

A SPECTRAL INTERPRETATION FOR THE ZEROS OF THE RIEMANN ZETA FUNCTION

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Abstract. Based on work of Alain Connes, I have constructed a spectral interpretation for zeros of L -functions. Here we specialise this construction to the Riemann ζ -function. We construct an operator on a nuclear Fréchet space whose spectrum is the set of non-trivial zeros of ζ . We exhibit the explicit formula for the zeros of the Riemann ζ -function as a character formula.

1. Introduction

The purpose of this note is to explain what the spectral interpretation for zeros of L -functions in [Mey] amounts to in the simple special case of the Riemann ζ -function. The article [Mey] is inspired by the work of Alain Connes in [Con99]. We will construct a nuclear Fréchet space $\mathcal{H}_-^0$ and an operator D_- on $\mathcal{H}_-^0$ whose spectrum is equal to the set of non-trivial zeros of the Riemann ζ -function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$. By definition, the non-trivial zeros of ζ are the zeros of the *complete ζ -function*

$$(1) \quad \xi(s) = \pi^{-s/2} \Gamma(s/2) \zeta(s).$$

In addition, the algebraic multiplicity of s as an eigenvalue of D_- is the zero order of ξ at s . Thus D_- is a spectral interpretation for the zeros of ξ .

November 11, 2004.

This research was supported by the EU-Network *Quantum Spaces and Noncommutative Geometry* (Contract HPRN-CT-2002-00280) and the *Deutsche Forschungsgemeinschaft* (SFB 478).

We construct D_- as the generator of a smooth representation ρ_- of $\mathbb{R}_+^\times \cong \mathbb{R}$ on $\mathcal{H}_-^0$. Although the single operator D_- is more concrete, it is usually better to argue with the representation ρ_- instead. Let $\mathcal{D}(\mathbb{R}_+^\times)$ be the convolution algebra of smooth, compactly supported functions on $\mathbb{R}_+^\times$. The integrated form of ρ_- is a bounded algebra homomorphism $\int \rho_- : \mathcal{D}(\mathbb{R}_+^\times) \rightarrow \text{End}(\mathcal{H}_-^0)$. We show that ρ_- is a summable representation in the notation of [Mey]. That is, $\int \rho_-(f)$ is a nuclear operator for all $f \in \mathcal{D}(\mathbb{R}_+^\times)$. The character $\chi(\rho_-)$ is the distribution on $\mathbb{R}_+^\times$ defined by $\chi(\rho_-)(f) = \text{tr} \int \rho_-(f)$. The representation ρ_- is part of a virtual representation $\rho = \rho_+ \ominus \rho_-$, where ρ_+ is a spectral interpretation for the poles of ξ . That is, ρ_+ is 2-dimensional and its generator D_+ has eigenvalues 0 and 1. We interpret ρ as a formal difference of ρ_+ and ρ_- and therefore define $\chi(\rho) := \chi(\rho_+) - \chi(\rho_-)$.

The spectrum of ρ consists exactly of the poles and zeros of ξ , and the spectral multiplicity (with appropriate signs) of $s \in \mathbb{C}$ is the order of ξ at s , which is positive at the two poles 0 and 1 and negative at the zeros of ξ . The spectral computation of the character yields

$$\chi(\rho)(f) = \sum_{s \in \mathbb{C}} \text{ord}_\xi(s) \hat{f}(s), \quad \text{where} \quad \hat{f}(s) := \int_0^\infty f(x) x^s \frac{dx}{x}.$$

If an operator has a sufficiently nice integral kernel, then we can also compute its trace by integrating its kernel along the diagonal. This recipe applies to $\int \rho_-(f)$ and yields another formula for $\chi(\rho_-)$. Namely,

$$\chi(\rho) = W = \sum_{p \in \mathcal{P}} W_p + W_\infty,$$

where $\mathcal{P}$ is the set of primes,

$$(2) \quad W_p(f) = \sum_{e=1}^{\infty} f(p^{-e}) p^{-e} \ln(p) + \sum_{e=1}^{\infty} f(p^e) \ln(p),$$

$$(3) \quad W_\infty(f) = pv \int_0^\infty \frac{f(x)}{|1-x|} + \frac{f(x)}{1+x} dx.$$

The distribution W_∞ involves a principal value because the integrand may have a pole at 1. Equating the two formulas for $\chi(\rho)$, we get the well-known explicit formula that relates zeros of ξ and prime numbers.

We do not need the functions ζ and ξ to define our spectral interpretation. Instead we use an operator Z , called the Zeta operator, which is closely related to the ζ -function. This operator is the key ingredient in our construction. In addition, we have to choose the domain and target space of Z rather carefully.

It is possible to prove the Prime Number Theorem using the representation ρ instead of the ζ -function. The only input that we need is that the distribution W is quantised, that is, of the form $\sum n(s)\hat{f}(s)$ with some function $n: \mathbb{C} \rightarrow \mathbb{Z}$ (see [Mey]).

Our constructions for the ζ -function can be generalised to Dirichlet L -functions. We indicate how this is done in the last section. We run into problems, however, for number fields with more than one infinite place. There are also other conceptual and aesthetic reasons for preferring adelic constructions as in [Mey]. Our goal here is only to make these constructions more explicit in a simple special case.

2. The ingredients: some function spaces and operators

Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space of $\mathbb{R}$. Thus $f: \mathbb{R} \rightarrow \mathbb{C}$ belongs to $\mathcal{S}(\mathbb{R})$ if and only if all its derivatives $f^{(n)}$ are rapidly decreasing in the sense that $f^{(n)}(x) = O(|x|^{-s})$ for $|x| \rightarrow \infty$ for all $s \in \mathbb{R}_+$, $n \in \mathbb{N}$. We topologise $\mathcal{S}(\mathbb{R})$ in the usual fashion. The convolution turns $\mathcal{S}(\mathbb{R})$ into a Fréchet algebra.

We remark that we neither gain nor lose anything if we view $\mathcal{S}(\mathbb{R})$ as a bornological vector space as in [Mey]. All function spaces that we shall need are Fréchet spaces, so that bornological and topological analysis are equivalent. The bornological point of view only becomes superior if we mix $\mathcal{S}(\mathbb{R})$ with spaces like $\mathcal{S}(\mathbb{Q}_p)$, which are not Fréchet.

We use the natural logarithm $\ln$ to identify the multiplicative group $\mathbb{R}_+^\times$ with $\mathbb{R}$. This induces an isomorphism between the Schwartz algebras $\mathcal{S}(\mathbb{R}_+^\times)$ and $\mathcal{S}(\mathbb{R})$. The standard Lebesgue measure on $\mathbb{R}$ corresponds to the Haar measure $d^\times x = x^{-1} dx$ on $\mathbb{R}_+^\times$. We always use this measure in the following.

We let

$$\mathcal{S}(\mathbb{R}_+^\times)_s = \mathcal{S}(\mathbb{R}_+^\times) \cdot x^{-s} = \{f: \mathbb{R}_+^\times \rightarrow \mathbb{C} \mid (x \mapsto f(x)x^s) \in \mathcal{S}(\mathbb{R}_+^\times)\}$$

for $s \in \mathbb{R}$ and

$$\mathcal{S}(\mathbb{R}_+^\times)_I = \bigcap_{s \in I} \mathcal{S}(\mathbb{R}_+^\times)_s$$

for an interval $I \subseteq \mathbb{R}$. We will frequently use that

$$\mathcal{S}(\mathbb{R}_+^\times)_{[a,b]} = \mathcal{S}(\mathbb{R}_+^\times)_a \cap \mathcal{S}(\mathbb{R}_+^\times)_b.$$

The reason for this is that $\mathcal{S}(\mathbb{R}_+^\times)$ is closed under multiplication by $(x^\varepsilon + x^{-\delta})^{-1}$ for $\varepsilon, \delta \geq 0$.

Hence $\mathcal{S}(\mathbb{R}_+^\times)_{[a,b]}$ becomes a Fréchet space in a canonical way. Exhausting I by an increasing sequence of compact intervals, we may turn $\mathcal{S}(\mathbb{R}_+^\times)_I$ into a Fréchet space for general I . Since $x \mapsto x^s$ is a character of $\mathbb{R}_+^\times$, the spaces

$\mathcal{S}(\mathbb{R}_+^\times)_s$ for $s \in \mathbb{R}$ are closed under convolution. Hence $\mathcal{S}(\mathbb{R}_+^\times)_I$ is closed under convolution as well and becomes a Fréchet algebra. We are particularly interested in

$$\begin{aligned}\mathcal{H}_- &= \mathcal{O}(\mathbb{R}_+^\times) := \mathcal{S}(\mathbb{R}_+^\times)_{]-\infty, \infty[}, \\ \mathcal{S}_> &:= \mathcal{S}(\mathbb{R}_+^\times)_{]1, \infty[}, \\ \mathcal{S}_< &:= \mathcal{S}(\mathbb{R}_+^\times)_{]-\infty, 0[}.\end{aligned}$$

We also let

$$\mathcal{H}_+ := \{f \in \mathcal{S}(\mathbb{R}) \mid f(x) = f(-x) \text{ for all } x \in \mathbb{R}\}.$$

The spaces $\mathcal{H}_\pm$ will be crucial for our spectral interpretation; the spaces $\mathcal{S}_>$ and $\mathcal{S}_<$ only play an auxiliary role as sufficiently large spaces in which the others can be embedded.

Given topological vector spaces A and B , we write $A \prec B$ to denote that A is contained in B and that the inclusion is a continuous linear map. Clearly,

$$\mathcal{S}(\mathbb{R}_+^\times)_I \prec \mathcal{S}(\mathbb{R}_+^\times)_J \quad \text{if } I \supseteq J.$$

The group $\mathbb{R}_+^\times$ acts on $\mathcal{S}(\mathbb{R}_+^\times)_I$ and $\mathcal{H}_+$ by the *regular representation*

$$\lambda_t f(x) := f(t^{-1}x) \quad \text{for } t, x \in \mathbb{R}_+^\times.$$

Its integrated form is given by the same formula as the convolution:

$$f \lambda(h) f(x) := \int_0^\infty h(t) f(t^{-1}x) d^\times t.$$

We denote the projective complete topological tensor product by $\hat{\otimes}$ (see [Gro55]). If V and W are Fréchet spaces, so is $V \hat{\otimes} W$. We want to know $\mathcal{S}(\mathbb{R}_+^\times)_I \hat{\otimes} \mathcal{S}(\mathbb{R}_+^\times)_J$ for two intervals I, J . Since both tensor factors are nuclear Fréchet spaces, this is easy enough to compute. We find

$$(4) \quad \begin{aligned} &\mathcal{S}(\mathbb{R}_+^\times)_I \hat{\otimes} \mathcal{S}(\mathbb{R}_+^\times)_J \\ &\cong \{f: (\mathbb{R}_+^\times)^2 \rightarrow \mathbb{C} \mid f(x, y) \cdot x^s y^t \in \mathcal{S}((\mathbb{R}_+^\times)^2) \text{ for all } s \in I, t \in J\} \end{aligned}$$

with the canonical topology. This follows easily from $\mathcal{S}(\mathbb{R}) \hat{\otimes} \mathcal{S}(\mathbb{R}) \cong \mathcal{S}(\mathbb{R}^2)$ and the compatibility of $\hat{\otimes}$ with inverse limits.

We shall need the *Fourier transform*

$$(5) \quad \mathfrak{F}: \mathcal{S}(\mathbb{R}) \rightarrow \mathcal{S}(\mathbb{R}), \quad \mathfrak{F}f(y) := \int_{\mathbb{R}} f(x) \exp(2\pi i xy) dx.$$

Notice that $\mathfrak{F}f$ is even if f is. Hence $\mathfrak{F}$ restricts to an operator on $\mathcal{H}_+ \subseteq \mathcal{S}(\mathbb{R})$. In the following, we usually restrict $\mathfrak{F}$ to this subspace. It is well-known that

$\mathfrak{F}^{-1}f(y) = \mathfrak{F}f(-y)$ for all $f \in \mathcal{S}(\mathbb{R})$, $y \in \mathbb{R}$. Hence

$$(6) \quad \mathfrak{F}^2 = \text{id} \quad \text{as operators on } \mathcal{H}_+.$$

Since $\mathfrak{F}$ is unitary on $L^2(\mathbb{R}, dx)$, it is also unitary on the subspace of even functions, which is isomorphic to $L^2(\mathbb{R}_+^\times, x d^\times x)$.

We also need the involution

$$(7) \quad J: \mathcal{O}(\mathbb{R}_+^\times) \rightarrow \mathcal{O}(\mathbb{R}_+^\times), \quad Jf(x) := x^{-1}f(x^{-1}).$$

We have $J^2 = \text{id}$. One checks easily that J extends to a unitary operator on $L^2(\mathbb{R}_+^\times, x d^\times x)$ and to an isomorphism of topological vector spaces

$$(8) \quad J: \mathcal{S}(\mathbb{R}_+^\times)_I \xrightarrow{\cong} \mathcal{S}(\mathbb{R}_+^\times)_{1-I}$$

for any interval I . Especially, J is an isomorphism between $\mathcal{S}_<$ and $\mathcal{S}_>$.

We have

$$(9) \quad f\lambda(h) \circ \mathfrak{F} = \mathfrak{F} \circ f\lambda(Jh), \quad f\lambda(h) \circ J = J \circ f\lambda(Jh)$$

for all $h \in \mathcal{D}(\mathbb{R}_+^\times)$. Hence the composites $\mathfrak{F}J$ and $J\mathfrak{F} = (\mathfrak{F}J)^{-1}$, which are unitary operators on $L^2(\mathbb{R}_+^\times, x d^\times x)$, commute with the regular representation λ .

A *multiplier* of $\mathcal{S}(\mathbb{R}_+^\times)_I$ is a continuous linear operator on $\mathcal{S}(\mathbb{R}_+^\times)_I$ that commutes with the regular representation.

Proposition 2.1. *Viewing $\mathcal{H}_+ \subseteq L^2(\mathbb{R}_+^\times, x d^\times x)$, we have*

- (10)
- $\mathcal{H}_+ = \{f \in L^2(\mathbb{R}_+^\times, x d^\times x) \mid f \in \mathcal{S}(\mathbb{R}_+^\times)_{]0, \infty[} \text{ and } J\mathfrak{F}(f) \in \mathcal{S}(\mathbb{R}_+^\times)_{]-\infty, 1[}\}$.
 - The operator $\mathfrak{F}J$ is a multiplier of $\mathcal{S}(\mathbb{R}_+^\times)_I$ for $I \subseteq]0, \infty[$.
 - The operator $J\mathfrak{F}$ is a multiplier of $\mathcal{S}(\mathbb{R}_+^\times)_I$ for $I \subseteq]-\infty, 1[$.
 - Hence $\mathfrak{F}J$ and $J\mathfrak{F}$ are invertible multipliers of $\mathcal{S}(\mathbb{R}_+^\times)_I$ for $I \subseteq]0, 1[$.

Proof. We first have to describe $\mathcal{S}(\mathbb{R}_+^\times)_I$ more explicitly. For simplicity, we assume the interval I to be open. Let $Df(x) := x \cdot f'(x)$. This differential operator is the generator of the representation λ of $\mathbb{R}_+^\times \cong \mathbb{R}$. Let $f \in L^2(\mathbb{R}_+^\times, x d^\times x)$. Then $f \in \mathcal{S}(\mathbb{R}_+^\times)_I$ if and only if $D^m(f \cdot x^s) \cdot (\ln x)^k \in L^2(\mathbb{R}_+^\times, d^\times x)$ for all $m, k \in \mathbb{N}$, $s \in I$. Using the Leibniz rule, one shows that this is equivalent to $D^m(f) \cdot x^s \cdot (\ln x)^k \in L^2(\mathbb{R}_+^\times, d^\times x)$ for all $m, k \in \mathbb{N}$, $s \in I$. Since I is open, we may replace x^s by $x^{s+\varepsilon} + x^{s-\varepsilon}$ for some $\varepsilon > 0$. This dominates $x^s (\ln x)^k$ for any $k \in \mathbb{N}$, so that it suffices to require $(D^m f) \cdot x^s \in L^2(\mathbb{R}_+^\times, d^\times x)$ for all $m \in \mathbb{N}$, $s \in I$.

This description of $\mathcal{S}(\mathbb{R}_+^\times)_I$ easily implies $\mathcal{H}_+ \subseteq \mathcal{S}(\mathbb{R}_+^\times)_{]0, \infty[}$. Since $\mathfrak{F}$ maps $\mathcal{H}_+$ to itself and J maps $\mathcal{S}(\mathbb{R}_+^\times)_{]0, \infty[}$ to $\mathcal{S}(\mathbb{R}_+^\times)_{]-\infty, 1[}$, we get “ $\subseteq$ ” in (10).

Conversely, $f \in \mathcal{H}_+$ if and only if f and $\mathfrak{F}f$ are both $O(x^{-s})$ for $|x| \rightarrow \infty$ for all $s \in \mathbb{N}$. This yields “ $\supseteq$ ” in (10) and finishes the proof of (10).

In the following computation, we describe $\mathcal{O}(\mathbb{R}_+^\times) \hat{\otimes} \mathcal{S}(\mathbb{R}_+^\times)_I$ as in (4). Choose any $\psi \in \mathcal{O}(\mathbb{R}_+^\times)$ with $\int_0^\infty \psi(x) d^\times x = 1$. Then $\sigma f(x, y) := \psi(x)f(xy)$ defines a continuous linear map from $\mathcal{S}(\mathbb{R}_+^\times)_I$ to $\mathcal{O}(\mathbb{R}_+^\times) \hat{\otimes} \mathcal{S}(\mathbb{R}_+^\times)_I$ by (4). This is a section for the convolution map

$$f\lambda: \mathcal{O}(\mathbb{R}_+^\times) \hat{\otimes} \mathcal{S}(\mathbb{R}_+^\times)_I \rightarrow \mathcal{S}(\mathbb{R}_+^\times)_I, \quad (f\lambda f)(x) = \int_0^\infty f(t, t^{-1}x) d^\times t.$$

We have $\mathcal{O}(\mathbb{R}_+^\times) = J\mathcal{O}(\mathbb{R}_+^\times) \subset \mathcal{H}_+ = \mathfrak{F}\mathcal{H}_+ \subset \mathcal{S}(\mathbb{R}_+^\times)_I$ if $I \subseteq]0, \infty[$. Hence we get a continuous linear operator

$$\mathcal{S}(\mathbb{R}_+^\times)_I \xrightarrow{\sigma} \mathcal{O}(\mathbb{R}_+^\times) \hat{\otimes} \mathcal{S}(\mathbb{R}_+^\times)_I \xrightarrow{\mathfrak{F}J \hat{\otimes} \text{id}} \mathcal{S}(\mathbb{R}_+^\times)_I \hat{\otimes} \mathcal{S}(\mathbb{R}_+^\times)_I \xrightarrow{f\lambda} \mathcal{S}(\mathbb{R}_+^\times)_I.$$

The last map exists because $\mathcal{S}(\mathbb{R}_+^\times)_I$ is a convolution algebra. If we plug $f_0 * f_1$ with $f_0, f_1 \in \mathcal{O}(\mathbb{R}_+^\times)$ into this operator, we get $\mathfrak{F}J(f_0 * f_1)$ because σ is a section for $f\lambda$ and because of (9). Since products $f_0 * f_1$ are dense in $\mathcal{O}(\mathbb{R}_+^\times)$, the above operator on $\mathcal{S}(\mathbb{R}_+^\times)_I$ extends $\mathfrak{F}J$ on $\mathcal{O}(\mathbb{R}_+^\times)$.

Now (8) implies the continuity of $J\mathfrak{F} = J(\mathfrak{F}J)J$ on $\mathcal{S}(\mathbb{R}_+^\times)_I$ for $I \subseteq]-\infty, 1[$. Hence both $J\mathfrak{F}$ and $\mathfrak{F}J$ are multipliers of $\mathcal{S}(\mathbb{R}_+^\times)_I$ for $I \subseteq]0, 1[$. They are inverse to each other on $\mathcal{S}(\mathbb{R}_+^\times)_I$ because they are inverse to each other on $L^2(\mathbb{R}_+^\times, x d^\times x)$. $\square$

3. The Zeta operator and the Poisson Summation Formula

In this section we study the properties of the following operator:

Definition 3.1. The *Zeta operator* is defined by

$$Zf(x) := \sum_{n=1}^{\infty} f(nx) = \sum_{n=1}^{\infty} \lambda_n^{-1} f(x)$$

for $f \in \mathcal{H}_+$, $x \in \mathbb{R}_+^\times$.

Let $\check{\zeta}$ be the distribution $\sum_{n=1}^{\infty} \delta_n^{-1}: \psi \mapsto \sum_{n=1}^{\infty} \psi(n^{-1})$. Then

$$\check{\zeta}(s) = \sum_{n=1}^{\infty} n^{-s} = \zeta(s), \quad Zf = f\lambda(\check{\zeta})(f).$$

Thus the data Z , ζ , and $\check{\zeta}$ are equivalent.

The Euler product expansion of the ζ -function takes the following form in this picture. Let $\mathscr{P}$ be the set of prime numbers in $\mathbb{N}^*$. We have

$$Z = \prod_{p \in \mathscr{P}} \sum_{e=0}^{\infty} \lambda_{p^{-e}} = \prod_{p \in \mathscr{P}} (1 - \lambda_p^{-1})^{-1}.$$

Hence we get a candidate for an inverse of Z :

$$Z^{-1}f(x) = \prod_{p \in \mathscr{P}} (1 - \lambda_p^{-1})f(x) = \sum_{n=1}^{\infty} \mu(n)f(nx).$$

Here $\mu(n)$ is the usual Möbius function; it vanishes unless n is square-free, and is $(-1)^j$ if n is a product of j different prime numbers.

The following assertion is equivalent to the absolute convergence in the region $\operatorname{Re} s > 1$ of the Euler product defining $\zeta(s)$.

Proposition 3.2. *Z and Z^{-1} are continuous linear operators on $\mathscr{S}_>$ that are inverse to each other.*

Proof. Let $Df(x) = xf'(x)$. We have observed above that $f \in \mathscr{S}_>$ if and only if $D^m f \in L^2(\mathbb{R}_+^\times, x^{2s} d^\times x)$ for all $m \in \mathbb{N}$, $s > 1$. We check that Z and Z^{-1} preserve this estimate. The operator λ_t for $t \in \mathbb{R}_+^\times$ has norm $\|\lambda_t\|^s = t^s$ on $L^2(\mathbb{R}_+^\times, x^{2s} d^\times x)$ for all $s \in \mathbb{R}$. Hence the same estimates that yield the absolute convergence of the Euler product for $\zeta(s)$ also show that the products $\prod_{p \in \mathscr{P}} (1 - \lambda_p^{-1})^{\pm 1}$ converge absolutely with respect to the operator norm on $L^2(\mathbb{R}_+^\times, x^{2s} d^\times x)$ for $s > 1$. Since all factors commute with D , this remains true if we replace $L^2(\mathbb{R}_+^\times, x^{2s} d^\times x)$ by the Sobolev space defined by the norm

$$\|f\|_2^{m,s} := \int_0^\infty |D^m f(x)|^2 x^{2s} d^\times x.$$

Thus Z and Z^{-1} are continuous linear operators on these Sobolev spaces for all $m \in \mathbb{N}$ and all $s > 1$. This yields the assertion. $\square$

Next we recall the *Poisson Summation Formula*. It asserts that

$$\sum_{n \in \mathbb{Z}} f(xn) = x^{-1} \sum_{n \in \mathbb{Z}} \mathfrak{F}f(x^{-1}n)$$

for all $x \in \mathbb{R}_+^\times$ and all $f \in \mathscr{S}(\mathbb{R})$. For $f \in \mathscr{H}_+$, this becomes

$$(11) \quad f(0)/2 + Zf(x) = JZ\mathfrak{F}f(x) + x^{-1} \cdot \mathfrak{F}f(0)/2.$$

Let

$$\mathscr{H}_\cap := \{f \in \mathscr{S}(\mathbb{R}) \mid f(0) = \mathfrak{F}f(0) = 0\}.$$

This is a closed, λ -invariant subspace of $\mathcal{S}(\mathbb{R})$. If $f \in \mathcal{H}_\cap$, then (11) simplifies to $Zf = JZ\mathfrak{F}f$.

Let $h_0: \mathbb{R}_+^\times \rightarrow [0, 1]$ be a smooth function with $h_0(t) = 1$ for $t \ll 1$ and $h_0(t) = 0$ for $t \gg 1$. Let $\mathcal{H}_\cup$ be the space of functions on $\mathbb{R}_+^\times$ that is generated by $\mathcal{H}_- = \mathcal{O}(\mathbb{R}_+^\times)$ and the two additional functions h_0 and $x^{-1} \cdot h_0$. The regular representation extends to $\mathcal{H}_\cup$. Writing $\mathbb{C}(x^s)$ for $\mathbb{C}$ equipped with the representation by the character x^s , we get an extension of representations

$$(12) \quad \mathcal{H}_- \hookrightarrow \mathcal{H}_\cup \rightarrow \mathbb{C}(x^0) \oplus \mathbb{C}(x^1).$$

Theorem 3.3. *The Zeta operator is a continuous linear map $Z: \mathcal{H}_+ \rightarrow \mathcal{H}_\cup$. Even more, this map has closed range and is a topological isomorphism onto its range. We have $Zf \in \mathcal{H}_-$ if and only if $f \in \mathcal{H}_\cap$.*

Proof. Proposition 2.1 yields $\mathcal{H}_+ \prec \mathcal{S}_>$. By Proposition 3.2, Z is continuous on $\mathcal{S}_>$. Hence we get continuity of $Z: \mathcal{H}_+ \rightarrow \mathcal{S}_>$. Similarly, $JZ\mathfrak{F}$ is a continuous linear operator $\mathcal{H}_+ \rightarrow \mathcal{S}_<$. By (11), Z restricts to a continuous linear map

$$\mathcal{H}_\cap \rightarrow \mathcal{S}_> \cap \mathcal{S}_< = \mathcal{H}_-.$$

Equation (11) also implies that Zf still belongs to $\mathcal{H}_\cup$ for arbitrary $f \in \mathcal{H}_+$ and that $Zf \in \mathcal{H}_-$ if and only if $f \in \mathcal{H}_\cap$.

It remains to prove that Z is a topological isomorphism onto its range. This implies that the range is closed because all spaces involved are complete. It suffices to prove that the restriction $Z: \mathcal{H}_\cap \rightarrow \mathcal{H}_-$ is an isomorphism onto its range. Equivalently, a sequence (f_n) in $\mathcal{H}_\cap$ converges if and only if (Zf_n) converges in $\mathcal{H}_-$. One implication is contained in the continuity of Z . Suppose that the sequence (Zf_n) converges in $\mathcal{H}_-$. Hence it converges in both $\mathcal{S}_>$ and $\mathcal{S}_<$. Equation (11) yields $Zf_n = JZ\mathfrak{F}f_n$. Using (8), we get that both (Zf_n) and $(Z\mathfrak{F}f_n)$ converge in $\mathcal{S}_>$. Proposition 3.2 yields that (f_n) and $(\mathfrak{F}f_n)$ converge in $\mathcal{S}_>$.

Therefore, $(D^m f_n \cdot x^s)$ and $(D^m \mathfrak{F}f_n \cdot x^s)$ converge in $L^2(\mathbb{R}, dx)$ for all $s > 1/2$, where $Df(x) = xf'(x)$. Hence $x^k(d/dx)^l f_n$ and $(d/dx)^k(x^l f_n)$ converge if $k, l \in \mathbb{N}$ satisfy $k > l + 1$. The first condition implies convergence in $\mathcal{S}(\mathbb{R} \setminus]-1, 1[)$ because $x \geq 1$ in this region. The second condition contains convergence of $(d/dx)^k f_n$ in $L^2([-1, 1], dx)$. Hence both conditions together imply convergence in $\mathcal{S}(\mathbb{R})$ as desired. $\square$

We now discuss the close relationship between the above theorem and the meromorphic continuation of the ζ -function and the functional equation (see also [Tat67]). Recall that $\hat{f}(s) := \int_0^\infty f(x)x^s dx$. This defines an entire

function for $f \in \mathcal{H}_-$. We have described Z as the convolution with the distribution $\check{\zeta}$ on $\mathbb{R}_+^\times$, which satisfies $\check{\zeta}(s) = \zeta(s)$. Therefore,

$$(13) \quad (Zf)\check{\gamma}(s) = \zeta(s)\hat{f}(s)$$

for all $f \in \mathcal{S}_>$, $s \in \mathbb{C}$ with $\operatorname{Re} s > 1$. The Poisson Summation Formula implies $Zf \in \mathcal{H}_-$ for $f \in \mathcal{H}_\cap$, so that $\zeta(s)\hat{f}(s)$ extends to an entire function. Especially, this holds if $f \in \mathcal{H}_-$ satisfies $\hat{f}(1) = 0$. For such f , the function $\hat{f}(s)$ is an entire function on $\mathbb{C}$ as well. Therefore, ζ has a meromorphic continuation to all of $\mathbb{C}$. For any $s \neq 1$, there exists $f \in \mathcal{H}_-$ with $\hat{f}(1) = 0$ and $\hat{f}(s) \neq 0$. Therefore, the only possible pole of ζ is at 1.

It is easy to see that $(Jf)\check{\gamma}(s) = \hat{f}(1-s)$. Hence (11) implies

$$(14) \quad \zeta(1-s)(J\mathfrak{F}f)\check{\gamma}(s) = \zeta(s)\hat{f}(s)$$

for all $s \in \mathbb{C}$, $f \in \mathcal{H}_\cap$. This equation still holds for $f \in \mathcal{H}_+$ by $\mathbb{R}_+^\times$ -equivariance. Now we plug in the special function $f(x) = 2\exp(-\pi x^2)$, which satisfies $\mathfrak{F}f = f$ and $\hat{f}(s) = \pi^{-s/2}\Gamma(s/2)$. Thus $\zeta(s)\hat{f}(s) = \xi(s)$ is the complete ζ -function. Equation (14) becomes the functional equation $\xi(1-s) = \xi(s)$.

4. The spectral interpretation

Let $Z\mathcal{H}_+ \subseteq \mathcal{H}_\cup$ be the range of Z . This is a closed subspace of $\mathcal{H}_\cup$ and topologically isomorphic to $\mathcal{H}_+$ by Theorem 3.3. Moreover,

$$Z\mathcal{H}_\cap = Z\mathcal{H}_+ \cap \mathcal{H}_-, \quad \mathcal{H}_\cup = Z\mathcal{H}_+ + \mathcal{H}_-.$$

We define

$$\begin{aligned} \mathcal{H}_+^0 &:= \mathcal{H}_+ / \mathcal{H}_\cap \cong \mathcal{H}_\cup / \mathcal{H}_- \\ \mathcal{H}_-^0 &:= \mathcal{H}_- / Z\mathcal{H}_\cap \cong \mathcal{H}_\cup / Z\mathcal{H}_+. \end{aligned}$$

We equip $\mathcal{H}_\pm^0$ with the quotient topology from $\mathcal{H}_\pm$ or from $\mathcal{H}_\cup$ (both topologies on $\mathcal{H}_\pm^0$ coincide) and with the representations $\rho_\pm$ of $\mathbb{R}_+^\times$ induced by λ on $\mathcal{H}_\pm$ or $\mathcal{H}_\cup$. We view the pair of representations (ρ_+, ρ_-) as a formal difference $\rho_+ \ominus \rho_-$, that is, as a virtual representation of $\mathbb{R}_+^\times$.

A smooth representation of $\mathbb{R}_+^\times \cong \mathbb{R}$ is determined uniquely by the action of the generator of the Lie algebra of $\mathbb{R}_+^\times$. This generator corresponds to the scaling invariant vector field $Df(x) = xf'(x)$ on $\mathcal{H}_\cup$. We let $D_\pm$ be the operators on $\mathcal{H}_\pm^0$ induced by D on $\mathcal{H}_\cup$. We claim that the operators $D_\pm$ are spectral interpretations for the poles and zeros of the complete ζ -function ξ defined in (1). This is rather trivial for D_+ . Since

$$\mathcal{H}_+^0 \cong \mathbb{C}(x^0) \oplus \mathbb{C}(x^1),$$

the operator D_+ is equivalent to the diagonal 2×2 -matrix with eigenvalues 0, 1. Hence it is a spectral interpretation for the two poles of ξ . To treat $\mathcal{H}_-^0$, we identify the Fourier-Laplace transforms of $\mathcal{H}_-$ and $Z\mathcal{H}_\cap$. Recall that $\hat{f}(s) = \int_0^\infty f(x)x^s d^\times x$.

Theorem 4.1. *The operator $f \mapsto \hat{f}$ identifies $\mathcal{H}_-$ with the space of entire functions $h: \mathbb{C} \rightarrow \mathbb{C}$ for which $t \mapsto h(s+it)$ is a Schwartz function on $\mathbb{R}$ for each $s \in \mathbb{R}$.*

The subspace $Z\mathcal{H}_\cap$ is mapped to the space of those entire functions h for which

$$t \mapsto \frac{h(s+it)}{\zeta(s+it)} \quad \text{and} \quad t \mapsto \frac{h(s+it)}{\zeta(1-s-it)}$$

are Schwartz functions for $s \geq 1/2$ and $s \leq 1/2$, respectively. (In particular, this means that $h(z)/\zeta(z)$ has no poles with $\operatorname{Re} z \geq 1/2$ and $h(z)/\zeta(1-z)$ has no poles with $\operatorname{Re} z \leq 1/2$.)

Proof. It is well-known that the Fourier transform is an isomorphism (of topological vector spaces) $\mathcal{S}(\mathbb{R}_+^\times) \cong \mathcal{S}(i\mathbb{R})$. Hence $f \mapsto \hat{f}$ is an isomorphism $\mathcal{S}(\mathbb{R}_+^\times)_s \cong \mathcal{S}(s+i\mathbb{R})$ for all $s \in \mathbb{R}$. It is clear that $\hat{f}$ is an entire function on $\mathbb{C}$ for $f \in \mathcal{H}_-$. Since $\mathcal{H}_- = \bigcap_{s \in \mathbb{R}} \mathcal{S}(\mathbb{R}_+^\times)_s$, we also get $\hat{f} \in \mathcal{S}(s+i\mathbb{R})$ for all $s \in \mathbb{R}$. Conversely, if h is entire and $h \in \mathcal{S}(s+i\mathbb{R})$ for all $s \in \mathbb{R}$, then for each $s \in \mathbb{R}$ there is $f_s \in \mathcal{S}(\mathbb{R}_+^\times)_s$ with $h(s+it) = \hat{f}_s(s+it)$ for all $t \in \mathbb{R}$. Using the Cauchy-Riemann differential equation for the analytic function $h(s+it)$, we conclude that f_s is independent of s . Hence we get a function f in $\mathcal{H}_-$ with $\hat{f} = h$ on all of $\mathbb{C}$. This yields the desired description of $(\mathcal{H}_-)^{\wedge}$.

The same argument shows that $\mathcal{S}(\mathbb{R}_+^\times)_I^{\wedge}$ for an open interval I is the space of all holomorphic functions $h: I+i\mathbb{R} \rightarrow \mathbb{C}$ with $h \in \mathcal{S}(s+i\mathbb{R})$ for all $s \in I$.

Proposition 2.1 asserts that $f \in \mathcal{H}_+$ if and only if $f \in \mathcal{S}(\mathbb{R}_+^\times)_{]0,\infty[}$ and $J\mathfrak{F}f \in \mathcal{S}(\mathbb{R}_+^\times)_{]-\infty,1[}$. Since $J\mathfrak{F}$ is invertible on $\mathcal{S}(\mathbb{R}_+^\times)_s$ for $0 < s < 1$, this is equivalent to $f \in \mathcal{S}(\mathbb{R}_+^\times)_s$ for $s \geq 1/2$ and $J\mathfrak{F}f \in \mathcal{S}(\mathbb{R}_+^\times)_s$ for $s \leq 1/2$. Moreover, these two conditions are equivalent for $s = 1/2$. We get $f \in Z\mathcal{H}_+$ if and only if $Z^{-1}f \in \mathcal{S}(\mathbb{R}_+^\times)_s$ for $s \geq 1/2$ and $J\mathfrak{F}Z^{-1}f \in \mathcal{S}(\mathbb{R}_+^\times)_s$ for $s \leq 1/2$.

Now let $h: \mathbb{C} \rightarrow \mathbb{C}$ be an entire function with $h \in \mathcal{S}(s+i\mathbb{R})$ for all $s \in \mathbb{R}$. Thus $h = \hat{f}$ for some $f \in \mathcal{H}_-$. Equation (13) implies $f \in Z\mathcal{H}_\cap$ if and only if $h/\zeta \in \mathcal{S}(s+i\mathbb{R})$ for $s \geq 1/2$ and $(J\mathfrak{F}f)^{\wedge}/\zeta \in \mathcal{S}(s+i\mathbb{R})$ for $s \leq 1/2$. The functional equation (14) yields $(J\mathfrak{F}f)^{\wedge}(z)/\zeta(z) = h(z)/\zeta(1-z)$. $\square$

Let $(\mathcal{H}_-^0)'$ be the space of continuous linear functionals $\mathcal{H}_-^0 \rightarrow \mathbb{C}$ and let ${}^tD_- \in \operatorname{End}((\mathcal{H}_-^0)')$ be the transpose of D_- .

Corollary 4.2. *The eigenvalues of the transpose ${}^tD_- \in \text{End}((\mathcal{H}^0_-)')$ are exactly the zeros of the complete Riemann ζ -function ξ . The algebraic multiplicity of λ as an eigenvalue of ${}^tD_-$ is the zero order of ξ at λ . Here we define the algebraic multiplicity as the dimension of $\bigcup_{k \in \mathbb{N}} \ker({}^tD_- - \lambda)^k$.*

Proof. We can identify $(\mathcal{H}^0_-)'$ with the space of continuous linear functionals on $\mathcal{H}_-$ that annihilate $Z\mathcal{H}_\cap$. Let $l: \mathcal{H}_- \rightarrow \mathbb{C}$ be such a linear functional. In the Fourier-Laplace transformed picture, we have $D_-h(s) = s \cdot h(s)$ for all $s \in \mathbb{C}$. Hence l is an eigenvector for the transpose ${}^tD_-$ if and only if it is of the form $h \mapsto h(s)$ for some $s \in \mathbb{C}$. Similarly, $({}^tD_- - s)^k(l) = 0$ if and only if $l(h) = \sum_{j=0}^{k-1} a_j h^{(j)}(s)$ for some $a_0, \dots, a_{k-1} \in \mathbb{C}$. It follows from the functional equation $\xi(s) = \xi(1-s)$ that the zero orders of ξ at s and $1-s$ agree for all $s \in \mathbb{C}$. Moreover, ξ and ζ have the same zeros and the same zero orders for $\text{Re } s > 0$. Hence the assertion follows from Theorem 4.1. $\square$

5. A geometric character computation

Our next goal is to prove that the representation ρ is summable and to compute its character geometrically.

Definition 5.1 ([Mey]). Let G be a Lie group and let $\mathcal{D}(G)$ be the space of smooth, compactly supported functions on G . A smooth representation ρ of G on a Fréchet space is called *summable* if $\int \rho(f)$ is nuclear for all $f \in \mathcal{D}(G)$ and if these operators are uniformly nuclear for f in a bounded subset of $\mathcal{D}(G)$.

The theory of nuclear operators is due to Alexandre Grothendieck and rather deep. Nuclear operators are analogues of trace class operators on Hilbert spaces. It follows easily from the definition that $f \circ g$ is nuclear if at least one of the operators f and g is nuclear. That is, the nuclear operators form an *operator ideal*. For the purposes of this article, we do not have to recall the definition of nuclearity because of the following simple criterion:

Theorem 5.2. *An operator between nuclear Fréchet spaces is nuclear if and only if it can be factored through a Banach space.*

The spaces $\mathcal{S}(\mathbb{R}_+^\times)_I$ and $\mathcal{H}_+$ are nuclear because $\mathcal{S}(\mathbb{R})$ is nuclear and nuclearity is hereditary for subspaces and inverse limits. Hence Theorem 5.2 applies to all operators between these spaces.

The *character of a summable representation* ρ is the distribution on G defined by $\chi_\rho(f) := \text{tr} \int \rho(f)$ for all $f \in \mathcal{D}(G)$. The uniform nuclearity of $\int \rho(f)$ for f in bounded subsets of $\mathcal{D}(G)$ ensures that χ_ρ is a bounded linear functional on $\mathcal{D}(G)$. This is equivalent to continuity because $\mathcal{D}(G)$ is an LF-space.

If ρ is a virtual representation as in our case, we let $\text{tr} \int \rho(f)$ be the supertrace $\text{tr} \int \rho_+(f) - \text{tr} \int \rho_-(f)$.

One can see from the above arguments and definitions that summability of representations really has to do with bounded subsets of $\mathcal{D}(G)$ and bounded maps, not with open subsets and continuous maps. The same is true for the concept of a nuclear operator. That is, the theory of nuclear operators and summable representations is at home in bornological vector spaces. We can still give definitions in the context of topological vector spaces if we turn them into bornological vector spaces using the standard bornology of (von Neumann) bounded subsets. Nevertheless, topological vector spaces are the wrong setup for studying nuclearity. The only reason why I use them here is because they are more familiar to most readers and easier to find in the literature.

We need uniform nuclearity because we want $\chi_\rho(f)$ to be a bounded linear functional of f . In the following, we will only prove nuclearity of various operators. The same proofs yield uniform nuclearity as well. We leave it to the reader to add the remaining details. Suffice it to say that there are analogues of Theorem 5.2 and Theorem 5.5 below for uniformly nuclear sets of operators.

In order to prove the summability of our spectral interpretation ρ , we define several operators between $\mathcal{H}_\pm$ and the space $\mathcal{S}_> \oplus \mathcal{S}_<$. As auxiliary data, we use a smooth function $\varphi: \mathbb{R}_+ \rightarrow [0, 1]$ with $\varphi(t) = 0$ for $t \ll 1$ and $\varphi(t) = 1$ for $t \gg 1$. Let M_φ be the operator of multiplication by φ . We assume for simplicity that $\varphi(t) + \varphi(t^{-1}) = 1$, so that

$$M_\varphi + JM_\varphi J = \text{id}.$$

It is easy to check that M_φ is a continuous map from $\mathcal{S}_>$ into $\mathcal{O}(\mathbb{R}_+^\times)$. We warn the reader that our notation differs from that in [Mey]: there the auxiliary function $1 - \varphi$ is used and denoted φ .

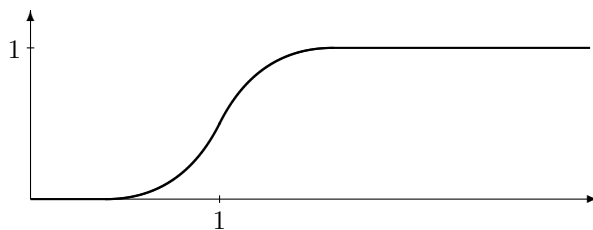


FIGURE 1. The auxiliary function φ

Now we can define our operators:

$$\begin{aligned}
\iota_+ : \mathcal{H}_+ &\rightarrow \mathcal{S}_> \oplus \mathcal{S}_<, & \iota_+ f &:= (Zf, JZ\mathfrak{F}f); \\
\iota_- : \mathcal{H}_- &\rightarrow \mathcal{S}_> \oplus \mathcal{S}_<, & \iota_- f &:= (f, f); \\
\pi_+ : \mathcal{S}_> \oplus \mathcal{S}_< &\rightarrow \mathcal{H}_+, & \pi_+(f_1, f_2) &:= (M_\varphi Z^{-1}f_1, \mathfrak{F}M_\varphi Z^{-1}Jf_2); \\
\pi_- : \mathcal{S}_> \oplus \mathcal{S}_< &\rightarrow \mathcal{H}_-, & \pi_-(f_1, f_2) &:= (M_\varphi f_1, JM_\varphi Jf_2).
\end{aligned}$$

It follows from Equation (8) and Proposition 3.2 that these operators are well-defined and continuous. The operators $\iota_\pm$ are λ -equivariant, the operators $\pi_\pm$ are not. We compute

$$\begin{aligned}
\pi_- \iota_- &= M_\varphi + JM_\varphi J = \text{id}_{\mathcal{H}_-}, \\
\pi_+ \iota_+ &= M_\varphi + \mathfrak{F}M_\varphi \mathfrak{F} = \text{id}_{\mathcal{H}_+} + M_\varphi - \mathfrak{F}JM_\varphi J\mathfrak{F}, \\
\iota_- \pi_- &= \begin{pmatrix} M_\varphi & JM_\varphi J \\ M_\varphi & JM_\varphi J \end{pmatrix}, \\
\iota_+ \pi_+ &= \begin{pmatrix} ZM_\varphi Z^{-1} & Z\mathfrak{F}M_\varphi Z^{-1}J \\ JZ\mathfrak{F}M_\varphi Z^{-1} & JZM_\varphi Z^{-1}J \end{pmatrix}.
\end{aligned}$$

Thus π_- is a section for ι_- and $\iota_- \pi_- \in \text{End}(\mathcal{S}_> \oplus \mathcal{S}_<)$ is a projection onto a subspace isomorphic to $\mathcal{H}_-$. The proof of Theorem 3.3 shows that ι_+ has closed range and is a topological isomorphism onto its range. Although π_+ is not a section for ι_+ , it is a near enough miss for the following summability arguments. I do not know whether there exists an honest section for ι_+ , that is, whether the range of ι_+ is a complemented subspace of $\mathcal{S}_> \oplus \mathcal{S}_<$.

Lemma 5.3. *The operator $\int \lambda(f)(M_\varphi - ZM_\varphi Z^{-1})$ on $\mathcal{S}_>$ is nuclear for $f \in \mathcal{S}_>$.*

The operator $\int \lambda(f)M_\varphi : \mathcal{S}_> \rightarrow \mathcal{S}_<$ is nuclear for $f \in \mathcal{S}_<$.

The operator $\int \lambda(f)(M_\varphi - \mathfrak{F}JM_\varphi J\mathfrak{F})$ on $\mathcal{H}_+$ is nuclear for $f \in \mathcal{O}(\mathbb{R}_+^\times)$.

Proof. We have $\int \lambda(f)Z(h) = f * \check{\zeta} * h = (Zf) * h = \int \lambda(Zf)(h)$ for all $f, h \in \mathcal{S}_>$. Hence

$$\begin{aligned}
\int \lambda(f)(M_\varphi - ZM_\varphi Z^{-1}) &= [\int \lambda(f), M_\varphi] - [\int \lambda(f)Z, M_\varphi]Z^{-1} \\
&= [\int \lambda(f), M_\varphi] - [\int \lambda(Zf), M_\varphi]Z^{-1},
\end{aligned}$$

We are going to show that $[\int \lambda(f), M_\varphi]$ is a nuclear operator on $\mathcal{S}_>$ for all $f \in \mathcal{S}_>$. Together with the above computation, this implies the first assertion of the lemma because Z^{-1} is continuous on $\mathcal{S}_>$ and $f, Zf \in \mathcal{S}_>$.

One checks easily that

$$[\lambda(f), M_\varphi](h)(x) = \int_0^\infty f(xy^{-1})(\varphi(y) - \varphi(x))h(y) d^\times y.$$

Thus our operator has the smooth integral kernel $f(xy^{-1})(\varphi(y) - \varphi(x))$. If f had compact support, this integral kernel would also be compactly supported. If only $f \in \mathcal{S}_>$, we can still estimate that our kernel lies in $\mathcal{S}_> \hat{\otimes} \mathcal{S}(\mathbb{R}_+^\times)_{-s}$ for any $s \in]1, \infty[$. Thus $[\lambda(f), M_\varphi]$ factors through the embedding $\mathcal{S}_> \subseteq L^2(\mathbb{R}_+^\times, x^{2s} d^\times x)$ for any $s > 1$. This implies nuclearity by Theorem 5.2 and finishes the proof of the first assertion of the lemma.

To prove the second assertion, we let $L_I^2 = \bigcap_{s \in I} L^2(\mathbb{R}_+^\times, x^{2s} d^\times x)$. This is a Fréchet space. We claim that $\lambda(f)$ is a continuous linear operator $L_I^2 \rightarrow \mathcal{S}_I$ for any open interval I . (Actually, $\mathcal{S}_I$ is the Gårding subspace for the representation λ on L_I^2 .) This follows from the description of $\mathcal{S}_I$ in the proof of Proposition 2.1 and $D^m(f * h) = (D^m f) * h$. Therefore, we have continuous linear operators

$$\mathcal{S}_> \prec L_s^2 \xrightarrow{M_\varphi} L_{]-\infty, s]}^2 \prec L_{]-\infty, 0[}^2 \xrightarrow{\lambda(f)} \mathcal{S}_<$$

for any $s > 1$. Thus $\lambda(f)M_\varphi$ factors through the Hilbert space L_s^2 . This yields the assertion by our criterion for nuclear operators, Theorem 5.2.

We claim that $\lambda(f)(M_\varphi - \mathfrak{F}JM_\varphi J\mathfrak{F})$ as an operator on $\mathcal{H}_+$ factors continuously through $L_{1/2}^2 \cong L^2(\mathbb{R}_+^\times, x d^\times x)$. This implies the third assertion. To prove the claim, we use $\mathcal{H}_+ = \mathcal{S}_> \cap \mathfrak{F}J(\mathcal{S}_<)$. That is, the map

$$\mathcal{H}_+ \rightarrow \mathcal{S}_> \oplus \mathcal{S}_<, \quad f \mapsto (f, J\mathfrak{F}f)$$

is a topological isomorphism onto its range. We have already seen this during the proof of Theorem 3.3. Hence we merely have to check the existence of continuous extensions

$$\lambda(f)(M_\varphi - \mathfrak{F}JM_\varphi J\mathfrak{F}): L_{1/2}^2 \rightarrow \mathcal{S}_>, \quad J\mathfrak{F} \circ \lambda(f)(M_\varphi - \mathfrak{F}JM_\varphi J\mathfrak{F}): L_{1/2}^2 \rightarrow \mathcal{S}_<.$$

We write

$$\begin{aligned} \lambda(f)(M_\varphi - \mathfrak{F}JM_\varphi J\mathfrak{F}) &= -\lambda(f)(M_{1-\varphi} - \mathfrak{F}JM_{1-\varphi}J\mathfrak{F}) \\ &= -\lambda(f)M_{1-\varphi} + \lambda(f)(\mathfrak{F}Jf)M_{1-\varphi}J\mathfrak{F}. \end{aligned}$$

Proposition 2.1 yields $\mathfrak{F}Jf \in \mathcal{S}_>$. Moreover, J and $\mathfrak{F}$ are unitary on $L_{1/2}^2$. Hence we get bounded extensions $L_{1/2}^2 \rightarrow \mathcal{S}_>$ of both summands by the same argument as for the second assertion of the lemma. Similarly, both summands in

$$J\mathfrak{F} \circ \lambda(f)(M_\varphi - \mathfrak{F}JM_\varphi J\mathfrak{F}) = \lambda(f)(J\mathfrak{F}f)M_\varphi - \lambda(f)M_\varphi J\mathfrak{F}$$

have bounded extensions $L_{1/2}^2 \rightarrow \mathcal{S}_<$ as desired. $\square$

Corollary 5.4. *The operator $\mathfrak{f}\lambda(f) \circ (\iota_- \pi_- - \iota_+ \pi_+)$ on $\mathcal{S}_> \oplus \mathcal{S}_<$ is nuclear for all $f \in \mathcal{O}(\mathbb{R}_+^\times)$.*

The operator $\mathfrak{f}\lambda(f) \circ (\text{id} - \pi_+ \iota_+)$ on $\mathcal{H}_+$ is nuclear for all $f \in \mathcal{O}(\mathbb{R}_+^\times)$.

Proof. We have

$$\begin{aligned} & \mathfrak{f}\lambda(f) \circ (\iota_- \pi_- - \iota_+ \pi_+) \\ &= \begin{pmatrix} \mathfrak{f}\lambda(f)(M_\varphi - ZM_\varphi Z^{-1}) & \mathfrak{f}\lambda(f)J(M_\varphi - JZ\mathfrak{F}M_\varphi Z^{-1})J \\ \mathfrak{f}\lambda(f)(M_\varphi - JZ\mathfrak{F}M_\varphi Z^{-1}) & \mathfrak{f}\lambda(f)J(M_\varphi - ZM_\varphi Z^{-1})J \end{pmatrix}. \end{aligned}$$

The upper left corner is nuclear by the first assertion of Lemma 5.3. Since

$$\mathfrak{f}\lambda(f)(J(M_\varphi - ZM_\varphi Z^{-1})J) = J(\mathfrak{f}\lambda(Jf)(M_\varphi - ZM_\varphi Z^{-1}))J,$$

we also get the nuclearity of the lower right corner. We have $\mathfrak{f}\lambda(f)JZ\mathfrak{F} = \mathfrak{f}\lambda(JZ\mathfrak{F}f)$ because $JZ\mathfrak{F}$ is λ -invariant. Proposition 2.1 and Proposition 3.2 yield $JZ\mathfrak{F}(f) \in \mathcal{S}_<$. Hence the two summands $\mathfrak{f}\lambda(f)M_\varphi$ and $\mathfrak{f}\lambda(f)JZ\mathfrak{F}M_\varphi Z^{-1}$ in the lower left corner are nuclear by the second assertion of Lemma 5.3. The assertion for the upper right corner follows by a symmetric argument. The nuclearity of $\mathfrak{f}\lambda(f) \circ (\text{id} - \pi_+ \iota_+)$ on $\mathcal{H}_+$ is exactly the third assertion of Lemma 5.3. $\square$

In order to apply this to the representation ρ , we need a general fact about nuclear operators. Let W_1, W_2 be Fréchet spaces and let $V_1 \subseteq W_1$ and $V_2 \subseteq W_2$ be closed subspaces. If $T: W_1 \rightarrow W_2$ maps V_1 into V_2 , we write $T|_{V_1, V_2}$ and $T|^{W_1/V_1, W_2/V_2}$ for the operators $V_1 \rightarrow V_2$ and $W_1/V_1 \rightarrow W_2/V_2$ induced by T .

Theorem 5.5 ([Gro55]). *If T is nuclear, so are $T|_{V_1, V_2}$ and $T|^{W_1/V_1, W_2/V_2}$. If $V_1 = V_2 = V$ and $W_1 = W_2 = W$, then $\text{tr } T = \text{tr } T|_V + \text{tr } T|^{W/V}$.*

Proposition 5.6. *The representation $\rho: \mathbb{R}_+^\times \rightarrow \text{Aut}(\mathcal{H}^0)$ is summable and*

$$\chi(\rho)(f) = \text{tr } \mathfrak{f}\lambda(f)(\text{id} - \pi_+ \iota_+) - \text{tr } \mathfrak{f}\lambda(f)(\iota_- \pi_- - \iota_+ \pi_+).$$

Proof. The embeddings of $\mathcal{H}_+$ and $\mathcal{H}_-$ in $\mathcal{S}_> \oplus \mathcal{S}_<$ agree on the common subspace $\mathcal{H}_\cap \cong Z\mathcal{H}_\cap$ and hence combine to an embedding of $\mathcal{H}_\cup$. Thus we identify $\mathcal{H}_\cup$ with the subspace $\iota_+ \mathcal{H}_+ + \iota_- \mathcal{H}_-$ of $\mathcal{S}_> \oplus \mathcal{S}_<$. Let $T := \mathfrak{f}\lambda(f)(\iota_- \pi_- - \iota_+ \pi_+)$. The range of T is contained in $\mathcal{H}_\cup$. Therefore, its trace as an operator on $\mathcal{S}_> \oplus \mathcal{S}_<$ agrees with its trace as an operator on $\mathcal{H}_\cup$ by Theorem 5.5. Write

$$T|_{\mathcal{H}_\cup} = \mathfrak{f}\lambda(f)(\text{id}_{\mathcal{H}_\cup} - \iota_+ \pi_+|_{\mathcal{H}_\cup}) - \mathfrak{f}\lambda(f)(\text{id}_{\mathcal{H}_\cup} - \iota_- \pi_-|_{\mathcal{H}_\cup}).$$

Since $\iota_- \pi_-$ is a projection onto $\mathcal{H}_- \subset \mathcal{H}_\cup$, Theorem 5.5 yields

$$\mathrm{tr} \int \lambda(f) (\mathrm{id}_{\mathcal{H}_\cup} - \iota_- \pi_-|_{\mathcal{H}_\cup}) = \mathrm{tr} \int \lambda(f)|_{\mathcal{H}_\cup/\mathcal{H}_-} = \chi(\rho_+)(f).$$

Similarly, since $\iota_+ \pi_+$ maps $\mathcal{H}_\cup$ into $\mathcal{H}_+$ we get

$$\begin{aligned} \mathrm{tr} \int \lambda(f) (\mathrm{id}_{\mathcal{H}_\cup} - \iota_+ \pi_+|_{\mathcal{H}_\cup}) &= \mathrm{tr} \int \lambda(f)|_{\mathcal{H}_\cup/\mathcal{H}_+} + \mathrm{tr} \int \lambda(f) (\mathrm{id}_{\mathcal{H}_+} - \iota_+ \pi_+|_{\mathcal{H}_+}) \\ &= \chi(\rho_-)(f) + \mathrm{tr} \int \lambda(f) (\mathrm{id}_{\mathcal{H}_+} - \pi_+ \iota_+). \end{aligned}$$

Hence

$$\mathrm{tr} \int \lambda(f) (\iota_- \pi_- - \iota_+ \pi_+) = \chi(\rho_-)(f) + \mathrm{tr} \int \lambda(f) (\mathrm{id}_{\mathcal{H}_+} - \pi_+ \iota_+) - \chi(\rho_+)(f).$$

Along the way, we see that the operators whose trace we take are nuclear. That is, ρ_+ and ρ_- are summable representations. $\square$

It remains to compute the traces in Proposition 5.6 explicitly. We need the following definitions. For a continuous function $f: \mathbb{R}_+^\times \rightarrow \mathbb{C}$, let $\tau(f) := f(1)$ and $\partial f(x) = f(x) \ln x$. This defines a bounded derivation ∂ on $\mathcal{S}(\mathbb{R}_+^\times)_I$ for any interval I , that is, $\partial(f_1 * f_2) = \partial(f_1) * f_2 + f_1 * \partial(f_2)$. This derivation is the generator of the dual action $t \cdot f(x) := x^{it} f$ of $\mathbb{R}$. Notice that $\tau(\partial f) = 0$. The obvious extension of ∂ to distributions is still a derivation.

Lemma 5.7. *Let $f_0, f_1 \in \mathcal{S}(\mathbb{R}_+^\times)_s$ for some $s \in \mathbb{R}$. Then $\int \lambda(f_0)[M_\varphi, \int \lambda(f_1)]$ is a nuclear operator on $L^2(\mathbb{R}_+^\times, x^{2s} d^\times x)$ and $\mathcal{S}(\mathbb{R}_+^\times)_s$ and*

$$\mathrm{tr} \int \lambda(f_0)[M_\varphi, \int \lambda(f_1)] = \tau(f_0 * \partial f_1).$$

Proof. The operators of multiplication by $x^{\pm s}$ are unitary operators between L_s^2 and L_0^2 . We can use them to reduce the general case to the special case $s = 0$. We assume this in the following. We have checked above that $[M_\varphi, \int \lambda(f_1)]$ has an integral kernel in $\mathcal{S}(\mathbb{R}_+^\times) \hat{\otimes} \mathcal{S}(\mathbb{R}_+^\times)$. Therefore, so has $\int \lambda(f_0)[M_\varphi, \int \lambda(f_1)]$. This implies nuclearity as an operator from $L^2(\mathbb{R}_+^\times, d^\times x)$ to $\mathcal{S}(\mathbb{R}_+^\times)$ by Theorem 5.2. Moreover, the operator has the same trace on both spaces. Explicitly, the integral kernel is

$$(x, y) \mapsto \int_0^\infty f_0(xz^{-1}) f_1(zy^{-1}) (\varphi(z) - \varphi(y)) d^\times z.$$

We get

$$\begin{aligned} \mathrm{tr} \int \lambda(f_0)[M_\varphi, \int \lambda(f_1)] &= \int_0^\infty \int_0^\infty f_0(xz^{-1}) f_1(zx^{-1}) (\varphi(z) - \varphi(x)) d^\times z d^\times x \\ &= \int_0^\infty f_0(x) f_1(x^{-1}) \int_0^\infty \varphi(z) - \varphi(xz) d^\times z d^\times x. \end{aligned}$$

We compute $\int_0^\infty \varphi(z) - \varphi(xz) d^\times z$. If φ had compact support, the λ -invariance of $d^\times z$ would force the integral to vanish. Therefore, we may replace φ by any function φ' with the same behaviour at 0 and ∞ . We choose φ' to be the characteristic function of $[1, \infty[$. If $x \leq 1$, then $\varphi'(z) - \varphi'(xz)$ is the characteristic function of the interval $[1, x^{-1}[$, so that the integral is $\ln(x^{-1})$. We get the same value for $x \geq 1$ as well. Hence $\text{tr} \int \lambda(f_0)[M_\varphi, \int \lambda(f_1)] = \int_0^\infty f_0(x) f_1(x^{-1}) \ln(x^{-1}) d^\times x = \tau(f_0 * \partial f_1)$. $\square$

Theorem 5.8. *Define the distributions W_p for $p \in \mathcal{P}$ and $p = \infty$ as in (2) and (3). Then*

$$\sum_{z \in \mathbb{C}} \text{ord}_\xi(z) \hat{f}(z) = \chi(\rho)(f) = \sum_{p \in \mathcal{P}} W_p(f) + W_\infty(f)$$

for all $f \in \mathcal{O}(\mathbb{R}_+^\times)$. Here $\text{ord}_\xi(z)$ denotes the order at z of the complete ζ -function ξ , which is positive at poles and negative at zeros of ξ .

Proof. The trace of a nuclear operator on a nuclear Fréchet space is equal to the sum of its eigenvalues counted with algebraic multiplicity (see [Gro55]). Since $\text{tr}(A) = \text{tr}({}^t A)$ for any nuclear operator A , the first equality follows from Corollary 4.2. It remains to show $\chi(\rho)(f) = \sum_{p \in \mathcal{P}} W_p(f) + W_\infty(f)$. Proposition 5.6 yields

$$\begin{aligned} (15) \quad \chi(\rho)(f) &= -\text{tr} \int \lambda(f)(\iota_- \pi_- - \iota_+ \pi_+) + \text{tr} \int \lambda(f)(\text{id} - \pi_+ \iota_+) \\ &= -\text{tr} \int \lambda(f)(M_\varphi - Z M_\varphi Z^{-1})|_{\mathcal{S}_>} - \text{tr} \int \lambda(f) J(M_\varphi - Z M_\varphi Z^{-1}) J|_{\mathcal{S}_<} \\ &\quad - \text{tr} \int \lambda(f)(M_\varphi - \mathfrak{F} J M_\varphi J \mathfrak{F})|_{\mathcal{H}_+}. \end{aligned}$$

It suffices to check that this agrees with $W(f)$ if $f = f_0 * f_1$ because such elements are dense in $\mathcal{O}(\mathbb{R}_+^\times)$. We compute

$$\begin{aligned} &-\text{tr} \int \lambda(f_0 * f_1)(M_\varphi - Z M_\varphi Z^{-1})|_{\mathcal{S}_>} \\ &= \text{tr} \int \lambda(f_0)[M_\varphi, \int \lambda(f_1)] - \text{tr} \int \lambda(f_0)[M_\varphi, \int \lambda(Z f_1)] Z^{-1} \\ &= \text{tr} \int \lambda(f_0)[M_\varphi, \int \lambda(f_1)] - \text{tr} \int \lambda(Z^{-1} f_0)[M_\varphi, \int \lambda(Z f_1)] \end{aligned}$$

because $\text{tr}(AB) = \text{tr}(BA)$ if A is nuclear. This is a nuclear operator $L_s^2 \rightarrow \mathcal{S}_>$ for any $s > 1$. Hence it has the same trace as an operator on $\mathcal{S}_>$ and L_s^2 . Lemma 5.7 yields

$$\begin{aligned} &-\text{tr} \int \lambda(f_0 * f_1)(M_\varphi - Z M_\varphi Z^{-1})|_{\mathcal{S}_>} = \tau(f_0 * \partial f_1) - \tau(Z^{-1} f_0 * \partial(Z f_1)) \\ &= -\tau(f_0 * f_1 * Z^{-1} \partial(Z)) = \tau(f_0 * f_1 * Z \partial(Z^{-1})) \end{aligned}$$

where $\partial(Z)$ is defined in the obvious way. We also use $\partial(Z^{-1}) = -Z^{-2}\partial(Z)$, which follows from the derivation property. Now we use the Euler product for the Zeta operator and the derivation rule:

$$\begin{aligned} Z * \partial(Z^{-1}) &= Z * \partial\left(\prod_{p \in \mathcal{P}} (1 - \lambda_p^{-1})\right) = \sum_{p \in \mathcal{P}} (1 - \lambda_p^{-1})^{-1} \partial(1 - \lambda_p^{-1}) \\ &= \sum_{p \in \mathcal{P}} \ln(p) \lambda_p^{-1} (1 - \lambda_p^{-1})^{-1} = \sum_{p \in \mathcal{P}} \sum_{e=1}^{\infty} \ln(p) \lambda_p^{-e}. \end{aligned}$$

Hence

$$-\operatorname{tr} f\lambda(f)(M_\varphi - ZM_\varphi Z^{-1})|_{\mathcal{S}_>} = \tau\left(f * \sum_{p \in \mathcal{P}} \sum_{e=1}^{\infty} \ln(p) \lambda_p^{-e}\right) = \sum_{p \in \mathcal{P}} \sum_{e=1}^{\infty} \ln(p) f(p^e).$$

The second summand in (15) is reduced to this one by

$$\begin{aligned} \operatorname{tr} f\lambda(f)J(M_\varphi - ZM_\varphi Z^{-1})J|_{\mathcal{S}_<} &= \operatorname{tr} Jf\lambda(f)J(M_\varphi - ZM_\varphi Z^{-1})|_{\mathcal{S}_>} \\ &= \operatorname{tr} f\lambda(Jf)(M_\varphi - ZM_\varphi Z^{-1})|_{\mathcal{S}_>}. \end{aligned}$$

Hence

$$-\operatorname{tr} f\lambda(f)J(M_\varphi - ZM_\varphi Z^{-1})J|_{\mathcal{S}_<} = \sum_{p \in \mathcal{P}} \sum_{e=1}^{\infty} \ln(p) p^{-e} f(p^{-e}).$$

These two summands together equal $\sum_{p \in \mathcal{P}} W_p$.

Now we treat the third summand in (15). The same arguments as above yield

$$\begin{aligned} &-\operatorname{tr} f\lambda(f_0 * f_1)(M_\varphi - \mathfrak{F}JM_\varphi J\mathfrak{F})|_{\mathcal{H}_+} \\ &= \operatorname{tr} f\lambda(f_0)[M_\varphi, f\lambda(f_1)] - \operatorname{tr} f\lambda(f_0)[M_\varphi, f\lambda(\mathfrak{F}Jf_1)]J\mathfrak{F} \\ &= \tau(f_0 * \partial f_1) - \tau(J\mathfrak{F}(f_0) * \partial(\mathfrak{F}Jf_1)) = -\tau(f_0 * f_1 * J\mathfrak{F}\partial(\mathfrak{F}J)) = \tau(\mathfrak{F}J\partial(J\mathfrak{F}(f_0 * f_1))). \end{aligned}$$

Here we use $\tau(\partial f) = 0$. Explicitly,

$$\tau(\mathfrak{F}J\partial(J\mathfrak{F}f)) = \mathfrak{F}M_{\ln(x^{-1})}\mathfrak{F}^{-1}f(1) = -\mathfrak{F}(\ln x)\dagger f(1) = -\langle \mathfrak{F}(\ln x), y \mapsto f(1-y) \rangle.$$

Here $M_{\ln(x^{-1})}$ denotes the operator of multiplication by $\ln(x^{-1}) = -\ln x$ and $\dagger$ denotes convolution with respect to the additive structure on $\mathbb{R}$. Thus it remains to compute the Fourier transform of $\ln x$. Since $\psi \mapsto \int_{\mathbb{R}} \psi(x) \ln x \, dx$ defines a tempered distribution on $\mathbb{R}$, $\mathfrak{F}(\ln x)$ is a well-defined tempered distribution on $\mathbb{R}$. The covariance property $\ln(tx) = \ln(t) + \ln(x)$ for $t, x \in \mathbb{R}_+^\times$ implies

$$(16) \quad \langle \mathfrak{F}(\ln x), \lambda_t \psi \rangle = \langle \mathfrak{F}(\ln x), \psi \rangle - \ln(t) \psi(0).$$

Especially, $\mathfrak{F}(\ln x)$ is λ -invariant on the space of $\psi \in \mathcal{S}(\mathbb{R})$ with $\psi(0) = 0$. Thus

$$\langle \mathfrak{F}(\ln x), \psi \rangle = c \int_{\mathbb{R}^\times} \psi(x) d^\times x$$

for some constant $c \in \mathbb{R}$ for all $\psi \in \mathcal{S}(\mathbb{R})$ with $\psi(0) = 0$.

We claim that $c = -1$. To see this, pick $\psi \in \mathcal{S}(\mathbb{R})$ with $\psi(0) \neq 0$ and consider $\psi - \lambda_t \psi$ for some $t \neq 1$. Equation (16) yields

$$c \int_{\mathbb{R}^\times} \psi(x) - \psi(t^{-1}x) d^\times x = \ln(t)\psi(0).$$

As in the proof of Lemma 5.7, this implies $c = -1$. Thus the distribution $\mathfrak{F}(\ln x)$ is some principal value for the integral $-\int_{\mathbb{R}^\times} \psi(x)|x|^{-1} dx$. This principal value can be described uniquely by the condition that $\mathfrak{F}^2(\ln x)(1) = 0$. See also [Con99] for a comparison between this principal value and the one that usually occurs in the explicit formulas.

Finally, we compute

$$\begin{aligned} -\mathrm{tr} \, f\lambda(f)(M_\varphi - \mathfrak{F}JM_\varphi J\mathfrak{F})|_{\mathcal{H}_+} &= -\langle \mathfrak{F}(\ln x), y \mapsto f(1-y) \rangle = pv \int_{\mathbb{R}^\times} f(1-y) \frac{dy}{|y|} \\ &= pv \int_{-\infty}^{\infty} f(x) \frac{dx}{|1-x|} = pv \int_0^{\infty} \frac{f(x)}{|1-x|} + \frac{f(x)}{1+x} dx = W_\infty(f). \end{aligned}$$

Plugging this into (15), we get the desired formula for $\chi(\rho)$. $\square$

6. Generalisation to Dirichlet L -functions

We recall the definition of Dirichlet L -functions. Fix some $d \in \mathbb{N}_{\geq 2}$ and let $(\mathbb{Z}/d\mathbb{Z})^\times$ be the group of invertible elements in the finite ring $\mathbb{Z}/d\mathbb{Z}$. Let χ be a character of $(\mathbb{Z}/d\mathbb{Z})^\times$. Define $\chi: \mathbb{N} \rightarrow \mathbb{C}$ by $\chi(n) := \chi(n \bmod d)$ if $(n, d) = 1$ and $\chi(n) := 0$ otherwise. The associated Dirichlet L -function is defined by

$$L_\chi(s) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

We suppose that d is equal to the conductor of χ , that is, χ does not factor through $(\mathbb{Z}/d'\mathbb{Z})^\times$ for any proper divisor $d' \mid d$. In particular, $\chi \neq 1$.

The constructions for the Riemann ζ -function that we have done above work similarly for such L -functions. We define the space $\mathcal{H}_-$ as above and let

$$\mathcal{H}_+ := \{f \in \mathcal{S}(\mathbb{R}) \mid f(-x) = \chi(-1)f(x)\}$$

be the space of even or odd functions, depending on $\chi(-1) \in \{\pm 1\}$. The Fourier transform on $\mathcal{S}(\mathbb{R})$ preserves the subspace $\mathcal{H}_+$, and the assertions of

Proposition 2.1 remain true. However, now $\mathfrak{F}^2 = \chi(-1)$, so that we have to replace $\mathfrak{F}$ by $\mathfrak{F}^* = \chi(-1)\mathfrak{F}$ in appropriate places to get correct formulas.

Of course, the L -function analogue of the Zeta operator is defined by

$$\mathcal{L}_\chi f(x) := \sum_{n=1}^{\infty} \chi(n) \cdot f(nx),$$

for $f \in \mathcal{H}_+$. We now have the Euler product expansion

$$\mathcal{L}_\chi = \sum_{n=1}^{\infty} \chi(n) \lambda_{n^{-1}} = \prod_{p \in \mathcal{P}} \sum_{e=0}^{\infty} \chi(p)^e \lambda_{p^{-e}} = \prod_{p \in \mathcal{P}} (1 - \chi(p) \lambda_{p^{-1}})^{-1}.$$

The same estimates as for the Zeta operator show that this product expansion converges on $\mathcal{S}_>$ (compare Proposition 3.2).

The Poisson Summation Formula looks somewhat different now: we have

$$\mathcal{L}_\chi(f) = \kappa \cdot d^{1/2} \lambda_d^{-1} J \mathcal{L}_\chi \mathfrak{F}(f),$$

where κ is some complex number with $|\kappa| = 1$. The proof of Theorem 3.3 then carries over without change. We also get the holomorphic continuation and the functional equation for L_χ . If $\chi(-1) = -1$, we have to use the special function $2x \exp(-\pi x^2)$ instead of $2 \exp(-\pi x^2)$ to pass from L_χ to the complete L -function. The results in Section 4 carry over in an evident way. Now $\mathcal{H}_+^0 = \{0\}$ because L_χ does not have poles, and the eigenvalues of ${}^t D_-$ are the non-trivial zeros of L_χ , with correct algebraic multiplicity.

Some modifications are necessary in Section 5. We define ι_- and π_- as above. Since we want the embeddings $\iota_\pm$ to agree on $\mathcal{H}_\cap \cong \mathcal{L}_\chi \mathcal{H}_\cap$, we should put

$$\iota_+(f) := (\mathcal{L}_\chi(f), \kappa \cdot d^{1/2} \lambda_d^{-1} J \mathcal{L}_\chi \mathfrak{F} f)$$

and modify π_+ accordingly so that $\pi_+ \iota_+ = M_\varphi + \mathfrak{F}^* M_\varphi \mathfrak{F}$. With these changes, the remaining computations carry over easily. Of course, we get different local summands W_p in the explicit formula for $\mathcal{L}_\chi$. You may want to compute them yourself as an exercise to test your understanding of the arguments above.

Even more generally, we can replace the rational numbers $\mathbb{Q}$ by an imaginary quadratic extension like $\mathbb{Q}[i]$ and study L -functions attached to characters of the idele class group of this field extension. Such fields have only one infinite place, which is complex. A character of the idele class group restricts to a character of the circle group inside $\mathbb{C}^\times$. The space $\mathcal{H}_+$ is now replaced by the homogeneous subspace of $\mathcal{S}(\mathbb{C})$ defined by that character.

Once there is more than one infinite place, we need the more general setup of [Mey]. In addition, the adelic constructions in [Mey] provide a better understanding even for $\mathbb{Q}$ because they show the similarity of the analysis at

the finite and infinite places. The explicit formula takes a much nicer form if we put together all characters of the idele class group. The resulting local summands that make up the Weil distribution are of the same general form

$$W_v(f) = pv \int_{\mathbb{Q}_v^\times} \frac{f(x)}{|1 - x^{-1}|} d^\times x$$

at all places v (with $\mathbb{Q}_\infty = \mathbb{R}$) and can also be interpreted geometrically as a generalised Lefschetz trace formula (see [Con99]).

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