EQUIVARIANT BURNSIDE GROUPS AND TORIC VARIETIES

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Abstract. We study G-equivariant birational geometry of toric varieties, where G is a finite group.

1. Introduction

In this paper we continue our studies of actions of finite groups on algebraic varieties, up to equivariant birational equivalence. Our main tool is the new invariant introduced in [24]: to an n-dimensional smooth projective variety X with a regular action of a finite group G we assigned a class

$$[X \circlearrowleft G] \in \mathrm{Burn}_n(G),$$

taking values in the *equivariant Burnside group*, which is defined by certain symbols and relations; this class is an equivariant birational invariant. In [19], [26] we studied various structural properties of this new invariant and provided first applications. In [25] we presented an algorithm to compute the classes of linear actions and presented examples of nonbirational such actions. Here, we turn to algebraic tori.

Recall that an algebraic torus of dimension n over a field k is a linear algebraic group T which is a k-form of $\mathbb{G}_m^n$. The absolute Galois group of k acts on the geometric character group $M := \mathfrak{X}^*(T_{\bar{k}})$ via a finite subgroup

$$G \subset \mathrm{Aut}(M) \cong \mathrm{GL}_n(\mathbb{Z}).$$

A torus T over k is uniquely determined up to isomorphism by its splitting field, Galois over k, with Galois group G, and this representation of G.

Rationality properties of tori over nonclosed fields have been extensively studied, see, e.g., [38], [16], [7], [21], [28]. The relevant cohomological obstruction results from the Galois module $\mathrm{Pic}(X_{\bar{k}})$, for a smooth projective compactification X of T. For any subgroup $G' \subseteq G$ the group

$$\mathrm{H}^1(G', \mathrm{Pic}(X_{\bar{k}})) \tag{1.1}$$

is independent of the choice of X; its nontriviality is an obstruction to stable k-rationality of T. This is the only obstruction in dimensions ≤ 3;

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moreover, every stably rational torus in dimension ≤ 3 is rational [27]. The Zariski problem for algebraic tori, i.e., the question of whether or not stably rational tori over k are rational over k is still open, in particular, for 4-dimensional tori identified in [28, Prop. 4.15], with G a subgroup of $C_2 \times \mathfrak{A}_5$ or $C_2 \times \mathfrak{S}_4$.

Here, over an algebraically closed field k of characteristic zero, we study the G-equivariant version of this question, for a finite group G:

Problem: *Is a given G-action on an algebraic torus linearizable, i.e., G-equivariantly birational to a linear action on projective space?*

There are many similarities but also subtle distinctions between these points of view, highlighted, e.g., in [20]. One of the similarities is that the cohomological obstruction (1.1) applies also as an obstruction to (stable) linearizability of the G-action, since for linear actions of G, the invariant (1.1) vanishes (see, e.g., [3, Prop. 2.2] and references therein). On the other hand, all tori in dimension 2, over any field, are rational, while there is an action of $G := C_2 \times \mathfrak{S}_3$ on $\mathbb{G}_m^2$, which is not linearizable [23], but is stably linearizable [29, Prop. 9.11]; the corresponding class in $\mathrm{Burn}_2(G)$ is distinct from those of linear actions [19, Section 7.6].

In the case of surfaces both points of view have received ample attention, going back to [32], [22], with further developments in [15], [3], [36], and in many other papers. The main approach there is via the (equivariant) Minimal Model Program, i.e., classification of all birational models and (equivariant) birational transformations between those models. Much less is known about linearizability of G-actions in higher dimensions, in particular, for tori, see [5].

Our main results in this paper are:

- We give a procedure (Theorem 5.3) to compute the class
$$[X \circlearrowleft G] \in \mathrm{Burn}_n(G). \tag{1.2}$$
This uses the De Concini-Procesi formalism to construct a suitable equivariant birational model of the torus T, combined with a recursive process, which takes into account equivariant birational types of the boundary strata $X \setminus T$, together with normal bundle information.
- We present an example of such a computation (Proposition 6.1).
- We discuss the relation between the class (1.2) and existing (stable) G-birational invariants, such as group cohomology (1.1): there exist actions that can be distinguished by one invariant but not the other (Proposition 6.1 and Proposition 7.1).

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2. Toric varieties: generalities

We work over an algebraically closed field k of characteristic zero.

2.1. **Fans.** Let $T = \mathbb{G}_m^n$ be an algebraic torus of dimension n,

$$M := \mathfrak{X}^*(T), \quad \text{respectively} \quad N := \mathfrak{X}_*(T),$$

the lattice of its algebraic characters, respectively, co-characters. A smooth projective toric variety

$$X = X_\Sigma$$

of dimension n is an equivariant compactification of T. It is uniquely determined the combinatorial structure of a fan

$$\Sigma = \{\sigma\},$$

a finite collection of strongly convex rational polyhedral cones σ in $N_\mathbb{R} := N \otimes_\mathbb{Z} \mathbb{R}$ (see, e.g., [18] for basic definitions concerning toric varieties). We let

$$\Sigma(d), \quad d = 0, \dots, n,$$

denote the collection of d-dimensional cones in Σ.

The fan is subject to various conditions to insure smoothness and projectivity of X; see, e.g., [1]:

- every cone $\sigma \in \Sigma$ is simplicial and is generated by a part of a basis of N,
- the union of cones is all of $N_\mathbb{R}$, and
- Σ admits a piecewise linear convex support function.

Such a fan Σ is called a smooth projective fan.

2.2. **Subtori and their closures.** A primitive sublattice $N' \subseteq N$ gives rise to a *subtorus* $T' \subset T$, and an induced equivariant compactification X' of T'. The corresponding fan Σ' is the fan in $N'_\mathbb{R}$ induced by Σ, that is, given by intersecting the cones of Σ with $N'_\mathbb{R}$. The interesting case for us is when T' satisfies the following

Property (E): $\sigma \cap N'_\mathbb{R}$ is a face of σ, for all $\sigma \in \Sigma$ (see [9]).

Equivalently, every $\sigma \in \Sigma$ has strongly convex image under the projection

$$N_\mathbb{R} \to (N/N')_\mathbb{R}.$$

By [9, Thm. 3.1], property (E) for T', with respect to Σ, implies that X' is nonsingular, isomorphic to the toric variety $X_{\Sigma'}$. Then we have a closed immersion

$$X_{\Sigma'} \to X, \tag{2.1}$$

given on affine charts, for $\sigma \in \Sigma'$, by ring homomorphisms

$$k[\sigma^\vee \cap M] \to k[(\sigma^\vee \cap M)/(N'^\perp \cap M)].$$

Generally, by [9, Thm. 4.1], after a suitable finite subdivision of Σ, one can insure property (E) for a given subtorus, or any finite collection of subtori of T.

2.3. **Quotient tori and orbit closures.** A primitive sublattice $N' \subseteq N$ also gives rise to a quotient torus T/T'. The case of interest to us is the quotient torus T^σ, associated with the sublattice N_σ spanned by generators of a cone $\sigma \in \Sigma$; by the standing smoothness assumption on fans, σ is generated by a part of a basis of N. Furthermore, σ determines an orbit D_σ°, the closed T-orbit in the corresponding affine chart $\mathrm{Spec}(k[\sigma^\vee \cap M])$ of X. We denote its closure in X by D_σ. This is a smooth projective toric variety, whose fan is obtained as follows [8, Thm. 3.2.6]:

- $\mathfrak{X}^*(T^\sigma) = \sigma^\perp \cap M$, which is dual to N/N_σ,
- for cones $\tau \supseteq \sigma$ let

  $$\bar\tau := (\tau + \mathbb{R}\sigma)/\mathbb{R}\sigma \subseteq (N/N_\sigma)_\mathbb{R}$$

  be the induced cone in the quotient, these form a smooth projective fan Σ^σ.
- We have a closed immersion $X_{\Sigma^\sigma} \hookrightarrow X$ with image D_σ. On respective affine charts $\mathrm{Spec}(k[\sigma^\perp \cap \tau^\vee \cap M])$ and $\mathrm{Spec}(k[\tau^\vee \cap M])$, for $\tau \supseteq \sigma$, this is given by the surjective ring homomorphism

  $$k[\tau^\vee \cap M] \to k[\sigma^\perp \cap \tau^\vee \cap M],$$

  with kernel generated by characters in $\tau^\vee$, not in $\sigma^\perp$. In particular, we have a canonical isomorphism

  $$T^\sigma = \mathrm{Spec}(k[\sigma^\perp \cap M]) \cong D_\sigma^\circ$$

- The projection $T \to T^\sigma$ determines a rational map $X \dashrightarrow X_{\Sigma^\sigma}$, which is defined as a morphism on the union U^σ of affine charts of X corresponding to cones $\tau \supseteq \sigma$. This morphism

  $$U^\sigma \to X_{\Sigma^\sigma} \cong D_\sigma \tag{2.2}$$

  is smooth, given by the injective ring homomorphisms

  $$k[\sigma^\perp \cap \tau^\vee \cap M] \to k[\tau^\vee \cap M].$$

2.4. **Transversality of intersections.** By our assumption we have

$$X \setminus T = \bigcup_{\rho \in \Sigma(1)} D_\rho,$$

a simple normal crossing divisor in X. For $\sigma \in \Sigma(d)$, we have D_σ of codimension d in X, with transverse intersection

$$D_\sigma = \bigcap_{j=1}^{d} D_{\rho_j},$$

where $\rho_j \in \Sigma(1)$, $j = 1, \ldots, d$, are the rays spanning σ.

Lemma 2.1. *Let $T' \subset T$ be a subtorus satisfying property* (E) *with respect to Σ, and let X' be the closure of T' in $X = X_\Sigma$. The morphism $T \to T/T'$ extends to a smooth T-equivariant morphism to T/T' from a T-invariant neighborhood of X' in X, with fiber X' over $1 \in T/T'$. As well, X' has transverse intersection with the boundary $X \setminus T$.*

Proof. Let $N' \subseteq N$ be the corresponding primitive sublattice, with corresponding fan Σ' in $N'_{\mathbb{R}}$. By property (E), the cones of Σ' are already in Σ. The variety X' is contained in the union of affine charts of X associated with cones of Σ, that belong to Σ', by the algebraic description of the closed immersion (2.1). This union is a T-invariant neighborhood of X'. Let σ be a maximal cone of Σ'. Now we have a T-equivariant morphism

$$\operatorname{Spec}(k[\sigma^\vee \cap M]) \to \operatorname{Spec}(k[\sigma^\perp \cap M]),$$

extending $T \to T/T'$, and these patch to give the desired morphism, which is smooth. We have

$$\operatorname{Spec}(k[\sigma^\vee \cap M]) \cong \mathbb{A}^{n'} \times \mathbb{G}_m^{n-n'},$$

where n' denotes the rank of N'. Now

$$X' \cap \operatorname{Spec}(k[\sigma^\vee \cap M]) \cong \mathbb{A}^{n'} \times \{1\}$$

meets the complement of $\mathbb{G}_m^n$ transversely. □

Proposition 2.2. *Let $T' \subset T$ be a subtorus satisfying property* (E) *with respect to Σ, and let X' be the closure of T' in X. Let $\sigma \in \Sigma$ be such that the sublattice $N' \subset N$ corresponding to T' satisfies $\sigma \subset N'_{\mathbb{R}}$. Then:*

- *The intersection of X' with $D_\sigma^\circ \cong T^\sigma$ is the subtorus $T'^\sigma \subset T^\sigma$ associated with*
$$N'/N_\sigma \subseteq N/N_\sigma.$$
- *The subtorus $T'^\sigma \subset T^\sigma$ satisfies property* (E) *with respect to Σ^σ, and the intersection of X' with $D_\sigma \cong X_{\Sigma^\sigma}$ is $X_{\Sigma'^\sigma}$, where Σ' is the fan in $N'_{\mathbb{R}}$, induced from Σ, and $\sigma \in \Sigma'$ determines the fan Σ'^σ in $(N'/N_\sigma)_{\mathbb{R}}$.*

- *We have $X' \cap U^\sigma$ equal to the pre-image of $X_{\Sigma'^\sigma}$ under the smooth morphism* (2.2).

Proof. In the affine chart $\mathrm{Spec}(k[\sigma^\vee \cap M])$ of X associated with σ, we have closed subvarieties D°_σ, defined by the characters in $\sigma^\vee \cap M$, not in $\sigma^\perp$, and the intersection with X', given by equating characters whose difference lies in $N'^\perp \cap M$. So the coordinate ring of the intersection is obtained by setting to zero all characters not in $\sigma^\perp$ and equating characters whose difference lies in $N'^\perp \cap M$; this gives $k[(\sigma^\perp \cap M)/(N'^\perp \cap M)]$. The first statement is established, since the subtorus of $T^\sigma = \mathrm{Spec}(k[\sigma^\perp \cap M])$ associated with N'/N_σ is $\mathrm{Spec}(k[(\sigma^\perp \cap M)/(N'^\perp \cap M)])$.

For the second statement, we want to show, for $\tau \in \Sigma$ with $\tau \supset \sigma$, that $\bar\tau \cap (N'/N_\sigma)_\mathbb{R}$ is equal to $\bar\omega$ for some $\omega \in \Sigma$ with $\omega \supset \sigma$. We take $\omega := \tau \cap N'_\mathbb{R}$; since T' satisfies property (E) we have $\omega \in \Sigma$. Since $\sigma \subset N'_\mathbb{R}$, we have $\omega \supset \sigma$. Now

$$(\tau + (N_\sigma)_\mathbb{R}) \cap N'_\mathbb{R} = \tau \cap N'_\mathbb{R} + (N_\sigma)_\mathbb{R} = \omega + (N_\sigma)_\mathbb{R}.$$

So $\bar\tau \cap (N'/N_\sigma)_\mathbb{R} = \bar\omega$, as desired.

We treat the remainder of the second statement, and the third, by an analysis on coordinate charts. Let $\tau \in \Sigma'$, with $\tau \supset \sigma$. In the affine chart $\mathrm{Spec}(k[\tau^\vee \cap M])$, we have D_σ defined by characters in $\tau^\vee \cap M$, not in $\sigma^\perp$, and the intersection with X', given by equating characters whose difference lies in $N'^\perp \cap M$. This gives a description of the coordinate ring of the intersection as

$$k[(\sigma^\perp \cap \tau^\vee \cap M)/(N'^\perp \cap M)].$$

The same coordinate ring arises when we apply the description of the subtorus closure to $T'^\sigma \subset T^\sigma$. By considering the same coordinate rings, and the injective ring homomorphisms corresponding to (2.2) for $\tau \in \Sigma$ and $\tau \in \Sigma'$, we obtain a commutative diagram of affine schemes which we see easily to be cartesian, and thereby obtain the third statement. □

2.5. **Equivariant combinatorics.** Let G be a finite group. A regular (right) G-action on a toric variety $X = X_\Sigma$ determines a representation

$$G \to \mathrm{GL}_n(\mathbb{Z}) = \mathrm{Aut}(M). \tag{2.3}$$

We are mainly interested in cases when this is *injective*. The homomorphism (2.3) determines a right action on the cocharacter lattice N. The induced action on $N_\mathbb{R}$ leaves the fan Σ invariant, i.e., $\sigma \cdot g \in \Sigma$, for all $\sigma \in \Sigma$ and $g \in G$.

Since we are interested in equivariant birational types, we can start with a faithful representation (2.3). By [6], there exists a smooth projective fan Σ, that is invariant under the G-action. Correspondingly, G acts on X_Σ, with associated representation (2.3).

Suppose we are given a finite collection of subtori of T. By combining [9, Thm. 4.1] and [6], we achieve, after a suitable subdivision of Σ, property (E) for all of the subtori, still requiring G to act regularly on the toric variety. Notice that any further subdivision of Σ preserves property (E), for the given collection of subtori.

After a further G-equivariant subdivision of Σ, we may suppose, that the boundary $X \setminus T$ may be written as a union

$$X \setminus T = \bigcup_{i \in \mathcal{I}} D_i, \qquad \mathcal{I} := \{1, \ldots, \ell\},$$

where D_i is a nonsingular G-invariant divisor, for all i. This is achieved by iterated star subdivision along cones, where the collection of generators is contained in an orbit of $\Sigma(1)$ and is maximal, with this property.

3. Stabilizer stratification and subdivisions

3.1. **Background.** We recall basic terminology concerning *toric arrangements*, studied in, e.g., [14], [12], [10], [9], [11], [4], [2].

The main motivation for the introduction of the combinatorial formalism below was the computation of the cohomology ring of the *complement* of the arrangement; the relevant notions are:

- Morgan differential algebra [34],
- Orlik-Solomon algebra [35].

We start with a torus T. The ingredients are:

- *Layers*, i.e., cosets of subtori of T.
- *Toric arrangement*, i.e., finite collections of layers.
- *Saturations* of toric arrangements, obtained by adding all connected components of intersections of layers.

The main result of [14] and [11] is that the *rational homotopy type* of the complement of a toric arrangement only depends on discrete data associated with the toric arrangement.

Projective *wonderful models* for toric arrangements are constructed in [9], building on the wonderful models of subspace arrangements [13] and conical arrangements [31]. These are defined as the closures of complements of toric arrangements in products of

- a nonsingular projective toric variety $X = X_\Sigma$, such that the subtori associated with the given layers satisfy property (E) with respect to Σ, and
- the blow-ups of X along the closures of the layers.

3.2. **Arrangements of diagonalizable groups.** The arrangements, relevant for our applications, are on the one hand less general than those in §3.1, in that layers that arise are translates of subtori by *torsion* elements of T. On the other hand our inductive scheme requires arrangements in more general *diagonalizable algebraic groups*, than just tori.

Let Δ be a diagonalizable algebraic group over k. The identity component $T \subset \Delta$ fits into an exact sequence

$$1 \to T \to \Delta \to \Delta/T \to 1, \tag{3.1}$$

which is (noncanonically) split. The subgroup T is an algebraic torus, and the quotient Δ/T is a finite abelian group.

A (right) action of a finite group G on Δ, by automorphisms, is determined uniquely by the data of a group homomorphism

$$\mu\colon G \to \mathrm{Aut}(M), \quad M := \mathfrak{X}^*(\Delta).$$

The induced action of G on T is determined by the induced homomorphism

$$\nu\colon G \to \mathrm{Aut}(M/M_{\mathrm{tors}}).$$

There is also an induced action of G on Δ/T. A k-point $\delta \in \Delta$ is given by a homomorphism $M \to k^\times$, that we also denote by δ, and the corresponding class $\bar\delta \in \Delta/T$ is given by the restriction of δ to M_{tors}; so,

$$\ker(\bar\delta) = \ker(\delta) \cap M_{\mathrm{tors}}.$$

There is an induced homomorphism

$$\nu_{\bar\delta}\colon G_{\bar\delta} \to \mathrm{Aut}(M/\ker(\bar\delta)),$$

where $G_{\bar\delta} = \mathrm{Stab}(\bar\delta) \subseteq G$ denotes the stabilizer of $\bar\delta \in \Delta/T$. We also remark that the G-action on Δ/T always fixes the identity; consequently, when Δ has nontrivial group of connected components, the action of G cannot be transitive on components of Δ.

That G need not act transitively on components, represents a departure from the convention in [25, §2]. For instance, then, different orbits of components can have different isomorphism types of generic stabilizer groups. Nevertheless, we might agree to call the G-action on Δ *generically free* when it satisfies the following equivalent conditions.

Lemma 3.1. *There exists a G-invariant dense open subscheme of Δ, on which G acts freely, if and only if G acts generically freely on the identity component T of Δ.*

Proof. A dense open subscheme of Δ, on which G acts freely, has nontrivial intersection with T and exhibits the G-action on T as generically free. For the reverse implication, we let r be a positive integer, such that the group of connected components of Δ is r-torsion. Then the rth power

endomorphism of Δ is G-equivariant and has image T. The pre-image of a nonempty invariant open subscheme of T, on which G acts freely, is an invariant dense open subscheme of Δ, on which G acts freely. $\square$

3.3. **Lattice structure.** In the following discussion we do not assume that G act generically freely on Δ.

We call a subgroup $\Gamma \subseteq G$ *distinguished* if Γ is the largest subgroup, acting trivially on some algebraic subgroup $\Theta \subset \Delta$. We call an algebraic subgroup $\Theta \subset \Delta$ *distinguished* if Θ is maximal, on which Γ acts trivially, for some subgroup $\Gamma \subseteq G$. The operations, associating to Θ the distinguished Γ, and to Γ, the distinguished Θ, restrict to inverse order-reversing bijections between distinguished subgroups of G and distinguished algebraic subgroups of T. The set of distinguished subgroups of G has a structure of lattice, with $\Gamma_1 \wedge \Gamma_2 = \Gamma_1 \cap \Gamma_2$. To describe $\Gamma_1 \vee \Gamma_2$ we let Θ_i denote the algebraic subgroup of Δ associated with Γ_i for $i = 1, 2$. Then $\Gamma_1 \vee \Gamma_2$ is the subgroup, containing both Γ_1 and Γ_2, with associated algebraic subgroup $\Theta_1 \cap \Theta_2$.

Let Γ be a distinguished subgroup, with associated algebraic subgroup $\Theta \subset \Delta$. Then Γ is the intersection of the generic stabilizer groups of the components of Θ. We might have Γ as generic stabilizer of some component of Θ, indeed this arises when we use the stabilizer of a point of Δ to define Θ, whereby we see: *every stabilizer group is distinguished*. As the next example shows, a distinguished subgroup is not necessarily the stabilizer group of any point of Δ.

Example 3.2. Consider $G = C_2 \times \mathfrak{S}_3 \subset \mathrm{GL}_2(\mathbb{Z})$. Its center C_2 is distinguished, associated with $(\mu_2)^2$. However, every point of $(\mu_2)^2$ has stabilizer of order 4 or 12.

We modify the discussion above, by considering *subtori* $T' \subset T$ and their associated distinguished subgroups $\Gamma' \subseteq G$. The maximal algebraic subgroup $\Theta' \subset \Delta$, on which Γ' acts trivially, contains T' and has the property, that every connected component has the same generic stabilizer. The above bijection restricts to one, between distinguished subgroups of $\Gamma' \subseteq G$ associated with subtori of T and distinguished algebraic subgroups of Δ whose components all have the same generic stabilizer; we write $T_{\Gamma'}$ for the identity component of the associated algebraic subgroup of Δ. There is a lattice structure, with $\Gamma'_1 \wedge \Gamma'_2 = \Gamma'_1 \cap \Gamma'_2$, but now $\Gamma'_1 \vee \Gamma'_2$ is the subgroup associated with the identity component of $T_{\Gamma'_1} \cap T_{\Gamma'_2}$.

We write

$$\mathcal{L}' = \mathcal{L}'(T)$$

for the lattice of distinguished subgroups of G, associated with subtori of T. The lattice $\mathcal{L}'$ has maximal element G, and minimal element $\ker(\nu)$. To each $\Gamma' \in \mathcal{L}'$, there is the subtorus $T_{\Gamma'} \subset T$, associated with a primitive sublattice $N_{\Gamma'} \subset N$; concretely, $N_{\Gamma'}$ is the sublattice of N, where Γ' acts trivially. We introduce

$$\mathcal{T}_{\mathcal{L}'} := \{T_{\Gamma'} \,|\, \Gamma' \in \mathcal{L}'\}, \qquad \mathcal{G}_{\mathcal{L}'} := \mathcal{T}_{\mathcal{L}'} \setminus \{T\}.$$

Lemma 3.3. *Every distinguished algebraic subgroup $\Theta \subset \Delta$ has connected component of the identity in $\mathcal{T}_{\mathcal{L}'}$.*

Proof. We let $\Gamma \subseteq G$ denote the distinguished subgroup associated with Θ, and T', the identity component of Θ. The generic stabilizer Γ' of T' contains Γ, and we have $\Gamma' \in \mathcal{L}'$. We have T' contained in $T_{\Gamma'}$, the identity component of the associated algebraic subgroup Θ'. Since $T_{\Gamma'} \subset \Theta' \subset \Theta$, we have $\dim T_{\Gamma'} \leq \dim T'$. It follows that $T' = T_{\Gamma'}$. □

Notice, by Lemma 3.3, the locus in Δ with nontrivial stabilizers is contained in a finite union of translates of tori $T_{\Gamma'}$ for $\Gamma' \in \mathcal{L}'$, where the translates are by torsion elements of Δ. (For any stabilizer group, by Lemma 3.3 the associated distinguished algebraic subgroup will be such a finite union.) As well, $T_{\Gamma'}$ is invariant under the action of $g \in G$ if and only if $g \in N_G(\Gamma')$. (The generic stabilizer gets conjugated by g.)

3.4. **Equivariant compactifications of diagonalizable groups.** Let Δ be a diagonalizable algebraic group over k, with an action of a finite group G. Since the character groups $\mathfrak{X}^*(T)$ and $\mathfrak{X}^*(\Delta)$ have a common dual N, a smooth projective fan Σ in $N_{\mathbb{R}}$ determines, besides the equivariant compactification

$$T \subset X = X_\Sigma,$$

also an equivariant compactification

$$\Delta \subset \mathbb{X} = \mathbb{X}_\Sigma.$$

The smooth projective scheme $\mathbb{X}$ has components indexed by Δ/T:

$$\mathbb{X} = \bigsqcup_{\bar{\delta} \in \Delta/T} X\bar{\delta}.$$

Let r be a positive integer, such that Δ/T is r-torsion. A primitive sublattice $N' \subseteq N$ now gives rise to:

- The subtorus $T' \subset T$, its r-torsion translate

$$T'_{[r]} := T'\Delta[r] \subset \Delta,$$

and the induced surjective homomorphism

$$\vartheta_{[r]} \colon T'_{[r]}/T' \to \Delta/T.$$

- The corresponding equivariant compactification
$$X'_{[r]} := X'\Delta[r] \subset \mathbb{X}_\Sigma,$$
with components indexed by $T'_{[r]}/T'$.
- The quotient diagonalizable algebraic group
$$\Delta/T',$$
which as in Lemma 2.1 is the target of a smooth Δ-equivariant morphism from a Δ-invariant neighborhood of $X'_{[r]}$ in $\mathbb{X}_\Sigma$, with $X'_{[r]}$ as fiber over $(\Delta/T')[r]$.
- In case $N' = N_\sigma$, notation Δ^σ for the quotient diagonalizable algebraic group, with Δ^σ identified with a corresponding Δ-orbit in $\mathbb{X}_\Sigma$ as in §2.3, and closure of Δ^σ in $\mathbb{X}_\Sigma$ identified with $\mathbb{X}_{\Sigma^\sigma}$.

3.5. **Constructing the model.** Let $\mathcal{L}' = \mathcal{L}'(T)$ be the lattice of distinguished subgroups of G, associated with subtori of T, as above. As we have seen in Section 2, there exists a smooth projective fan Σ that meets the following criteria:

- Property (E) holds for all subtori in $\mathcal{T}_{\mathcal{L}'}$, with respect to Σ.
- Σ is G-invariant.
- No pair of rays of Σ, in a single G-orbit, spans a cone of Σ.

For $\sigma \in \Sigma$, the stabilizer $\mathrm{Stab}(\sigma)$ acts on the toric variety $D_\sigma \cong X_{\Sigma^\sigma}$, an equivariant compactification of the quotient torus T^σ.

Lemma 3.4. *We have* $\mathrm{Stab}(\sigma) \in \mathcal{L}'$. *The lattice* $\mathcal{L}'(T^\sigma)$, *associated with the* $\mathrm{Stab}(\sigma)$*-action on* T^σ, *is equal to the sublattice of* $\mathcal{L}'$, *of elements bounded above by* $\mathrm{Stab}(\sigma)$. *For* $\Gamma' \in \mathcal{L}'(T^\sigma)$, *the pre-image of* $(T^\sigma)_{\Gamma'}$ *under the projection morphism*
$$\mathrm{pr}^\sigma : T \to T^\sigma$$
is equal to $T_{\Gamma'}$.

Proof. By the third criterion on Σ, the action of $\mathrm{Stab}(\sigma)$ on N_σ is trivial. Consequently, if $\Gamma' \in \mathcal{L}'(T^\sigma)$, so Γ' is the generic stabilizer of $(T^\sigma)_{\Gamma'}$, then Γ' acts trivially on $(\mathrm{pr}^\sigma)^{-1}((T^\sigma)_{\Gamma'})$. Using Proposition 2.2, we see that Γ' is the generic stabilizer of $(\mathrm{pr}^\sigma)^{-1}((T^\sigma)_{\Gamma'})$, and $T_{\Gamma'} = (\mathrm{pr}^\sigma)^{-1}((T^\sigma)_{\Gamma'})$. In the other direction, if $\Gamma' \in \mathcal{L}'$, $\Gamma' \subseteq \mathrm{Stab}(\sigma)$, then $T_{\Gamma'}$ is the pre-image under pr^σ of a subtorus of T^σ, whose generic stabilizer is Γ'. □

Each Δ-orbit of $\mathbb{X}_\Sigma$ gives an instance of §3.2, with $\mathrm{Stab}(\sigma)$ acting on Δ^σ, and the locus with nontrivial stabilizer contained in some translates of the subtori in $\mathcal{T}_{\mathcal{L}'(T^\sigma)}$. As noted above, the following are valid:

- For a suitable positive integer r, the translates of the subtori in $\mathcal{T}_{\mathcal{L}'(T^\sigma)}$ are by r-torsion elements of Δ^σ.

- The subtori in $\mathcal{T}_{\mathcal{L}'(T^\sigma)}$ have pre-images in T, belonging to $\mathcal{T}_{\mathcal{L}'}$.
- Property (E) holds for the subtori in $\mathcal{T}_{\mathcal{L}'(T^\sigma)}$, with respect to Σ^σ.

Since Σ has finitely many cones, a single positive integer r may be chosen, so that translation is by r-torsion elements of Δ^σ, in the first item above, for all $\sigma \in \Sigma$. We suppose, as well, that Δ/T is r-torsion.

Consequently, the intersection of any pair of subtori in $\mathcal{T}_{\mathcal{L}'}$ is a diagonalizable algebraic group, whose group of connected components is r-torsion. Indeed, if Γ', $\Gamma'' \in \mathcal{L}'$, with respective associated subtori $T' := T_{\Gamma'}$ and $T'' := T_{\Gamma''}$, then $T''' := T_{\Gamma' \vee \Gamma''}$ is the identity component of $T' \cap T''$. Any non-identity component of $T' \cap T''$ has generic stabilizer contained in $\Gamma' \vee \Gamma''$ and associated distinguished algebraic subgroup $\Theta \subset T$, that satisfies

$$T''' \subset \Theta \subset T'''\Delta[r].$$

So $(T' \cap T'')/T'''$ is r-torsion.

Inside $\mathbb{X} = \mathbb{X}_\Sigma$ there is the union of r-torsion translates of the closures of the subtori in $\mathcal{G}_{\mathcal{L}'}$. The complement, the "open part"

$$\mathbb{X}^\circ \subset \mathbb{X},$$

has the stabilizer distinguishing property, for the G-action with respect to the toric boundary.

The projective model

$$\mathbb{X}_{\Sigma,\mathcal{L}',[r]}$$

is obtained as in [9], by applying the De Concini-Procesi iterated blowup procedure, as developed in [30], to the r-torsion translates of the closures in X_Σ of the subtori in $\mathcal{G}_{\mathcal{L}'}$. So, $\mathbb{X}_{\Sigma,\mathcal{L}',[r]}$ is the closure of $\mathbb{X}^\circ$, in the product of $\mathbb{X}$ with all blow-ups

$$B\ell_{X'_{[r]}}\mathbb{X}$$

for $T' \in \mathcal{G}_{\mathcal{L}'}$, where $X'_{[r]} \subset \mathbb{X}$ denotes the corresponding r-torsion translate compactification (§3.4). There is a projection morphism

$$\pi \colon \mathbb{X}_{\Sigma,\mathcal{L}',[r]} \to \mathbb{X}, \tag{3.2}$$

which is an isomorphism over $\mathbb{X}^\circ$. As is the case for $\mathbb{X}$, the projective model has connected components indexed by Δ/T:

$$\mathbb{X}_{\Sigma,\mathcal{L}',[r]} = \bigsqcup_{\bar\delta \in \Delta/T} X_{\Sigma,\mathcal{L}',[r]}\bar\delta.$$

In particular, when Δ is a torus already, we have $\mathbb{X}_{\Sigma,\mathcal{L}',[r]} = X_{\Sigma,\mathcal{L}',[r]}$.

The complement of $\mathbb{X}^\circ$ in $\mathbb{X}_{\Sigma,\mathcal{L}',[r]}$ is a normal crossing divisor

$$\mathbb{D} = \bigcup_{\ker(\nu) \neq \Gamma' \in \mathcal{L}'} \mathbb{D}_{\Gamma'}.$$

For $\Lambda \subset \mathcal{L}' \setminus \{\ker(\nu)\}$, there is the stratum

$$\mathbb{D}_\Lambda := \bigcap_{\Gamma' \in \Lambda} \mathbb{D}_{\Gamma'},$$

with $\mathbb{D}_\emptyset := \mathbb{X}_{\Sigma,\mathcal{L}',[r]}$ by convention. We have $\mathbb{D}_\Lambda \neq \emptyset$ if and only if Λ is a chain in $\mathcal{L}'$, i.e., letting t denote the cardinality of Λ we have

$$\Lambda = \{\Gamma^1, \dots, \Gamma^t\} \subset \mathcal{L}' \setminus \{\ker(\nu)\}, \qquad \Gamma^1 \supset \cdots \supset \Gamma^t. \tag{3.3}$$

Suppose Λ is a nonempty chain. Then with

$$T' := T_{\Gamma^1}, \tag{3.4}$$

the connected components of $\mathbb{D}_\Lambda$ are indexed by $T'_{[r]}/T'$:

$$\mathbb{D}_\Lambda = \bigsqcup_{\bar\tau \in T'_{[r]}/T'} D_\Lambda \bar\tau. \tag{3.5}$$

Example 3.5. Let $G = \mathbb{Z}/2\mathbb{Z}$ act on $T = \mathbb{G}_m^2$, swapping the two factors. With Σ the complete fan in $N_\mathbb{R}$, for $N = \mathbb{Z}^2 = \mathbb{Z}\langle e_1, e_2\rangle$, with rays generated by e_1, $e_1 + e_2$, e_2, $-e_1 - e_2$, so X_Σ is the blow-up of $\mathbb{P}^2$ at a point, the origin in $\mathbb{A}^2 \subset \mathbb{P}^2$. We have $\mathcal{L}' = \{\mathrm{triv}, G\}$, with respective subtori $\mathbb{G}_m^2$ and $\Delta_{\mathbb{G}_m}$ (the diagonal). The stabilizer locus in T is precisely $\Delta_{\mathbb{G}_m}$. However, we need to take r divisible by 2, since $\sigma = \mathbb{R}_{\geq 0} \cdot (e_1 + e_2)$ (and also $\sigma = \mathbb{R}_{\geq 0} \cdot (-e_1 - e_2)$) leads to $\mathrm{Stab}(\sigma) = G$ acting nontrivially on a one-dimensional torus; this fixes the 2-torsion subgroup. We only blow up divisors, so

$$X_{\Sigma,\mathcal{L}',[2]} \cong X_\Sigma,$$

and X° is the complement of the proper transforms of two lines in $\mathbb{P}^2$, intersecting at the origin in $\mathbb{A}^2 \subset \mathbb{P}^2$, with slopes ± 1.

3.6. **Properties of the model.**

Proposition 3.6. *Suppose that G acts generically freely on Δ. Then, for Σ and r as above, the projective variety*

$$\mathbb{X}_{\Sigma,\mathcal{L}',[r]}$$

is in standard form with respect to the union of the strict transform of the toric boundary and the exceptional divisors of the De Concini-Procesi iterated blowup procedure.

To be in *standard form* with respect to a simple normal crossing divisor, means that the divisor has smooth G-orbits of components and that G acts freely on the complement [37].

Proof. The exceptional divisors of the De Concini-Procesi blowup form a simple normal crossing divisor. The centers of blowup $X'_{[r]}$ have transverse intersection with the toric boundary by Lemma 2.1. By Proposition 2.2, the exceptional divisors, together with the proper transform of the toric boundary, form a simple normal crossing divisor. We conclude by Lemma 3.3. □

By analogy with the treatment in [17], we describe a point $p \in \mathbb{X}_{\Sigma,\mathcal{L}',[r]}$ as a pair

$$p = (x, V_1 \subset W_1 \subset \cdots \subset V_t \subset W_t)$$

consisting of a point $x \in \mathbb{X}$, say in the Δ-orbit identified with Δ^σ (§3.4), and a flag of subspaces of $N_k := N \otimes k$ such that

- $V_i = N_{\Gamma^i} \otimes k$ for some $\Gamma^i \in \mathcal{L}'$, for all i.
- We have $\dim(W_i) = \dim(V_i) + 1$ for all i.
- The maximal $\Gamma' \in \mathcal{L}'(T^\sigma)$ with $x \in (T^\sigma)_{\Gamma'}\Delta^\sigma[r]$ is Γ^1, when $t \geq 1$, otherwise is $\ker(\nu)$.
- The maximal $\Gamma' \in \mathcal{L}'$ with $W_i \subset N_{\Gamma'} \otimes k$, is Γ^{i+1}, for $i < t$.
- If $t \geq 1$, the maximal $\Gamma' \in \mathcal{L}'$ with $W_t \subset N_{\Gamma'} \otimes k$, is $\ker(\nu)$.

The point p lies in the stratum $\mathbb{D}_\Lambda$ indexed by the chain

$$\Gamma^1 \supset \cdots \supset \Gamma^t,$$

and not in any deeper stratum. The stabilizer of p is

$$\{g \in G \,|\, x \cdot g = x, \text{ and } V_i \cdot g = V_i \text{ and } W_i \cdot g = W_i \text{ for all } i\}.$$

We take T' as in (3.4), with corresponding sublattice $N' := N_{\Gamma^1}$. So, the tangent space to T/T' at the identity is naturally identified with $(N/N')_k$. The normalizer $N_G(\Gamma^1)$ acts on the quotient Δ/T', and the smooth morphism from a Δ-invariant neighborhood of $X'_{[r]}$ to Δ/T' with fiber $X'_{[r]}$ over $(\Delta/T')[r]$, mentioned in §3.4, is $N_G(\Gamma^1)$-equivariant. Denoting such a Δ-invariant neighborhood by Q, we have the composite

$$Q \to \Delta/T' \xrightarrow{r} \Delta/T', \tag{3.6}$$

with image T/T', and pre-image $X'_{[r]}$ of the identity element.

Lemma 3.7. *Let the notation be as above.*

(i) *The projection morphism π, of introduced in* (3.2)*, factors through $B\ell_{X'_{[r]}}\mathbb{X}$.*

(ii) *There exist an $N_G(\Gamma^1)$-invariant neighborhood $W \subset T/T'$ of the identity and an étale $N_G(\Gamma^1)$-equivariant morphism from W to the vector space $(N/N')_k$, with fiber over* 0 *consisting just of the identity element of T/T', where the corresponding map of tangent spaces gives the natural identification with $(N/N')_k$.*

(iii) *If we let U denote the pre-image of W under the composite morphism* (3.6)*, then by following the composite* (3.6) *with the morphism of* (ii)*, we get a smooth $N_G(\Gamma^1)$-equivariant morphism*

$$U \to (N/N')_k,$$

with pre-image of 0 *equal to $X'_{[r]}$.*

(iv) *For the induced equivariant morphism $B\ell_{X'_{[r]}}U \to \mathbb{P}((N/N')_k)$, the class of the exceptional divisor in $\mathrm{Pic}^{N_G(\Gamma^1)}(B\ell_{X'_{[r]}}U)$ is the pullback of $\mathcal{O}_{\mathbb{P}((N/N')_k)}(-1)$. The pullback of $\mathcal{O}_{\mathbb{P}((N/N')_k)}(-1)$ to $\pi^{-1}(U)$ is the class of the divisor*

$$\bigcup_{\substack{\Gamma'\in\mathcal{L}'\\ \Gamma^1\subseteq\Gamma'}} \mathbb{D}_{\Gamma'},$$

with all components of multiplicity 1*.*

Proof. With $\mathbb{X}_{\Sigma,\mathcal{L}',[r]}$ as the closure of $\mathbb{X}^\circ$ in a product of blow-ups, projection to $B\ell_{X'_{[r]}}\mathbb{X}$ yields the factorization in (i).

We get (ii) from an equivariant version of a construction of Moci [33]. The coordinate ring $k[(N'^{\perp}\cap M)/M_{\mathrm{tors}}]$ of T/T' has a maximal ideal $\mathfrak{m}$, corresponding to the identity element of T/T'. A splitting of the surjective $N_G(\Gamma^1)$-equivariant homomorphism

$$\mathfrak{m} \to \mathfrak{m}/\mathfrak{m}^2$$

determines $N_G(\Gamma^1)$-equivariant $T/T' \to (N/N')_k$, sending the identity to 0. The corresponding map of tangent spaces gives the natural identification of the tangent space to T/T' at the identity with $(N/N')_k$. Then for suitable W we have (ii); an immediate consequence is (iii).

Since blowing up commutes with smooth base change, we get the first assertion in (iv) from the standard description of $B\ell_{\{0\}}(N/N')_k$ as the closure of the complement of 0 in $(N/N')_k\times\mathbb{P}((N/N')_k)$. For the remaining assertion, we use the treatment of iterated blow-ups in [30], which lets us express π as an iterated blow-up of $\mathbb{X}$ in the following manner. The first step is the blow-up

$$\pi_1\colon B\ell_{X'_{[r]}}\mathbb{X} \to \mathbb{X}.$$

In subsequent steps we blow up the (proper transforms of the) pre-images under π_1 of r-torsion translate compactifications $X''_{[r]} \subsetneq X'_{[r]}$ in any order of weakly increasing dimensions. Finally we blow up the proper transforms of the remaining r-torsion translate compactifications, in any order of weakly increasing dimensions. Examination of the behavior of the exceptional divisor of π_1 under these blow-ups gives what we need. □

4. Equivariant Burnside group

In this section we recall the equivariant Burnside group, introduced in [24], and state the formula, that we will use for our computation in the equivariant Burnside group.

The *equivariant Burnside group*

$$\mathrm{Burn}_n(G) = \mathrm{Burn}_{n,k}(G)$$

is an abelian group, generated by symbols

$$(H, Y \circlearrowright K, \beta),$$

where

- $H \subseteq G$ is an abelian subgroup,
- K is a field, finitely generated over k, with faithful action over k of a subgroup $Y \subseteq Z := Z_G(H)/H$, where $Z_G(H)$ denotes the centralizer of H in G,
- β is a sequence of length $r := n - \mathrm{trdeg}_{K/k}$ of nontrivial characters of H, that generates $H^\vee$.

The symbols are subject to relations, labeled

$$(\mathbf{O}), (\mathbf{C}), (\mathbf{B1}), \text{ and } (\mathbf{B2})$$

(which stand for ordering, conjugation, and blowup relations), e.g., the equivalence of symbols that differ by a re-ordering of the sequence of characters. Symbols are also permitted in which K is a Galois algebra for some $Y \subseteq Z$ over a field that is finitely generated over k; we identify $(H, Y \circlearrowright K, \beta)$ with $(H, Z \circlearrowright \mathrm{Ind}_Y^Z(K), \beta)$. See [25] for a complete description of relations.

In a symbol, β is determined uniquely, up to order, by the similarity type of a faithful $(n - \mathrm{trdeg}_{K/k})$-dimensional representation of H over k, or any field containing k. Furthermore, we declare a symbol to be trivial in case the trivial character occurs, i,e., if the given representation has nontrivial space of invariants.

We record a frequently used consequence of defining relations [24, Prop. 4.7]: If $\beta = (b_1, \dots, b_r)$ is such that for some

$$I \subseteq [1, \dots, r], \quad |I| \geq 2,$$

one has

$$\sum_{i \in I} b_i = 0,$$

then

$$(H, Y \circlearrowright K, \beta) = 0 \in \mathrm{Burn}_n(G).$$

As an immediate application, we obtain:

Proposition 4.1. *Consider the symbol*

$$(H, Y \circlearrowright K, \beta) \in \mathrm{Burn}_n(G), \quad \beta = (b_1, \dots, b_r), \quad 1 \le r \le n.$$

Let

$$\ell := \min_{1 \le j \le r} (\mathrm{ord}(b_j))$$

be the smallest order of a character appearing in β. We have

$$(H, Y \circlearrowright K(t_1, \dots, t_{\ell-1}), \beta) = 0 \in \mathrm{Burn}_{n+\ell-1}(G).$$

Here Y acts trivially on the variables $t_1, \dots t_{\ell-1}$.

Proof. We may assume that the minimum is at b_1. Consider the symbol

$$(H, Y \circlearrowright K, (\underbrace{b_1, \dots, b_1}_{\ell \text{ times}}, b_2, \dots, b_r)) \in \mathrm{Burn}_{n+\ell-1}(G).$$

This symbol vanishes. Applying relation **(B2)** iteratively, we obtain the claim. □

A nonsingular projective variety X with generically free G-action determines, as G-equivariant birational invariant, a class

$$[X \circlearrowleft G] \in \mathrm{Burn}_n(G).$$

This is particularly easy to describe, when X is in standard form with respect to a G-invariant simple normal crossing divisor, or more generally satisfies Assumption 2 of [24]. Then

$$[X \circlearrowleft G] = \sum_{\substack{x_0 \in X/G \\ x_0 = [x],\, x \in X}} \big(\text{generic stabilizer of } \overline{\{x\}}, \mathrm{Gal}(k(x)/k(x_0)), (\mathcal{I}_{\overline{\{x\}}}/\mathcal{I}^2_{\overline{\{x\}}})_x\big).$$

Points of the quotient X/G are in bijective correspondence with G-orbits of points of X; the sum is over $x_0 \in X/G$, and x, in the sum, denotes an orbit representative. Importantly, not only k-points, but all points are taken in the sum. The residue field $k(x)$ is a Galois extension of $k(x_0)$. The ideal sheaf $\mathcal{I}_{\overline{\{x\}}}$ of the closure $\overline{\{x\}}$ defines the coherent sheaf $\mathcal{I}_{\overline{\{x\}}}/\mathcal{I}^2_{\overline{\{x\}}}$ on $\overline{\{x\}}$, whose stalk at x gives a faithful representation of the generic stabilizer of $\overline{\{x\}}$. In order to get a representation with trivial space of invariants, x has to be a maximal point with this generic stabilizer, so only finitely many points $x_0 \in X/G$ yield nontrivial symbols.

Remark 4.2. An immediate corollary of Proposition 4.1 is that the symbols invariant is stably trivial: For any G-variety X of dimension n, there exists a $d \in \mathbb{N}$ such that class

$$[X \times \mathbb{P}^d \circlearrowleft G] - (\mathrm{triv}, G \circlearrowright k(X)(t_1, \dots, t_d), ()) = 0 \in \mathrm{Burn}_{n+d}(G);$$

here G-acts trivially $\mathbb{P}^d$, respectively, on the variables $t_1, \dots, t_d$.

This is in stark contrast with invariants originating in unramified cohomology.

When X is replaced by a G-invariant open subvariety U, the same formula leads to a class in $\mathrm{Burn}_n(G)$, that is denoted by

$$[U \circlearrowleft G]^{\mathrm{naive}}.$$

(A class $[U \circlearrowleft G]$ is also defined in [24], but plays no role in this paper.)

Let

$$D = D_1 \cup \cdots \cup D_\ell$$

be a simple normal crossing divisor on X, where each D_i is G-invariant and nonsingular. Then, with $\mathcal{I} := \{1, \dots, \ell\}$, we have

$$[X \circlearrowleft G] = [U \circlearrowleft G]^{\mathrm{naive}} + \sum_{\emptyset \neq I \subseteq \mathcal{I}} \sum_{j \in \mathcal{J}_I} \mathrm{ind}_{G_{I,j}}^{G} \big(\psi_I([\underline{D}^\circ_{I,j} \circlearrowleft G_{I,j}]^{\mathrm{naive}}_{(\mathcal{N}_{D_i/X})_{i\in I}})\big).$$

Here,

- $\mathcal{J}_I$ indexes G-orbits of components of $D_I := \bigcap_{i\in I} D_i$, with notation $D_{I,j}$ for the G-orbit of components corresponding to $j \in \mathcal{J}_I$, and $\underline{D}_{I,j}$ for a chosen component of $D_{I,j}$.
- $G_{I,j}$ is the maximal subgroup of G, for which $\underline{D}_{I,j}$ is invariant.
- We denote by $D^\circ_I \subset D_I$, by $D^\circ_{I,j} \subset D_{I,j}$, and by $\underline{D}^\circ_{I,j} \subset \underline{D}_{I,j}$, the complement of all D_j, $j \notin I$.
- The class

  $$[\underline{D}^\circ_{I,j} \circlearrowleft G_{I,j}]^{\mathrm{naive}}_{(\mathcal{N}_{D_i/X})_{i\in I}} \in \mathrm{Burn}_{n,I}(G_{I,j}),$$

  in the *indexed equivariant Burnside group* [26, §4] [25, §4] is defined like a naive class in $\mathrm{Burn}_n(G_{I,j})$, but with additional data of characters associated with each of the indicated line bundles [25, §5] (in this case, the restrictions of the normal bundles $\mathcal{N}_{D_i/X}$).
- The homomorphism

  $$\psi_I \colon \mathrm{Burn}_{n,I}(G_{I,j}) \to \mathrm{Burn}_n(G_{I,j})$$

  appends the characters associated with the line bundles to the characters β [26, Rem. 4.1].
- The induction homomorphism

  $$\mathrm{ind}_{G_{I,j}}^{G} : \mathrm{Burn}_n(G_{I,j}) \to \mathrm{Burn}_n(G)$$

  is defined in [25, Defn. 3.1].

This formula holds by [25, Prop. 4.8]; see, also [25, Exa. 5.12].

The additional characters, attached to symbols in the indexed equivariant Burnside group, may be manipulated, e.g., by applying any element

of $\mathrm{Aut}(\mathbb{Z}^I)$, where $\mathbb{Z}^I$ denotes $\bigoplus_{i\in I}\mathbb{Z}$. For instance, when $J \subseteq I$ and we apply the element

$$\tau_{I,J} \in \mathrm{Aut}(\mathbb{Z}^I), \qquad \tau_{I,J}(e_j) := \begin{cases} \sum_{\substack{i\in I,\, i\le j \\ i>j' \,\forall\, j'\in J,\, j'<j}} e_i, & \text{if } j \in J, \\ e_j, & \text{if } j \notin J, \end{cases}$$

of [25, Exa. 4.1], to a symbol $(H \subseteq H', Y \circlearrowright K, \beta, \gamma) \in \mathrm{Burn}_{n,I}(G)$ with the additional characters

$$\gamma = (c_i)_{i\in I},$$

we get the symbol $\tau_{I,J}(H \subseteq H', Y \circlearrowright K, \beta, \gamma) := (H \subseteq H', Y \circlearrowright K, \beta, \tilde{\gamma})$,

$$\tilde{\gamma} = (\tilde{c}_i)_{i\in I},$$

where $\tilde{c}_j$ is the sum of c_i over $i \le j$ with $i > j'$ for all $j' \in J$, $j' < j$ for $j \in J$, and $\tilde{c}_j = c_j$ otherwise.

As a generalization of the map ψ_I there is the map

$$\psi_{I,J}\colon \mathrm{Burn}_{n,I}(G) \to \mathrm{Burn}_{n,J}(G),$$

which appends just the characters indexed by elements of $I \setminus J$ to the characters β; see [25, Defn. 4.2].

5. Computing the class in the Burnside group

We wish to compute a class in an equivariant Burnside group, associated with a given G-action on Δ. We have the exact sequence of algebraic groups (3.1) and the notation introduced in §3.2. We fix $\bar{\delta} \in \Delta/T$, the class of some $\delta \in \Delta$. Besides the stabilizer $G_{\bar{\delta}}$ of $\bar{\delta}$, there is

$$G_\delta := \ker(\nu_{\bar{\delta}}),$$

which is equal to the stabilizer of $\delta \in \Delta$, provided that δ is a suitably general lift of $\bar{\delta} \in \Delta/T$. The group G_δ is the generic stabilizer of the induced action of $G_{\bar{\delta}}$ on the component $T\bar{\delta}$ of Δ.

We recall, $\bar{\delta} \in \Delta/T$ indexes a component $X\bar{\delta}$ of $\mathbb{X}$. This has a class

$$[X\bar{\delta} \circlearrowleft G_{\bar{\delta}}/G_\delta] \in \mathrm{Burn}_n(G_{\bar{\delta}}/G_\delta). \tag{5.1}$$

Our goal is to compute the class of $\mathbb{X} \circlearrowleft G$, which we understand to mean the collection of classes (5.1) for all $\bar{\delta} \in \Delta/T$.

We recall the lattice $\mathcal{L}' = \mathcal{L}'(T)$ from §3.3 and make a choice of fan Σ and positive integer r as in §3.5; we work on the model $\mathbb{X}_{\Sigma,\mathcal{L}',[r]}$. Exactly as in Proposition 3.6, for the action of $G_{\bar{\delta}}/G_\delta$ we have $X_{\Sigma,\mathcal{L}',[r]}\bar{\delta}$ in standard form, with respect to the union of the strict transform of the toric boundary and the exceptional divisors of the De Concini-Procesi iterated blowup procedure.

We introduce $G_{\bar\delta}$-invariant divisors. Let

$$\Gamma'_1, \dots, \Gamma'_{\ell_{\bar\delta}} \in \mathcal{L}'$$

be a choice of conjugacy class representatives of elements of $\mathcal{L}' \setminus \{\ker(\nu)\}$. We set

$$\mathbb{D}_i := \bigcup_{\Gamma' \text{ conjugate to } \Gamma'_i} \mathbb{D}_{\Gamma'}.$$

Then $\mathbb{D} = \mathbb{D}_1 \cup \dots \cup \mathbb{D}_{\ell_{\bar\delta}}$ is a simple normal crossing divisor on $\mathbb{X}_{\Sigma,\mathcal{L}',[r]}$, with each $\mathbb{D}_i$ invariant under $G_{\bar\delta}$. Now suppose

$$I \subseteq \mathcal{I}_{\bar\delta} := \{1, \dots, \ell_{\bar\delta}\},$$

with $\mathbb{D}_I$ nonempty. We take $\mathcal{J}_I$ to be the set of conjugacy classes of chains in $\mathcal{L}'$, with one element from each conjugacy class, indexed by an element of I. Then $j \in \mathcal{J}_I$ indexes an orbit $\mathbb{D}_{I,j}$ of $\mathbb{D}_\Lambda$ for a representative chain Λ of the conjugacy class of j. The maximal subgroup, under which $\mathbb{D}_\Lambda$ is invariant, is

$$N_{G_{\bar\delta}}(\Lambda) := N_{G_{\bar\delta}}(\Gamma^1) \cap \dots \cap N_{G_{\bar\delta}}(\Gamma^t),$$

the stabilizer of Λ under the conjugation action of $G_{\bar\delta}$.

With the subspaces $V_i = N_{\Gamma^i} \otimes k$ of

$$V := N \otimes k$$

for $1 \le i \le t$, appearing in the description of a point of $\mathbb{D}_\Lambda$, not in any deeper stratum, we have from Lemma 3.7, for $1 \le i \le k$, a $N_{G_{\bar\delta}}(\Lambda)$-equivariant morphism $\mathbb{D}_\Lambda \to \mathbb{P}(V/V_i)$. This is surjective when $i = t$ and has image $\mathbb{P}(V_{i+1}/V_i)$ for $i < t$.

For $\bar\delta \in \Delta/T$ the stabilizer $N_{G_{\bar\delta}}(\Lambda)$ acts on the fiber $\vartheta_{[r]}^{-1}(\bar\delta)$ over $\bar\delta$ in $T'_{[r]}/T'$, where T' is as in (3.4). We let $\mathcal{K}_j$ denote the set of $N_{G_{\bar\delta}}(\Lambda)$-orbits for this action.

Proposition 5.1. *Let G act on Δ, and let $\bar\delta \in \Delta/T$. Let us write*

$$[X\bar\delta \circlearrowleft G_{\bar\delta}/G_\delta] = A_{\bar\delta} + B_{\bar\delta}$$

in

$$\mathrm{Burn}_n(G_{\bar\delta}/G_\delta),$$

where $A_{\bar\delta}$ records the contribution from $\mathbb{X}^\circ \cap X\bar\delta$, and $B_{\bar\delta}$, the contribution from strata obtained from exceptional divisors in the De Concini-Procesi model for X. Then

$$A_{\bar\delta} = \sum_{[\sigma] \in \Sigma/G_{\bar\delta}} (\mathrm{Stab}(\sigma)_\delta, \mathrm{Stab}(\sigma)/\mathrm{Stab}(\sigma)_\delta \circlearrowright k(T^\sigma\bar\delta), \rho_\sigma),$$

where

- *the sum is over $G_{\bar\delta}$-orbits of Σ; for each orbit an orbit representative $\sigma \in \Sigma$ is chosen,*
- *the action of* $\mathrm{Stab}(\sigma)$ *on* $\mathbb{X}^\circ \cap T^\sigma \bar\delta$ *has constant stabilizer* $\mathrm{Stab}(\sigma)_\delta$,
- *the representation*

$$\rho_\sigma \colon \mathrm{Stab}(\sigma)_\delta \to \mathrm{GL}(\mathfrak{J}/\mathfrak{J}^2)$$

is defined, using the ideal

$$\mathfrak{J} := (x)_{x \in (\sigma^\vee \cap M)\setminus(\sigma^\perp \cap M)}(k[\sigma^\vee \cap M]/(x - \delta(x))_{x \in \sigma^\perp \cap M} k[\sigma^\vee \cap M]),$$

and

$$B_{\bar\delta} = \sum_{\emptyset \neq I \subseteq \mathcal{I}_{\bar\delta}} \sum_{[\Lambda] \in \mathcal{J}_I} \sum_{[\bar\tau] \in \mathcal{K}_j} \mathrm{ind}_{N_{G_{\bar\delta}}(\Lambda)_{\bar\tau}/G_\delta}^{G_{\bar\delta}/G_\delta} \big(\psi_{\{1,\dots,t\}}\big([D_\Lambda^\circ \bar\tau \circlearrowleft N_{G_{\bar\delta}}(\Lambda)_{\bar\tau}/G_\delta]^{\mathrm{naive}}_{(\mathcal{O}(-1))}\big)\big).$$

By analogy with [25, Conv. 8.1], here $(\mathcal{O}(-1))$ denotes the following collection of line bundles, indexed by $\{1, \dots, t\}$:

$$\mathcal{O}_{\mathbb{P}(V_2/V_1)}(-1), \mathcal{O}_{\mathbb{P}(V_2/V_1)}(1) \otimes \mathcal{O}_{\mathbb{P}(V_3/V_2)}(-1), \dots. \tag{5.2}$$

Proof. The contribution from $\mathbb{X}^\circ \cap X\bar\delta$ is obtained directly from the formula for the naive class in $\mathrm{Burn}_n(G_{\bar\delta}/G_\delta)$, from Section 4. The term $B_{\bar\delta}$ is taken from the formula in Section 4, where the additional sum over orbit representatives of $\mathcal{K}_j$ accounts for the components in (3.5). □

Let us fix a chain (3.3), which indexes a stratum $\mathbb{D}_\Lambda \subset \mathbb{X}_{\Sigma,\mathcal{L}',[r]}$. Let $\bar\delta \in \Delta/T$, and with T' as in (3.4), let $\bar\tau \in \vartheta_{[r]}^{-1}(\bar\delta)$. The group $N_{G_{\bar\delta}}(\Lambda)_{\bar\tau}$ acts on

$$X'\bar\tau, \quad \mathbb{P}(V_2/V_1), \quad \dots, \quad \mathbb{P}(V_t/V_{t-1}), \quad \mathbb{P}(V/V_t), \tag{5.3}$$

and we have an $N_{G_{\bar\delta}}(\Lambda)_{\bar\tau}$-equivariant birational morphism from $D_\Lambda\bar\tau$ to the product of the varieties in (5.3). Therefore:

Lemma 5.2. *For $\bar\tau \in T'_{[r]}/T'$, mapping to $\bar\delta \in \Delta/T$, we have*

$$[D_\Lambda\bar\tau \circlearrowleft N_{G_{\bar\delta}}(\Lambda)_{\bar\tau}/G_\delta]_{(\mathcal{O}(-1))} = [X'\bar\tau \times \mathbb{P}(V_2/V_1) \times \cdots \times \mathbb{P}(V/V_t) \circlearrowleft N_{G_{\bar\delta}}(\Lambda)_{\bar\tau}/G_\delta]_{(\mathcal{O}(-1))}.$$

in $\mathrm{Burn}_{n,\{1,\dots,t\}}(N_{G_{\bar\delta}}(\Lambda)_{\bar\tau}/G_\delta)$.

For $\Lambda \neq \emptyset$, the expression on the right-hand side in Lemma 5.2 may be taken as known, in the recursive determination of the classes (5.1).

Theorem 5.3. *The class*

$$[X\bar\delta \circlearrowleft G_{\bar\delta}/G_\delta] = A_{\bar\delta} + B_{\bar\delta}$$

in $\mathrm{Burn}_n(G_{\bar\delta}/G_\delta)$ *may be computed by applying the formula for* $A_{\bar\delta}$ *from Proposition 5.1 directly, and by computing the classes*

$$[D^\circ_\Lambda \bar\tau \circlearrowleft N_{G_{\bar\delta}}(\Lambda)_{\bar\tau}/G_\delta]^{\mathrm{naive}}_{(\mathcal{O}(-1))} \in \mathrm{Burn}_{n,\{1,\dots,t\}}(N_{G_{\bar\delta}}(\Lambda)_{\bar\tau}/G_\delta)$$

appearing in the formula for $B_{\bar\delta}$ *from Proposition 5.1 in a recursive fashion, by proceeding from large* $t = |I|$ *to small, and applying the formula*

$$\begin{aligned}[D^\circ_\Lambda \bar\tau \circlearrowleft N_{G_{\bar\delta}}(\Lambda)_{\bar\tau}/G_\delta]^{\mathrm{naive}}_{(\mathcal{O}(-1))} &= [D_\Lambda \bar\tau \circlearrowleft N_{G_{\bar\delta}}(\Lambda)_{\bar\tau}/G_\delta]_{(\mathcal{O}(-1))} \\ &- \sum_{[\Lambda']} \mathrm{ind}^{N_{G_{\bar\delta}}(\Lambda)_{\bar\tau}/G_\delta}_{N_{G_{\bar\delta}}(\Lambda')_{\bar\tau}/G_\delta}\big(\psi_{I',J}\big(\tau_{I',J}[D^\circ_{\Lambda'}\bar\tau \circlearrowleft N_{G_{\bar\delta}}(\Lambda')_{\bar\tau}/G_\delta]^{\mathrm{naive}}_{(\mathcal{O}(-1))}\big)\big) \\ &- \sum_{[\Lambda'']}\sum_{[\bar\tau'']} \mathrm{ind}^{N_{G_{\bar\delta}}(\Lambda)_{\bar\tau}/G_\delta}_{N_{G_{\bar\delta}}(\Lambda'')_{\bar\tau''}/G_\delta}\big(\psi_{I'',J}\big(\tau_{I'',J}[D^\circ_{\Lambda''}\bar\tau'' \circlearrowleft N_{G_{\bar\delta}}(\Lambda'')_{\bar\tau''}/G_\delta]^{\mathrm{naive}}_{(\mathcal{O}(-1))}\big)\big).\end{aligned}$$

The first sum is over $N_G(\Lambda)_{\bar\tau}$*-conjugacy classes of chains*

$$\Lambda' : \quad \Gamma^1 = \Gamma'^1 \supset \Gamma'^2 \supset \cdots \supset \Gamma'^{t'}$$

strictly containing Λ *with the same largest member* $\Gamma^1 = \Gamma'^1$*; we put* $I' := \{1,\dots,t'\}$*. The second sums are over* $N_G(\Lambda)_{\bar\tau}$*-conjugacy classes of chains*

$$\Lambda'' : \quad \Gamma''^1 \supset \Gamma''^2 \supset \cdots \supset \Gamma''^{t''}$$

containing Λ*, with* $\Gamma''^1 \supsetneq \Gamma^1$*, and* $N_G(\Lambda'')_{\bar\tau}$*-orbit representatives* $\bar\tau''$ *of the fiber of*

$$T''_{[r]}/T'' \to T'_{[r]}/T'$$

over $\bar\tau$*, where* T'' *denotes* $T_{\Gamma''^1}$*; we put* $I'' := \{1,\dots,t''\}$*. In each sum,* J *records the indices of the members of* Λ *and is identified in an order-preserving fashion with* $\{1,\dots,t\}$ *to land in* $\mathrm{Burn}_{n,\{1,\dots,t\}}(N_{G_{\bar\delta}}(\Lambda)_{\bar\tau}/G_\delta)$*.*

Proof. This follows from the formula in §4, and the evident analogous formula for indexed equivariant Burnside groups. By Lemma 3.7, application of $\tau_{I',J}$ and $\tau_{I'',J}$ corrects the divisor characters in the first, respectively second sums in the formula. □

6. Dimension 2

There is only one nontrivial action on $\mathbb{G}_m$, namely $t \mapsto t^{-1}$, and it is linearizable.

In dimension 2, it is known [38, II.4.9, Exa. 7] that every action on $\mathbb{G}_m^2$ factors through a subgroup of

$$\mathfrak{D}_4 \text{ or } C_2 \times \mathfrak{S}_3 \subset \mathrm{GL}_2(\mathbb{Z}).$$

Subgroups of $\mathfrak{D}_4$ give rise to a regular action on $\mathbb{P}^1 \times \mathbb{P}^1$. Projecting from the identity of $\mathbb{G}_m^2 \subset \mathbb{P}^1 \times \mathbb{P}^1$ gives an induced regular action on $\mathbb{P}^2$, and this is linear. So, we focus on

$$G := C_2 \times \mathfrak{S}_3,$$

whose action we realize by inverse and permutation on the coordinates of

$$T \subset \mathbb{G}_m^3, \qquad T := \{(t_1, t_2, t_3) \,|\, t_1 t_2 t_3 = 1\}.$$

This extends to a regular G-action on a del Pezzo surface X of degree 6; the action is obtained by regularizing the $\mathfrak{S}_3$ permutation action on the standard coordinates of $\mathbb{P}^2$ together with the Cremona involution. As mentioned in the introduction:

- This action is not linearizable, by [23]; there, the proof relied on the (equivariant) Minimal Model Program, specifically, on the classification of Sarkisov links. In [19] we provided an alternative proof, using partial information about the class of the action in the equivariant Burnside group.
- There are no cohomological obstructions to stable linearizability,

  $$\mathrm{H}^1(G', \mathrm{Pic}(X)) = 0,$$

  for all subgroups $G' \subseteq G$.
- For every subgroup $G' \subsetneq G$, the G'-action on T is linearizable [29, Section 9].
- The G-action is stably linearizable [29, Prop. 9.11].

We apply the procedure from Section 5 to compute the class of the G-action on an equivariant compactification of T in the Burnside group. The result is:

Proposition 6.1. *The class in* $\mathrm{Burn}_2(G)$ *of the* G*-action on* X *is:*

$$\begin{aligned}
[X \circlearrowleft G] = (&\mathrm{triv}, G \circlearrowright k(X), ()) \\
&+(\mathfrak{S}_2, C_2 \circlearrowright k(\mathbb{P}^1), (1)) \\
&+(\textit{diagonal in } C_2 \times \mathfrak{S}_2, C_2 \circlearrowright k(\mathbb{P}^1), (1)) \\
&+(C_2, \mathfrak{S}_3 \circlearrowright k(\mathbb{P}^1), (1)) \\
&+(C_2, \mathfrak{S}_2 \circlearrowright k(\mathbb{P}^1), (1)) \\
&+2(C_2 \times \mathfrak{S}_2, \mathrm{triv} \circlearrowright k, (e_1, e_2)) \\
&+2(C_2 \times \mathfrak{S}_2, \mathrm{triv} \circlearrowright k, (e_1 + e_2, e_2)) \\
&+(C_2 \times C_3, \mathrm{triv} \circlearrowright k, ((0,1), (1,1))) \\
&+(C_3, \mathrm{triv} \circlearrowright k, (1,1)).
\end{aligned}$$

For comparison, we display the class of the linear action

$$[\mathbb{P}(1 \oplus V_\chi) \circlearrowleft G] \in \mathrm{Burn}_2(G),$$

computed in [26, Exa. 5.3]; here V_χ is the standard 2-dimensional representation of $\mathfrak{S}_3$, twisted by a nontrivial character of the central C_2. This, in essence the only possible linear action, yields the class

$$\begin{aligned}
&(\mathrm{triv}, G \circlearrowright k(\mathbb{P}(1 \oplus V_\chi)), ()) \\
+&(\mathfrak{S}_2, C_2 \circlearrowright k(\mathbb{P}^1), (1)) \\
+2&(C_2, \mathfrak{S}_3 \circlearrowright k(\mathbb{P}^1), (1)) \\
+2&(C_2 \times \mathfrak{S}_2, \mathrm{triv} \circlearrowright k, (e_1, e_2)) \\
+2&(C_2 \times \mathfrak{S}_2, \mathrm{triv} \circlearrowright k, (e_1 + e_2, e_2)) \\
+&(C_2 \times C_3, \mathrm{triv} \circlearrowright k, ((0,1), (1,1))) \\
+&(C_2 \times C_3, \mathrm{triv} \circlearrowright k, ((0,1), (1,2))).
\end{aligned}$$

The term

$$(C_2, \mathfrak{S}_3 \circlearrowright k(\mathbb{P}^1), (1)) \tag{6.1}$$

is *incompressible* (see Defn. 3.3 and Prop. 3.6 in [25]). It appears with different coefficients in the two expressions above. By [25, Prop. 3.4], the actions are not equivariantly birational. However, there are other terms in these formulas that distinguish the two actions, e.g., the terms with $C_2 \times C_3$-stabilizer.

Recall that $X \times \mathbb{P}^r$ and $\mathbb{P}(1 \oplus V_\chi) \times \mathbb{P}^r$, with trivial G-action on $\mathbb{P}^r$, are equivariantly birational for $r \geq 2$, by [29, Prop. 9.11]. The case of $r = 1$ is unknown. Computing $\mathrm{Burn}_3(G)$ we find nothing to distinguish the classes

$$[X \times \mathbb{P}^1 \circlearrowleft G], [\mathbb{P}(1 \oplus V_\chi) \times \mathbb{P}^1 \circlearrowleft G] \in \mathrm{Burn}_3(G).$$

Indeed, all contributions from nontrivial stabilizers vanish in $\mathrm{Burn}_3(G)$. For terms with C_2- and C_3-stabilizers this follows immediately from Proposition 4.1, while for terms with $C_2 \times C_3$-stabilizer this requires further analysis.

The rest of this section consists of a proof of Proposition 6.1. It is an application of the algorithm from Section 5, which we carry out in detail. We have:

- $T = \Delta$,
- $\bar{\delta} = 1$,
- $G_{\bar{\delta}} = G$, $G_\delta = \mathrm{triv}$,

and the formula from Proposition 5.1 simplifies to

$$[X \circlearrowleft G] = A + B.$$

We use the coordinates t_1 and t_2 to identify T with $\mathbb{G}_m^2$. There is a corresponding basis e_1, e_2 of $N \cong \mathbb{Z}^2$.

Step 1. We compute $\mathcal{L}' = \mathcal{L}'(T)$, the lattice of distinguished subgroups of G, associated with subtori of T:

Γ'	$T_{\Gamma'}$
G	$\{(1,1,1)\}$
$\langle(0,(1,2))\rangle$	$\{(t,t,t^{-2})\}$
$\langle(0,(1,3))\rangle$	$\{(t,t^{-2},t)\}$
$\langle(0,(2,3))\rangle$	$\{(t^{-2},t,t)\}$
$\langle(1,(1,2))\rangle$	$\{(t,t^{-1},1)\}$
$\langle(1,(1,3))\rangle$	$\{(t,1,t^{-1})\}$
$\langle(1,(2,3))\rangle$	$\{(1,t,t^{-1})\}$
triv	T

Step 2: We construct a smooth projective G-invariant fan, with respect to which property (E) holds for every $T_{\Gamma'}$; this has rays generated by

$$(1,0),(1,1),(0,1),(-1,0),(-1,-1),(0,-1).$$

Step 3: We subdivide to obtain a fan Σ satisfying the additional property, that no pair of rays in a single G-orbit spans a cone of Σ; the ray generators are

$$(1,0),(2,1),(1,1),(1,2),(0,1),(-1,1),(-1,0),(-2,-1),\\ (-1,-1),(-1,-2),(0,-1),(1,-1).$$

Step 4. We find a positive integer r, such that the stabilizer locus in T is in the union of the r-torsion translates of subtori in $\mathcal{G}_{\mathcal{L}'}$, and the same holds for the $\mathrm{Stab}(\sigma)$-action on T^σ, for all $\sigma \in \Sigma$. In T, the stabilizer locus consists of the one-dimensional subtori above, together with

$$(1,1,1),(-1,-1,1),(-1,1,-1),(1,-1,-1),(\zeta,\zeta,\zeta),(\zeta^2,\zeta^2,\zeta^2).$$

For $\sigma \in \Sigma(1)$ we have $\mathrm{Stab}(\sigma) \cong \mathbb{Z}/2\mathbb{Z}$ acting on $T^\sigma \cong \mathbb{G}_m$, fixing ± 1. So we take

$$r = 6.$$

Step 5. We carry out the De Concini-Procesi blow-up procedure, which in this case amounts to blowing up the 6-torsion of T in $X = X_\Sigma$ to obtain

$$X_{\Sigma,\mathcal{L}',[6]} \cong B\ell_{36 \text{ points}} X.$$

Step 6. We compute A directly as

$$A = (\mathrm{triv}, G \curvearrowright k(X), ()).$$

This is the contribution from the zero cone in the formula from $\mathbf{\Sigma}_1$ in Proposition 5.1. The two orbits of 1-dimensional cones σ lead to action of $\mathrm{Stab}(\sigma) \cong \mathbb{Z}/2\mathbb{Z}$ on $T^\sigma \cong \mathbb{G}_m$ with trivial generic stabilizer, hence no contribution to the equivariant Burnside group. As well there is no contribution from the 2-dimensional cones, which form a single orbit with trivial stabilizer.

Step 7. We compute B by the procedure of Theorem 5.3. We have $\ell = 3$:

$$\mathbb{D} = \mathbb{D}_1 \cup \mathbb{D}_2 \cup \mathbb{D}_3 \subset X_{\Sigma,\mathcal{L}',[6]},$$

with respective conjugacy class representatives of $\mathcal{L}' \setminus \{\mathrm{triv}\}$ from the table in Step 1:

$$\Gamma'_1 = G, \qquad \Gamma'_2 = \langle (0,(1,2)) \rangle, \qquad \Gamma'_3 = \langle (1,(1,2)) \rangle.$$

Together, Γ'_2 and Γ'_3 generate $C_2 \times \mathfrak{S}_2 \cong \mathfrak{K}_4$.

With $\mathcal{I} = \{1,2,3\}$ we have nonempty D_I corresponding to the following subsets $I \subseteq \mathcal{I}$:

$$\{1\}, \quad \{2\}, \quad \{3\}, \quad \{1,2\}, \quad \{1,3\}.$$

For each I, there is exactly one conjugacy class of chains, with representative

$$\{\Gamma'_i \,|\, i \in I\}.$$

We list representative chains Λ, with corresponding $N_G(\Lambda)$ and $N_G(\Lambda)$-orbits of $T'_{[6]}/T'$:

Λ	$N_G(\Lambda)$	T'	$T'_{[6]}/T'$	$N_G(\Lambda)$-orbits
G	G	$\{1\}$	$T[6]$	see Table 1
$\langle (0,(1,2)) \rangle$	$\mathfrak{K}_4$	$\{(t,t,t^{-2})\}$	μ_6 (via $t_1^{-1}t_2$)	see below
$\langle (1,(1,2)) \rangle$	$\mathfrak{K}_4$	$\{(t,t^{-1},1)\}$	μ_6 (via t_1t_2)	see below
$G \supset \langle (0,(1,2)) \rangle$	$\mathfrak{K}_4$	$\{1\}$	$T[6]$	see Table 1
$G \supset \langle (1,(1,2)) \rangle$	$\mathfrak{K}_4$	$\{1\}$	$T[6]$	see Table 1

When T' has dimension 1, we identify $T'_{[6]}/T'$ with μ_6 by the indicated coordinate function. The action by $N_G(\Lambda) = \mathfrak{K}_4$ has orbits

$$\{1\}, \quad \{-1\}, \quad \{\zeta, \zeta^2\}, \quad \{-\zeta, -\zeta^2\}.$$

When $\Lambda = \{\langle (0,(1,2)) \rangle\}$, the elements in orbits of size 2 have stabilizer $\langle (1,(1,2)) \rangle$. When $\Lambda = \{\langle (1,(1,2)) \rangle\}$, the elements in orbits of size 2 have stabilizer $\langle (0,(1,2)) \rangle$.

As indicated in Theorem 5.3, we start the computation of B by looking at contributions with $t = 2$; for these, we have $\mathbb{D}^\circ_\Lambda = \mathbb{D}_\Lambda$.

stabilizer	orbit [G-stabilizer]
$\mathfrak{K}_4$	$(1,1,1)$ [G]
	$(-1,-1,1)$ [$\mathfrak{K}_4$]
$\langle(0,(1,2))\rangle$	$(\zeta,\zeta,\zeta)\,(\zeta^2,\zeta^2,\zeta^2)$ [$\mathfrak{S}_3$]
	$(-\zeta,-\zeta,\zeta)\,(-\zeta^2,-\zeta^2,\zeta^2)$ [$\langle(0,(1,2))\rangle$]
$\langle(1,(1,2))\rangle$	$(\zeta,\zeta^2,1)\,(\zeta^2,\zeta,1)$ [$\langle(1,(1,2))\rangle$]
	$(-\zeta,-\zeta^2,1)\,(-\zeta^2,-\zeta,1)$ [$\langle(1,(1,2))\rangle$]
$\langle(1,\mathrm{id})\rangle$	$(1,-1,-1)\,(-1,1,-1)$
triv	$(\zeta,-\zeta,-\zeta)\,(\zeta^2,-\zeta^2,-\zeta^2)\,(-\zeta,\zeta,-\zeta)\,(-\zeta^2,\zeta^2,-\zeta^2)$
	$(1,\zeta,\zeta^2)\,(1,\zeta^2,\zeta)\,(\zeta,1,\zeta^2)\,(\zeta^2,1,\zeta)$
	$(1,-\zeta,-\zeta^2)\,(1,-\zeta^2,-\zeta)\,(-\zeta,1,-\zeta^2)\,(-\zeta^2,1,-\zeta)$
	$(\zeta,-\zeta^2,-1)\,(\zeta^2,-\zeta,-1)\,(-\zeta^2,\zeta,-1)\,(-\zeta,\zeta^2,-1)$ [triv]
	$(-1,\zeta,-\zeta^2)\,(-1,\zeta^2,-\zeta)\,(\zeta,-1,-\zeta^2)\,(\zeta^2,-1,-\zeta)$
	$(-1,-\zeta,\zeta^2)\,(-1,-\zeta^2,\zeta)\,(-\zeta,-1,\zeta^2)\,(-\zeta^2,-1,\zeta)$

TABLE 1. Orbits of $T[6]$ under $\mathfrak{K}_4 \subset G$. Orbits under G are unions of $\mathfrak{K}_4$-orbits; for each a representative is identified, with G-stabilizer displayed in brackets [ ].

- $\Lambda = \{G \supset \langle(0,(1,2))\rangle\}$: We have $N_G(\Lambda) = \mathfrak{K}_4$, with

$$V_1 = 0 \qquad \text{and} \qquad V_2 = k\cdot(1,1).$$

Following Lemma 5.2, we have

$$[D_\Lambda \bar{\tau} \circlearrowleft (\mathfrak{K}_4)_{\bar{\tau}}]_{(\mathcal{O}(-1))} = [\{1\}\bar{\tau} \times \mathbb{P}(V_2/V_1) \times \mathbb{P}(V/V_2) \circlearrowleft (\mathfrak{K}_4)_{\bar{\tau}}]_{(\mathcal{O}(-1))},$$

as a point, with pair of characters e_1 determined by V_2/V_1, and e_1+e_2 determined by V/V_2. So, (5.2) gives e_1 and e_2 as characters of $(\mathcal{O}(-1))$. Applying $\psi_{\{1,2\}}$ to get an element of $\mathrm{Burn}_2((\mathfrak{K}_4)_{\bar{\tau}})$, we only get something nontrivial when $(\mathfrak{K}_4)_{\bar{\tau}} = \mathfrak{K}_4$. There are two contributions from Table 1; when we apply induction to $\mathrm{Burn}_2(G)$ we obtain

$$\psi_{\{1,2\}}\big([D_\Lambda \bar{\tau} \circlearrowleft (\mathfrak{K}_4)_{\bar{\tau}}]_{(\mathcal{O}(-1))}\big) = \begin{cases} (C_2 \times \mathfrak{S}_2, \mathrm{triv} \circlearrowright k, (e_1,e_2)), & \text{if } (\mathfrak{K}_4)_{\bar{\tau}} = \mathfrak{K}_4, \\ 0, & \text{otherwise.} \end{cases}$$

- $\Lambda = \{G \supset \langle(1,(1,2))\rangle\}$: The computation is similar, with

$$V_2 = k\cdot(1,-1),$$

and we obtain

$$\psi_{\{1,2\}}\big([D_\Lambda\bar\tau \leftarrow (\mathfrak{K}_4)_{\bar\tau}]_{(\mathcal{O}(-1))}\big) = \begin{cases}(C_2\times\mathfrak{S}_2, \mathrm{triv}\curvearrowright k, (e_1+e_2, e_2)), & \text{if } (\mathfrak{K}_4)_{\bar\tau}=\mathfrak{K}_4,\\ 0, & \text{otherwise.}\end{cases}$$

We proceed to cases with $t=1$.

- $\Lambda=\{G\}$: We have $V_1=0$. So,

$$\psi_{\{1\}}\big([D^\circ_\Lambda\bar\tau \leftarrow G_{\bar\tau}]_{(\mathcal{O}(-1))}\big) = \begin{cases}\psi_{\{1\}}\big([D_\Lambda\bar\tau \leftarrow G_{\bar\tau}]_{(\mathcal{O}(-1))}\big) - \mathrm{C}_0 - \mathrm{C}_1, & \text{if } G_{\bar\tau}\supseteq\mathfrak{K}_4,\\ \psi_{\{1\}}\big([D_\Lambda\bar\tau \leftarrow G_{\bar\tau}]_{(\mathcal{O}(-1))}\big), & \text{if } |G_{\bar\tau}|\in\{1,2,6\},\end{cases}$$

where for $i\in\{0,1\}$,

$$\mathrm{C}_i := \mathrm{ind}_{\mathfrak{K}_4}^{G_{\bar\tau}}\big(\psi_{\{1,2\}}\big([D_{\{G\supset\langle(i,(1,2))\rangle\}}\bar\tau \leftarrow \mathfrak{K}_4]_{(\mathcal{O}(-1))}\big)\big);$$

by Lemma 5.2, we have

$$[D_\Lambda\bar\tau \leftarrow G_{\bar\tau}]_{(\mathcal{O}(-1))} = [\{1\}\bar\tau\times\mathbb{P}(V) \leftarrow G_{\bar\tau}]_{(\mathcal{O}(-1))}.$$

The nontrivial contributions come from three values of $\bar\tau$. When $\bar\tau=(1,1,1)$, we get

$$\psi_{\{1\}}\big([D^\circ_\Lambda\bar\tau \leftarrow G_{\bar\tau}]_{(\mathcal{O}(-1))}\big) = (C_2, \mathfrak{S}_3\curvearrowright k(\mathbb{P}^1), (1)) + (C_2\times C_3, \mathrm{triv}\curvearrowright k, ((0,1),(1,1))).$$

The cases $\bar\tau=(-1,-1,1)$ and $\bar\tau=(\zeta,\zeta,\zeta)$ give

$$(C_2, \mathfrak{S}_2\curvearrowright k(\mathbb{P}^1), (1)), \quad \text{respectively} \quad (C_3, \mathrm{triv}\curvearrowright k, (1,1)).$$

- $\Lambda=\{\langle(0,(1,2))\rangle\}$: The only nontrivial contribution is from $\bar\tau=1$. Then,

$$\psi_1\big([D_\Lambda\bar\tau \leftarrow \mathfrak{K}_4]_{(\mathcal{O}(-1))}\big) = (\mathfrak{S}_2, C_2\curvearrowright k(\mathbb{P}^1), (1)) + 2(\mathfrak{K}_4, \mathrm{triv}\curvearrowright k, (e_1, e_1+e_2))\in \mathrm{Burn}_2(\mathfrak{K}_4). \tag{6.2}$$

We get $\psi_1\big([D^\circ_\Lambda\bar\tau \leftarrow \mathfrak{K}_4]^{\mathrm{naive}}_{(\mathcal{O}(-1))}\big)$, according to Theorem 5.3, by subtracting contributions from

$$\Lambda''=\{G, \langle 0,(1,2)\rangle\}, \quad \bar\tau''\in\{(1,1,1),(-1,-1,1),(\zeta,\zeta,\zeta),(-\zeta,-\zeta,\zeta)\}.$$

When $\bar\tau''\in\{(1,1,1),(-1,-1,1)\}$, we apply $\tau_{\{1,2\},\{2\}}$ to the indexed equivariant Burnside group element

$$(\mathfrak{K}_4\subseteq\mathfrak{K}_4, \mathrm{triv}\curvearrowright k, (), (e_1,e_2))\in \mathrm{Burn}_{2,\{1,2\}}(\mathfrak{K}_4)$$

to yield, in each case, weights e_1 and e_1+e_2, thereby cancelling the term in (6.2) with coefficient 2. When $\bar{\tau}'' \in \{(\zeta,\zeta,\zeta),(-\zeta,-\zeta,\zeta)\}$ the contribution is trivial in $\mathrm{Burn}_2(\mathfrak{K}_4)$. So

$$\psi_1\big([D^\circ_\Lambda \bar{\tau} \circlearrowleft \mathfrak{K}_4]^{\mathrm{naive}}_{(\mathcal{O}(-1))}\big) = (\mathfrak{S}_2, C_2 \circlearrowright k(\mathbb{P}^1), (1)).$$

- $\Lambda = \{\langle (1,(1,2))\rangle\}$: The computation is similar. There is only a nontrivial computation for $\bar{\tau} = 1$, and we obtain

$$\psi_1\big([D^\circ_\Lambda \bar{\tau} \circlearrowleft \mathfrak{K}_4]^{\mathrm{naive}}_{(\mathcal{O}(-1))}\big) = (\text{diagonal in } C_2 \times \mathfrak{S}_2, C_2 \circlearrowright k(\mathbb{P}^1), (1)).$$

Combining the contributions, we obtain the formula in the statement of Proposition 6.1.

7. Dimension 3

In this section, we analyze 3-dimensional tori, following [27]. This is the smallest dimension where cohomology can obstruct rationality and linearizability. We have two motivating problems:

- Find nonlinearizable actions with vanishing $\mathrm{H}^1(G', \mathrm{Pic}(X))$, for all $G' \subseteq G$, this is the analogue of [23].
- Investigate the relation between

$$[X \circlearrowleft G] \quad \text{and} \quad \mathrm{H}^1(G', \mathrm{Pic}(X)),$$

where X is a smooth G-equivariant projective compactification of T and $G' \subseteq G$.

Any action on a torus $T = \mathbb{G}_m^3$ factors through a subgroup of

$$C_2 \times \mathfrak{S}_3 \times C_2 \quad \text{or} \quad C_2 \times \mathfrak{S}_4,$$

and the second group admits 3 different actions, labeled C, S, and P in [27]. The first group is realized on a product of a del Pezzo surface of degree 6 with $\mathbb{P}^1$, with the natural action of $G' := C_2 \times \mathfrak{S}_3$ on the DP6 (described in Section 6) and C_2 on $\mathbb{P}^1$. As mentioned in Section 6 and [29, Rem. 9.13], it is already an open problem, whether the G'-action is linearizable.

The other actions are realized as follows [27, Section 2]:

(C) on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$,

(S) on the blowup of $\mathbb{P}^3$ in the four coordinate points and the six lines through these points; here $\mathfrak{S}_4$ permutes the coordinates in $\mathbb{P}^3$ and C_2 is the Cremona involution;

(P) on the (singular) hypersurface

$$\{x_1x_2x_3x_4 = y_1y_2y_3y_4\} \subset (\mathbb{P}^1)^4,$$

where $\mathfrak{S}_4$ acts by permuting the factors and C_2 switches x_i and y_i, for all i.

The $G = C_2 \times \mathfrak{S}_4$-action in type (C) is linearizable. The following proposition covers the types (S) and (P), see [27, Fig. 4].

Proposition 7.1. *Assume that T admits the action of the Klein group $G := \mathbb{Z}/2 \oplus \mathbb{Z}/2$ such that $G \subset \mathrm{GL}(M)$ is generated by*

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} -1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Let X be a smooth projective G-equivariant compactification of T. Then

(1) $\mathrm{H}^1(G, \mathrm{Pic}(X)) = \mathbb{Z}/2$, *and the action is not stably linearizable,*
(2) $[X \leftmapsto G] = (\mathrm{triv}, G \curvearrowright k(X), ()) \in \mathrm{Burn}_3(G)$.

Proof. The first statement is one of the key results in [27, Section 4]. The second follows by an argument as in the proof of Proposition 4.1: any symbol with a nontrivial stabilizer that arises, vanishes in $\mathrm{Burn}_3(G)$. □

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