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UNIVERSALITY OF L-FUNCTIONS

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Abstract. We survey recent results on the value-distribution of L-functions with emphasis on aspects of universality.

1. Voronin's theorem

In 1975 Voronin [**Vor75b**] discovered a remarkable analytical property of the Riemann zeta-function $\zeta(s)$. Roughly speaking, he proved that any non-vanishing analytic function can be approximated uniformly by certain purely imaginary shifts of the zeta-function in the right half of the critical strip. After significant improvements due to Reich [**Rei77**] and Bagchi [**Bag81**] the strongest version of Voronin's theorem has the form (see [**Lau96**]):

Theorem 1. *Suppose that $\mathscr{K}$ is a compact subset of the strip $\mathscr{D} := \{s \in \mathbb{C} : \frac{1}{2} < \operatorname{Re} s < 1\}$ with connected complement, and let $g(s)$ be a non-vanishing continuous function on $\mathscr{K}$ which is analytic in the interior of $\mathscr{K}$. Then, for any $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas} \left\{ \tau \in [0,T] \, : \, \max_{s\in\mathscr{K}} |\zeta(s+i\tau) - g(s)| < \varepsilon \right\} > 0.$$

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We may interpret the absolute value of an analytic function as an analytic landscape over the complex plane. Then the universality theorem states that any (finite) analytic landscape can be found (up to an arbitrarily small error) in the analytic landscape of $\zeta(s)$. This remarkable property has several interesting consequences on the value-distribution of the zeta-function. For instance, for any $n \in \mathbb{N}$ and any fixed s with $\operatorname{Re} s \in (\frac{1}{2}, 1)$ the set

$$\{(\zeta(s+i\tau), \zeta'(s+i\tau), \ldots, \zeta^{(n-1)}(s+i\tau)) : \tau \in \mathbb{R}\}$$

is dense in $\mathbb{C}^n$. Moreover, it follows that the zeta-function does not satisfy any algebraic differential equation.

In the first half of the twentieth century, Harald Bohr applied probabilistic methods in order to study the value distribution of $\zeta(s)$. Voronin's proof of his universality theorem relies heavily on Bohr's ideas. Here we sketch a variant of the proof in the language of weakly convergent probability measures due to Bagchi [**Bag81**].

Denote by $\gamma = \{s \in \mathbb{C} : |s| = 1\}$ the unit circle in the complex plane and put

$$\Omega = \prod_p \gamma_p,$$

where $\gamma_p = \gamma$ for each prime number p. With product topology and pointwise multiplication this infinite dimensional torus Ω is a compact topological abelian group, and hence the normalized Haar measure m on the metric space $(\Omega, \mathscr{B}(\Omega))$ exists; here $\mathscr{B}(\Omega)$ denotes the class of Borel sets of Ω. This induces a probability space $(\Omega, \mathscr{B}(\Omega), \mathrm{m})$. Let $\omega(p)$ denote the projection of $\omega \in \Omega$ on the coordinate space γ_p. Since the Haar measure m on Ω is the product of the Haar measures m_p on the coordinate spaces,$\{\omega(p) : p \text{ prime}\}$ is a sequence of independent complex-valued random variables defined on $(\Omega, \mathscr{B}(\Omega), \mathrm{m})$. Now denote by $\mathscr{H}(\mathscr{D})$ the set of analytic functions defined on the strip $\mathscr{D}$, equipped with the topology of uniform convergence on compacta. For $\operatorname{Re} s > \frac{1}{2}$ and $\omega \in \Omega$, let

$$\zeta(s, \omega) = \prod_p \left(1 - \frac{\omega(p)}{p^s}\right)^{-1}.$$

This defines an $\mathscr{H}(\mathscr{D})$-valued random element on the probability space $(\Omega, \mathscr{B}(\Omega), \mathrm{m})$. It can be shown that for almost all $\omega \in \Omega$ the infinite product defining $\zeta(s, \omega)$ converges uniformly on compact subsets of $\mathscr{D}$.

The first step in the proof of Theorem 1 is to establish the weak convergence of the probability measure $\mathbf{P}_T$, defined by

$$\mathbf{P}_T(A) = \frac{1}{T} \operatorname{meas}\{\tau \in [0, T] : \zeta(s+i\tau) \in A\} \qquad \text{for} \quad A \in \mathscr{B}(\mathscr{H}(\mathscr{D})),$$

to the distribution $\mathbf{P}$ of the random element $\zeta(s,\omega)$, given by

$$\mathbf{P}(A) = \mathrm{m}\{\omega \in \Omega \,:\, \zeta(s+i\tau) \in A\} \qquad \text{for} \quad A \in \mathscr{B}(\mathscr{H}(\mathscr{D})),$$

as $T \to \infty$. For short,

$$\mathbf{P}_T \quad \Rightarrow \quad \mathbf{P} \qquad (T \to \infty). \tag{1}$$

The proof of this limit theorem relies on fundamental results from probabilty theory, e.g. Prokhorov's theorems, ergodic theory for random processes, and the simple but important fact that the logarithms of different prime numbers are linearly independent.

The support of the random element

$$\log \zeta(s,\omega) = \sum_p \log \left(1 - \frac{\omega(p)}{p^s}\right)^{-1}$$

is the closure of the set of all convergent series

$$\sum_p \log \left(1 - \frac{a(p)}{p^s}\right)^{-1} \qquad \text{with} \quad a(p) \in \gamma.$$

The second step is to prove that the set of all these convergent series is dense in $\mathscr{H}(\mathscr{D}_M)$, where $\mathscr{D}_M := \{s \in : \frac{1}{2} < \mathrm{Re}\, s < 1, |\mathrm{Im}\, s| < M\}$ and M is an arbitrary positive constant. This involves the theory of entire functions of exponential type, a rearrangement theorem in Hilbert spaces, and the prime number theorem. The map $f \mapsto \exp f$ sends $\mathscr{H}(\mathscr{D}_M)$ to

$$\left\{ g \in \mathscr{H}(\mathscr{D}_M) \,:\, g(s) \neq 0 \quad \text{for} \quad \frac{1}{2} < \mathrm{Re}\, s < 1 \right\}.$$

Now, roughly speaking, the limit theorem (1) ties both ends together. Since $\mathscr{K}$ is a compact subset of $\mathscr{D}$, there exists some M for which $\mathscr{K} \subset \mathscr{D}_M$. It follows that any g which is contained in the support of the random element $\zeta(s,\omega)$ and has a non-vanishing analytic continuation to $\mathscr{D}_M$ can be approximated uniformly by some shift $\zeta(s+i\tau)$ for $s \in \mathscr{K}$: if Φ denotes the set of $\varphi \in \mathscr{H}$ such that

$$\max_{s \in \mathscr{K}} |\varphi(s) - g(s)| < \varepsilon,$$

then

$$\liminf_{T \to \infty} \frac{1}{T} \operatorname{meas} \left\{ \tau \in [0,T] \,:\, \max_{s \in \mathscr{K}} |\zeta(s+i\tau) - g(s)| < \varepsilon \right\} \geqslant \mathbf{P}(\Phi) > 0.$$

The case of functions g with zeros in $\mathscr{D}_M \setminus \mathscr{K}$ follows from an application of Mergelyan's approximation theorem). This yields the statement of Theorem 1.

All known proofs of universality results for Dirichlet series, like the one for the zeta-function above or those covered by Theorem 2 below, depend on some arithmetical conditions. But is universality really an arithmetic phenomenon or not? Reviewing the proof one might understand universality as a kind of ergodicity on function spaces. It seems reasonable that the universality of Dirichlet series is a common phenomenon in analysis, that it is related to Julia rays in value-distribution theory and to ergodical dynamical systems as well.

2. Zeros and the Riemann hypothesis

Since any non-vanishing analytic function possesses an analytic logarithm, it follows that $\log\zeta(s)$ is *strongly universal*, i.e., $\log\zeta(s)$ can uniformly approximate functions having zeros; in fact, we have almost given a proof of this fact in the previous section. It is natural to ask whether $\zeta(s)$ is also strongly universal. The answer is negative. We give a heuristic argument which can be made waterproof with a bit more effort by the techniques of Section 5.

Assume that $g(s)$ is an analytic function on $|s|\leqslant r$, where $0<r<\frac{1}{4}$, which has a zero ξ with $|\xi|<r$ but which is non-vanishing on the boundary. Then, whenever the inequality

$$\max_{|s|\leqslant r}\left|\zeta\left(s+\frac{3}{4}+i\tau\right)-g(s)\right|<\varepsilon<\min_{|s|\leqslant r}|g(s)|;$$

holds, $\zeta\left(s+\frac{3}{4}+i\tau\right)$ has to have a zero inside $|s|\leqslant r$. This can be seen as follows. By the maximum principle the maximum on the left hand side of the inequality above is taken on the boundary. The second inequality holds for sufficiently small ε (since the zeros of an analytic function form a discrete set or the function vanishes identically). Consequently, an application of Rouché's theorem yields the existence of a zero of $\zeta(s+\frac{3}{4}+i\tau)$ inside $|s|\leqslant r$. If now for any $\varepsilon>0$

$$\liminf_{T\to\infty}\frac{1}{T}\,\mathrm{meas}\left\{\tau\in[0,T]\,:\,\max_{|s|\leqslant r}\left|\zeta\left(s+\frac{3}{4}+i\tau\right)-g(s)\right|<\varepsilon\right\}>0,$$

then we expect $\gg T$ many complex zeros of $\zeta(s)$ in the strip $\frac{3}{4}-r<\operatorname{Re}s<\frac{3}{4}+r$ up to height T. This contradicts classical density estimates: for any $\sigma>\frac{1}{2}$, the number of zeros $\rho=\beta+i\gamma$ satisfying $\beta>\sigma, 0<\gamma\leqslant T$ is known to be $o(T)$ as $T\to\infty$. Thus, uniform approximation of a function $g(s)$ with a zero by the zeta-function is impossible.

Bohr [**Boh22**] discovered an interesting relation between the Riemann hypothesis and almost periodicity. He showed that if χ is non-principal character, then the Riemann hypothesis for the associated Dirichlet L-function $L(s,\chi)$

(i.e., the non-vanishing of $L(s,\chi)$ for $\mathrm{Re}\, s > \frac{1}{2}$) is equivalent to the almost periodicity of $L(s,\chi)$ in the half-plane $\mathrm{Re}\, s > \frac{1}{2}$. Because of the restriction on non-principal characters χ this result does not cover the case of $\zeta(s)$. More than half a century later Bagchi [**Bag81**] proved that the same criterion holds also for $\zeta(s)$, namely that Riemann's hypothesis is true if and only if for any compact subset $\mathscr{K}$ of the strip $\frac{1}{2} < \mathrm{Re}\, s < 1$ with connected complement and for any $\varepsilon > 0$

$$\liminf_{T\to\infty} \frac{1}{T} \mathrm{meas} \left\{ \tau \in [0,T] \,:\, \max_{s\in\mathscr{K}} |\zeta(s+i\tau) - \zeta(s)| < \varepsilon \right\} > 0.$$

The crucial implication of Bagchi's proof relies essentially on Voronin's universality theorem, which, of course, was unknown to Bohr.

3. The Selberg class

Meanwhile, it is known that there exists a *rich zoo* of universal Dirichlet series; for a list we refer to [**Lau96**], [**Mat04**], [**Ste04**]. It was conjectured by Linnik and Ibragimov that all functions given by Dirichlet series and analytically continuable to the left of the half plane of absolute convergence, which satisfy some natural growth conditions, are universal. In this section we are interested in the universality of L-functions.

In 1989 Selberg [**Sel92**] defined a general class $\mathscr{S}$ of Dirichlet series having an Euler product, analytic continuation and a functional equation of Riemann-type, and formulated some fundamental conjectures concerning them. His aim was to study the value-distribution of linear combinations of L-functions. In the meantime this so-called Selberg class became an important object of research. All known examples of functions in the Selberg class are automorphic (or at least conjecturally automorphic) L-functions, and for all of them it turns out that the related Euler factors are the inverse of a polynomial in p^{-s}. This special shape of the Euler product is related to Langlands' reciprocity conjecture. In the sequel we will consider a subclass of polynomial Euler products.

The class $\tilde{\mathscr{S}}$ consists of Dirichlet series

$$\mathscr{L}(s) := \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$$

satisfying the following axioms:

– **Polynomial Euler product**: for $1 \leqslant j \leqslant m$ and each prime p there exist complex numbers $\alpha_j(p)$ with $|\alpha_j(p)| \leqslant 1$ such that

$$\mathscr{L}(s) = \prod_p \prod_{j=1}^m \left(1 - \frac{\alpha_j(p)}{p^s}\right)^{-1};$$

– **Mean-square:** there exists a positive constant κ such that

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{p\leqslant x} |a(p)|^2 = \kappa;$$

– **Analytic continuation**: there exists a non-negative integer k such that $(s-1)^k \mathscr{L}(s)$ is an entire function of finite order;

– **Functional equation**: there are positive real numbers Q, λ_j, and there are complex numbers $\mu_j,\ \omega$ with $\operatorname{Re}\mu_j \geqslant 0$ and $|\omega| = 1$, such that

$$\Lambda_{\mathscr{L}}(s) = \omega\overline{\Lambda_{\mathscr{L}}(1-\overline{s})},$$

where

$$\Lambda_{\mathscr{L}}(s) := \mathscr{L}(s) Q^s \prod_{j=1}^f \Gamma(\lambda_j s + \mu_j).$$

It should be noted that the axiom on the mean-square is intimately related to Selberg's conjectures (see [**Sel92**]). We expect that $\tilde{\mathscr{S}}$ contains all non-constant functions from the Selberg class: $\tilde{\mathscr{S}} = \mathscr{S} \setminus \{1\}$.

The degree of any non-constant function $\mathscr{L} \in \mathscr{S}$ (and so in $\tilde{\mathscr{S}}$) is defined by

$$\mathrm{d}_{\mathscr{L}} = 2\sum_{j=1}^f \lambda_j.$$

This quantity is well-defined. If $N_{\mathscr{L}}(T)$ counts the number of zeros of $\mathscr{L} \in \mathscr{S}$ in the rectangle $0 \leqslant \operatorname{Re} s \leqslant 1, |\operatorname{Im} s| \leqslant T$ (counting multiplicities) one can show by standard contour integration

$$N_{\mathscr{L}}(T) \sim \frac{\mathrm{d}_{\mathscr{L}}}{\pi} T \log T,$$

in analogy to the classical Riemann-von Mangoldt formula for $\zeta(s)$. It is conjectured that the degree is always a positive integer (provided that $\mathscr{L}$ is not constant one).

The functions of degree one in $\tilde{\mathscr{S}}$ are the Riemann zeta-function and shifts of Dirichlet L-functions $L(s+i\theta,\chi)$ attached to primitive characters χ with $\theta \in \mathbb{R}$. Examples of degree two are normalized L-functions associated with holomorphic newforms; normalized L-functions attached to non-holomorphic newforms are expected to lie in $\tilde{\mathscr{S}}$. The Rankin-Selberg L-function of any two holomorphic newforms is an element of the Selberg class of degree 4. Further examples are Dedekind zeta-functions to number fields $\mathbb{K}$; their degree is equal to the degree of the field extension $\mathbb{K}/\mathbb{Q}$.

In [**Ste04**] the following generalization of Voronin's universality theorem was proved.

Theorem 2. *Let $\mathscr{L} \in \tilde{\mathscr{S}}$ and $\mathscr{K}$ be a compact subset of the strip*

$$\mathscr{D}_{\mathscr{L}} := \left\{ s \in \mathbb{C} \,:\, \max\left\{\frac{1}{2}, 1 - \frac{1}{\mathrm{d}_{\mathscr{L}}}\right\} < \mathrm{Re}\, s < 1 \right\}$$

with connected complement, and let $g(s)$ be a non-vanishing continuous function on $\mathscr{K}$ which is analytic in the interior of $\mathscr{K}$. Then, for any $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{T} \,\mathrm{meas} \left\{ \tau \in [0, T] \,:\, \max_{s\in\mathscr{K}} |\mathscr{L}(s + i\tau) - g(s)| < \varepsilon \right\} > 0.$$

This theorem may be regarded as a verification of the Linnik-Ibragimov conjecture for Dirichlet series in the Selberg class.

4. The strip of universality

Besides the arithmetic axioms on the polynomial Euler product and on the mean-square, a further important ingredient in the proof of Theorem 2 is the second moment bound

$$\limsup_{T\to\infty} \frac{1}{T} \int_1^T |\mathscr{L}(\sigma + it)|^2 \,\mathrm{d}t < \infty.$$

In [**Ste04**] an asymptotic formula for $\mathscr{L} \in \tilde{\mathscr{S}}$ in the range $\mathscr{D}_{\mathscr{L}}$ was proved. If $\mathrm{d}_{\mathscr{L}} > 2$, the strip $\mathscr{D}_{\mathscr{L}}$ does not cover the right half of the critical strip; any extension of this strip to the left would imply universality in this extended strip. However, the critical line is a natural boundary for universality of $\mathscr{L} \in \tilde{\mathscr{S}}$ (at least in the sense of Theorem 2).

In particular cases of functions $\mathscr{L} \in \tilde{\mathscr{S}}$ with degree $\mathrm{d}_{\mathscr{L}} > 2$ the existence of the mean-square covering the strip $\mathscr{D}_{\mathscr{L}}$ is known. For instance, let $L(s, \chi)$ be an arbitrary Dirichlet L-function to a primitive character χ. Then $\zeta(s)^2 L(s, \chi)$ is an element of $\tilde{\mathscr{S}}$ of degree 3, so Theorem 2 gives universality for $\frac{2}{3} < \mathrm{Re}\, s < 1$. Using Montgomery's estimates for the fourth moment of Dirichlet L-functions and Ivić's eigth-moment estimate for $\zeta(s)$, the Cauchy-Schwarz inequality yields

$$\int_1^T |\zeta(\sigma + it)^2 L(\sigma + it, \chi)|^2 \,\mathrm{d}t \ll T^{\frac{11-8\sigma}{12} + \frac{1}{2} + \varepsilon} \ll T$$

for any $\sigma > \frac{5}{8}$. Thus $\zeta(s)^2 L(s,\chi)$ is universal in the strip $\frac{5}{8} < \mathrm{Re}\, s < 1$. If the generalized Lindelöf hypothesis is true for $\mathscr{L} \in \tilde{\mathscr{S}}$, i.e.,

$$\mathscr{L}\left(\frac{1}{2} + it\right) \ll t^{\varepsilon}$$

for any $\varepsilon > 0$ as $t \to \infty$, then the strip of universality can be extended to the full open right half of the critical strip.

5. Effectivity

The known proofs of universality theorems are ineffective, giving neither an estimate for the first approximating shift τ nor bounds for the positive lower density. There are some remarkable attempts due to Garunkštis [**Gar03**], Good [**Goo81**], and Laurinčikas [**Lau00**], however, their results are either restricted to rather small classes of functions or conditional subject to certain unproved hypotheses. Following [**Ste03**] we now consider the problem of effective upper bounds for the upper density of universality.

Denote by B_r the closed disc of radius $r > 0$ with center in the origin. We define for a meromorphic function $L(s)$, an analytic function $g : B_r \to \mathbb{C}$ with fixed $r \in \left(0, \frac{1}{4}\right)$, and positive ε the densities

$$\underline{\mathrm{d}}(\varepsilon, g, L) = \liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{\tau \in [0,T] \, : \, \max_{|s|\leqslant r}\left|L\left(s + \frac{3}{4} + i\tau\right) - g(s)\right| < \varepsilon\right\},$$

and

$$\overline{\mathrm{d}}(\varepsilon, g, L) = \limsup_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{\tau \in [0,T] \, : \, \max_{|s|\leqslant r}\left|L\left(s + \frac{3}{4} + i\tau\right) - g(s)\right| < \varepsilon\right\}.$$

We consider analytic isomorphisms $g : B_r \to B_1$, i.e., the inverse g^{-1} exists and is analytic. Obviously, such a function g has exactly one simple zero ξ in the interior of B_r. By the Schwarz lemma any such g has a representation

$$g(s) = r \exp(i\varphi) \frac{\xi - s}{r^2 - \bar{\xi} s} \qquad \text{with} \quad \varphi \in \mathbb{R} \quad \text{and} \quad |\xi| < r.$$

Denote by $\mathscr{A}_r$ the class of analytic isomorphisms from B_r (with fixed $0 < r < \frac{1}{4}$) to the unit disc. Further, let $N_L(\sigma_1, \sigma_2, T)$ count the number of zeros of $L(s)$ in $\frac{1}{2} < \sigma_1 < \mathrm{Re}\, s < \sigma_2 < 1,\ 0 \leqslant t < T$ (counting multiplicities).

Theorem 3. *Suppose that $g \in \mathscr{A}_r$. Assume that $L(s)$ is analytic in $\mathrm{Re}\, s \geqslant \frac{3}{4} - r$ except for at most $o(T)$ many singularities inside $\mathrm{Re}\, s \geqslant \frac{3}{4} - r, 0 \leqslant$*

$\operatorname{Im} s \leqslant T$, *as* $T \to \infty$, *and that* $\underline{\mathrm{d}}(\varepsilon, g, L) > 0$ *for all* $\varepsilon > 0$. *Then, for any* $\varepsilon \in \left(0, \frac{1}{2r}\left(\frac{1}{4} + \operatorname{Re}|\xi|\right)\right)$,

$$\overline{\mathrm{d}}(\varepsilon, g, L) \leqslant \frac{8r^3\varepsilon}{r^2 - |\xi|^2} \limsup_{T\to\infty} \frac{1}{T} N_L\left(\frac{3}{4} + \operatorname{Re}\xi - 2r\varepsilon, \frac{3}{4} + \operatorname{Re}\xi + 2r\varepsilon, T\right).$$

We sketch the proof (which is a bit in the spirit of Section 2). The zero ξ of g is related to some zeros of $L(s)$ in $\frac{1}{2} < \operatorname{Re} s < 1$. Since g maps the boundary of B_r onto the unit circle, Rouché's theorem implies the existence of one simple zero λ of $L(z)$ in

$$K_\tau := \left\{z = s + \frac{3}{4} + i\tau \,:\, s \in B_r\right\},$$

whenever

$$\max_{s\in B_r}\left|L\left(s + \frac{3}{4} + i\tau\right) - g(s)\right| < \varepsilon < 1 = \min_{s\in B_r}|g(s)|.$$

We may say that the zero λ of $L(s)$ is *generated* by the zero ξ of $g(s)$. Universality is a phenomenon that happens in intervalls. Suppose that a zero λ of $L(s)$, generated by ξ, lies in two different sets K_{τ_1} and K_{τ_2}. Then one can show that

$$|\tau_1 - \tau_2| < \frac{8r^4\varepsilon}{r^2 - |\xi|^2}.$$

Now denote by $\mathscr{I}_j(T)$ the disjoint intervalls in $[0, T]$ such that (1) is valid exactly for $\tau \in \bigcup_j \mathscr{I}_j(T) =: \mathscr{I}(T)$. By the latter estimate, in every intervall $\mathscr{I}_j(T)$, there lie at least

$$1 + \left[\frac{r^2 - |\xi|^2}{8r^3\varepsilon} \operatorname{meas}\mathscr{I}_j(T)\right] \geqslant \frac{r^2 - |\xi|^2}{8r^3\varepsilon} \operatorname{meas}\mathscr{I}_j(T)$$

zeros λ of $L(s)$ in the strip $\frac{1}{2} < \operatorname{Re} s < 1$. Therefore, the number $\mathscr{N}(T)$ of such zeros λ satisfies the estimate

$$\frac{8r^3\varepsilon}{r^2 - |\xi|^2}\mathscr{N}(T) \geqslant \operatorname{meas}\mathscr{I}(T). \tag{2}$$

The value distribution of $L(z)$ in K_τ is ruled by that of $g(s)$ in B_r. This gives a restriction on the real parts of the zeros λ. One can show that $\left|\operatorname{Re}\lambda - \frac{3}{4} - \operatorname{Re}\xi\right| < 2r\varepsilon$. This yields

$$\mathscr{N}(T) \leqslant N_L\left(\frac{3}{4} + \operatorname{Re}\xi - 2r\varepsilon, \frac{3}{4} + \operatorname{Re}\xi + 2r\varepsilon, T\right).$$

Since $\underline{\mathrm{d}}(\varepsilon, g, L) > 0$, this leads via (2) to the estimate of the theorem. Note that the set of singularities of $L(s)$ in $\sigma \geqslant \frac{3}{4} - r$ has zero density but $\underline{\mathrm{d}}(\varepsilon, g, L) > 0$. So the singularities do not affect the above observations.

Theorem 3 relates the density of universality to the value-distribution of L. In the case of the Riemann zeta-function we can be more explicit. In view of classical density theorems the set of singularities of $\log\zeta(s)$ has density zero. Hence, we may apply Theorem to $L(s) = \log\zeta(s)$. By Bohr and Jessen [**BJ32**], Hilfssatz 6, the limit

$$\lim_{T\to\infty} \frac{1}{T} N_{\log\zeta}\left(\frac{3}{4} + \operatorname{Re}\xi - 2r\varepsilon, \frac{3}{4} + \operatorname{Re}\xi + 2r\varepsilon, T\right)$$

exists, and tends to zero as $\varepsilon \to 0$. Hence, under the above assumptions

$$\overline{\mathrm{d}}\,(\varepsilon, \exp g, \zeta(s)) = o(\varepsilon).$$

Thus, the decay of $\overline{\mathrm{d}}\,(\varepsilon, \exp g, \zeta)$ with $\varepsilon \to 0$ is more than linear in ε.

6. Joint universality

We conclude with another interesting problem concerning universality of L-functions.

Voronin [**Vor75a**] also obtained joint universality for Dirichlet L-functions, that is simultaneous uniform approximation by a family of L-functions associated with non-equivalent characters; the non-equivalence of the characters assures a certain independence of the related L-functions, and this independence is necessary for joint universality. Recently, Laurinčikas & Matsumoto [**LM04**] proved a joint universality theorem for L-functions associated with newforms twisted by characters. It is natural to ask for joint universality in the Selberg class. However, all known jointly universal families are given by (multiplicative or additive) twists of a single universal Dirichlet series by characters. In some sense, Selberg's Conjecture B (see [**Sel92**]) states that primitive functions form an orthonormal system in the Selberg class. As proved by Bombieri & Hejhal [**BH95**], this implies the statistical independence of primitive functions. There is some hope that this can be used as substitute for the independence induced by non-equivalent characters in order to prove joint universality for distinct primitive L-functions from the Selberg class.

For $1 \leqslant j \leqslant m$, assume that the L-functions

$$\mathscr{L}_j(s) = \sum_{n=1}^{\infty} \frac{a_{\mathscr{L}_j}(n)}{n^s}$$

from $\tilde{\mathscr{S}}$ satisfy the orthogonality condition

$$\sum_{p\leqslant x} \frac{a_{\mathscr{L}_j}(p)\overline{a_{\mathscr{L}_k}(p)}}{p} = \delta_{jk}\kappa_j \log\log x + O(1), \tag{3}$$

where κ_j is a positive constant depending on $\mathscr{L}_j$, and $\delta_{jk} = 1$ if $j = k$ and $\delta_{jk} = 0$ otherwise. This condition is known to hold for several families of L-functions in $\tilde{\mathscr{S}}$, for example for Dirichlet L-functions associated with pairwise non-equivalent characters (in which case it is nothing else than the orthogonality relation for characters); it is expected to hold for any two distinct primitive L-functions from the Selberg class (Selberg's Conjecture B). Moreover, (3) may be regarded as an extension of the axiom on the mean square in the definition of $\tilde{\mathscr{S}}$.

Conjecture. *Suppose that $\mathscr{L}_1, \ldots, \mathscr{L}_m$ are elements of $\tilde{\mathscr{S}}$ satisfying condition (3). For $1 \leqslant j \leqslant m$ let $g_j(s)$ be a continuous function on $\mathscr{K}_j$ which is non-vanishing in the interior, where $\mathscr{K}_j$ is a compact subset of the strip*

$$\mathscr{D} := \left\{ s \,:\, \max\left\{\frac{1}{2}, 1 - \frac{1}{\mathrm{d}}\right\} < \operatorname{Re} s < 1 \right\}$$

with connected complement, and d *is the maximum of the degrees of the $\mathscr{L}_j$ (a quantity determined by the functional equation for $\mathscr{L}_j$). Then, for any $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{ \tau \in [0,T] \,:\, \max_{1\leqslant j\leqslant m} \max_{s\in\mathscr{K}_j} |\mathscr{L}_j(s+i\tau) - g_j(s)| < \varepsilon \right\} > 0.$$

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