m}$ naturally correspond to integral $\mathfrak{gl}_N$ and $\mathfrak{gl}_M$ weights, respectively.

Let $\boldsymbol{L_m}$ and $\boldsymbol{L_n}$ be $\mathfrak{gl}_N$ and $\mathfrak{gl}_M$ modules, respectively, defined by the formulas

$$\boldsymbol{L_m} = \otimes_{a=1}^M L_{m_a}^{(N)}, \qquad \boldsymbol{L_n} = \otimes_{b=1}^N L_{n_b}^{(M)}.$$

The isomorphisms (13) induce an isomorphism of the weight subspaces,

$$\boldsymbol{L_n}[\boldsymbol{m}] \simeq \boldsymbol{L_m}[\boldsymbol{n}]. \tag{14}$$

Under the isomorphism (14) the Gaudin and dynamical Hamiltonians interchange,

$$\begin{aligned} \pi^{(M)} H_a(M,N,\boldsymbol{z},\boldsymbol{\lambda}) &= \pi^{(N)} H_a^\vee(N,M,\boldsymbol{\lambda},\boldsymbol{z}), \\ \pi^{(M)} H_b^\vee(M,N,\boldsymbol{z},\boldsymbol{\lambda}) &= \pi^{(N)} H_b(N,M,\boldsymbol{\lambda},\boldsymbol{z}), \end{aligned}$$

for $a = 1,\dots,N$, $b = 1,\dots,M$, see [TV].

We prove a stronger statement that the images of $\mathfrak{gl}_M$ and $\mathfrak{gl}_N$ Bethe subalgebras in $\mathrm{End}(L_\bullet^{(M,N)})$ are the same.

Theorem 6. *We have*

$$\pi^{(M)}(\mathcal{G}(M,N,\boldsymbol{z},\boldsymbol{\lambda})) = \pi^{(N)}(\mathcal{G}(N,M,\boldsymbol{\lambda},\boldsymbol{z})).$$

Moreover, we have

$$\prod_{a=1}^{N}(u-z_a)\pi^{(M)}\,\mathrm{rdet}(\widetilde{G}(M,N,\boldsymbol{z},\boldsymbol{\lambda},u)) = \sum_{a=1}^{N}\sum_{b=1}^{M} A_{ab}^{(M)}\ u^a \frac{\partial^b}{\partial u^b},$$
$$\prod_{b=1}^{M}(v-\lambda_b)\pi^{(N)}\,\mathrm{rdet}(\widetilde{G}(N,M,\boldsymbol{\lambda},\boldsymbol{z},v)) = \sum_{a=1}^{N}\sum_{b=1}^{M} A_{ab}^{(N)}\ v^b \frac{\partial^a}{\partial v^a},$$

where $A_{ab}^{(M)}$, $A_{ab}^{(N)}$ are linear operators independent on $u, v, \partial/\partial u, \partial/\partial v$ and

$$A_{ab}^{(M)} = A_{ab}^{(N)}.$$

Proof. We obviously have

$$\pi^{(M)}(\widetilde{G}(M,N,\boldsymbol{z},\boldsymbol{\lambda},u)) = \bar{\imath}_{h=1}(G_{h=1}),$$
$$\pi^{(N)}(\widetilde{G}(N,M,\boldsymbol{\lambda},\boldsymbol{z},v)) = \bar{\imath}_{h=1}(H_{h=1}),$$

where $G_{h=1}$ and $H_{h=1}$ are matrices defined in (6) and (8). Then the coefficients of the differential operators $\prod_{a=1}^{N}(u-z_a)\pi^{(M)}\,\mathrm{rdet}(\widetilde{G}(M,N,\boldsymbol{z},\boldsymbol{\lambda},u))$ and $\prod_{b=1}^{M}(v-\lambda_b)\pi^{(N)}\,\mathrm{rdet}(\widetilde{G}(N,M,\boldsymbol{\lambda},\boldsymbol{z},v))$ are polynomials in u and v of degrees N and M, respectively, by Theorem 1. The rest of the theorem follows directly from Corollary 3. □

3.3 Scalar differential operators

Let $w \in \boldsymbol{L_n}[\boldsymbol{m}]$ be a common eigenvector of the Bethe subalgebra $\mathcal{G}(M,N,\boldsymbol{z},\boldsymbol{\lambda})$. Then the operator $\mathrm{rdet}(\widetilde{G}(M,N,\boldsymbol{z},\boldsymbol{\lambda},u))$ acting on w defines a monic scalar differential operator of order M with rational coefficients in variable u. Namely, let $D_w(M,N,\boldsymbol{\lambda},\boldsymbol{z})$ be the differential operator given by

$$D_w(M,N,\boldsymbol{z},\boldsymbol{\lambda},u) = \frac{\partial^M}{\partial u^M} + \widetilde{G}_1^w(M,N,\boldsymbol{z},\boldsymbol{\lambda},u)\frac{\partial^{M-1}}{\partial u^{M-1}} + \cdots + \widetilde{G}_M^w(M,N,\boldsymbol{z},\boldsymbol{\lambda},u),$$

where $\widetilde{G}_i^w(M,N,\boldsymbol{z},\boldsymbol{\lambda},u)$ is the eigenvalue of the ith transfer matrix acting on the vector w:

$$\widetilde{G}_i(M,N,\boldsymbol{z},\boldsymbol{\lambda},u)w = \widetilde{G}_i^w(M,N,\boldsymbol{z},\boldsymbol{\lambda},u)w.$$

Using isomorphism (14), consider w as a vector in $\boldsymbol{L_m}[\boldsymbol{n}]$. Then by Theorem 6, w is also a common eigenvector for algebra $\mathcal{G}(N,M,\boldsymbol{\lambda},\boldsymbol{z})$. Thus, similarly, the operator $\mathrm{rdet}(\widetilde{G}(N,M,\boldsymbol{\lambda},\boldsymbol{z},v))$ acting on w defines a monic scalar differential operator of order N, $D_w(N,M,\boldsymbol{\lambda},\boldsymbol{z},v)$.

Corollary 7. *We have*

$$\prod_{a=1}^{N}(u-z_a)D_w(M,N,\boldsymbol{z},\boldsymbol{\lambda},u)=\sum_{a=1}^{N}\sum_{b=1}^{M}A^{(M)}_{ab,w}\ u^a\,\frac{\partial^b}{\partial u^b},$$
$$\prod_{b=1}^{M}(v-\lambda_b)D_w(N,M,\boldsymbol{\lambda},\boldsymbol{z},v)=\sum_{a=1}^{N}\sum_{b=1}^{M}A^{(N)}_{ab,w}\ v^b\,\frac{\partial^a}{\partial v^a},$$

where $A^{(M)}_{ab,w}$, $A^{(N)}_{ab,w}$ *are numbers independent on* $u, v, \partial/\partial u, \partial/\partial v$. *Moreover,*

$$A^{(M)}_{ab,w}=A^{(N)}_{ab,w}.$$

Proof. The corollary follows directly from Theorem 6. □

Corollary 7 was essentially conjectured in Conjecture 5.1 in [MTV2].

Remark 8. The operators $D_w(M,N,\boldsymbol{z},\boldsymbol{\lambda})$ are useful objects, see [MV1], [MTV2], [MTV3]. They have the following three properties.

(i) The kernel of $D_w(M,N,\boldsymbol{z},\boldsymbol{\lambda})$ is spanned by the functions $p_i^w(u)e^{\lambda_i u}$, $i=1,\dots,M$, where $p_i^w(u)$ is a polynomial in u of degree m_i.
(ii) All finite singular points of $D_w(M,N,\boldsymbol{z},\boldsymbol{\lambda})$ are $z_1,\dots,z_N$.
(iii) Each singular point z_i is regular and the exponents of $D_w(M,N,\boldsymbol{z},\boldsymbol{\lambda})$ at z_i are $0, n_i+1, n_i+2, \dots, n_i+M-1$.

A converse statement is also true. Namely, if a linear differential operator of order M has properties (i-iii), then the operator has the form $D_w(M,N,\boldsymbol{z},\boldsymbol{\lambda})$ for a suitable eigenvector w of the Bethe subalgebra. This statement may be deduced from Proposition 9 below.

We discuss the properties of such differential operators in [MTV4], cf. also [MTV2] and Appendix A in [MTV3].

3.4 The simple joint spectrum of the Bethe subalgebra

It is proved in [R], that for any tensor product of irreducible $\mathfrak{gl}_M$ modules and for generic $\boldsymbol{z},\boldsymbol{\lambda}$ the Bethe subalgebra has a simple joint spectrum. We give here a proof of this fact in the special case of the tensor product $\boldsymbol{L_n}$.

Proposition 9. *For generic values of* $\boldsymbol{\lambda}$*, the joint spectrum of the Bethe subalgebra* $\mathcal{G}(M,N,\boldsymbol{z},\boldsymbol{\lambda})$ *acting in* $\boldsymbol{L_n}[\boldsymbol{m}]$ *is simple.*

Proof. We claim that for generic values of $\boldsymbol{\lambda}$, the joint spectrum of the Gaudin Hamiltonians $H_a(M,N,\boldsymbol{z},\boldsymbol{\lambda})$, $a=1,\dots,N$, acting in $\boldsymbol{L_n}[\boldsymbol{m}]$ is simple. Indeed fix $\boldsymbol{z}$ and consider $\boldsymbol{\lambda}$ such that $\lambda_1 \gg \lambda_2 \gg \dots \gg \lambda_M \gg 0$. Then the eigenvectors of the Gaudin Hamiltonians in $\boldsymbol{L_n}[\boldsymbol{m}]$ will have the form

$v_1 \otimes \cdots \otimes v_N + o(1)$, where $v_i \in L_{n_i}[\boldsymbol{m}^{(i)}]$ and $\boldsymbol{m} = \sum_{i=1}^{N} \boldsymbol{m}^{(i)}$. The corresponding eigenvalue of $H_a(M, N, \boldsymbol{z}, \boldsymbol{\lambda})$ will be $\sum_{j=1}^{M} \lambda_j m_j^{(a)} + O(1)$.

The weight spaces $L_{n_i}^{(M)}[\boldsymbol{m_i}]$ all have dimension at most 1 and therefore the joint spectrum is simple in this asymptotic zone of parameters. Therefore it is simple for generic values of $\boldsymbol{\lambda}$. □

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