EQUIVARIANT UNIRATIONALITY OF FANO THREEFOLDS

IVAN CHELTSOV, YURI TSCHINKEL, AND ZHIJIA ZHANG

Abstract. We study unirationality of actions of finite groups on Fano threefolds with indices greater than one.

1. Introduction

In this paper, we consider generically free regular actions of finite groups G on smooth projective varieties X over an algebraically closed field k of characteristic zero. The simplest such actions arise from linear representations, i.e., as projectivizations $\mathbb{P}(V)$ of linear representations V of G. Note that not every faithful representation gives rise to a generically free action – one also requires that the representation intersects the central $\mathbb{G}_m$ trivially. We are interested in birational properties of G-actions, e.g.,

- **(L)** *linearizability*: equivariant birationality to a linear action,
- **(SL)** *stable linearizability*: linearizability of $X \times \mathbb{P}(W)$, where W is a G-representation; when the G-action on X is generically free, we may take W to be the trivial representation, by the no-name lemma.

These notions should be viewed as equivariant analogs of rationality and stable rationality in geometry over nonclosed fields: there are many similarities but also striking differences between these theories, see, e.g., [28], [29] for an introduction to this circle of problems.

Here, we explore an equivariant version of unirationality:

- **(U)** *G-unirationality*: existence of a dominant equivariant rational map
$$\mathbb{P}(V) \dashrightarrow X,$$
where V is a G-representation.

Clearly,
$$(\mathbf{L}) \Rightarrow (\mathbf{SL}) \Rightarrow (\mathbf{U}).$$

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In the definition of **(U)**, one can replace $\mathbb{P}(V)$ by V, a G-representation, i.e., **(U)** is equivalent to *very versality* of the G-action, in the terminology of, e.g., [18].

A necessary condition for G-unirationality is existence of fixed points upon restriction to abelian subgroups:

- **Condition (A)**: for every abelian subgroup $A \subseteq G$ one has

$$X^A \neq \emptyset.$$

This should be viewed as analogous to the existence of rational points, in the framework of birational geometry over nonclosed fields. Note that there do exist linearizable G-actions without G-fixed points, moreover, the existence of G-fixed points is not an equivariantly birational property, for nonabelian G. Thus, some standard unirationality constructions in geometry over nonclosed fields fail to apply in the equivariant setting.

Duncan proved that Condition **(A)** is also sufficient for G-unirationality of del Pezzo surfaces of degree ≥ 3, with generically free actions [17, Theorem 1.4]. The cases of del Pezzo surfaces of degree 2 and 1 remain open.

We turn to smooth Fano threefolds. These are classified by

- $r \in \{1, \dots, 10\}$ – rank of the Picard group,
- $i \in \{1, 2, 3, 4\}$ – Fano index,
- $d = (-K_X)^3/i^3$ – degree.

Their rationality over algebraically closed fields is *almost* settled. There are still outstanding open problems concerning stable rationality and unirationality, e.g., unirationality of general quartic threefolds. Failure of stable rationality of very general cubic threefolds has been recently established [20]. Rationality properties of geometrically rational Fano threefolds have received a lot of attention, see [21], [22], [32], [31]. There is also a wealth of results concerning (stable) linearizability of group actions on (possibly singular) Fano threefolds, see, e.g., [34], [11], [10], [8], [24], [6], [13].

In this paper, we focus on equivariant unirationality of Fano threefolds of index ≥ 2; these should be viewed as 3-dimensional analogs of del Pezzo surfaces. As in the work of Duncan on del Pezzo surfaces [17], Fano threefolds of *small* degree are currently out of reach – we do not even know the unirationality of any smooth Fano threefold of index 2 and degree 1 (all other Fano threefolds of index ≥ 2 are known to be unirational, over algebraically closed fields).

Our principal results are:

- A generically free G-action on a smooth quadric threefold is stably linearizable if and only if it satisfies Condition **(A)**, Theorem 4.1.

- A generically free G-action on a smooth cubic threefold satisfying Condition **(A)** is G-unirational, with the possible exception of $G = C_9 \rtimes C_3$, acting on the Fermat cubic threefold, and $C_{11} \rtimes C_5, \mathsf{PSL}_2(\mathbb{F}_{11})$, acting on the Klein cubic threefold, and a 1-parameter family of cubics with an $\mathfrak{A}_5$-action, Theorem 5.1.
- A generically free action on a cubic threefold with isolated singularities, satisfying Condition **(A)**, is G-unirational, Theorem 6.1. This class contains cubic threefolds with cohomological obstructions to stable linearizability, see [10], [8].
- A generically free G-action on a smooth complete intersection of two quadrics $X \subset \mathbb{P}^5$ is G-unirational if and only if it satisfies Condition **(A)**, which turns out to be equivalent to $X^G \neq \emptyset$, Theorem 7.1.
- A generically free G-action on the (unique) smooth del Pezzo threefold V_5 of degree 5 is linearizable except for $G = \mathfrak{A}_5$, which is not linearizable, by [9], but stably linearizable, by [5, Example 7], see Remark 7.2.

Here is the roadmap of the paper: in Section 2 we recall notions of G-birational geometry, with a focus on G-unirationality. In Section 3 we present several general unirationality constructions, applicable in the equivariant context. In Section 4 we specialize to quadric threefolds. Then we proceed to treat smooth cubics in Section 5, singular cubics in Section 6, and intersections of two quadrics in Section 7.

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2. Generalities

Throughout, we work over an algebraically closed field k of characteristic zero; all varieties are assumed to be irreducible over k. A G-variety is a projective variety with a generically free regular G-action; we will also consider situations when the G-action is *not* generically free. The G-fixed point locus of a G-variety X will be denoted by X^G. For smooth projective G-varieties X and *abelian* subgroups $A \subseteq G$, existence of A-fixed

points is a birational property; this is not the case for nonabelian groups! Note that cyclic actions on rationally connected G-varieties always have fixed points, but this need not be the case for actions of abelian groups, e.g., the C_2^2-action on $\mathbb{P}^1$ generated by $t \to -t$ and $t \to 1/t$ where t is the affine coordinate on $\mathbb{P}^1$.

If $\tilde{X} \to X$ is a dominant morphism of G-varieties and $\tilde{X}$ has G-fixed points, then so does X. Since *linear* actions of abelian groups always have fixed points, a necessary condition for G-unirationality of an action of a finite group G on a smooth projective variety X is Condition **(A)**: existence of fixed points upon restriction to every abelian subgroup.

Linearizability and stable linearizability of generically free actions of finite groups on rational surfaces and rational Fano threefolds have been considered in, e.g., [24], [5], [11], [8], [25]. The related notion of G-unirationality, or, equivalently very versality, plays an important role in the study of *essential dimension*, see, e.g., [18], [35]. However, only few concrete examples of G-unirational varieties failing (stable) linearizability are known, see [17] for a systematic treatment of del Pezzo surfaces of degree ≥ 3.

The bridge between equivariant geometry and geometry over nonclosed fields is provided via *torsors*: For a field extension K/k and a G-torsor T over K, let ${}^TX$ be the T-twist of X over K, defined as the quotient ${}^TX := (X_K \times T)/G$ by the diagonal G-action. A key result is:

Theorem 2.1. [18, Theorem 1.1] *Let X be a G-variety over k. Then*

- *the G-action on X is stably linearizable if and only if for every field extension K/k and every G-torsor T over K, the twist ${}^TX$ is stably rational over K,*
- *the G-action on X is unirational if and only if every twist ${}^TX$ as above is unirational over K.*

We will apply this in subsequent sections to quadric and cubic hypersurfaces.

A related notion that plays a role in equivariant geometry is the *Amitsur* group $\mathrm{Am}(X, G)$, defined in [3, Section 6] as the image of G-invariant divisor classes

$$\mathrm{Pic}(X)^G \xrightarrow{\delta_2} \mathrm{H}^2(G, k^\times),$$

where δ_2 is the functorial homomorphism arising from the Leray spectral sequence, see, e.g., [27, Section 3]. By [3, Theorem 6.1], this is a G-birational invariant, which vanishes for linear actions and as soon as $X^G \neq \emptyset$. The vanishing of $\mathrm{Am}(X, G)$ implies that every G-invariant divisor class in $\mathrm{Pic}(X)$ is represented by a G-linearized line bundle on X.

All possibilities for $\mathrm{Am}(X,G)$ for G-surfaces X have been determined in [3, Proposition 6.7]. In particular, this group is trivial for del Pezzo surfaces S of degree ≤ 6 when $\mathrm{Pic}(S)^G = \mathbb{Z}$.

Lemma 2.2. *Let X be a G-variety with nontrivial Amitsur invariant. Then X is not G-unirational.*

Proof. Assume that X is G-unirational. There exists a G-equivariant dominant rational map $\pi : \mathbb{P}(V) \dashrightarrow X$. Resolving the indeterminancy by blowups, we have a G-equivariant morphism $\tilde{\pi} : \widetilde{\mathbb{P}}(V) \to X$ which gives rise to an inclusion

$$\mathrm{Am}(X,G) \subseteq \mathrm{Am}(\widetilde{\mathbb{P}}(V),G) = \mathrm{Am}(\mathbb{P}(V),G).$$

This is clear from the functoriality of the map δ_2 in the basic exact sequence (see [39] or [3, Section 6])

$$0 \to \mathrm{Hom}(G,k^\times) \to \mathrm{Pic}(X,G) \to \mathrm{Pic}(X)^G \xrightarrow{\delta_2} \mathrm{H}^2(G,k^\times),$$

where $\mathrm{Pic}(X,G)$ is the group of isomorphism classes of G-linearized line bundles on X, see, e.g., [30, Section 3]. Since $\mathrm{Am}(\mathbb{P}(V),G) = 0$, we obtain a contradiction. □

Example 2.3. Note that the invariant $\mathrm{Am}(X,G)$ does not only depend on the effective action on X. It also depends on the generic stabilizer. Let $X = \mathbb{P}^1$ with the action of the Klein four group $\bar{G} = C_2^2$; it is fixed point free. The Amitsur invariant $\mathrm{Am}(X,\bar{G}) = \mathbb{Z}/2$ is nontrivial, and the $\bar{G}$-action is not stably linearizable. However, X admits a regular *non-generically free* action of the dihedral group $G = \mathfrak{D}_4$ (of order 8), with generic stabilizer C_2. As a variety with G-action (not necessarily generically free), X is clearly G-unirational: we have a dominant G-equivariant map $V \dashrightarrow X$, where V is an irreducible 2-dimensional representation of G. Of course, every abelian subgroup of G has fixed points on X.

Recall the definition of *Bogomolov multiplier*:

$$\mathrm{B}_0(G) := \mathrm{Ker}\left(\mathrm{H}^2(G,k^\times) \to \oplus_A \mathrm{H}^2(A,k^\times)\right),$$

where A runs over all abelian subgroups of G. In fact, it suffices to consider *bicyclic* subgroups [4], and to treat Sylow subgroups, one prime at a time. There is an extensive literature on this invariant, see, e.g., [30, Section 3]. In Example 2.3, we have $\mathrm{B}_0(G) = 0$. More generally, the Bogomolov multiplier vanishes when

- G is a p-group of order $|G| \leq p^4$,
- G is an extension of a cyclic group by an abelian group [4, Lemma 4.9], [30, Lemma 3.1],

- G is a split extension of a bicyclic group by an abelian group [30, Lemma 3.2].

For a smooth projective G-variety X, Condition **(A)** implies that

$$\mathrm{Am}(X,G) \subseteq \mathrm{B}_0(G).$$

Lemma 2.4. *Let G be a finite group with $\mathrm{B}_0(G) = 0$. Assume that G acts regularly on $\mathbb{P}^n$. This G-action is unirational if and only if it satisfies Condition* **(A)**.

Proof. Under our assumption, one has that $\mathrm{Am}(\mathbb{P}^n, G) \subseteq \mathrm{B}_0(G) = 0$. This implies that $\mathcal{O}_{\mathbb{P}^n}(1)$ is G-linearized, and the corresponding homomorphism $G \to \mathsf{PGL}_n$ lifts to $G \to \mathsf{GL}_n$, i.e., the G-action on $\mathbb{P}^n$ arises from a linear representation. The action is thus G-unirational by definition. □

Here we do not assume that the G-action on $\mathbb{P}^n$ is generically free. Note that for G with $\mathrm{B}_0(G) \neq 0$, there exist regular generically free actions on projective spaces, which satisfy Condition **(A)** but are *not* linear, and not G-unirational.

Lemma 2.5. *Let*

$$X := \mathbb{P}_B(\mathcal{E})$$

be the projectivization of a vector bundle $\mathcal{E} \to B$, where B is a smooth projective G-variety. Assume that X carries a generically free regular G-action such that the canonical projection

$$\pi : X \to B$$

is G-equivariant. Assume that

(1) $\mathrm{H}^1(G, \mathrm{Pic}(B)) = 0$,
(2) *X satisfies Condition* **(A)**, *and*
(3) $\mathrm{B}_0(G) = 0$.

Then the G-action lifts to $\mathcal{E}$.

Proof. Consider the sequence of G-modules

$$0 \to \mathrm{Pic}(B) \xrightarrow{\pi^*} \mathrm{Pic}(X) \to \mathbb{Z} \to 1.$$

Note that we do not assume that the G-action on B is generically free.

From the long exact sequence

$$0 \to \mathrm{Pic}(B)^G \xrightarrow{\pi^*} \mathrm{Pic}(X)^G \to \mathbb{Z} \to \mathrm{H}^1(G, \mathrm{Pic}(B))$$

and assumption (1), we obtain that there is a G-invariant class in $\mathrm{Pic}(X)$ surjecting onto the class of the relative $\mathcal{O}(1)$. By assumption (2), every line bundle on X admits a linearization upon restriction to abelian subgroups, i.e., $\mathrm{Am}(X,G) \subseteq \mathrm{B}_0(G)$. By assumption (3), that latter

group is trivial, which implies that every invariant line bundle on X is G-linearizable, in particular, the G-action lifts to $\mathcal{E}$. □

3. Unirationality constructions

General results linking G-unirationality of actions on quadric and cubic hypersurfaces to geometry over nonclosed fields are in [18, Section 10]. We complement these considerations by providing explicit equivariant unirationality constructions in many cases; this allows us to essentially settle unirationality of G-actions on quadric and cubic threefolds, in subsequent sections.

We start with obvious observations:

- G-unirationality is compatible with taking products;
- existence of a dominant G-equivariant rational map $X \dashrightarrow Y$ from a G-unirational X implies G-unirationality of Y.

The following is an analog of a standard result in the theory of quadrics over nonclosed fields, see, e.g., [19, Chapter 4].

Proposition 3.1. *Let G be a finite group and V a representation of G giving rise to a generically free G-action on a smooth quadric hypersurface $X \subset \mathbb{P}(V)$, of dimension ≥ 2. Assume that there is a G-invariant irreducible subvariety $S \subset X$, which is G-unirational. Then X is G-unirational.*

Here, we allow S to have a nontrivial generic stabilizer, i.e., we require that there is a dominant rational map $\mathbb{P}(W) \dashrightarrow S$, for some G-representation W.

Proof. By assumption, there is a dominant G-equivariant rational map $\mathbb{P}(W) \dashrightarrow S$. Consider the rational map

$$S \times \mathbb{P}(V) \dashrightarrow X,$$

sending a pair of points $(s, p) \in S \times \mathbb{P}(V)$ to the second intersection point of X with the line $\mathfrak{l}(s, p)$ through s and p. This is clearly dominant and G-equivariant, ensuring the G-unirationality of X. □

We continue with general G-unirationality constructions for cubic hypersurfaces of dimensions ≥ 2. Over fields K of characteristic zero unirationality of smooth cubic hypersurfaces with $X(K) \neq \emptyset$ is classical; generalizations to singular cubic hypersurfaces that are not cones can be found in [26].

Proposition 3.2. *Let V be a representation of a finite group G. Let $X \subset \mathbb{P}(V)$ be a G-invariant irreducible cubic hypersurface of dimension*

≥ 3 *which is not a cone. Assume that the* G*-action on* X *is generically free and that* $X^G \neq \emptyset$*. Then* X *is* G*-unirational.*

Proof. This is essentially [18, Corollary 10.6], extended to singular cubic hypersurfaces. In particular, if G fixes a singular point, then the G-action is linearizable. If G fixes a smooth point, then the same argument in [18, Corollary 10.6] applies. □

Proposition 3.3. *Let* V *be a representation of a finite group* G*. Let* $X \subset \mathbb{P}(V)$ *be a* G*-invariant cubic hypersurface which is not a cone. Assume that the* G*-action on* X *is generically free and that* G *has normal subgroup* H *such that* X *is* H*-unirational and* G/H *is a 2-group. Then* X *is* G*-unirational.*

Proof. Since G/H is a 2-group, we know there exists a tower

$$H \subset G_1 \subset G_2 \subset \cdots G$$

where each inclusion has index 2. By induction, it suffices to consider when $|G/H| = 2$. In this case, the assertion essentially follows from [17, Theorem 3.2], see also [17, Remark 3.3]: G-unirationality is equivalent to K-unirationality of all twists of X over K (via G-torsors), and for cubic hypersurfaces, unirationality over quadratic extensions of K is equivalent to unirationality over K. □

Corollary 3.4. *Let* V *be a representation of a finite group* G*. Let* $X \subset \mathbb{P}(V)$ *be a* G*-invariant cubic hypersurface which is not a cone. Assume that the* G*-action on* X *is generically free and there exists a* G*-orbit of points on* X *of length one or two. Then the* G*-action on* X *is* G*-unirational.*

Proof. First we consider when X has a G-fixed point $\mathfrak{p}$. It is G-unirational by Proposition 3.2 if $\mathfrak{p}$ is smooth. If $\mathfrak{p}$ is singular, X is G-linearizable, and thus G-unirational by projection from $\mathfrak{p}$. When X has a G-orbit $\{\mathfrak{p}_1, \mathfrak{p}_2\}$ of points of length 2, there exists an index-2 subgroup $H \subset G$ fixing $\mathfrak{p}_1$ and $\mathfrak{p}_2$. By the argument in the previous case, we know that X is H-unirational. The G-unirationality of X then follows from Proposition 3.3. □

Proposition 3.5. *Let* V *be a faithful* n*-dimensional representation of a finite group* G *with* $n \geq 5$*. Let* $X \subset \mathbb{P}(V)$ *be a* G*-invariant cubic hypersurface which is not a cone. Assume that the* G*-action on* X *is generically free and that there is a* G*-invariant subvariety* $S \subset X$*, which is* G*-unirational. Then* X *is* G*-unirational.*

Here we do not require that the G-action on S is generically free.

Proof. For any field extension K/k and every G-torsor T over K, the twist ${}^TS$ is a K-unirational subvariety of ${}^TX$ by Theorem 2.1, which implies that ${}^TX$ has a K-point. By assumptions, ${}^TX$ is a cubic hypersurface in ${}^T\mathbb{P}(V) = \mathbb{P}^n$. It follows from [26] that ${}^TX$ is K-unirational, and thus X is G-unirational by Theorem 2.1 again.

□

Remark 3.6. One can also find explicit G-unirational constructions in many cases under the assumptions of Proposition 3.5. For example, assume that S is an irreducible hyperplane section, i.e., a cubic surface. Let $\mathcal{T} \to X^\circ$ be the tangent bundle over the smooth locus X° of X, and $\mathcal{T}|_{S^\circ}$ the restriction of $\mathcal{T}$ to the smooth locus of S. Then $\mathcal{T}|_{S^\circ}$ is a rank-$(n-2)$ vector bundle and the G-action lifts to $\mathcal{T}|_{S^\circ}$. By [14, Theorem 4.2] and [15, Theorem 2.5], we know that the intersection of the tangent space at the generic point of S° with X is not a cone. Thus, each point in the fiber above a general point $\mathfrak{p} \in S^\circ$ canonically corresponds to a line in $\mathbb{P}(V)$ that is tangent to X at $\mathfrak{p}$ with multiplicity 2, and thus intersects X at a unique point $\mathfrak{q} \neq \mathfrak{p}$. Sending each point in $\mathcal{T}|_{S^\circ}$ to the corresponding point $\mathfrak{q} \in X$, we obtain a G-equivariant rational map

$$\varphi : \mathcal{T}|_{S^\circ} \dashrightarrow X. \tag{3.1}$$

We show that φ is dominant. Assume that

$$X = \{f_3(x_1, \dots, x_n) = 0\} \quad \text{and} \quad S = X \cap \{x_1 = 0\}.$$

Given a general point $\mathfrak{q} = [q_1 : \cdots : q_n] \in X$, we can find general $p_1, \dots, p_{n-1} \in k$ such that

$$\sum_{i=1}^{n} \frac{\partial f_3}{\partial x_i}(0, p_1, \dots, p_{n-1}) q_i = f_3(0, p_1, \dots, p_{n-1}) = 0.$$

Then the line passing through $\mathfrak{q}$ and $\mathfrak{p} = [0 : p_1 : \dots : p_{n-1}]$ is tangent to X at $\mathfrak{p}$. It follows that (3.1) is dominant.

It remains to show that $\mathcal{T}|_{S^\circ}$ is G-unirational. We apply the no-name lemma to the vector bundle

$$\mathcal{T}|_{S^\circ} \times V \to S^\circ \times V,$$

which yields a G-equivariant birationality

$$\mathcal{T}|_{S^\circ} \times V \sim_G S \times V \times \mathbb{A}^{n-2},$$

with trivial action on $\mathbb{A}^{n-2}$. By our assumptions, $\mathcal{T}|_{S^\circ} \times V$ is G-unirational, implying the G-unirationality of $\mathcal{T}|_{S^\circ}$.

Remark 3.7. When S is a G-invariant linear subspace, e.g., a line, contained in X, the assumption that S is G-unirational follows from the assumption that G-acts on the ambient projective space via a linear

representation V. Thus, any cubic hypersurface of dimension ≥ 3 with a linear G-action containing a G-invariant line or a G-invariant plane is G-unirational by Proposition 3.5.

4. Quadrics

Stable linearizability of certain G-quadric threefolds $X \subset \mathbb{P}^4$ has been considered in [25, Sections 5 and 6]: all but one subgroup G of the Weyl group $W(\mathsf{D}_5)$, acting via signed permutations on the diagonal quadric

$$\sum_{i=1}^{5} x_i^2 = 0,$$

are stably linearizable, provided Condition **(A)** is satisfied. The only open case was $G = \mathfrak{D}_4$, acting on

$$\mathbb{P}(\chi_1 \oplus \chi_2 \oplus \chi_3 \oplus V_2),$$

where V_2 is the unique faithful 2-dimensional representation, and χ_1, χ_2, χ_3 are distinct characters of G. There are also many actions that do not arise from subgroups of $W(\mathsf{D}_5)$, e.g., an action of $\mathfrak{D}_8$ on the quadric

$$x_1^2 + x_2x_3 + x_4x_5 = 0,$$

arising from an action on

$$\mathbb{P}(\mathbf{1} \oplus V_2 \oplus V_2'),$$

where V_2 is a faithful representation and V_2' a nonfaithful 2-dimensional representation, see [25, Remark 6.6]. The following theorem covers these and other actions, settling the stable linearizability problem for actions of finite groups on smooth quadric threefolds. The key observation is that, when Condition **(A)** is satisfied, there always exists a G-invariant unirational quadric surface or conic in X and Proposition 3.1 applies.

Theorem 4.1. *Let $X \subset \mathbb{P}^4$ be a smooth quadric threefold, with a generically free action of a finite group G. Then the G-action is stably linearizable if and only if it satisfies Condition* **(A)**.

The rest of this section is devoted to the proof of this theorem.

By Theorem 2.1, stable linearizability of the G-action is equivalent to stable rationality of every twist ${}^TX$ over every field extension K/k. For quadrics, (stable) rationality and unirationality are equivalent over any field. Again by Theorem 2.1, this is equivalent to G-unirationality of the action. By [18, Theorem 10.2], G-unirationality is equivalent to G_2-unirationality, where G_2 is the 2-Sylow subgroup of G. We may also assume that G is nonabelian: abelian group actions on X satisfying Condition **(A)** are linearizable via the projection from a fixed point.

Since $X \subset \mathbb{P}^4$ is a G-invariant quadric, we know that $\mathcal{O}_X(1)$ is G-linearized. It follows that the G-action on the ambient $\mathbb{P}^4$ arises from the projectivization of a linear representation of G. Thus, to show Theorem 4.1, we now assume that G is a nonabelian 2-group and $X \subset \mathbb{P}(V)$, where V is a faithful representation of G. Since G is a 2-group, we have the following cases:

(1) $V = \chi_1 \oplus \chi_2 \oplus \chi_3 \oplus V_2$, where V_2 is an irreducible G-representation, and χ_j are characters of G for $j = 1, 2, 3$,
(2) $V = \chi_1 \oplus V_2 \oplus V_2'$, where V_2 and V_2' are irreducible G-representations,
(3) $V = \chi_1 \oplus V_4$, where V_4 is an irreducible representation of G.

We proceed with a case-by-case analysis.

Case (1): Up to isomorphism, we may assume that X is given by

$$X = \{x_1^2 + x_2^2 + x_3^2 + x_4 x_5 = 0\} \subset \mathbb{P}^4,$$

with G acting via characters on x_1, x_2, x_3, and via the irreducible representation V_2 on x_4, x_5. We claim that Condition **(A)** for X implies Condition **(A)** for the G-action on the conic $C \subset \mathbb{P}^2 = \mathbb{P}(\chi_1 \oplus \chi_2 \oplus \chi_3)$, given by $x_4 = x_5 = 0$. Note that the G-action on this $\mathbb{P}^2$ has a nontrivial generic stabilizer, since G is nonabelian, and the effective action on the conic C may not satisfy Condition **(A)**.

Since V_2 is irreducible, there exist elements of the form

$$\operatorname{diag}(\pm 1, \pm 1, \pm 1, a, a^{-1}) \in G,$$

where a is a 2-power root of unity of order $\operatorname{ord}(a) \geq 4$. Then

$$\varepsilon_1 = \operatorname{diag}(1, 1, 1, -1, -1) \in G.$$

Note that

$$X^{\langle \varepsilon_1 \rangle} = C \cup \mathfrak{p}_1 \cup \mathfrak{p}_2,$$

where

$$\mathfrak{p}_4 = [0 : 0 : 0 : 1 : 0], \quad \mathfrak{p}_5 = [0 : 0 : 0 : 0 : 1].$$

Let $A \subset G$ be an abelian subgroup containing ε_1, fixing points on X but not on C. Then A fixes $\mathfrak{p}_4, \mathfrak{p}_5$, and acts via C_2^2 on C. It follows that A contains

$$\varepsilon_2 = \operatorname{diag}(-1, 1, 1, a_1, a_1^{-1}), \quad \varepsilon_3 = \operatorname{diag}(1, -1, 1, b_1, b_1^{-1}),$$

for some 2-power roots of unity a_1, b_1. Depending on a_1 and b_1, the elements ε_2 and ε_3 generate at least one of the following elements

$$\operatorname{diag}(-1, 1, 1, -1, -1), \quad \operatorname{diag}(1, -1, 1, -1, -1), \quad \operatorname{diag}(1, 1, -1, -1, -1).$$

Without loss of generality, assume that A contains

$$\varepsilon_4 = \operatorname{diag}(-1, 1, 1, -1, -1).$$

By the irreducibility of V_2, G also contains an element switching $\mathfrak{p}_4$ and $\mathfrak{p}_5$. Up to multiplying by ε_2 and ε_3, we may assume that G contains

$$\sigma : (x_1, \dots, x_5) \mapsto (x_1, -x_2, x_3, a_2 x_5, a_2^{-1} x_4),$$

for some a_2. The subgroup

$$\langle \varepsilon_1, \varepsilon_4, \sigma \rangle \simeq C_2^3$$

does not fix points on X.

Let $A \subset G$ be an abelian subgroup not containing ε_1, fixing points on X but not on C. Then $\langle A, \varepsilon_1 \rangle$ is an abelian group whose fixed locus is $\{\mathfrak{p}_4, \mathfrak{p}_5\}$ or empty. In the first case, we repeat the argument above. We conclude that Condition **(A)** for the G-action on X implies Condition **(A)** for the (not generically free) G-action on C.

Next, we observe that G preserves the set $\{\mathfrak{p}_4, \mathfrak{p}_5\}$, which yields an exact sequence

$$0 \to N \to G \to C_2 \to 0,$$

where N is the maximal subgroup fixing $\mathfrak{p}_4$ and $\mathfrak{p}_5$, acting diagonally on $x_1, \dots, x_5$. Thus N is abelian. By [30, Lemma 3.1], $\mathrm{B}_0(G) = 0$, since G is an extension of a cyclic group by an abelian group. By Lemma 2.4, C is G-unirational since Condition **(A)** is satisfied on C. By Proposition 3.1, X is G-unirational.

Case (2): We may assume that G acts on

$$X = \{x_1^2 + x_2 x_3 + x_4 x_5 = 0\},$$

via χ_1 on x_1, via V_2 on x_2, x_3, and via V_2' on x_4, x_5. By the irreducibility of V_2 and V_2', G acts on $\mathbb{P}(V_2)$ and $\mathbb{P}(V_2')$ via dihedral groups. Let

$$\mathfrak{p}_2 = [0:1:0:0:0], \quad \mathfrak{p}_3 = [0:0:1:0:0],$$

$$\mathfrak{p}_4 = [0:0:0:1:0], \quad \mathfrak{p}_5 = [0:0:0:0:1].$$

Since G leaves invariant the sets $\{\mathfrak{p}_2, \mathfrak{p}_3\}$, and $\{\mathfrak{p}_4, \mathfrak{p}_5\}$ respectively, there is an exact sequence

$$0 \to N \to G \to Q \to 0, \tag{4.1}$$

where $Q = C_2$ or C_2^2, depending on the induced action on $\{\mathfrak{p}_2, \mathfrak{p}_3, \mathfrak{p}_4, \mathfrak{p}_5\}$, and $N \subset G$ is the maximal subgroup fixing these points; N consists of elements

$$\varepsilon_1 = \mathrm{diag}(1, t_1, t_1^{-1}, t_2, t_2^{-1}), \quad t_1, t_2 \text{ are 2-power roots of unity.} \tag{4.2}$$

The (nontrivial) center of G acts via scalars on V_2, V_2', i.e., it must contain one of the following

$$\varepsilon_2 = \mathrm{diag}(1, -1, -1, 1, 1), \quad \varepsilon_3 = \mathrm{diag}(1, 1, 1, -1, -1), \quad \varepsilon_2\varepsilon_3.$$

If $Q = C_2^2 = \langle (1,2),(3,4)\rangle$, we may assume that G also contains

$$m_1 : (x_1,\dots,x_5) \mapsto (x_1, a_1x_3, a_1^{-1}x_2, a_2x_4, a_2^{-1}x_5),$$
$$m_2 : (x_1,\dots,x_5) \mapsto (x_1, b_1x_2, b_1^{-1}x_3, b_2x_5, b_2^{-1}x_4),$$

where m_1 is a lift of $(1,2)$ and m_2 is a lift of $(3,4)$. After a change of variables, we have $a_1 = b_2 = 1$, and a_2 and b_1 are 2-power roots of unity:

$$m_1^2 = \mathrm{diag}(1,1,1,a_2^2,1/a_2^2), \quad m_2^2 = \mathrm{diag}(1,b_1^2,1/b_1^2,1,1).$$

If $a_2^2 \neq 1$ and $b_1^2 \neq 1$, then G contains ε_2 and ε_3. The abelian group

$$\langle \varepsilon_2, \varepsilon_3, m_1m_2\rangle \simeq C_2^3 \tag{4.3}$$

does not fix points on X. If $a_2^2 = 1$ and $b_1^2 \neq 1$, then $\varepsilon_2 \in G$, and N contains an element ε_1 as in (4.2) with $t_2 \neq \pm 1$ since otherwise the representation on x_4, x_5 is reducible. If $\mathrm{ord}(b_1) \geq \mathrm{ord}(t_1)$, or $\mathrm{ord}(t_2) \geq \mathrm{ord}(t_1)$, then ε_3 is generated by m_2^2 and ε_1, and G contains the C_2^3 from (4.3) with no fixed points. If $\mathrm{ord}(t_1) > \mathrm{ord}(b_1), \mathrm{ord}(t_2)$, one can find $r_1, r_2, r_3 \in \mathbb{Z}$ such that

$$m_1^{r_1}\varepsilon_1^{r_2} : (x_1,\dots,x_5) \mapsto (x_1, c_1x_3, c_1^{-1}x_2, -x_4, -x_5),$$
$$m_2\varepsilon_1^{r_3} : (x_1,\dots,x_5) \mapsto (x_1, x_2, x_3, c_2x_5, c_2^{-1}x_4)$$

for some c_1, c_2. Then

$$\langle m_1^{r_1}\varepsilon_1^{r_2}, m_2\varepsilon_1^{r_3}, \varepsilon_2\rangle \simeq C_2^3$$

does not fix points on X. By symmetry, the same applies when $b_1^2 = 1, a_2^2 \neq 1$.

We are reduced to the case $a_2^2 = b_1^2 = 1$, with $\langle m_1, m_2\rangle \simeq C_2^2$. By the irreducibility of V_2, N contains an ε_1 as in (4.2) of order at least 4. We repeat the argument above to find an abelian group failing Condition **(A)**, after multiplying m_1 and m_2 by ε_1.

Thus, if the G-action on X satisfies Condition **(A)**, we have

$$Q = C_2 = \langle (1,2)(3,4)\rangle$$

and G contains a lift of $(1,2)(3,4)$ of the form

$$m_3 : (x_1,\dots,x_5) \mapsto (x_1, d_1x_3, d_1^{-1}x_2, d_2x_5, d_2^{-1}x_4)$$

for some d_1, d_2. We discuss cases based on the center of G:

- If $\varepsilon_2, \varepsilon_3 \in G$, then
$$\langle \varepsilon_2, \varepsilon_3, m_3\rangle \simeq C_2^3$$
has no fixed points on X.
- If $\varepsilon_2\varepsilon_3 \in G$, then every abelian subgroup of G fixes a point on the quadric $S = X \cap \{x_1 = 0\}$. Otherwise, we can find an abelian group without fixed points by adjoining $\varepsilon_2\varepsilon_3$.

- If $\varepsilon_2 \in G$ and $\varepsilon_3 \notin G$, then every abelian subgroup $A \subset G$ fixes a point on $C = X \cap \{x_2 = x_3 = 0\}$. Otherwise, $A' = \langle A, \varepsilon_2 \rangle$ fixes $\{\mathfrak{p}_2, \mathfrak{p}_3\}$, and does not surject onto Q, which implies that it also fixes $\mathfrak{p}_4, \mathfrak{p}_5 \in C$, contradiction.
- If $\varepsilon_3 \in G$ and $\varepsilon_2 \notin G$, the same argument shows that every abelian subgroup of G fixes a point on $C = X \cap \{x_4 = x_5 = 0\}$.

Note that we have $\mathrm{B}_0(G) = 0$, since G is an extension of C_2 by an abelian group. In the last two cases, the corresponding conic is G-unirational by Lemma 2.4, and X is G-unirational by Proposition 3.1. In the second case, the G-action on the quadric $S = \mathbb{P}^1 \times \mathbb{P}^1$ does not switch the two rulings. By Lemma 2.4, we can lift the G-action on each $\mathbb{P}^1$ factor to $\mathbb{A}^2$, i.e., the G-action on S to $\mathbb{A}^2 \times \mathbb{A}^2 = \mathbb{A}^4$. It follows that S is G-unirational. By Proposition 3.1, X is also G-unirational.

Case (3): We show that Condition **(A)** is never satisfied. Since V_4 is irreducible, there are no fixed points in $\mathbb{P}(V_4)$, and $S := \mathbb{P}(V_4) \cap X$ is smooth. We may assume that

$$X = \{x_1^2 + x_2x_5 - x_3x_4 = 0\} \subset \mathbb{P}^4 = \mathbb{P}(\mathbf{1} \oplus V_4),$$

with G acting trivially on x_1, and via V_4 on x_2, x_3, x_4, x_5. Since V_4 is faithful and G preserves X, the center of G is C_2, generated by

$$\varepsilon_1 = \mathrm{diag}(1, -1, -1, -1, -1).$$

Finite 2-subgroups of $\mathrm{Aut}(S) = \mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ are subgroups of $\mathfrak{D}_n \wr C_2$, for n some power of 2. This yields 4 distinguished points

$$\mathfrak{p}_2 = [0:1:0:0:0], \quad \mathfrak{p}_3 = [0:0:1:0:0],$$

$$\mathfrak{p}_4 = [0:0:0:1:0], \quad \mathfrak{p}_5 = [0:0:0:0:1],$$

and an exact sequence

$$0 \to N \to G \to Q \to 0,$$

where N fixes each $\mathfrak{p}_i$. Since V_4 is irreducible, the image $Q \subset \mathfrak{S}_4$, acting via permutations on $\{\mathfrak{p}_2, \mathfrak{p}_3, \mathfrak{p}_4, \mathfrak{p}_5\}$, can be C_2^2, C_4, or $\mathfrak{D}_4$. The subgroup N is generated by elements of the form

$$\mathrm{diag}(1, t_2, t_1, t_1^{-1}, t_2^{-1}), \quad t_1, t_2 \text{ are 2-power roots of unity.}$$

We proceed to analyze each case.

- $Q = C_2^2$, with lifts of $(1,2)(3,4)$ and $(1,3)(2,4)$:

$$m_1 : (x_1, \dots, x_5) \mapsto (x_1, a_2x_3, a_1x_2, -a_1^{-1}x_5, -a_2^{-1}x_4), \tag{4.4}$$
$$m_2 : (x_1, \dots, x_5) \mapsto (x_1, b_2x_4, b_1x_5, -b_1^{-1}x_2, -b_2^{-1}x_3),$$

for some a_1, a_2, b_1, b_2. Put $s_1 = a_1a_2, s_2 = b_2/b_1$ so that

$$m_1^2 = \mathrm{diag}(1, s_1, s_1, 1/s_1, 1/s_1),$$
$$m_2^2 = \mathrm{diag}(1, -s_2, -1/s_2, -s_2, -1/s_2).$$

Since G is a 2-group, s_1 and s_2 are 2-power roots of unity. When the orders $\mathrm{ord}(s_1), \mathrm{ord}(s_2) \geq 4$, we can find integers r_1, r_2 such that

$$s_2^{r_2} = \zeta_4, \quad s_1^{r_1} = \zeta_4^3.$$

Setting

$$\varepsilon_3 := \varepsilon_1^{r_2} m_1^{2r_1} m_2^{2r_2} = \mathrm{diag}(1, 1, -1, -1, 1),$$

we have $S^{\langle \varepsilon_1, \varepsilon_3 \rangle} = \{\mathfrak{p}_2, \dots, \mathfrak{p}_5\}$. The abelian group

$$\langle \varepsilon_3, m_1 m_2, \varepsilon_1 \rangle \simeq C_2^3 \tag{4.5}$$

has no fixed points on X. Thus, s_1 or s_2 equals ± 1. By symmetry, we may assume that $s_2 = \pm 1$. Then the line

$$\mathfrak{l} := \{x_1 = x_2 + x_3\sqrt{a_2/a_1} = x_4 \pm x_5\sqrt{a_2/a_1} = 0\} \subset \mathbb{P}(V)$$

is $\langle m_1, m_2 \rangle$-invariant. By the irreducibility of V_4, the kernel N contains an element ε_2 not leaving $\mathfrak{l}$ invariant. In particular, ε_2 takes the form

$$\varepsilon_2 = \mathrm{diag}(1, t_2, t_1, t_1^{-1}, t_2^{-1}) \neq \varepsilon_1, \tag{4.6}$$

for some 2-power roots of unity t_1, t_2 such that $t_1 \neq t_2$. If $\mathrm{ord}(t_2/t_1) \geq 4$, we repeat the argument above, replacing m_1 by $\varepsilon_2 m_1$ to find a C_2^3 with no fixed points on X. Thus we are reduced to $t_1 = -t_2$.

Similarly, if $s_1 = \pm 1$, we are reduced to $t_1 = -1/t_2$. In particular, for $s_1, s_2 \in \{\pm 1\}$, we are reduced to $t_1, t_2 \in \{\pm 1\}$. In this case, G contains ε_3 and a C_2^3 with no fixed points as in (4.5).

We consider the case when $s_1 \neq \pm 1$. As is discussed, we may assume $t_1 = -t_2$. If $\mathrm{ord}(t_1) \leq \mathrm{ord}(s_1)$, then $\varepsilon_3 \in \langle \varepsilon_2, m_1^2 \rangle \subset G$, and G contains a C_2^3 with no fixed points. In the opposite case, there is a $d_1 \in \mathbb{Z}$ such that

$$t_1^{2d_1} = (-1)^{d_1}/s_1.$$

Setting $d_2 = (3 - s_2)/2$, we find that

$$\langle m_1 \varepsilon_2^{d_1}, m_2 \varepsilon_2^{d_2}, \varepsilon_1 \rangle \simeq C_2^3$$

has no fixed points on X.

- $Q = \mathfrak{D}_4$, with G containing m_1, m_2 as in (4.4). If G contains

$$\varepsilon_3 = \mathrm{diag}(1, 1, -1, -1, 1),$$

there is a C_2^3 with no fixed points on X. To show this, we are reduced to the case when $s_1 = \pm 1$ or $s_2 = \pm 1$, as above. Consider the lift of $(1,3,4,2) \in \mathfrak{S}_4$:

$$m_3 : (x_1, \dots, x_5) \mapsto (x_1, c_2 x_4, c_1 x_2, -c_1^{-1} x_5, -c_2^{-1} x_3), \tag{4.7}$$

for some c_1, c_2. Observe that

$$\begin{aligned} (m_2 m_3)^2 &= \mathrm{diag}(1, c_2^2/b_1^2, 1, 1, b_1^2/c_2^2), \\ (m_1 m_3)^2 &= \mathrm{diag}(1, 1, a_2^2 c_1^2, 1/(a_2^2 c_1^2), 1). \end{aligned} \tag{4.8}$$

We may assume that $c_1 = \pm 1/a_2$ and $c_2 = \pm b_1$, otherwise ε_3 can be generated by elements in (4.8). Assume $c_1 = 1/a_2$ and $c_2 = b_1$ (the other cases are similar). If $s_1 \neq -s_2$, ε_3 is some power of $m_1 m_2 m_3^2$. When $s_1 = -s_2 = \pm 1$, we find an $\langle m_1, m_2, m_3 \rangle$-invariant line:

$$\mathfrak{l} = \{x_1 = x_2 + \frac{b_1}{a_1} x_5 = x_3 \mp b_1 a_1 x_4 = 0\} \subset V_4.$$

Thus N contains an ε_2 as in (4.6) which does not leave $\mathfrak{l}$ invariant. This implies that $t_1^2, t_2^2 \neq 1$. As above, when $t_1 \neq \pm t_2$ or $t_1 t_2 \neq \pm 1$, we have $\varepsilon_3 \in G$. When $t_1 = \pm t_2$ and $t_1 t_2 = \pm 1$, we know $t_1^2 = t_2^2 = -1$. Then $\varepsilon_3 = (m_1 m_3 \varepsilon_2)^2 \in G$.

• $Q = C_4$, with G containing a lift m_3 of $(1,3,4,2) \in \mathfrak{S}_4$ as in (4.7). Once we have $\varepsilon_3 \in G$, the abelian group

$$\langle \varepsilon_1, \varepsilon_3, m_3^2 \rangle \simeq C_2^3$$

has no fixed points on X. If follows from the irreducibility of V_4 that N contains an element ε_2 as in (4.6). When $\mathrm{ord}(t_1) \neq \mathrm{ord}(t_2)$, we know that ε_3 is a power of ε_2. When $t_1^2 = t_2^2 = 1$, $\varepsilon_2 = \varepsilon_3$. We are reduced to

$$t_1 = \zeta_{2^n}^{2d_1+1}, \quad t_2 = \zeta_{2^n}^{2d_2+1}, \quad \text{for some} \quad n \geq 2, \quad d_1, d_2 \in \mathbb{Z}.$$

We have

$$\varepsilon_2 m_3 \varepsilon_2^{-1} m_3^{-1} = \mathrm{diag}(1, t_2/t_1, t_1 t_2, 1/(t_1 t_2), t_1/t_2).$$

Observe that $\mathrm{ord}(t_2/t_1) \neq \mathrm{ord}(t_1 t_2)$. Indeed,

$$t_2/t_1 = \zeta_{2^n}^{2(d_2-d_1)}, \quad t_1 t_2 = \zeta_{2^n}^{2(d_1+d_2+1)},$$

and $d_2 - d_1$ has a different parity from $d_1 + d_2 + 1$. Thus, either ε_3 or $\varepsilon_1 \varepsilon_3$ is a power of $\varepsilon_2 m_3 \varepsilon_2^{-1} m_3^{-1}$, and thus is in G.

This completes the proof of Theorem 4.1.

We include an alternative proof of Theorem 4.1 which requires less computation; it was suggested by the referee. Recall the Goursat lemma

from group theory (see, e.g., [16, Lemma 4.1]): a subgroup of a product group $G_1 \times G_2$ which surjects to G_1 and G_2 via the natural projections has the form

$$G_1 \times_H G_2 = \{(g_1, g_2) \in G_1 \times G_2 : \varphi_1(g_1) = \varphi_2(g_2)\},$$

where H is a group admitting surjective homomorphisms

$$\varphi_1 : G_1 \to H, \quad \varphi_2 : G_2 \to H.$$

One can use this fact to simplify the proof of Theorem 4.1. As above, there are three cases based on the splitting of V, and we retain the corresponding notation. Note that we may assume that χ_1 is the trivial character.

Case (1): We have homomorphisms

$$\pi_1 : G \to \mathrm{Aut}(C), \quad \pi_2 : G \to \mathsf{GL}(V_2).$$

We may assume that $\pi_1(G) = C_2^2$ since otherwise G fixes a point on C. Since χ_1 is the trivial character, we know that $\pi_2(G)$ leaves invariant the form x_4x_5, which implies that $\pi_2(G) = \mathfrak{D}_{2^n}$ for some $n \geq 2$. Similarly, we also have that $\mathrm{Ker}(\pi_1) \cap \mathrm{Ker}(\pi_2) = 1$. It follows that

$$G = C_2^2 \times_H \mathfrak{D}_{2^n},$$

for $H = 1, C_2$, or C_2^2. If $H = 1$ or C_2, then G contains a subgroup C_2^3 generated by the elements

$$(a, 1), \quad (1, c), \quad (b, s),$$

where $a \neq b \in C_2^2$, c is the nontrivial element in the center of $\mathfrak{D}_{2^n}$, and $s \in \mathfrak{D}_{2^n}$ is a reflection (a non-central involution). The fixed locus of $(1, c)$ on X is $C \cup \mathfrak{p}_2 \cup \mathfrak{p}_3$, but s permutes $\mathfrak{p}_2 \cup \mathfrak{p}_3$ and a, b act on C via C_2^2 with no fixed points. Thus the G-action on X fails Condition **(A)**.

Case (2): Similarly, we have homomorphisms $\pi_1 : G \to \mathsf{GL}(V_2)$ and $\pi_2 : G \to \mathsf{GL}(V_2')$ such that $\pi_1(G) = \mathfrak{D}_{2^n}$, $\pi_2(G) = \mathfrak{D}_{2^m}$ for some integers $n, m \geq 1$, and $\mathrm{Ker}(\pi_1) \cap \mathrm{Ker}(\pi_2) = 1$. We may assume that $n \leq m$. It follows that

$$G \simeq \mathfrak{D}_{2^n} \times_H \mathfrak{D}_{2^m},$$

where $H = 1, C_2$, or $\mathfrak{D}_{2^r}$ for some $1 \leq r \leq n$. If $H \neq \mathfrak{D}_{2^n}$, then G contains a subgroup C_2^3 generated by elements

$$(c_1, 1), \quad (1, c_2), \quad (s_1, s_2),$$

where c_1 and c_2 are in the center of $\mathfrak{D}_{2^n}$ and $\mathfrak{D}_{2^m}$, respectively, and s_1 and s_2 are reflections in $\mathfrak{D}_{2^n}$ and $\mathfrak{D}_{2^m}$, respectively. The fixed locus of

$\langle (c_1, 1), (1, c_2)\rangle$ is $\mathfrak{p}_2 \cup \mathfrak{p}_3 \cup \mathfrak{p}_4 \cup \mathfrak{p}_5$, and (s_1, s_2) switches $\mathfrak{p}_2 \leftrightarrow \mathfrak{p}_3$, $\mathfrak{p}_4 \leftrightarrow \mathfrak{p}_5$. The G-action fails Condition **(A)**.

When $H = \mathfrak{D}_{2^n}$, the natural surjection $G \to \mathfrak{D}_{2^m}$ is an isomorphism. If $n < m$, then the image of every abelian subgroup of G under the projection $G \to \mathfrak{D}_{2^n}$ is a cyclic group. This implies that every abelian subgroup of G fixes a point on the conic $C = X \cap \{x_4 = x_5 = 0\}$. Since $\mathrm{B}_0(G) = 0$, we know that C is G-unirational and thus X is G-unirational, by Proposition 3.1. If $n = m$, then $(c, c) \in G$, where c is the center of $\mathfrak{D}_{2^n}$. We know that (c, c) acts via $\mathrm{diag}(1, -1, -1, -1, -1)$. Arguing as in the last paragraph of the previous proof of Case (2), we conclude that X is G-unirational if Condition **(A)** is satisfied.

Case (3): Recall that $\varepsilon_1 \in G$, where $\varepsilon_1 = \mathrm{diag}(1, -1, -1, -1, -1)$, and $G' = G/\langle \varepsilon_1\rangle$ acts generically freely on $S = X \cap \{x_1 = 0\}$. We have

$$\mathrm{Aut}(S) = (\mathsf{PGL}_2(k) \times \mathsf{PGL}_2(k)) \rtimes C_2.$$

It follows that

$$G = \mathfrak{D}_{2^n} \times_H \mathfrak{D}_{2^m}, \quad \text{or} \quad (\mathfrak{D}_{2^n} \times_H \mathfrak{D}_{2^n}).C_2,$$

where $1 \le n \le m$, $H = 1, C_2, \mathfrak{D}_{2^r}$, for some $r \le n$. Here the action of $\mathfrak{D}_{2^n} \times \mathfrak{D}_{2^m}$ on $\mathbb{P}(V_4)$ is given by

$$\begin{aligned}
&r_1 : (x_2, \dots, x_5) \mapsto (\zeta_{2^{n+1}} x_2, \zeta_{2^{n+1}} x_3, \zeta_{2^{n+1}}^{-1} x_4, \zeta_{2^{n+1}}^{-1} x_5),\\
&s_1 : (x_2, \dots, x_5) \mapsto (x_4, x_5, x_2, x_3),\\
&r_2 : (x_2, \dots, x_5) \mapsto (\zeta_{2^{m+1}} x_2, \zeta_{2^{m+1}}^{-1} x_3, \zeta_{2^{m+1}} x_4, \zeta_{2^{m+1}}^{-1} x_5),\\
&s_2 : (x_2, \dots, x_5) \mapsto (x_3, x_2, x_5, x_4),
\end{aligned}$$

and the C_2 switching two factors of $S = \mathbb{P}^1 \times \mathbb{P}^1$ is given by

$$t : (x_2, \dots, x_5) \mapsto (x_2, x_4, x_3, x_5).$$

As above, when $H \ne \mathfrak{D}_{2^n}$, G' contains elements

$$(c_1, 1), \quad (1, c_2), \quad (s_1', s_2'),$$

where c_1 and c_2 are in the center of $\mathfrak{D}_{2^n}$ and $\mathfrak{D}_{2^m}$ respectively, and s_1' and s_2' are reflections in $\mathfrak{D}_{2^n}$ and $\mathfrak{D}_{2^m}$ respectively. Let c_i' be a lift of $(c_i, 1)$ in G, for $i = 1, 2$, and s' be a lift of (s_1', s_2') in G. Then up to multiplying by ε_1, we have that

$$c_1' = \mathrm{diag}(1, \zeta_4, \zeta_4, -\zeta_4, -\zeta_4), \quad c_2' = \mathrm{diag}(1, \zeta_4, -\zeta_4, \zeta_4, -\zeta_4),$$

$$s' : (x_1, \dots, x_5) \mapsto (x_1, a x_5, b x_4, b^{-1} x_3, a^{-1} x_2),$$

where a, b are roots of unity. It follows that the subgroup C_2^3 generated by $(c_1')^2, c_1' c_2'$, and s' has no fixed points on X, failing Condition **(A)**.

If $H = \mathfrak{D}_{2^n}$ and $G' = \mathfrak{D}_{2^n} \times_H \mathfrak{D}_{2^m} \simeq \mathfrak{D}_{2^m}$, then G' is generated by $r_1^u r_2$ and $s_1' s_2$, for some odd integer u and s_1' is a reflection in $\mathfrak{D}_{2^n}$. It follows that the line $\{x_1 = x_2 = x_5 = 0\}$ is G-invariant, contradicting the irreducibility of V_4.

When $H = \mathfrak{D}_{2^n}$ and $G' = (\mathfrak{D}_{2^n} \times_H \mathfrak{D}_{2^n}).C_2$, then G' contains elements (c, c) and (s_1', s_2') where c is in the center of $\mathfrak{D}_{2^n}$, and s_1' and s_2' are reflections. Let c' be the lift of (c, c) in G, and s' be the lift of (s_1', s_2') in G. As above, the abelian subgroup generated by c', s' and ε_1 does not fix a point on X, failing Condition **(A)**.

5. G-unirationality of smooth cubic threefolds

In this section, we prove:

Theorem 5.1. *Let $X \subset \mathbb{P}^4$ be a smooth cubic threefold with a generically free regular action of a finite group G. Assume that the action satisfies Condition* **(A)**. *Then X is G-unirational, with the possible exception of the following actions:*

- *$G = C_9 \rtimes C_3$, and X is the Fermat cubic threefold.*
- *$G = \mathsf{PSL}_2(\mathbb{F}_{11})$, or $G = C_{11} \rtimes C_5 \subset \mathsf{PSL}_2(\mathbb{F}_{11})$, and X is the Klein cubic threefold.*
- *$G = \mathfrak{A}_5$, acting on $X \subset \mathbb{P}(V_5)$, where V_5 is the irreducible 5-dimensional representation. There is a 1-parameter family of such cubics, this is the pencil generated by the Klein and the Segre cubic threefolds.*

For applications of results of Section 3 to cubic threefolds, we need to address G-unirationality of *singular* cubic surfaces, a case not considered in [17]. The proof of the following proposition is suggested by the referee.

Proposition 5.2. *Let $S \subset \mathbb{P}^3$ be a normal rational cubic surface, with a generically free regular action of a finite group G. If the G-action satisfies condition* **(A)** *then it is G-unirational.*

Proof. Let $\pi : \tilde{S} \to S$ be a G-equivariant minimal resolution of singularities. Let S' be the output of G-MMP applied to $\tilde{S}$. Then S' is a G-minimal surface with $K_{S'}^2 \geq 4$. In particular, S' is either a G-minimal del Pezzo surface or a G-minimal conic bundle. If S' is a del Pezzo surface, it is G-unirational by [17, Theorem 1.4]. If S' is a conic bundle which is not a del Pezzo surface, it is either the Hirzebruch surface F_2, or an Iskovskikh surface with two disjoint (-2) curves. The preimage of (-2)-curve(s) in S is an orbit of singular points of length 1 or 2. Thus, S is G-unirational by Corollary 3.4. □

Remark 5.3. One can also prove Proposition 5.2 case by case. All combinations of singularities of cubic surfaces are known classically, see [7]; their automorphisms are partially classified in [38]. Examining possible combinations of singularities [38, Table 1] and applying Corollary 3.4, we obtain G-unirationality in all cases, with the possible exception of $3\mathsf{A}_1$, $3\mathsf{A}_2$, and $4\mathsf{A}_1$ singularities. In the last case, the Cayley cubic surface, the action is birational to a linear action, via the standard Cremona involution, see [10, Section 5] and [8, Section 7]. The $3\mathsf{A}_2$-cubic surface is toric, and G-equivariantly birational to a del Pezzo surface of degree 6, thus G-unirational when it satisfies Condition **(A)**, by [17].

Any $3\mathsf{A}_1$-cubic surface S with an action of a subgroup $G \subseteq \mathrm{Aut}(S)$ is G-unirational. Indeed, we may assume the singular points of S are

$$[1:0:0:0], \quad [0:1:0:0], \quad [0:0:1:0].$$

By linear algebra, S is given by

$$\{x_1x_2x_3 + (x_1 + x_2 + ax_3)x_4^2 + x_4^3 = 0\}, \tag{5.1}$$

for some $a \neq 0$, with $\mathrm{Aut}(S) = \mathfrak{S}_3$, generated by

$$\sigma_1 : (x_1, x_2, x_3, x_4) \mapsto (x_2, x_1, x_3, x_4),$$
$$\sigma_2 : (x_1, x_2, x_3, x_4) \mapsto (ax_3, x_1, \frac{x_2}{a}, x_4).$$

Indeed, it is clear that $\sigma_1, \sigma_2 \in \mathrm{Aut}(S)$. From the equation, one sees that there are no nontrivial elements in $\mathrm{Aut}(S)$ fixing all three nodes. Therefore, $\mathrm{Aut}(S) = \langle \sigma_1, \sigma_2 \rangle$. Furthermore, $\mathrm{Aut}(S)$ fixes three smooth points on S, implying that S is $\mathrm{Aut}(S)$-unirational, by Proposition 3.2.

We also need the following lemmas.

Lemma 5.4. *Let $S \subset \mathbb{P}^3$ be a normal rational cubic surface, with a generically free regular action of a finite abelian group G. If G does not fix any point on S, then $G \simeq C_3^2$ or C_3^3.*

Proof. The G-action on $\mathbb{P}^3$ lifts to $\mathsf{GL}_4(k)$, and thus we may assume that it is diagonal. When G does not fix points on S, we know that the defining equation of S in $\mathbb{P}^3_{x_1,\dots,x_4}$ contains terms $x_1^3, x_2^3, x_3^3, x_4^3$, since otherwise one of the coordinate points

$$[1:0:0:0], \quad [0:1:0:0], \quad [0:0:0:1], \quad [0:0:0:1]$$

is a G-fixed point on S. It follows that any element $g \in G$ has order 3, i.e., $G \simeq C_3, C_3^2, C_3^3$. A cyclic group action on S always has fixed points. Thus $G \simeq C_3^2$ or C_3^3. □

Lemma 5.5. *Let $X \subset \mathbb{P}^4$ be a smooth cubic threefold threefold with a generically free regular action of a finite group G. Assume that G does*

not contain a subgroup isomorphic to C_3^2, and there is a G-invariant hyperplane section S in X. Then the G-action on X is equivariantly unirational.

Proof. Since X is smooth, we know that S is normal and not a cone. When S has no generic stabilizer, it follows from Lemma 5.4 that the G-action on S satisfies Condition **(A)**, and thus is equivariantly unirational by Proposition 5.2. From Proposition 3.5, the G-action on X is equivariantly unirational.

When S has a generic stabilizer, necessarily a cyclic group C_n. We may assume that

$$S = X \cap \{x_1 = 0\} \subset \mathbb{P}^4_{x_1,\dots,x_5},$$

and G acts trivially on x_1. We know that C_n is a subgroup of the center of G. The group $\bar{G} = G/C_n$ acts generically freely on the cubic surface $S \subset \mathbb{P}^3_{x_2,\dots,x_5}$. It follows that the $\bar{G}$-action lifts to $\mathsf{GL}_4(k)$, i.e., G is a split central extension of $\bar{G}$ by C_n. Thus $G \simeq C_n \times \bar{G}$. By assumptions, $\bar{G}$ does not contain a subgroup isomorphic to C_3^2. Then the $\bar{G}$-action on S is equivariantly unirational by Proposition 5.2, and thus the G-action on S is equivariantly unirational. Proposition 3.5 implies that the G-action on X is equivariantly unirational. □

The rest of this section is devoted to a proof of Theorem 5.1. Actions of finite groups on smooth cubic threefolds $X \subset \mathbb{P}^4$ have been classified by Wei and Yu in [40]. More precisely, they find all groups that can act, and give examples of cubic threefolds that realize such actions, [40, Theorem 1.1 and Section 3]. We recall the maximal nonabelian groups, and the unique cubic threefolds admitting such actions, in the notation of [40]:

	Cubic	Automorphism
X_1	$\sum_{i=1}^5 x_i^3 = 0$	$C_3^4 \rtimes \mathfrak{S}_5$
X_2	$3(\sqrt{3}-1)x_1x_2x_3 + \sum_{i=1}^5 x_i^3 = 0$	$((C_3^2 \rtimes C_3) \rtimes C_4) \times \mathfrak{S}_3$
X_5	$x_1^2x_2 + x_2^2x_3 + x_3^2x_4 + x_4^2x_5 + x_5^2x_1 = 0$	$\mathsf{PSL}_2(\mathbb{F}_{11})$
X_6	$\sum_{i=1}^6 x_i^3 = \sum_{i=1}^5 x_i = 0$	$C_3 \times \mathfrak{S}_5$

TABLE 1

Using `magma`, we find that there are 82 *nonabelian* subgroups of the maximal groups appearing in Table 1 in total.

A table of abelian actions is also in [40, Appendix B]. In particular, they provide a list of all abelian groups that admit a unique action on a smooth cubic threefold. There are 12 isomorphism classes of such groups. We compile a list of nonabelian groups which contain one of these abelian groups in [12]: these groups also admit unique actions on smooth cubic threefolds. There are 41 such groups, up to isomorphism. We note that several of them admit nonconjugated actions on the same cubic – there are 50 conjugacy classes of actions. All the actions are realized on one of the cubic threefolds in Table 1.

Now, we study the 82 isomorphism classes of nonabelian subgroups. We divide them into three types:

(1) groups whose orders are divisible by 5,
(2) groups where their orders are not divisible by 5 and do not contain a subgroup isomorphic to C_3^2,
(3) the remaining groups.

Type (1): First we consider groups G where $5 \mid |G|$. These groups can possibly have 5-dimensional irreducible representations. The isomorphism classes are

- $\mathfrak{D}_5, \mathfrak{F}_5, C_3 \times \mathfrak{D}_5$: the corresponding cubics are G-unirational by Proposition 3.3, since in each case, G contains an index-2 cyclic subgroup which fixes a point on the cubic.
- $\mathfrak{C}_3^4 \rtimes C_5, C_3^4 \rtimes \mathfrak{D}_5, C_3^4 \rtimes \mathfrak{F}_5, C_3^4 \rtimes \mathfrak{A}_5, C_3^4 \rtimes \mathfrak{S}_5$: these groups contain C_3^4, which only acts on the Fermat cubic threefold with no fixed points. The corresponding actions are not equivariantly unirational.
- $C_3 \times \mathfrak{F}_5, C_3 \times \mathfrak{A}_5, C_3 \times \mathfrak{S}_5$: these groups act on a unique smooth cubic threefold, X_6. Each action leaves invariant a hyperplane section, which is isomorphic to the Clebsch cubic surface, with generic stabilizer C_3, and a residual action of $\mathfrak{S}_5$, $\mathfrak{A}_5$, and $\mathfrak{F}_5$, respectively. These actions on the Clebsch cubic surface are unirational. By Proposition 3.5, the respective action on X_6 is unirational.
- $\mathfrak{S}_5$: from representation theory, one can check that any $\mathfrak{S}_5$-action on a smooth cubic threefold leaves invariant a hyperplane section. The action is equivariantly unirational by Lemma 5.5.
- $\mathfrak{A}_5$: from representation theory, an $\mathfrak{A}_5$-action on a smooth cubic threefold either leaves invariant a hyperplane section, or is irreducible on $\mathbb{P}^4$. In the former case, the action is equivariantly unirational by Lemma 5.5. In the latter case, its equivariant unirationality remains open. There is a unique such action on $\mathbb{P}^4$, and

a pencil of invariant cubic threefolds such that a general member of the pencil is smooth.
- $C_{11} \rtimes C_5, \mathsf{PSL}_2(\mathbb{F}_{11})$: these groups act on a unique smooth cubic threefold, the Klein cubic threefold X_5. Their equivariant unirationalities are open.

Type (2): Next, we turn to groups G where $5 \nmid |G|$ and G does not contain a subgroup isomorphic to C_3^2. We show that such actions are always equivariantly unirational.

Lemma 5.6. *Let $X \subset \mathbb{P}(V) = \mathbb{P}^4$ be a smooth cubic threefold with a generically free regular action of a finite group G, where V is a linear representation of G. Assume that $5 \nmid |G|$ and G does not contain a subgroup isomorphic to C_3^2. Then the G-action on X is equivariantly unirational.*

Proof. Since $5 \nmid |G|$, we know that V is reducible. If V contains a 1-dimensional subrepresentation, the G-action on X is equivariantly unirational by Lemma 5.5. It suffices to consider when $V = V_2 \oplus V_3$ where V_2 and V_3 are irreducible 2 and 3-dimensional representations of G. Thus $l = \mathbb{P}(V_2)$ is a invariant line in $\mathbb{P}^4$. If $l \subset X$, then the G-action on X is equivariantly unirational by Proposition 3.5. When $l \cap X$ consists of a G-orbit of 1 or 2 points, the G-action on X is equivariantly unirational by Proposition 3.3. When $l \cap X$ consists of a G-orbit of 3 points, G acts on l via C_3 or $\mathfrak{S}_3$. There exists a subgroup $G' \subseteq G$ of index 1 or 2 respectively such that G' fixes a point on l, i.e., G' leaves invariant a hyperplane section on X. It follows from Lemma 5.5 that the G'-action on X is equivariantly unirational. Corollary 3.4 then implies that the G-action on X is equivariantly unirational. □

Type (3): We consider the remaining groups, i.e., those groups G containing a subgroup isomorphic to C_3^2 and $5 \nmid |G|$. There are 66 isomorphism classes. Among them, 33 groups have a normal abelian subgroup G' such that G/G' is a 2-group. Actions of these groups on smooth cubic threefolds are equivariantly unirational if Condition **(A)** is satisfied, by Proposition 3.3.

Among the remaining 33 isomorphism classes, there are 20 groups which act on a unique cubic threefold: 17 of them act on the Fermat cubic X_1, and 3 of them act on X_2. For actions of these 20 groups G on the corresponding cubics X, we check via computations that one of the following holds:

- the G-action fixes a point on X,

- the G-action on X fails Condition **(A)**,
- $X = X_1$ is the Fermat cubic threefold and $G = C_9 \rtimes C_3$ is generated by

$$\mathrm{diag}(1, 1, \zeta_3, \zeta_3^2, 1), \quad \mathrm{diag}(1, 1, 1, \zeta_3, \zeta_3^2), \quad (x_1, \zeta_3 x_2, x_4, \zeta_3 x_5, x_3).$$

The equivariant unirationality of the last action remains open.

There are 13 isomorphism classes left:

$$C_3 \times \mathrm{He}_3, \ C_3 \times C_3^2 \rtimes \mathfrak{S}_3, \ \mathrm{He}_3 \rtimes \mathfrak{S}_3, \ \mathfrak{S}_3 \times \mathrm{He}_3, \quad \mathfrak{S}_3 \times C_3^2 \rtimes \mathfrak{S}_3, \ (C_3 \times \mathrm{He}_3) \rtimes C_4,$$
$$\mathrm{He}_3, \quad C_3^2 \rtimes \mathfrak{S}_3, \quad C_2 \times \mathrm{He}_3, \quad C_2 \times C_3^2 \rtimes \mathfrak{S}_3, \quad C_3^2 \rtimes \mathfrak{S}_3.C_2,$$
$$C_3 \times \mathfrak{A}_4, \quad C_3 \times \mathfrak{S}_4,$$

All groups in the first row contain a subgroup isomorphic to $C_3 \times \mathrm{He}_3$; all groups G in the second row contain a normal subgroup isomorphic to He_3 and G/He_3 is a 2-group.

We focus on four groups: $G = \mathrm{He}_3$, $C_3 \times \mathrm{He}_3$, $C_3 \times \mathfrak{A}_4$ and $C_3 \times \mathfrak{S}_4$. Using `magma`, we can find all G-actions on $\mathbb{P}^4$, and their invariant cubics, see [12].

- When $G = \mathrm{He}_3$, we check that a G-action on a smooth cubic threefold either fixes a point, or fails Condition **(A)**. Thus the G-action in equivariantly unirational if and only if Condition **(A)** is satisfied. It follows that the same holds for all groups in the second row above.
- When $G = C_3 \times \mathrm{He}_3$, we check that all G-actions on smooth cubic threefolds fail Condition **(A)**, and thus the same holds for all groups in the first row above.
- When $G = C_3 \times \mathfrak{A}_4$ or $C_3 \times \mathfrak{S}_4$, we check that for any G-action on a smooth cubic threefold, there are two invariant hyperplanes. One of them has generic stabilizer C_3 and the G/C_3-action on the corresponding cubic surface S satisfies Condition **(A)**. The G-actions on both S and the cubic threefold are equivariantly unirational.

This completes the proof of Theorem 5.1.

Remark 5.7. Assuming [18, Conjecture 10.4 and Theorem 10.5], G-unirationality of X *would follow* from G_3-unirationality of X, where G_3 is a 3-Sylow subgroup of G. There are six nonabelian 3-groups contained in Table 1:

$$\mathrm{He}_3, C_9 \rtimes C_3, C_3 \wr C_3, C_3 \times \mathrm{He}_3, (C_3 \times C_9) \rtimes C_3, C_3 \times (C_3 \wr C_3).$$

The following, obtained via `magma` computations, *would* imply that all G-actions on X, with the possible exception of $C_9 \rtimes C_3$, are G-unirational: Let X be a smooth cubic threefold and $G \subset \mathrm{Aut}(X)$ a 3-group. Then:

- the G-action fixes a point on X, or
- the G-action fails condition **(A)**, or
- X is the Fermat cubic threefold and $G = C_9 \rtimes C_3$ is generated by

$$\mathrm{diag}(1, 1, \zeta_3, \zeta_3^2, 1), \quad \mathrm{diag}(1, 1, 1, \zeta_3, \zeta_3^2), \quad (x_1, \zeta_3 x_2, x_4, \zeta_3 x_5, x_3).$$

6. G-unirationality of singular cubic threefolds

In this section, we prove:

Theorem 6.1. *Let X be a cubic threefold with isolated singularities which is not a cone, carrying a generically free regular action of a finite group G. Assume that the G-action on X satisfies Condition* **(A)**. *Then X is G-unirational.*

The proof is based on the classification of configurations of singularities and possible group actions in [11], [10], [8]. In most of the cases, we can apply Propositions 3.2, 3.5, and Corollary 3.4. The remaining cases,

- $3\mathsf{A}_1$, G is a finite subgroup of $\mathrm{Aut}(X) = \mathsf{GL}_2(\mathbb{F}_3)$,
- $3\mathsf{D}_4$, G is a finite subgroup of $\mathrm{Aut}(X) = (\mathbb{G}_m^2 \times \mathfrak{S}_3) \rtimes \mathfrak{S}_3$,
- $6\mathsf{A}_1$, $G = \mathfrak{S}_4, \mathfrak{S}_5$,

require a separate analysis.

The linearizability properties of singular cubic threefolds have been studied in [10] and [8]; there are linearizable actions, as well as actions failing stable linearizability. The classification results of that paper allow us to address G-unirationality.

By Corollary 3.4, cubic threefolds X with a distinguished singularity or with the following singularity types are G-unirational, for all $G \subseteq \mathrm{Aut}(X)$:

$$2\mathsf{A}_n, n = 1, \dots, 5, \quad 2\mathsf{D}_4, \quad 2\mathsf{A}_2 + 2\mathsf{A}_1, \quad 2\mathsf{A}_3 + 2\mathsf{A}_1,$$
$$2\mathsf{D}_4 + 2\mathsf{A}_1, \quad 2\mathsf{A}_2 + 3\mathsf{A}_1, \quad 2\mathsf{A}_3 + 3\mathsf{A}_1, \quad 3\mathsf{A}_2 + 2\mathsf{A}_1, \quad 2\mathsf{D}_4 + 3\mathsf{A}_1.$$

We summarize the analysis of the remaining cases:

$3\mathsf{A}_n, 3\mathsf{D}_4, n = 1, 2, 3$. With `magma`, we verify that in all cases, except

(1) $3\mathsf{A}_1$, $G \subseteq \mathsf{GL}_2(\mathbb{F}_3)$,
(2) $3\mathsf{D}_4$, G is a finite subgroup of $\mathrm{Aut}(X) = (\mathbb{G}_m^2 \times \mathfrak{S}_3) \rtimes \mathfrak{S}_3$,

the cubic threefold X contains a G-invariant G-unirational hyperplane section $S \subset X$, a (possibly singular) cubic surface. Indeed, we check that S has fixed points upon restriction of the action to abelian subgroups of the quotient of G by the generic stabilizer of S. Proposition 5.2 implies

that S is equivariantly unirational for the action of the quotient, and thus for the action of G. It then follows from Proposition 3.5 that X is G-unirational.

Case 1: $X \subset \mathbb{P}(V)$ is the cubic threefold with $3\mathsf{A}_1$-singularities given by

$$x_1x_2x_3 + x_1q_1 + x_2q_2 + x_3q_3 = 0,$$

where

$$\begin{aligned} q_1 &= x_4^2 + bx_4x_5 + x_5^2,\\ q_2 &= x_4^2 + \zeta_3 bx_4x_5 + \zeta_3^2x_5^2,\\ q_3 &= x_4^2 + \zeta_3^2 bx_4x_5 + \zeta_3 x_5^2, \end{aligned}$$

for $b^2 = \sqrt{-2}$, and $V = V_2 \oplus V_3$, for irreducible representations V_2 and V_3 of $G = \mathsf{GL}_2(\mathbb{F}_3)$ generated by

$$\begin{aligned} &\operatorname{diag}(1,1,1,-1,-1),\\ (x_1,\dots,x_5) &\mapsto (x_2,x_3,x_1,x_4,\zeta_3x_5),\\ (x_1,\dots,x_5) &\mapsto (\zeta_3x_2,\zeta_3^2x_1,x_3,x_4,\zeta_6x_5),\\ (x_1,\dots,x_5) &\mapsto (x_2,\zeta_6^5x_1,\zeta_6x_3,\frac{\zeta_6bx_4+x_5}{1-\zeta_3},\frac{\zeta_6x_4+bx_5}{1-\zeta_3}). \end{aligned}$$

The G-invariant line $\mathfrak{l} = \mathbb{P}(V_2) = \{x_1 = x_2 = x_3 = 0\}$ is contained in the smooth locus of X. Note that $\mathfrak{l}$ is G-unirational, but the G-action is not generically free on $\mathfrak{l}$. The G-unirationality of X then follows from Proposition 3.5.

Case 2: X is given by

$$x_1x_2x_3 + x_4^3 + x_5^3 = 0 \tag{6.1}$$

and

$$\operatorname{Aut}(X) = N \times \mathfrak{S}_3, \quad N := \mathbb{G}_m^2 \rtimes \mathfrak{S}_3,$$

where N is acting via $\mathfrak{S}_3$-permutation and torus actions on x_1, x_2, x_3, and $\mathfrak{S}_3$ via the irreducible 2-dimensional representation on x_4, x_5. We use notation from [8, Proposition 6.6]

$$N = \langle \tau_{a,b}, \sigma_{(123)}, \sigma_{(12)}\rangle, \quad \mathfrak{S}_3 = \langle \eta, \sigma_{(45)}\rangle, \tag{6.2}$$

where

$$\begin{aligned} \tau_{a,b} &: \operatorname{diag}(a,b,(ab)^{-1},1,1),\\ \eta &: \operatorname{diag}(1,1,1,\zeta_3,\zeta_3^2), \end{aligned}$$

and σ are permutations of the variables as is labeled. Consider the subgroup

$$H := \langle \tau_{a,b}, \sigma_{(123)} \cdot \eta \rangle \subset \mathrm{Aut}(X). \tag{6.3}$$

By [8, Proposition 6.6], X is H-equivariantly birational to $S \times \mathbb{P}^1$, where S is the del Pezzo surface of degree 6. In particular, for every $G \subset H$ satisfying Condition **(A)**, we obtain G-unirationality of X.

Lemma 6.2. *Let X be the cubic* (6.1)*, $G \subset \mathrm{Aut}(X)$ a finite subgroup. Then X is G-unirational if and only if Condition* **(A)** *is satisfied.*

Proof. As discussed, Condition **(A)** is necessary for G-unirationality. We now assume that the G-action on X satisfies Condition **(A)** and proceed to show that X is G-unirational.

Observe that there are two distinguished G-orbits of three points

$$P_1 = \{[1:0:0:0:0], [0:1:0:0:0], [0:0:1:0:0]\},$$

$$P_2 = \{[0:0:0:1:-1], [0:0:0:1:-\zeta_3], [0:0:0:1:-\zeta_3^2]\}.$$

Using Propositions 3.2 and 3.3, we are reduced to the case when G acts on both P_1 and P_2 via C_3. Up to conjugation in $\mathrm{Aut}(X)$, G contains

$$m_1 := \sigma_{(123)} \cdot \eta.$$

The G-action on P_1 induces a homomorphism $G \to C_3$. Let N denote its kernel so that we have

$$G = \langle N, m_1 \rangle.$$

If $N \subset \mathbb{G}_m^2$, then $G \subset H$, from (6.3), and X is G-unirational.

If $N \not\subset \mathbb{G}_m^2$, then N contains

$$m_2 := \tau_{a,b} \cdot \eta,$$

for some $\tau_{a,b} \in \mathbb{G}_m^2$, and G is generated by m_1, m_2, and a subgroup

$$M = \mathbb{G}_m^2 \cap G.$$

Up to replacing m_2 by its power, we may assume that $\mathrm{ord}(\tau_{a,b})$ is a power of 3, so that both a and b are 3-power roots of unity. Condition **(A)** implies that G does not contain either of

$$\eta \quad \text{or} \quad \tau_{\zeta_3,\zeta_3} = \mathrm{diag}(\zeta_3, \zeta_3, \zeta_3, 1, 1),$$

since both abelian groups

$$\langle m_1, \eta \rangle \simeq \langle m_1, \tau_{\zeta_3,\zeta_3} \rangle \simeq C_3^2$$

do not fix points on X. In particular, $(a, b) \neq (1, 1)$.

If $|M|$ is not coprime to 3, then M contains either τ_{1,ζ_3} or $\tau_{\zeta_3,1}$, which is impossible since

$$\tau_{1,\zeta_3}\big(m_1\cdot\tau_{1,\zeta_3}\cdot m_1^{-1}\big)^2=\tau_{\zeta_3,1}\big(m_1^2\cdot\tau_{\zeta_3,1}\cdot m_1^{-2}\big)^2=\tau_{\zeta_3,\zeta_3}\notin G.$$

Hence, $|M|$ is coprime to 3, the order of $\tau_{a,b}$ must be 3, and

$$\tau_{a,b}\in\big\{\tau_{1,\zeta_3},\tau_{1,\zeta_3^2},\tau_{\zeta_3,1},\tau_{\zeta_3^2,1},\tau_{\zeta_3,\zeta_3},\tau_{\zeta_3^2,\zeta_3^2}\big\}.$$

Conjugating $\tau_{a,b}$ by m_1, we may assume that

- either $\tau_{a,b}=\tau_{\zeta_3,\zeta_3}$ and $m_2=\tau_{\zeta_3,\zeta_3}\cdot\eta$, or
- $\tau_{a,b}=\tau_{\zeta_3^2,\zeta_3^2}$ and $m_2=\tau_{\zeta_3^2,\zeta_3^2}\cdot\eta$.

Without loss of generality, we may assume that $m_2=\tau_{\zeta_3,\zeta_3}\cdot\eta$. Then G is generated by $m_1=\sigma_{(123)}\cdot\eta$, m_2, and a subgroup M of order coprime to 3.

Let $S:=X\cap\{x_5=0\}$, and $Q:=\langle m_1,M\rangle$. Then S is a cubic surface with $3\mathsf{A}_2$-singularities, with a generically free action of Q and generic stabilizer m_2, i.e., $S=X^{\langle m_2\rangle}$. Also note that $Q\simeq G/\langle m_2\rangle$. The Q-action on S satisfies Condition **(A)**. Indeed, if $A\subset Q$ is an abelian subgroup that does not fix any points on S, then $\langle m_2,A\rangle\subset G$ is an abelian subgroup that does not fix points in X. By Proposition 5.2, the Q-action, and thus the G-action on S is unirational, which implies that the G-action on X is unirational, by Proposition 3.2. The Q-unirationality of S also follows from the fact that S is Q-birational to $\mathbb{P}^2$ with a linear action of Q, combined with the classification of all possibilities for Q given in [16, Theorem 4.7]. □

$4\mathsf{A}_n, n=1,2$, in general position. The singular points are in linear general position. From the equations in [10, Section 5] and [8, Section 7], we see that $X\cap\{x_5=0\}$ is a G-invariant Cayley cubic surface, which is G-linearizable. Proposition 3.5 implies the G-unirationality of X.

$4\mathsf{A}_1$, contained in a plane. Let $\Pi\subset X$ be the plane containing the four nodes. Such X are equivariantly birational to a smooth intersection of two quadrics, discussed in Section 7, with their automorphisms and normal forms. Proposition 3.5 and Remark 3.7 imply the G-unirationality of X.

$5\mathsf{A}_n, n=1,2$. Cubics in this case are equivariantly birational to a smooth quadric threefold. The action is either a subgroup of the $\mathfrak{S}_5$-permutation action on coordinates, or a subgroup of the irreducible representation from $\mathfrak{A}_5$, which is stably linearizable by Theorem 4.1, see also [25, Proposition 6.2].

$6\mathsf{A}_1$. The classification of possible actions in [10, Section 7] yields subcases:

(1) **No plane:** $\mathrm{Aut}(X)$ is one of the following

$$C_2,\ \mathfrak{S}_3,\ \mathfrak{S}_4,\ \mathfrak{D}_4,\ C_2^2,\ \mathfrak{D}_6,\ \mathfrak{S}_3^2 \rtimes C_2,\ \mathfrak{S}_5.$$

All groups failing Condition **(A)** contain the unique subgroup $C_3^2 \subset \mathfrak{S}_3^2 \rtimes C_2$; they are:

$$C_3^2,\quad C_3 \rtimes \mathfrak{S}_3,\quad C_3 \times \mathfrak{S}_3,\quad \mathfrak{S}_3^2,\quad C_3 \rtimes \mathfrak{S}_3.C_2,\quad \mathfrak{S}_3^2 \rtimes C_2.$$

(2) **One plane:** all actions are linearizable, see [10, Section 7].

(3) **Three planes:** $\mathrm{Aut}(X)$ is one of the following

$$C_2^2,\quad C_2^3,\quad C_2 \times \mathfrak{S}_3,\quad C_2 \times \mathfrak{S}_4,$$

where X is given by

$$x_2x_3x_4 + ax_1^3 + x_1^2(b_1x_2 + b_2(x_3 + x_4)) + x_1(x_2^2 + x_3^2 + x_4^2 - x_5^2) = 0,$$

for some $a, b_1, b_2 \in k$.

In Case (3), the point $[0:0:0:0:1] \in X$ is fixed by any $G \subseteq \mathrm{Aut}(X)$; thus, X is G-unirational, by Proposition 3.2.

We turn to Case (1). Note that the action on X is not stably linearizable if the invariant class group has rank one, by [10, Proposition 7.5]. To prove unirationality, it suffices to consider the case when $\mathrm{Cl}^G(X) = \mathbb{Z}^2$, by passing to a suitable subgroup of index two, using Proposition 3.3.

This G leaves invariant each class of cubic scrolls on X. Recall from [10, Diagram 7.1] that an invariant divisor class of cubic scrolls shows that there exists a G-equivariant small resolution $X^+ \to X$ such that X^+ is a weak Fano 3-fold that admits a G-equivariant $\mathbb{P}^1$-bundle structure

$$p^+ : X^+ \to \mathbb{P}^2,$$

whose fibers are mapped to lines in the cubic X. Note that

$$X^+ \simeq \mathbb{P}(\mathcal{E}),$$

where $\mathcal{E}$ is a vector bundle of rank 2 on $\mathbb{P}^2$, described in [33, Theorem 3.2(7)]. The weak Fano 3-fold X^+ also appeared in [37, Theorem 3.6].

Note that $\mathrm{H}^1(G, \mathrm{Pic}(\mathbb{P}^2)) = 0$ and $\mathrm{B}_0(G) = 0$, for all G appearing in this case; the smallest G with nontrivial Bogomolov multiplier has order $|G| = p^5$, see, e.g., [36]. By Lemma 2.4, and Condition **(A)** for X, the base $\mathbb{P}^2$ is G-unirational. Lemma 2.5 then implies that the G-action lifts to $\mathcal{E}$. Note that the G-action on the base $\mathbb{P}^2$ may have generic stabilizers. Nevertheless, applying the no-name lemma as in Remark 3.6 yields G-equivariant birationality

$$\mathcal{E} \times V \sim_G \mathbb{P}^2 \times V \times \mathbb{A}^2,$$

where V is a faithful G-representation, with trivial action on $\mathbb{A}^2$. This implies G-unirationality of X.

$8\mathsf{A}_1$. In all cases, there is a G-invariant Cayley cubic surface $S \subset X$ (given by $x_3 = 0$ in the equations in [10, Proposition 8.2]), with a G-fixed point on S. It follows that X is G-unirational by Proposition 3.2.

$9\mathsf{A}_1$. We recall the normal form, see [10, Section 9]:

$$x_1x_2x_3 - x_4x_5x_6 = a(x_1 + x_2 + x_3) + x_4 + x_5 + x_6 = 0,$$

for $a^3 \neq 0, -1$, with G a subgroup of $\mathfrak{S}_3^2$, when $a^3 \neq 1$, and of $\mathfrak{S}_3^2 \rtimes C_2$, when $a = 1$. There is an invariant smooth cubic surface $S \subset X$, given by

$$x_1 + x_2 + x_3 = 0.$$

When the G-action on X satisfies Condition **(A)**, then so does the G-action on S. Thus S is G-unirational, and we apply Proposition 3.5.

$10\mathsf{A}_1$. We have a dominant rational map $\mathrm{Gr}(2, 6) \to X$, the Segre cubic threefold, equivariantly for the action of $G = \mathfrak{S}_6 = \mathrm{Aut}(X)$. By [24, Proposition 19], the $\mathfrak{S}_6$-action on $\mathrm{Gr}(2, 6)$ is stably linearizable, thus X is G-unirational.

7. Intersections of two quadrics

In this section, we prove:

Theorem 7.1. *Let $X \subset \mathbb{P}^5$ be a smooth complete intersection of two quadrics and $G \subseteq \mathrm{Aut}(X)$. The following are equivalent:*

- *the G-action on X is unirational,*
- *the G-action on X satisfies condition* **(A)**, *i.e., every abelian subgroup of G fixes a point on X,*
- *the G-action fixes a point on X.*

The problem of classification of actions of finite groups G on

$$X = X_{2,2} = Q_1 \cap Q_2 \subset \mathbb{P}^5,$$

a smooth complete intersection of two quadrics, has been addressed in [2]. There is a homomorphism $\pi : G \to \mathfrak{S}_6$, where the image is the induced action of a quotient of G on the pencil $\lambda_1 Q_1 + \lambda_2 Q_2$. We write down equations in each case:

- Type I, $\mathfrak{S}_4$: $\sum_{j=0}^4 x_j^2 = x_0^2 + \zeta_4 x_1^2 - x_2^2 - \zeta_4 x_3^2 + x_5^2 = 0$,
- Type II, $\mathfrak{D}_6$: $\sum_{j=0}^5 x_j^2 = \sum_{j=0}^5 \zeta_6^j x_j^2 = 0$,
- Type III, $\mathfrak{S}_3$: $\sum_{j=0}^5 x_j^2 = a(x_0^2 + \zeta_3 x_1^2 + \zeta_3^2 x_2^2) + x_3^2 + \zeta_3 x_4^2 + \zeta_3^2 x_5^2 = 0$,
- Type IV, C_5: $\sum_{j=0}^5 x_j^2 = \sum_{j=0}^4 \zeta_5^j x_j^2 = 0$,

- Type V, C_2^2: $\sum_{j=0}^4 x_j^2 = a(x_0^2 - x_1^2) + x_2^2 - x_3^2 + x_5^2 = 0$,
- Type VI, C_2: $\sum_{j=0}^5 x_j^2 = a(x_0^2 - x_1^2) + b(x_2^2 - x_3^2) + x_4^2 - x_5^2 = 0$.

We proceed with a list of maximal subgroups $G \subseteq \mathrm{Aut}(X)$ satisfying Condition **(A)**, up to conjugation in $\mathrm{Aut}(X)$. In each type, we list $\pi(G)$, the sign changes $\bar{G} = \ker(\pi)$, and indicate the number of conjugacy classes of each group in bracket.

Type	G	$\bar{G}$	$\pi(G)$
I	$\mathfrak{D}_4(3)$	C_2^2	C_2
I	$C_2^2 \rtimes C_4(1)$	C_2^3	C_2
I	$C_2 \times \mathfrak{A}_4(1)$	C_2^3	C_3
I	$C_2 \times C_8(1)$	C_2^2	C_4
I	$\mathfrak{S}_3(2)$	1	$\mathfrak{S}_3$

Type	G	$\bar{G}$	$\pi(G)$
II	$C_2^3(1)$	C_2^3	1
II	$\mathfrak{D}_4(4)$	C_2^2	C_2
II	$C_2 \times C_4(1)$	C_2^2	C_2
II	$C_2^2 \rtimes C_4(1)$	C_2^3	C_2
II	$C_6(1)$	1	C_6
II	$\mathfrak{S}_3(1)$	1	$\mathfrak{S}_3$
II	$\mathfrak{A}_4 \rtimes C_4(1)$	C_2^3	$\mathfrak{S}_3$

Type	G	$\bar{G}$	$\pi(G)$
III	$C_2^3(3)$	C_2^3	1
III	$\mathfrak{D}_4(6)$	C_2^2	C_2
III	$C_2 \times \mathfrak{A}_4(1)$	C_2^3	C_3
III	$\mathfrak{S}_3(2)$	1	$\mathfrak{S}_3$

Type	G	$\bar{G}$	$\pi(G)$
IV	$C_2^3(4)$	C_2^3	1
IV	$C_{10}(1)$	C_2	C_5

Type	G	$\bar{G}$	$\pi(G)$
V	$C_2^3(4)$	C_2^3	1
V	$\mathfrak{D}_4(6)$	C_2^2	C_2
V	$C_2 \times C_4(1)$	C_2^2	C_2
V	$C_2^2 \rtimes C_4(2)$	C_2^3	C_2

Type	G	$\bar{G}$	$\pi(G)$
VI	$C_2^3(10)$	C_2^3	1
VI	$\mathfrak{D}_4(6)$	C_2^2	C_2

TABLE 2

With `magma`, we find that every subgroup of $\mathrm{Aut}(X)$ satisfying Condition **(A)** has a fixed point on X. We may assume that X does not contain a G-invariant line, since otherwise, the G-action is linearizable, by [23, Theorem 24]. Projecting from the G-fixed point, we find that X is G-birational to a cubic threefold, with singularities of type

$$4\mathsf{A}_1, \quad \text{or} \quad 2\mathsf{A}_3,$$

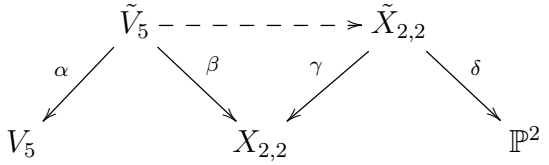
containing a unique plane, which must be G-invariant. By results of Section 6, all such cubic threefolds are G-unirational. This proves Theorem 7.1.

It would be interesting to classify actions of finite groups on singular intersections of two quadrics $X_{2,2} \subset \mathbb{P}^5$ and solve the linearization problems for this class of actions. Indeed, these naturally appear in related linearization problems:

Example 7.2. Let V_5 be the unique smooth Fano threefold of index 2 and degree 5 (the quintic del Pezzo threefold), and

$$G \subset \mathrm{Aut}(V_5) \simeq \mathsf{PGL}_2$$

a finite subgroup, not isomorphic to $\mathfrak{A}_5$. Then the G-action is linearizable. Indeed, for cyclic or dihedral groups, linearization follows from the construction in [1, Section 5.8]. If $G = \mathfrak{A}_4$, this follows from the proof of [9, Lemma 7.8.2]. If $G = \mathfrak{S}_4$, we have the following G-equivariant commutative diagram:



where

- α is the blowup of the unique G-fixed point,
- β is the contraction of the strict transform of 3 lines in V_5 that pass through the G-fixed point,
- $X_{2,2} \subset \mathbb{P}^5$ is a complete intersection of two quadrics that has three ordinary double points,
- γ is a small resolution of $X_{2,2}$,
- the dashed arrow is a composition of three Atiyah flops, and
- δ is a $\mathbb{P}^1$-bundle, with generically free G-action on the base $\mathbb{P}^2$.

This commutative diagram appears in [33, Theorem 3.2] and [37, Theorem 3.6]. Linearization of the G-action follows by applying the no-name lemma.

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Department of Mathematics, University of Edinburgh, UK
Email address: `I.Cheltsov@ed.ac.uk`

Courant Institute, 251 Mercer Street, New York, NY 10012, USA
Email address: `tschinkel@cims.nyu.edu`

Simons Foundation, 160 Fifth Avenue, New York, NY 10010, USA

Courant Institute, 251 Mercer Street, New York, NY 10012, USA
Email address: `zhijia.zhang@cims.nyu.edu`