HOMOLOGICAL STABILITY AND WEAK APPROXIMATION

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Abstract. We investigate homological stability for the space of sections of Fano fibrations over curves in the context of weak approximation, and establish it for projective bundles, as well as for conic and quadric surface bundles over curves.

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1. Introduction

Let B be a smooth projective geometrically irreducible curve over a field $\Bbbk$ and $F = \Bbbk(B)$ its function field. Let X be a smooth projective variety over F and

$$\pi : \mathcal{X} \to B \tag{1.1}$$

its smooth integral model, i.e., a flat proper morphism from a smooth projective $\mathcal{X}$ over $\Bbbk$, with generic fiber X. Our point of departure is the connection between arithmetic properties of X over F and geometric properties of spaces of sections of π, over $\Bbbk$. Of particular interest are cases when $\Bbbk = \mathbb{F}_q$, a finite field, or $\Bbbk = \mathbb{C}$.

Concretely, let ω_π^{-1} be the relative anticanonical class, assumed to be ample on the generic fiber X. We are interested in understanding the space of sections of π

$$\mathrm{Sect}(\mathcal{X}/B, h) := \{\sigma : B \to \mathcal{X}\}$$

of height

$$h := \deg(\sigma^*\omega_\pi^{-1}).$$

These spaces have been studied from different but related perspectives:

- *Manin's conjecture*, see, e.g., [Bat88],[FMT89], [BM90], [Pey95], [BT98], [Pey03], [Pey17], [LST22],
- *Homological stability* [CJS00], [DT24], [DLTT25].

Manin's conjecture concerns *asymptotics* of points (sections) of bounded height. When $\Bbbk = \mathbb{F}_q$, this translates into understanding the growth of

$$\#\mathrm{Sect}(\mathcal{X}/B, h)(\mathbb{F}_q), \quad h \to \infty.$$

In turn, these numbers can be accessed via Grothendieck's Lefschetz trace formula, which motivates the study of *topology* of complex points of spaces of sections of fibrations (1.1) defined over $\Bbbk = \mathbb{C}$. Homological stability asserts stabilization of homology of these spaces, as $h \to \infty$.

Applications of homological stability to arithmetic problems over function fields $F = \mathbb{F}_q(B)$ go back to [EVW16], and have been explored in a variety of contexts, e.g., Cohen-Lenstra heuristic, Malle's conjecture, and Manin's conjecture, see, e.g., [EVW16], [LL24b], [LL24a], [ETW23], [LL25], and [DLTT25].

It is natural to also consider *weak approximation*. In this context, weak approximation asserts the existence of sections matching a finite set of jet conditions, see [GHS03] for the existence of sections and [HT06] for weak approximation; this translates into *existence* of $\Bbbk$-points on spaces of such sections in the stable regime, when $h \to \infty$. Recall that *weak approximation* for X over F means that for any finite set of *admissible jets* there exists a section $\sigma : B \to \mathcal{X}$ matching these jets; an admissible N-th jet is the truncation of a local analytic section of π at $b \in B$ modulo $\mathfrak{m}_{B,b}^{N+1}$, power of the maximal ideal of the local ring at b. By [HT06, Theorem 3], a geometrically rationally connected variety over F satisfies *weak approximation* at places of good reduction; this has been extended to bad reduction places in some other situations, e.g., [HT09], [Xu12], [Tia15]. *Effective* weak approximation, considered in [HT12], seeks effective control over the height of sections matching these jets. This is also coupled to *equidistribution*, which can be viewed as a strong, quantitative form of weak approximation, and would require uniform control over the number of $\Bbbk$-points on the corresponding spaces, for $\Bbbk = \mathbb{F}_q$, as in (1).

Pursuing these analogies, we

- formulate the relevant version of homological stability, and
- prove it for fibrations arising from projectivizations of vector bundles, conic bundles, and quadric surface bundles over curves.

From now on, we assume that $\Bbbk = \mathbb{C}$ and identify varieties with their sets of complex points. Let

$$\alpha \in \mathrm{H}_2(\mathcal{X}(\mathbb{C}), \mathbb{Z})$$

be the class of a section of π; in particular, $\alpha \cdot \mathcal{X}_b = 1$, for all fibers $\mathcal{X}_b = \pi^{-1}(b)$, $b \in B$. Let

$$\beta \in \mathrm{H}_2(\mathcal{X}(\mathbb{C}), \mathbb{Z})$$

be the class of a very free rational curve in a smooth fiber of π. Clearly, $\alpha + \beta$ is also the class of a section of π. Let

$$\Sigma = \{\hat{\sigma}_j\}_{j \in J}$$

be a finite set of jets on $\mathcal{X}$ in distinct fibers $\mathcal{X}_{b_j}$, where each $\hat{\sigma}_j$ is an admissible N_j-th jet. Let

$$\mathrm{Sect}(\mathcal{X}/B, \alpha, \Sigma) :=$$
$$\{\sigma : B \to \mathcal{X} \mid \alpha(\sigma) = \alpha, \quad \sigma(b_j) \equiv \hat{\sigma}_j \pmod{\mathfrak{m}_{B,b_j}^{N_j+1}}, \quad \forall j \in J\}$$

be the space of sections of class α matching all jets in Σ; this is a complex algebraic variety. As explained in [HT06], an N-th jet condition in a fiber of $\mathcal{X}$ over $b \in B$ can be rephrased in terms of a 0-th jet condition on some component of a fiber over b of an iterated blowup of $\mathcal{X}$ with center in fibers over b. We know little about these spaces, e.g.,

- Are they irreducible and of expected dimension?
- Is there some stabilization in their homology?

The first question has been studied in absence of jet conditions, e.g., in [HRS04], [RY19], [LT19], [LT21],[LT24], [LT22], [BLRT22], [LRT25], [Oka24], [Oka25], and [BJ22]. The second question has not been addressed, to our knowledge. This motivates the introduction of

Condition (HS) (Homological stability, for the triple (α, β, Σ)): For α, β, and Σ as above, there exists a linear function ℓ, with positive leading term, such that, for all $i \leq \ell(m)$, one has

$$\mathrm{H}_i(\mathrm{Sect}(\mathcal{X}/B, \alpha + m\beta, \Sigma), \mathbb{Z}) \simeq \mathrm{H}_i(\mathrm{Sect}(\mathcal{X}/B, \alpha + (m+1)\beta, \Sigma), \mathbb{Z}).$$

Some geometric assumptions on X will be necessary. E.g., could this hold for X a del Pezzo surface? Or a toric variety? One of our main results is

Theorem 1. *Homological stability, for any (α, β, Σ) with nonempty Σ, holds for:*

- *$\mathbb{P}(\mathcal{E})$, projectivization of a vector bundle $\mathcal{E}$ of rank ≥ 2 over B,*
- *smooth conic bundles over B,*
- *smooth nonsplit quadric surface bundles over B, with at most A_1-singular fibers.*

Note that weak approximation and Manin's conjecture are known in many cases, e.g., for quadric surfaces over $F = \mathbb{F}_q(t)$. However, homological stability for spaces of sections (without jet matching conditions) has only been established for trivial families [Seg79], [Gue95], [DT24], [DLTT25].

In the same vein, when Σ is a set of 0-th jets, we consider the space

$$\mathrm{Sect}^{\mathrm{top}}(\mathcal{X}/B, \alpha, \Sigma)$$

of *topological sections* of corresponding classes; for N_j-th jets, with $N_j \geq 1$, we replace $\mathcal{X}$ with a birational model realizing the jet as a point on some component of the fiber. This allows to formulate the parallel condition:

Condition (HST) (Homological stability of topological sections): The homology

$$\mathrm{H}_i(\mathrm{Sect}^{\mathrm{top}}(\mathcal{X}/B, \alpha + m\beta, \Sigma), \mathbb{Z})$$

stabilizes, in the sense above.

We establish Condition **(HST)** in broad generality, see Corollary 10. In applications to projective bundles, we show that the inclusion

$$\mathrm{Sect}(\mathcal{X}/B, \alpha + m\beta, \Sigma) \hookrightarrow \mathrm{Sect}^{\mathrm{top}}(\mathcal{X}/B, \alpha + m\beta, \Sigma) \tag{1.2}$$

induces isomorphisms for low-degree homologies. This allows to establish Condition **(HS)** via Condition **(HST)**, see Theorem 15. The proofs rely on an explicit characterization of the relevant spaces of sections and topological gluing arguments, combined with tight control over the combinatorics of bar complexes, as in [DT24, DLTT25].

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2. Generalities

Notation. The cardinality of a finite set J is denoted by $|J|$. For an admissible set of jets $\Sigma = \{\hat{\sigma}_j\}_{j\in J}$, where σ_j are N_j-th jets, we let

$$\deg(\Sigma) = \sum_{j\in J}(N_j + 1),$$

be its total degree.

Schemes are separated of finite type over $\mathbb{C}$, and varieties are integral. For complex varieties, *dimension* refers to their complex dimension; for semi-algebraic sets, it is their real dimension. Isomorphisms of schemes or complex manifolds are denoted by $\cong$, homeomorphisms by $\approx$, and homotopy equivalences by $\sim$.

In this section, we introduce a *semi-topological* model of the space of topological sections

$$\mathrm{Sect}^{\mathrm{top}}(\mathcal{X}/B, \alpha, \Sigma),$$

based on ideas from [DT24, Section 6]. We make no assumptions on the geometry of the smooth projective variety X over $\mathbb{C}(B)$.

Compactly generated topologies. Let K be a compact space and $U \subset \mathbb{C}$ an open subset containing 0, with coordinate z. Let

$$\mathcal{C}(K, U) := \{f : K \times U \to \mathbb{C}\}$$

be the set of continuous functions, viewed as the set of continuous families, parametrized by K, of continuous functions $U \to \mathbb{C}$. This defines a compactly generated topology on $\mathcal{C}(U)$, the set of continuous functions on U: a continuous map

$$\bar{f} : K \to \mathcal{C}(U)$$

is equivalent to the condition that the corresponding family

$$f : K \times U \to \mathbb{C}$$

satisfies $f \in \mathcal{C}(K, U)$. Similar construction applies to

$$\mathcal{C}_0(K, U) := \{f : K \times U \to \mathbb{C} \mid f(\xi, 0) = 0, \quad \forall \xi \in K\} \subset \mathcal{C}(K, U).$$

Fix a positive integer N and a polynomial $\rho \in \mathbb{C}[z]$ of degree N. Let

$$\mathcal{C}(K, U, \rho, N) \subset \mathcal{C}(K, U)$$

be the set of continuous functions, vanishing to order $\rho(z)$ modulo $|z|^N$, i.e., for every $\xi \in K$,

$$f(\xi, z) - \rho(z) = o(|z|^N), \quad z \to 0,$$

uniformly in $\xi \in K$. As above, this induces a compactly generated topology on

$$\mathcal{C}(U, \rho, N).$$

The following is analogous to [DT24, Proposition 6.2]:

Lemma 2. *There is a natural bijection*

$$\mathcal{C}_0(K, U) \leftrightarrow \mathcal{C}(K, U, \rho, N). \tag{2.1}$$

Proof. Given an $f_0 \in \mathcal{C}_0(K, U)$, we have

$$f(\xi, z) = \rho(z) + z^N f_0(\xi, z) \in \mathcal{C}(K, U, \rho, N).$$

Conversely, for $f \in \mathcal{C}(K, U, \rho, N)$, we put

$$f_0(\xi, z) = \begin{cases} (f(\xi, z) - \rho(z))/z^N & \text{if } z \neq 0; \\ 0 & \text{if } z = 0, \end{cases}$$

establishing the claim. □

A corollary of (2.1) is a homeomorphism of topological vector spaces

$$\mathcal{C}(U) \approx \mathcal{C}(U, \rho, N). \tag{2.2}$$

Semi-topological models. The following is inspired by [DT24, Definition 6.5]:

- For $b \in B$, let $U_b \subset B$ be its open neighborhood, with local holomorphic coordinate z such that $z(b) = 0$.
- For a smooth $x \in \mathcal{X}_b$, let $\mathcal{U}_x \subset \mathcal{X}$ be its open neighborhood such that $\pi(\mathcal{U}_x) \subset U_b$.
- Assume that we have local holomorphic coordinates $z, z_1, \dots, z_n$ on $\mathcal{U}_x$ such that $\pi : \mathcal{U}_x \to U_b$ corresponds to

  $$(z, z_1, \dots, z_n) \mapsto z, \quad \text{and} \quad z_1(x) = 0, \dots, z_n(x) = 0.$$

- For $\hat{\sigma}$, an admissible N-th jet at x, let $\rho_i \in \mathbb{C}[z]$ be a polynomial of degree $\leq N$ induced by $\hat{\sigma}$ and z_i, for $i = 1, \dots, n$.

Let K be a compact set and

$$\sigma : K \times B \to \mathcal{X}$$

a family of continuous sections of π parametrized by K. Since K is compact, one can find an open $U_b' \subset U_b$ such that $\sigma(K \times U_b') \subset \mathcal{U}_x$.

Definition 3. We say that σ induces $\hat{\sigma}$ if for some (equivalently, every)

$$U_b, \mathcal{U}_x, z, \{z_i\}, \{\rho_i\}, U_b',$$

one has

$$z_i \circ \sigma|_{K \times U_b'} \in \mathcal{C}(K, U_b', \rho_i, N), \quad \text{for } i = 1, \dots, n.$$

Let $\Sigma = \{\hat{\sigma}_j\}_{j \in J}$ be a set of admissible jets, where $\hat{\sigma}_j$ is an N_j-th jet. Define

$$\mathrm{Sect}^{\mathrm{stop}}(K, \mathcal{X}/B, \alpha, \Sigma)$$

to be the set of continuous families, parametrized by K, of topological sections of π of class α constrained by Σ, i.e., sections inducing $\hat{\sigma}_j$, for all $j \in J$.

Combining the argument of Proposition 6.9 and Lemma 6.10 of [DT24], one verifies that this defines a compactly generated topology on

$$\mathrm{Sect}^{\mathrm{stop}}(\mathcal{X}/B, \alpha, \Sigma).$$

This is the *semi-topological model* of the complex algebraic variety

$$\mathrm{Sect}(\mathcal{X}/B, \alpha, \Sigma).$$

3. Properties of semi-topological models

Blowups. Consider a point $b \in B$ and an admissible N-th jet $\hat{\sigma}$, supported at a smooth point $x \in \mathcal{X}_b$, with $N \geq 1$. Let

$$\varphi : \tilde{\mathcal{X}} \to \mathcal{X}$$

be the blowup at x, with exceptional divisor E, and $\Sigma = \{\sigma\}$. Let $\tilde{\alpha}$ be the class of a section of $\tilde{\mathcal{X}} \to B$, with $\varphi_* \tilde{\alpha} = \alpha$ and $\tilde{\alpha} \cdot E = 1$.

Lemma 4. *The pushforward defines an isomorphism of algebraic varieties*

$$\varphi_* : \mathrm{Sect}(\tilde{\mathcal{X}}, \tilde{\alpha}, \tilde{\Sigma}) \cong \mathrm{Sect}(\mathcal{X}, \alpha, \Sigma),$$

where $\tilde{\Sigma}$ consists of an admissible $(N-1)$-th jet on $\tilde{X}$. Moreover, the pushforward induces a homeomorphism

$$\varphi_*^{\mathrm{stop}} : \mathrm{Sect}^{\mathrm{stop}}(\tilde{\mathcal{X}}, \tilde{\alpha}, \tilde{\Sigma}) \approx \mathrm{Sect}^{\mathrm{stop}}(\mathcal{X}, \alpha, \Sigma).$$

Proof. The first statement has been proved in [HT06, Section 2.3]. The argument below proves both statements simultaneously. As in Definition 3, choose

$$U_b,\, \mathcal{U}_x,\, z, \{z_i\},\, U_b'.$$

Let $\rho_i \in \mathbb{C}[z]$ be a polynomial of degree $\le N$ induced by $\hat{\sigma}$ and z_i, and put

$$\rho_i'(z) = \rho_i(z)/z.$$

Let σ be a (holomorphic or topological) section of π inducing $\hat{\sigma}$. Then

$$z_i(\sigma(z)) - \rho_i(z) \equiv 0 \pmod{z^{N+1}} \text{ or } o(|z|^N), \quad i = 1, \dots, n.$$

The blowup $\tilde{\mathcal{X}}$ is locally defined by

$$\varphi^{-1}(\mathcal{U}_x) \cong Z(t_i z = s z_i, t_i z_j = t_j z_i) \subset \mathcal{U}_x \times \mathbb{P}^n, \quad i, j = 1, \dots, n,$$

where $(s : t_1 : \dots : t_n)$ are homogeneous coordinates on $\mathbb{P}^n$. Let $U_b' \subset U_b$ be an open neighborhood of b such that $\sigma(U_b') \subset \mathcal{U}_x$. We define the strict transform $\tilde{\sigma} : B \to \tilde{\mathcal{X}}$ by

$$\tilde{\sigma}(b') = \begin{cases} \varphi^{-1} \circ \sigma(b') & \text{if } b' \notin U_b', \\ (s(b')) \times [z : z_1(\sigma(b')) : \dots : z_n(\sigma(b'))] & \text{if } b' \in U_b' \setminus \{b\}, \\ (\sigma(b)) \times [1 : \rho_1'(0) : \dots : \rho_n'(0)] & \text{if } b' = b. \end{cases}$$

Let $\tilde{x} = \tilde{\sigma}(b)$. Its local coordinates are $z, \tilde{z}_1, \dots, \tilde{z}_n$, where $\tilde{z}_i = t_i/s$. We have

$$\tilde{z}_i(\tilde{s}(z)) - \rho_i'(z) \equiv 0 \pmod{z^N} \text{ or } o(|z|^{N-1}).$$

Let $\hat{\sigma}^{(1)}$ be an admissible $(N-1)$-th jet of $\tilde{\mathcal{X}}$ determined by $\rho_i'(z)$. The above shows that $\tilde{\sigma}$ induces $\hat{\sigma}^{(1)}$. Conversely, if $\tilde{\sigma}$ induces $\hat{\sigma}^{(1)}$, then $\sigma = \varphi \circ \tilde{\sigma}$ satisfies

$$z_i(s(z)) \equiv z\rho_i'(z) \equiv \rho_i(z) \pmod{z^{N+1}} \text{ or } o(|z|^N).$$

□

We call $\tilde{\Sigma} = \{\hat{\sigma}^{(1)}\}$ the strict transform of $\Sigma = \{\hat{\sigma}\}$ on $\tilde{\mathcal{X}}$. Iterating this construction, for any set $\Sigma = \{\hat{\sigma}_j\}_{j \in J}$ of admissible N_j-th jets, we produce an iterated blowup

$$\varphi_\Sigma : \tilde{\mathcal{X}}_\Sigma \to \mathcal{X},$$

with a section class $\tilde{\alpha}_\Sigma$ and a set $\tilde{\Sigma}$ of admissible 0-th jets.

Proposition 5. *The pushforward map defines an isomorphism*

$$\varphi_{\Sigma *} : \mathrm{Sect}(\tilde{\mathcal{X}}_\Sigma, \tilde{\alpha}_\Sigma, \tilde{\Sigma}) \cong \mathrm{Sect}(\mathcal{X}, \alpha, \Sigma).$$

Moreover, the pushforward map also induces a homeomorphism

$$\varphi^{\mathrm{stop}}_{\Sigma *} : \mathrm{Sect}^{\mathrm{stop}}(\tilde{\mathcal{X}}_\Sigma, \tilde{\alpha}_\Sigma, \tilde{\Sigma}) \approx \mathrm{Sect}^{\mathrm{stop}}(\mathcal{X}, \alpha, \Sigma).$$

Moving the support of the jet. Let $\hat{\sigma}_0, \hat{\sigma}'_0$ be admissible 0-th jets above b_0 which are supported in the *same* component of the fiber $\mathcal{X}_{b_0}$, but with disjoint support. Let $\Sigma_0 = \{\hat{\sigma}_j\}_{j\in J}$ be a set of admissible N_j-th jets over points b_j distinct from b_0 and put

$$\Sigma := \{\hat{\sigma}_0\} \cup \Sigma_0, \quad \Sigma' := \{\hat{\sigma}'_0\} \cup \Sigma_0.$$

Proposition 6. *We have a homeomorphism*

$$\mathrm{Sect}^{\mathrm{stop}}(\mathcal{X}, \alpha, \Sigma) \approx \mathrm{Sect}^{\mathrm{stop}}(\mathcal{X}, \alpha, \Sigma').$$

Proof. It suffices to show that the evaluation map

$$\mathrm{ev} : \mathrm{Sect}^{\mathrm{stop}}(\mathcal{X}, \alpha, \Sigma_0) \to \mathcal{X}_{b_0}, \quad \sigma \mapsto \sigma(b_0)$$

is topologically locally fiberwise isotrivial on the smooth locus of $\mathcal{X}_{b_0}$. Consider the following data:

- U_{b_0} an open neighborhood of b_0 such that $\overline{U}_{b_0}$ does not contain b_j, for $j \in J$;
- a homeomorphism of closures $\overline{U}_{b_0} \approx \overline{\mathbb{D}}$, where $\mathbb{D}$ is the open unit disk with coordinate z, mapping U_{b_0} to $\mathbb{D}$ and b_0 to 0;
- $\mathcal{U}_{x_0}$ an open neighborhood of x_0 such that $\pi(\mathcal{U}_{x_0}) = U_{b_0}$,
- a homeomorphism $\overline{\mathcal{U}}_{x_0} \approx \overline{\mathbb{D}} \times \overline{\mathbb{B}}$, where $\mathbb{B}$ is a unit ball with coordinates $w = (w_1, \ldots, w_n)$, yielding a commutative diagram

$$\begin{array}{ccc} \overline{\mathcal{U}}_{x_0} & \xrightarrow{\approx} & \overline{\mathbb{D}} \times \overline{\mathbb{B}} \\ \downarrow{\scriptstyle \pi} & & \downarrow \\ U_{b_0} & \xrightarrow{\approx} & \overline{\mathbb{D}} \end{array}$$

  where the right vertical map is projection onto the first factor.
- x_0, x'_0, the supports of $\hat{\sigma}_0, \hat{\sigma}'_0$, contained in $\mathcal{X}_{b_0} \cap \mathcal{U}_{x_0}$, we assume that $x_0 = (0, 0)$ in the coordinates on $\overline{\mathbb{D}} \times \overline{\mathbb{B}}$, we suppose that $x'_0 = (0, w')$.

Our goal is to show that $\mathrm{ev}^{-1}(x_0) \approx \mathrm{ev}^{-1}(x'_0)$. Let

$$\rho_0 : \overline{\mathbb{B}} \to \overline{\mathbb{B}}$$

be a homeomorphism inducing the identity on the boundary and mapping 0 to w'. We construct a homotopy

$$\rho_t : [0,1] \times \overline{\mathbb{B}} \to \overline{\mathbb{B}}$$

such that

- ρ_1 is the identity and ρ_0 is the map above, and
- for any $t \in [0,1]$, $\rho_t : \overline{\mathbb{B}} \to \overline{\mathbb{B}}$ is a homeomorphism inducing identity on the boundary.

Then we define a π-homeomorphism $\Phi : \mathcal{X} \to \mathcal{X}$ by

$$\Phi(x) = \begin{cases} x & \text{if } x \notin \overline{\mathcal{U}}_{x_0}, \\ (z(\pi(x)), \rho_{|z(\pi(x)|}(w(x))) & \text{if } x \in \overline{\mathcal{U}}_{x_0}. \end{cases}$$

This realizes $\mathrm{ev}^{-1}(x_0) \approx \mathrm{ev}^{-1}(x_0')$, as it maps x_0 to x_0'. □

Gluing a rational curve. Now we assume that the fiber $\mathcal{X}_{b_0}$ is smooth and that it contains a rational curve of class β joining x_0, x_0'.

Proposition 7. *We have a homotopy equivalence:*

$$\mathrm{Sect}^{\mathrm{stop}}(\mathcal{X}/B, \alpha, \Sigma) \sim \mathrm{Sect}^{\mathrm{stop}}(\mathcal{X}/B, \alpha + \beta, \Sigma).$$

Proof. Let $f : \mathbb{P}^1 \to \mathcal{X}_{b_0}$ be a rational curve of class β, such that $f([1:0]) = x_0$ and $f([0:1]) = x_0'$. We introduce continuous functions:

$$\eta : [0,1] \to [0,1], \quad \eta(t) := \begin{cases} 0 & \text{if } t \in [0, \frac{1}{2}], \\ 2(t - \frac{1}{2}) & \text{if } t \in [\frac{1}{2}, 1], \end{cases}$$

and

$$\zeta : [0, 1/2) \to [0, +\infty), \quad \zeta(t) := \tan(\pi t).$$

Let U_{b_0} be an open neighborhood of b_0, with a fixed homeomorphism to $\mathbb{D}$, as above, with coordinate z and center b_0. Assume that

- $\overline{U}_{b_0}$ does not contain b_j, for all $j \in J$,
- there is a $\overline{\mathbb{D}}$-homeomorphism

$$\phi : \mathcal{X}|_{\overline{U}_{b_0}} \approx \overline{\mathbb{D}} \times \mathcal{X}_{b_0}$$

  such that $\phi(x_0) = (0, x_0)$.

Let $\sigma \in \mathrm{Sect}^{\mathrm{stop}}(\mathcal{X}/B, \alpha, \Sigma)$. We define $\sigma_1 \in \mathrm{Sect}^{\mathrm{stop}}(\mathcal{X}/B, \alpha, \Sigma)$ by (3.1)

$$\sigma_1(b) := \begin{cases} \sigma(b) & \text{if } b \notin \overline{\mathbb{D}}, \\ \phi^{-1}\left(z(b), \phi_2\left(\sigma\left(\frac{\eta(|z(b)|)}{|z(b)|} z(b)\right)\right)\right) & \text{if } b \in \overline{\mathbb{D}} \text{ and } b \neq b_0, \\ x_0 & \text{if } b = b_0, \end{cases}$$

where ϕ_2 is the composition of ϕ with the second projection. When $|z(b)| \leq 1/2$, we have

$$\phi_2(\sigma_1(b)) = x_0.$$

Next we define $\sigma' \in \mathrm{Sect}^{\mathrm{stop}}(\mathcal{X}/B, \alpha+\beta, \Sigma')$ by

$$\sigma'(b) := \begin{cases} \sigma_1(b) & \text{if } b \notin \overline{\mathbb{D}}, \\ \sigma_1(b) & \text{if } b \in \overline{\mathbb{D}} \text{ and } |z(b)| \geq 1/2, \\ \phi^{-1}(z(b), f([\zeta(|z(b)|)z : 1]) & \text{if } b \in \overline{\mathbb{D}} \text{ and } |z(b)| < 1/2. \end{cases}$$

One may think of σ' as obtained from σ by gluing the vertical rational curve

$$f : \mathbb{P}^1 \to \mathcal{X}_{b_0}$$

at x_0. By construction, $\sigma'(b_0) = x_0'$. This defines a continuous map

$$\Phi : \mathrm{Sect}^{\mathrm{stop}}(\mathcal{X}/B, \alpha, \Sigma) \to \mathrm{Sect}^{\mathrm{stop}}(\mathcal{X}/B, \alpha+\beta, \Sigma'), \quad \sigma \mapsto \sigma'.$$

We construct the homotopy inverse to Φ by gluing an inversely oriented sphere so that the composition is homotopic to the identity. Indeed, for $\tau' \in \mathrm{Sect}^{\mathrm{stop}}(\mathcal{X}/B, \alpha+\beta, \Sigma')$, we construct $\tau_1' \in \mathrm{Sect}^{\mathrm{top}}(\mathcal{X}/B, \alpha+\beta, \Sigma')$ in the same way we constructed σ_1. In particular, when $|z(b)| \leq 1/2$, we have

$$\phi_2(\tau_1'(b)) = x_0'.$$

We define $\tau \in \mathrm{Sect}^{\mathrm{top}}(\mathcal{X}/B, \alpha, \Sigma)$ by

$$\tau(b) = \begin{cases} \tau_1'(b) & b \notin \overline{\mathbb{D}}, \\ \tau_1'(b) & b \in \overline{\mathbb{D}} \text{ and } |z(b)| \geq 1/2, \\ \phi^{-1}(z(b), f([1 : \zeta(|z(b)|)\overline{z}]) & b \in \overline{\mathbb{D}} \text{ and } |z(b)| < 1/2. \end{cases}$$

Complex conjugation corresponds to gluing the inversely oriented sphere. This defines a continuous map

$$\Psi : \mathrm{Sect}^{\mathrm{stop}}(\mathcal{X}/B, \alpha+\beta, \Sigma') \to \mathrm{Sect}^{\mathrm{stop}}(\mathcal{X}/B, \alpha, \Sigma), \quad \tau' \mapsto \tau.$$

Since gluing of a sphere and the inversely oriented sphere is homotopic to a point, the compositions

$$\Psi \circ \Phi, \quad \Phi \circ \Psi,$$

are homotopic to identities. We conclude that

$$\Phi : \mathrm{Sect}^{\mathrm{stop}}(\mathcal{X}/B, \alpha, \Sigma) \sim \mathrm{Sect}^{\mathrm{stop}}(\mathcal{X}/B, \alpha+\beta, \Sigma')$$

is a homotopy equivalence. Proposition 6 implies that

$$\mathrm{Sect}^{\mathrm{stop}}(\mathcal{X}/B, \alpha, \Sigma) \sim \mathrm{Sect}^{\mathrm{stop}}(\mathcal{X}/B, \alpha+\beta, \Sigma).$$

□

Independence of jets in smooth fibers. Now we assume that the fiber $\mathcal{X}_0$ is smooth, that $\hat{\sigma}_0$ and $\hat{\sigma}'_0$ are arbitrary admissible jets such that the supports x_0, x'_0 of $\hat{\sigma}_0$ and $\hat{\sigma}'_0$ coincide, but the jets are distinct.

Proposition 8. *We have a homotopy equivalence*

$$\mathrm{Sect}^{\mathrm{stop}}(\mathcal{X}/B, \alpha, \Sigma) \sim \mathrm{Sect}^{\mathrm{stop}}(\mathcal{X}/B, \alpha, \Sigma').$$

We start with the following:

Lemma 9. *In the setting of Proposition 8, assume that $\hat{\sigma}_0$ and $\hat{\sigma}'_0$ have the same length. Then we have a homeomorphism*

$$\mathrm{Sect}^{\mathrm{stop}}(\mathcal{X}/B, \alpha, \Sigma) \approx \mathrm{Sect}^{\mathrm{stop}}(\mathcal{X}/B, \alpha, \Sigma').$$

Proof. Let $\tilde{\mathcal{X}}_\Sigma \to B$ be the blowup model associated to $(\mathcal{X}/B, \Sigma)$ as in Proposition 5. Let $\tilde{\Sigma}$ be the strict transform of Σ which is the set of admissible 0-th jets. Similarly we construct $\tilde{\mathcal{X}}_{\Sigma'} \to B$ and $\tilde{\Sigma}'$ associated to $(\mathcal{X}/B, \Sigma')$. By Proposition 5, we have

$$\mathrm{Sect}^{\mathrm{stop}}(\mathcal{X}/B, \alpha, \Sigma) \approx \mathrm{Sect}^{\mathrm{stop}}(\tilde{\mathcal{X}}_\Sigma/B, \alpha, \tilde{\Sigma}),$$

and

$$\mathrm{Sect}^{\mathrm{stop}}(\mathcal{X}/B, \alpha, \Sigma') \approx \mathrm{Sect}^{\mathrm{stop}}(\tilde{\mathcal{X}}_{\Sigma'}/B, \alpha, \tilde{\Sigma}').$$

Thus it suffices to show that

$$\mathrm{Sect}^{\mathrm{stop}}(\tilde{\mathcal{X}}_\Sigma/B, \alpha, \tilde{\Sigma}) \approx \mathrm{Sect}^{\mathrm{stop}}(\tilde{\mathcal{X}}_{\Sigma'}/B, \alpha, \tilde{\Sigma}').$$

Note that we can construct a proper smooth algebraic deformation over B from $\tilde{\mathcal{X}}_\Sigma$ to $\tilde{\mathcal{X}}_{\Sigma'}$, in particular, this shows that $\tilde{\mathcal{X}}_\Sigma$ is B-homeomorphic to $\tilde{\mathcal{X}}_{\Sigma'}$. Our assertion follows from Proposition 6. □

Proof of Proposition 8. It suffices to show the claim when $\hat{\sigma}_0$ is a 0-th jet and $\hat{\sigma}'_0$ is an N-th jet. We assume that both are supported at $x_0 \in \mathcal{X}_{b_0}$. Let $\mathbb{D}$ be an open neighborhood of b_0 such that

- $\overline{\mathbb{D}}$ does not contain any $b_j, j \in J$,
- the closure $\overline{\mathbb{D}}$ is homeomorphic to the closed unit disk with complex coordinate z and center corresponding to b_0, and;
- there is a $\overline{\mathbb{D}}$-homeomorphism

  $$\phi : \mathcal{X}|_{\overline{\mathbb{D}}} \approx \overline{\mathbb{D}} \times \mathcal{X}_{b_0}$$

  such that $\phi(x_0) = (0, x_0)$. By [Voi02, Proposition 9.5], we may assume that the fibers of the composition $\phi_2 = \mathrm{pr}_2 \circ \phi$ are complex manifolds.

Let $\mathcal{U}_x \subset \mathcal{X}$ be an open neighborhood of x with local holomorphic coordinates $z, z_1, \ldots, z_n$ such that $\pi : \mathcal{U}_x \to \mathbb{D}$ corresponds to mapping

$$(z, z_1, \ldots, z_n) \mapsto z \text{ and } z_1(x) = 0, \ldots, z_n(x) = 0.$$

We may assume that the section

$$z \mapsto (z, 0, \ldots, 0)$$

corresponds to the section

$$z \mapsto \phi^{-1}(z, x_0).$$

This is possible because the fibers of ϕ_2 are complex curves.

Let $\rho_i \in \mathbb{C}[z]$ be a polynomial of degree $\leq N$ induced by $\hat{\sigma}_0'$ and x_i. By Lemma 9, we may assume that $\rho_i(z) = 0$, for all $i = 1, \ldots, n$.

It is clear that

$$\mathrm{Sect}^{\mathrm{stop}}(\mathcal{X}/B, \alpha, \Sigma') \subset \mathrm{Sect}^{\mathrm{stop}}(\mathcal{X}/B, \alpha, \Sigma),$$

so we may define

$$\Phi : \mathrm{Sect}^{\mathrm{stop}}(\mathcal{X}/B, \alpha, \Sigma') \to \mathrm{Sect}^{\mathrm{stop}}(\mathcal{X}/B, \alpha, \Sigma), \quad \sigma \mapsto \sigma.$$

We construct the homotopy inverse. Let

$$\sigma \in \mathrm{Sect}^{\mathrm{stop}}(\mathcal{X}/B, \alpha, \Sigma).$$

Let $\sigma_1 : B \to \mathcal{X}$ be the section as constructed in (3.1). By construction, $\sigma_1 \in \mathrm{Sect}^{\mathrm{stop}}(\mathcal{X}/B, \alpha, \Sigma')$. The inverse map is defined by

$$\Psi : \mathrm{Sect}^{\mathrm{stop}}(\mathcal{X}/B, \alpha, \Sigma) \to \mathrm{Sect}^{\mathrm{stop}}(\mathcal{X}/B, \alpha, \Sigma'), \quad \sigma \mapsto \sigma_1.$$

It is easy to show that $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are homotopic to identities. Indeed, such a homotopy can be obtained using the function

$$\eta_t(s) : (0, 1] \times [0, 1] \to [0, 1] : (s, t) \mapsto \eta_t(s),$$

defined by

$$\eta_t(s) := \begin{cases} 0 & \text{if } s \in (0, \frac{t}{2}], \\ (1 - \frac{t}{2})^{-1}(s - \frac{t}{2}) & \text{if } s \in [\frac{t}{2}, 1]. \end{cases}$$

□

Combining Propositions 6, 7, and 8, we obtain:

Corollary 10. *Let Σ be an admissible jet datum with at least one jet supported in a smooth fiber $\mathcal{X}_{b_0}$. Let β be the class of a rational curve in $\mathcal{X}_{b_0}$. Then we have a homotopy equivalence*

$$\mathrm{Sect}^{\mathrm{stop}}(\mathcal{X}/B, \alpha, \Sigma) \sim \mathrm{Sect}^{\mathrm{stop}}(\mathcal{X}/B, \alpha + \beta, \Sigma).$$

4. Projective bundles over curves and the Abel–Jacobi map

Let B be a smooth projective irreducible curve of genus $g(B)$ and

$$\pi : \mathcal{X} = \mathbb{P}(\mathcal{E}) \to B$$

the projectivization of a vector bundle $\mathcal{E}$ over B of rank $(n+1) \geq 2$. A class α of sections is specified by the degree

$$d = d(\alpha) := [\mathcal{O}_{\mathbb{P}(\mathcal{E})/B}] \cdot \alpha;$$

we denote the corresponding space of sections by

$$\mathrm{Sect}(\mathcal{X}/B, d),$$

and the subspace of sections with prescribed admissible jet data Σ by

$$\mathrm{Sect}(\mathcal{X}/B, d, \Sigma) \subseteq \mathrm{Sect}(\mathcal{X}/B, d).$$

Abel–Jacobi map. There is a well-known bijection

$$\{\sigma : B \to \mathcal{X}\} \quad \leftrightarrow \quad \{\mathcal{E} \twoheadrightarrow L\},$$

induced by

$$\sigma \mapsto L := \sigma^* \mathcal{O}_{\mathbb{P}(\mathcal{E})/B}(1).$$

Consider the projections

$$\begin{array}{ccc} & B \times \mathrm{Pic}^d(B) & \\ {\scriptstyle \varpi}\swarrow & & \searrow{\scriptstyle \varpi_d} \\ B & & \mathrm{Pic}^d(B) \end{array}$$

and let

$$\mathcal{L}_d \to B \times \mathrm{Pic}^d(B)$$

be the universal line bundle. When $d \gg 0$ (depending on $\mathcal{E}$), the sheaf

$$\mathcal{V}_d := \varpi_{d*}(\varpi^*(\mathcal{E}^\vee) \otimes \mathcal{L}_d)$$

is locally free of rank

$$r = d(n+1) - \deg(\mathcal{E}) + (n+1)(1 - g(B)).$$

We have

$$\begin{array}{ccc} \mathrm{Sect}(\mathcal{X}/B, d) & \hookrightarrow & \mathbb{P}(\mathcal{V}_d^\vee) \\ & {\scriptstyle \mathrm{AJ}}\searrow & \downarrow \\ & & \mathrm{Pic}^d(B) \end{array}$$

as a Zariski open subset of a projective bundle over $\mathrm{Pic}^d(B)$, where AJ is the Abel–Jacobi map. Let

$$\tilde{S}_d \subset \mathcal{V}_d$$

be the Zariski open subset consisting of those $s \in \mathrm{H}^0(B, \mathcal{E}^\vee \otimes L_d)$ which do not vanish on B; here $L_d \in \mathrm{Pic}^d(B)$ is the restriction of the universal bundle $\mathcal{L}_d$. This is a $\mathbb{C}^\times$-torsor

$$\tilde{S}_d \to \mathrm{Sect}(\mathcal{X}/B, d).$$

We turn to $\mathrm{Sect}(\mathcal{X}/B, d, \Sigma)$. Note that a section $\sigma : B \to \mathcal{X}$ with $\deg(L) = d$ inducing Σ corresponds to a section

$$s \in \mathrm{H}^0(B, \mathcal{E}^\vee \otimes L_d)$$

satisfying the jet condition imposed by Σ, which is a linear condition, for $d \gg 0$, depending on $\mathcal{E}$ and $\deg(\Sigma)$. Such sections form a codimension $n \cdot \deg(\Sigma)$ vector subbundle

$$\mathcal{V}_{d,\Sigma} \subset \mathcal{V}_d.$$

Indeed, put $y_j = (N_j + 1)b_j$, an effective divisor on B. The homomorphism

$$\mathcal{E}^\vee \to \oplus_{j \in J}(\mathcal{E}^\vee|_{y_j}),$$

is surjective. The jet $\hat{\sigma}_j$ determines a codimension $(N_j + 1)n$ subspace $\mathcal{H}_j$ of $\mathcal{E}^\vee|_{y_j}$, and we denote the kernel of

$$\mathcal{E}^\vee \to \oplus_{j \in J}(\mathcal{E}^\vee|_{y_j}/\mathcal{H}_j)$$

by $\mathcal{E}^\vee(-\Sigma)$. Put

$$(\mathcal{V}_{d,\Sigma})_{L_d} := \mathrm{H}^0(B, \mathcal{E}^\vee(-\Sigma) \otimes L_d).$$

For $d \gg 0$, depending on $\mathcal{E}$ and $\deg(\Sigma)$,

$$\mathrm{H}^0(B, \mathcal{E}^\vee \otimes L_d) \to \oplus_{j \in J}(\mathcal{E}^\vee|_{y_j}),$$

is surjective. We conclude that $(\mathcal{V}_{d,\Sigma})_{L_d}$ has dimension independent of L_d, in this range. Thus we may define

$$\mathcal{V}_{d,\Sigma} = (\varpi_d)_*(\varpi^*(\mathcal{E}^\vee(-\Sigma)) \otimes \mathcal{L}_d).$$

We have

$$\begin{array}{ccc} \mathrm{Sect}(\mathcal{X}/B, d, \Sigma) & \hookrightarrow & \mathbb{P}(\mathcal{V}_{d,\Sigma}^\vee) \\ & \searrow^{\mathrm{AJ}} & \downarrow \\ & & \mathrm{Pic}^d(B). \end{array}$$

as a Zariski open subset. Similarly, we consider the Zariski open subset $\tilde{S}_{d,\Sigma} \subset \mathcal{V}_{d,\Sigma}$, yielding $\mathbb{C}^\times$-torsor

$$\tilde{S}_{d,\Sigma} \to \mathrm{Sect}(\mathcal{X}/B, d, \Sigma).$$

Proposition 11. *For $d \gg 0$, depending on $\mathcal{E}$ and $\deg(\Sigma)$, the space*

$$\mathrm{Sect}(\mathcal{X}/B, d, \Sigma)$$

is a Zariski open subset of a projective bundle over $\mathrm{Pic}^d(B)$, of relative dimension

$$d(n+1) - \deg(\mathcal{E}) + (n+1)(1-g(B)) - \sum_{j\in J} n(N_j+1) - 1.$$

In particular, it is irreducible, of expected dimension

$$d(n+1) - \deg(\mathcal{E}) + n(1-g(B)) - \sum_{j\in J} n(N_j+1).$$

Semi-topological counterparts. Put

$$\mathcal{V}_d^{\mathrm{stop}} := \{(s, L) \,|\, L \in \mathrm{Pic}^d(B), s \in \mathrm{H}^0_{\mathrm{cont}}(B, \mathcal{E}^\vee \otimes L)\},$$

where $\mathrm{H}^0_{\mathrm{cont}}$ denotes the space of continuous sections, with its compactly generated topology. Note that $\mathcal{V}_d^{\mathrm{stop}}$ is a locally trivial bundle of Banach spaces over $\mathrm{Pic}^d(B)$. Let

$$\tilde{S}_d^{\mathrm{stop}} \subset \mathcal{V}_d^{\mathrm{stop}}$$

be the open subset of (s, L) such that s is nowhere vanishing on B; it carries a $\mathbb{C}^\times$-action via

$$\lambda \cdot (s, L) = (\lambda s, L). \tag{4.1}$$

We have an inclusion

$$\mathrm{Sect}(\mathcal{X}/B, d) \hookrightarrow S_d^{\mathrm{stop}} := \tilde{S}_d^{\mathrm{stop}}/\mathbb{C}^\times.$$

We perform the same for jet conditions. Fix $\Sigma = \{\hat{\sigma}\}_{j\in J}$ such that $\hat{\sigma}_j$ is supported at $x_j \in \mathcal{X}_{b_j} = \mathbb{P}(\mathcal{E}_{b_j})$, the fiber at $b_j \in B$. Let $\epsilon_{b_j} \subset \mathcal{E}^\vee_{b_j}$ be the 1-dimensional subspace corresponding to x_j. Choose a Euclidean open neighborhood $U \subset \mathrm{Pic}^d(B)$ and a finite Euclidean open covering $\{U_\lambda\}$ of B with holomorphic trivializations

$$\mathcal{E}|_{U_\lambda} \cong \oplus_{i=1}^{n+1}\mathcal{O}, \quad \mathcal{L}_d|_{U\times U_\lambda} \cong \mathcal{O}. \tag{4.2}$$

We construct a Banach subbundle

$$\mathcal{V}_{d,\Sigma}^{\mathrm{stop}} \subset \mathcal{V}_d^{\mathrm{stop}},$$

parametrizing continuous sections matching jets by specifying the fiber over $L \in \mathrm{Pic}^d(B)$. Suppose we have a continuous section

$$s \in \mathrm{H}^0_{\mathrm{cont}}(B, \mathcal{E}^\vee \otimes L).$$

For each b_j, pick λ such that $b_j \in U_\lambda$. After shrinking U_λ if necessary, we pick a homogeneous coordinate z such that $z(b_j) = 0$ and z exhibits a holomorphic isomorphism $U_\lambda \cong \mathbb{D}$ to the unit disk.

We choose the trivialization in (4.2) so that ϵ_{b_j} corresponds to

$$\mathbb{C} \cdot (1, 0, \ldots, 0) \subset \oplus_{i=1}^{n+1} \mathcal{O}.$$

The trivialization (4.2) induces local holomorphic coordinates

$$z, t_0, \cdots, t_n,$$

of the bundle $\mathcal{E}^\vee|_{U_\lambda}$, and local coordinates

$$z, z_1 = t_1/t_0, \ldots, z_n = t_n/t_0,$$

in a neighborhood of x_j in $\mathbb{P}(\mathcal{E}|_{U_\lambda})$. The jet data define $\rho_i^j \in \mathbb{C}[z]$. We define a Banach bundle

$$\mathcal{V}^{\mathrm{stop}}_{d,\Sigma} \to \mathrm{Pic}^d(B)$$

so that the fiber

$$(\mathcal{V}^{\mathrm{stop}}_{d,\Sigma})_L, \quad L \in \mathrm{Pic}^d(B),$$

is the space of sections $s \in \mathrm{H}^0_{\mathrm{cont}}(B, \mathcal{E}^\vee \otimes L)$ such that for each j, s induces continuous functions $s_0^j, \ldots, s_n^j$, with respect to the trivialization (4.2), such that

$$\rho_i^j(z) s_0^j(z) = s_i^j(z) + o(|z|^N), \quad i = 1, \ldots, n.$$

These are linear conditions in

$$s \in \mathrm{H}^0_{\mathrm{cont}}(B, \mathcal{E}^\vee \otimes L).$$

Proof of Lemma 2 shows that the s_i^j correspond to functions $\check{s}_i^j$ such that $\check{s}_i^j(0) = 0$, for $i = 1, \ldots, n$, via

$$\begin{pmatrix} \check{s}_0^j \\ \check{s}_1^j \\ \vdots \\ \check{s}_n^j \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -\frac{\rho_1^j}{z^{N_j}} & \frac{1}{z^{N_j}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ -\frac{\rho_n^j}{z^{N_j}} & 0 & \cdots & \frac{1}{z^{N_j}} \end{pmatrix} \begin{pmatrix} s_0^j \\ s_1^j \\ \vdots \\ s_n^j \end{pmatrix}.$$

Using this matrix as a transition matrix, we obtain the twisted holomorphic vector bundle $\check{\mathcal{E}}^\vee(-\Sigma)$ over $\mathrm{Pic}^d(B)$ such that $(\mathcal{V}^{\mathrm{stop}}_{d,\Sigma})_L$ is identified with

$$\mathrm{H}^0_{\mathrm{cont}}(B, \check{\mathcal{E}}^\vee(-\Sigma)\otimes L)_\Sigma := \\ \{\check{s} \in \mathrm{H}^0_{\mathrm{cont}}(B, \check{\mathcal{E}}^\vee(-\Sigma)\otimes L) \mid \check{s}(b_j) \in \epsilon_{b_j} \subset \mathcal{E}^\vee_{b_j} \text{ for all } j \in J\}.$$

As this is a closed subspace of $\mathrm{H}^0_{\mathrm{cont}}(B, \check{\mathcal{E}}^\vee(-\Sigma)\otimes L)$ this is a Banach space. Moreover,

$$\mathrm{H}^0_{\mathrm{cont}}(B, \check{\mathcal{E}}^\vee(-\Sigma)\otimes L)_\Sigma \cap \mathrm{H}^0_{\mathrm{hol}}(B, \mathcal{E}^\vee\otimes L) = \mathrm{H}^0_{\mathrm{hol}}(B, \mathcal{E}^\vee(-\Sigma)\otimes L). \tag{4.3}$$

The local trivializations of $\mathcal{L}_d$ over $\mathrm{Pic}^d(B)$ enable us to realize

$$\mathcal{V}^{\mathrm{stop}}_{d,\Sigma} \to \mathrm{Pic}^d(B),$$

as a locally trivial bundle of Banach spaces. Let $\tilde{S}^{\mathrm{stop}}_{d,\Sigma} \subset \mathcal{V}^{\mathrm{stop}}_{d,\Sigma}$ be the open subset parametrizing (s, L) such that s is nowhere vanishing on B and $S^{\mathrm{stop}}_{d,\Sigma}$ its quotient by the $\mathbb{C}^\times$-action (4.1). We have an inclusion

$$\mathrm{Sect}(\mathcal{X}/B, d, \Sigma) \hookrightarrow S^{\mathrm{stop}}_{d,\Sigma}.$$

Comparison. A class $[s, L] \in S^{\mathrm{stop}}_{d,\Sigma}$ defines a continuous section $\sigma : B \to \mathcal{X}$ matching Σ such that $\sigma^*\mathcal{O}(1) \approx L$.

Proposition 12. *The continuous map*

$$\Phi_{d,\Sigma} : S^{\mathrm{stop}}_{d,\Sigma} \to \mathrm{Sect}^{\mathrm{stop}}(\mathcal{X}/B, d, \Sigma) \times \mathrm{Pic}^d(B), \quad [s, L] \mapsto (\sigma, L)$$

is a homeomorphism.

Proof. Since the line bundle $\mathcal{L}_d \to B \times \mathrm{Pic}^d(B)$ can be locally trivialized over $\mathrm{Pic}^d(B)$, the map $\Phi_{d,\Sigma}$ is a local homeomorphism over $\mathrm{Pic}^d(B)$. Since $\Phi_{d,\Sigma}$ is bijective, this proves our assertion. □

5. Comparison of algebraic and semi-topological models

In this section, we compare (co)homologies of

$$\mathcal{V}_{d,\Sigma} \setminus \tilde{S}_{d,\Sigma}, \quad \mathcal{V}^{\mathrm{stop}}_{d,\Sigma} \setminus \tilde{S}^{\mathrm{stop}}_{d,\Sigma},$$

following [DT24, DLTT25].

Stratifications. Let

$$\mathrm{Hilb}(B)$$

be the Hilbert scheme of 0-dimensional subschemes on B. We define a semi-topological *stratification*, in the sense of [DT24, Definition 2.4], via the introduction of

$$Z^{\mathrm{stop}}_{d,\Sigma} \subset \mathcal{V}^{\mathrm{stop}}_{d,\Sigma} \times \mathrm{Hilb}(B),$$

the closed subspace whose fiber at (L, y) is the subspace

$$(\mathrm{H}^0_{\mathrm{cont}}(B, \check{\mathcal{E}}^\vee(-\Sigma) \otimes L)_\Sigma)_y \subset \mathrm{H}^0_{\mathrm{cont}}(B, \check{\mathcal{E}}^\vee(-\Sigma) \otimes L)_\Sigma$$

of sections vanishing on the support of y:

$$\begin{array}{ccccc} Z^{\mathrm{stop}}_{d,\Sigma}|_{(L,y)} & \subset & Z^{\mathrm{stop}}_{d,\Sigma} & \subset & \mathcal{V}^{\mathrm{stop}}_{d,\Sigma} \times \mathrm{Hilb}(B) \\ \downarrow & & & & \downarrow \\ (L, y) \in & & & & \mathrm{Pic}^d(B) \times \mathrm{Hilb}(B) \end{array}$$

By definition,

$$Z^{\mathrm{stop}}_{d,\Sigma}|_{(L,y)} = Z^{\mathrm{stop}}_{d,\Sigma}|_{(L,\mathrm{red}(y))},$$

where $\mathrm{red}(y)$ is the reduced scheme of y. The inclusion $\mathcal{V}_{d,\Sigma} \hookrightarrow \mathcal{V}^{\mathrm{stop}}_{d,\Sigma}$, combined with observation (4.3), yields an algebraic stratification

$$Z^{\mathrm{alg}}_{d,\Sigma} = \mathcal{V}_{d,\Sigma} \times_{\mathcal{V}^{\mathrm{stop}}_{d,\Sigma}} Z^{\mathrm{stop}}_{d,\Sigma}.$$

Combinatorial types. Let $\Sigma = \{\hat{\sigma}\}_{j\in J}$ be a set of admissible N_j-th jets, supported in $x_j \in \mathcal{X}_{b_j}$. For an effective $y \in \mathrm{Hilb}(B)$, we express

$$\deg(y) = \sum_{j\in J} \ell_j + \sum_{i\in I} m_i,$$

where $m_i \geq 1$ are multiplicities of y in points c_i outside $\{b_j\}_{j\in J}$, and ℓ_j is the multiplicity of y at b_j. In particular, it is possible that $\ell_j = 0$. We define the *combinatorial type* of y by setting the multiset

$$T(y) := \{\boldsymbol{\ell}; \boldsymbol{m}\}, \qquad \boldsymbol{\ell} = \{\ell_j\}_{j\in J}, \quad \boldsymbol{m} = \{m_i\}_{i\in I}.$$

It is *essential*, in the sense of [DT24, Definition 3.11], if and only if y is reduced, i.e., all multiplicities are ≤ 1. In our setting, all combinatorial types are *saturated*, in the sense of [DT24, Definition 3.3]. Let

$$\mathcal{N}_T \subset \mathrm{Hilb}(B)$$

be the locally closed subset parametrizing effective divisors of combinatorial type T. This defines a stratification into locally closed subsets:

$$\mathrm{Hilb}(B) = \bigsqcup_T \mathcal{N}_T.$$

As in [DT24, Section 3], put

$$\gamma(y) := (n+1)\left(\sum_{i\in I} m_i\right) + \sum_{j\in J}(\ell_j + n\max\{\ell_j - N_j - 1, 0\}), \tag{5.1}$$

When y is reduced, this is the expected codimension of the incidence condition imposed by y, i.e., the expected codimension of

$$Z^{\mathrm{alg}}_{d,\Sigma}|_{(L,y)} \subset \mathcal{V}_{d,\Sigma}|_L.$$

We note that $\mathrm{rank}(y)$ from [DT24] equals $\deg(y)$, in our situation.

Semi-algebraic approximation. Following [DT24, Section 6.3], let M be a very ample line bundle on B and $\overline{M}$ its antiholomorphic bundle. Let

$$\mathcal{W}_k \subset \mathcal{V}^{\mathrm{stop}}_{d,\Sigma}$$

be the subbundle such that its fiber $(\mathcal{W}_k)_L$ over L is the image of

$$\mathrm{H}^0(B, \mathcal{E}^\vee(-\Sigma)\otimes L\otimes M^k)\otimes \mathrm{H}^0_{\mathrm{anti}}(B, \overline{M}^k) \to \mathrm{H}^0_{\mathrm{cont}}(B, \check{\mathcal{E}}^\vee(-\Sigma)\otimes L)_\Sigma,$$

where the map is given by a natural topological trivialization

$$\mathbb{C}\times B \cong M\otimes\overline{M}.$$

This map is injective, by [DT24, Lemma 6.17], thus its image has dimension independent of L. Hence $\mathcal{W}_k$ is a finite-dimensional semi-algebraic vector bundle over $\mathrm{Pic}^d(B)$, and we have inclusions

$$\mathcal{V}_{d,\Sigma} = \mathcal{W}_0 \subset \mathcal{W}_1 \subset \cdots \subset \mathcal{W}_k \subset \cdots \subset \mathcal{V}^{\mathrm{stop}}_{d,\Sigma}, \quad k\in\mathbb{Z}_{\geq 0}.$$

Using the argument of [Aum25, Lemma 7.2], one can prove that $\cup_k \mathcal{W}_k$ is dense in $\mathcal{V}^{\mathrm{stop}}_{d,\Sigma}$. Moreover,

$$\begin{array}{rcl} Z_{d,\Sigma,k} := \mathcal{W}_k \times_{\mathcal{V}^{\mathrm{stop}}_{d,\Sigma}} Z^{\mathrm{stop}}_{d,\Sigma} \subset & & \mathcal{W}_k \times \mathrm{Hilb}(B) \\ & & \downarrow \\ & & \mathrm{Pic}^d(B)\times \mathrm{Hilb}(B) \end{array}$$

is also a semi-algebraic stratification of $\mathcal{W}_k$, as the condition on $(\mathcal{W}_k)|_L$ imposed by $y\in\mathrm{Hilb}(B)$ is linear.

The bar complex. Put

$$Z_k := Z_{d,\Sigma,k},$$

and let $U \subset \mathrm{Pic}^d(B)$ be an open subset, in the Euclidean topology. Viewing Z_k as a bundle over $\mathrm{Pic}^d(B)$, we let $Z_{k,U}$ be its restriction to U. Consider

$$\mathrm{P} := \cup_T \mathcal{N}_T \subset \mathrm{Hilb}(B),$$

a downward closed proper union, over finitely many T. Following [DT24, Section 5], the *bar complex*

$$\mathrm{B}(\mathrm{P}, Z_{k,U})$$

is a simplicial space with r-simplices

$$\{(L, y_0 < \cdots < y_r, s) \,|\, L \in U, \quad y_i \in \mathrm{P}, \quad s \in (Z_{k,U})_{(L,y_r)})\}.$$

We denote its geometric realization by $\mathbf{B}(\mathrm{P}, Z_{k,U})$.

Unobstructedness.

Proposition 13. *Let $y \in \mathrm{Hilb}(B)$ be reduced, of combinatorial type*

$$T(y) := \{\boldsymbol{\ell}; \boldsymbol{m}\}, \qquad \boldsymbol{\ell} = \{\ell_j\}_{j\in J}, \quad \boldsymbol{m} = \{m_i\}_{i\in I}.$$

There exists a constant $A(\mathcal{E}, \deg(\Sigma))$ such that for

$$|I| \leq d - A(\mathcal{E}, \deg(\Sigma))$$

the real codimension of

$$(Z_{d,\Sigma,k})_{(L,y)} = (Z_k)_{(L,y)} \subset (\mathcal{W}_k)_L$$

is equal to $2\gamma(y)$, defined in (5.1).

Proof. Since y imposes a linear condition, the real codimension of

$$Z_k|_{(L,y)} \subset (\mathcal{W}_k)_L$$

is less than or equal $2\gamma(y)$. Equality holds if the real codimension of

$$(Z^{\mathrm{alg}}_{d,\Sigma})_{(L,y)} \subset (\mathcal{V}_{d,\Sigma})_L$$

is equal to $2\gamma(y)$, which we now prove. For $d \gg 0$, depending on $\mathcal{E}, \deg(\Sigma)$, we have a surjection

$$\mathrm{H}^0(B, \mathcal{E}^\vee \otimes L) \to \oplus_{j\in J} \mathcal{E}^\vee|_{y_j}, \quad y_j = (N_j + 1)b_j, \quad \forall j \in J,$$

and the conditions imposed by $b_j \in \mathrm{Supp}(y)$ are independent. There is a constant $A(\mathcal{E}, \deg(\Sigma))$ such that for

$$d - |I| \geq A(\mathcal{E}, \deg(\Sigma)),$$

the homomorphism

$$\mathrm{H}^0(B, \mathcal{E}^\vee \otimes L) \to \oplus_{j\in J}\mathcal{E}^\vee|_{y_j} \oplus \oplus_{i\in I}\mathcal{E}^\vee_{c_i},$$

is surjective. In that range,

$$(Z^{\mathrm{alg}}_{d,\Sigma})_{(L,y)} \subset (\mathcal{V}_{d,\Sigma})_L$$

has the expected real codimension $2\gamma(y)$. □

Approximation. One can show that

$$\deg(T) - |\Sigma| \le \kappa(T) := 2\gamma(T) - \deg(T) - 2|\mathrm{Supp}(T)| \tag{5.2}$$

Fix a positive integer R and define

$$\mathrm{P} = \{y \in \mathrm{Hilb}(B) \mid \deg(y) \le R + |\Sigma|\},$$

a downward closed proper union of finitely many $\mathcal{N}_T$ with $\deg(T) \le R + |\Sigma|$. One of the main applications of [DT24, Theorem 5.9] is:

Proposition 14. *Suppose that*

$$d \ge R + |\Sigma| + A(\mathcal{E}, \deg(\Sigma)) + 1.$$

Then the map

$$\mathrm{B}(\mathrm{P}, Z_{k,U}) \to \mathrm{im}(Z_{k,U}|_{\mathrm{P}} \to \mathcal{W}_k|_U)$$

induces a homomorphism

$$\mathrm{H}^i_c(\mathrm{im}(Z_{k,U}|_{\mathrm{P}} \to \mathcal{W}_k|_U), \mathbb{Z}) \to \mathrm{H}^i_c(\mathbf{B}(\mathrm{P}, Z_{k,U}), \mathbb{Z}),$$

which is an isomorphism when $i > \dim(\mathcal{W}_k) - R - 2$ and a surjection when $i = \dim(\mathcal{W}_k) - R - 2$, where $\dim(\mathcal{W}_k)$ is the real dimension of the semi-algebraic bundle $\mathcal{W}_k$.

Proof. This follows from a version of [DT24, Theorem 5.9]. To verify the assumptions, observe first that P is downward closed and proper. Furthermore, for $y \in \mathrm{P}$ and $y \prec y'$ such that y' is reduced, we have

$$|I(y')| \le \deg(y') \le R + |\Sigma| + 1.$$

By Proposition 13,

$$(Z_k)_{L,y} \subset (\mathcal{W}_k)_L$$

has expected real codimension $2\gamma(y)$. Finally, by (5.2), $\kappa(T) \le R$ implies that T is the type of P. □

The main result.

Theorem 15. *Assume that*

$$d \geq R + |\Sigma| + A(\mathcal{E}, \deg(\Sigma)) + 1.$$

Then the inclusion

$$\mathrm{Sect}(\mathcal{X}/B, d, \Sigma) \hookrightarrow S^{\mathrm{stop}}_{d,\Sigma},$$

is homology R-connected, i.e., the induced homomorphism

$$\mathrm{H}_i(\mathrm{Sect}(\mathcal{X}/B, d, \Sigma), \mathbb{Z}) \to \mathrm{H}_i(S^{\mathrm{stop}}_{d,\Sigma}, \mathbb{Z}),$$

is an isomorphism when $i < R$ and an injection when $i = R$.

Proof. We follow the proof of [DT24, Theorem 7.1]:

Step 1. Both $\tilde{S}_{d,\Sigma} \to \mathrm{Sect}(\mathcal{X}/B, d, \Sigma)$ and $\tilde{S}^{\mathrm{stop}}_{d,\Sigma} \to S^{\mathrm{stop}}_{d,\Sigma}$ are $\mathbb{C}^\times$-torsors, so they induce the following diagram:

$$\begin{array}{ccc} \mathrm{Sect}(\mathcal{X}/B, d, \Sigma) & \hookrightarrow & S^{\mathrm{stop}}_{d,\Sigma} \\ \downarrow & & \downarrow \\ B\mathbb{C}^\times & = & B\mathbb{C}^\times. \end{array} \tag{5.3}$$

Applying the Leray spectral sequence to both vertical maps, the theorem follows from homology R-connectedness of the inclusion

$$\tilde{S}_{d,\Sigma} \hookrightarrow \tilde{S}^{\mathrm{stop}}_{d,\Sigma}.$$

Step 2. Since the inclusion is a continuous map, compatible with projections to $\mathrm{Pic}^d(B)$, it follows from the two Leray spectral sequences over $\mathrm{Pic}^d(B)$ that it suffices to show that for a basis of open subsets $U \subset \mathrm{Pic}^d(B)$, the restriction

$$\tilde{S}_{d,\Sigma}|_U \hookrightarrow \tilde{S}^{\mathrm{stop}}_{d,\Sigma}|_U,$$

is homology R-connected.

Step 3. We have realized

$$\tilde{S}^{\mathrm{stop}}_{d,\Sigma}|_U \subset \mathcal{V}^{\mathrm{stop}}_{d,\Sigma}|_U$$

as an open subset of a Banach bundle over U. Taking a sufficiently small U, we trivialize this bundle. By [DT24, Proposition 6.16], the inclusion

$$\cup_k(\tilde{S}^{\mathrm{stop}}_{d,\Sigma}|_U \cap (\mathcal{W}_k|_U)) \hookrightarrow \tilde{S}^{\mathrm{stop}}_{d,\Sigma}|_U$$

is a weak homotopy equivalence. Thus, it suffices to show that

$$\tilde{S}_{d,\Sigma}|_U \hookrightarrow \tilde{S}_k|_U := \tilde{S}^{\mathrm{stop}}_{d,\Sigma}|_U \cap (\mathcal{W}_k|_U),$$

is homology R-connected, for sufficiently large k.

Step 4. Since $\tilde{S}_{d,\Sigma}|_U = \tilde{S}_k|_U \cap \mathcal{V}_{d,\Sigma}|_U$, by [DT24, Proposition 2.2], the induced homomorphism

$$\mathrm{H}_i(\tilde{S}_{d,\Sigma}|_U, \mathbb{Z}) \to \mathrm{H}_i(\tilde{S}_k|_U, \mathbb{Z}),$$

is Poincaré dual to the Gysin map

$$\mathrm{H}_c^{2\dim(\mathcal{V}_{d,\Sigma})-i}(\tilde{S}_{d,\Sigma}|_U, \mathbb{Z}) \to \mathrm{H}_c^{\dim(\mathcal{W}_k)-i}(\tilde{S}_k|_U, \mathbb{Z}).$$

It suffices to prove that this Gysin map is an isomorphism, when $i < R$, and a surjection, when $i = R$.

Step 5. Let

$$C^{\mathrm{alg}} = \mathcal{V}_{d,\Sigma}|_U \setminus \tilde{S}_{d,\Sigma}|_U, \quad C_k = \mathcal{W}_k|_U \setminus \tilde{S}_k|_U.$$

We have a commuting diagram of long exact sequences of cohomology with compact supports:

$$\begin{array}{ccccc}
\mathrm{H}_c^{2\dim(\mathcal{V}_{d,\Sigma})-i}(\tilde{S}_{d,\Sigma}|_U, \mathbb{Z}) & \twoheadrightarrow & \mathrm{H}_c^{2\dim(\mathcal{V}_{d,\Sigma})-i}(\mathcal{V}_{d,\Sigma}|_U, \mathbb{Z}) & \twoheadrightarrow & \mathrm{H}_c^{2\dim(\mathcal{V}_{d,\Sigma})-i}(C^{\mathrm{alg}}, \mathbb{Z}) \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \psi_i} \\
\mathrm{H}_c^{\dim(\mathcal{W}_k)-i}(\tilde{S}_k|_U, \mathbb{Z}) & \longrightarrow & \mathrm{H}_c^{\dim(\mathcal{W}_k)-i}(\mathcal{W}_k|_U, \mathbb{Z}) & \longrightarrow & \mathrm{H}_c^{\dim(\mathcal{W}_k)-i}(C_k, \mathbb{Z}).
\end{array}$$

Here the vertical maps are the Gysin maps. It suffices to show that ψ_i is an isomorphism when $i < R+1$ and a surjection when $i = R+1$.

Step 6. By Proposition 14, it suffices to show that the Gysin map

$$\mathrm{H}_c^{2\dim(\mathcal{V}_{d,\Sigma})-i}(\mathbf{B}(\mathrm{P}, Z^{\mathrm{alg}}|_U), \mathbb{Z}) \to \mathrm{H}_c^{\dim(\mathcal{W}_k)-i}(\mathbf{B}(\mathrm{P}, Z_{k,U}), \mathbb{Z})$$

is an isomorphism when $i < R+1$ and a surjection when $i = R+1$. In turn, this follows from a version of [DT24, Theorem 5.6], in our setting. □

6. Homological stability

Projective bundles over curves.

Theorem 16. *Let*

$$\pi : \mathcal{X} = \mathbb{P}(\mathcal{E}) \to B$$

be the projectivization of a vector bundle, of relative dimension $n \geq 1$. Let Σ, Σ' be non-empty sets of admissible jets for π, such that

$$\pi(\Sigma) = \pi(\Sigma') \quad \text{and} \quad \deg(\Sigma) \geq \deg(\Sigma').$$

Let

$$\ell(d) = d - |\Sigma| - A(\mathcal{E}, \deg(\Sigma)) - 2,$$

where $A(\mathcal{E}, \deg(\Sigma))$ is the constant from Proposition 13. Then, for all $i \leq \ell(d)$, we have isomorphisms

$$\begin{aligned} \mathrm{H}_i(\mathrm{Sect}(\mathcal{X}/B, d, \Sigma), \mathbb{Z}) &\cong \mathrm{H}_i(\mathrm{Sect}(\mathcal{X}/B, d+1, \Sigma), \mathbb{Z}) \\ &\cong \mathrm{H}_i(\mathrm{Sect}(\mathcal{X}/B, d, \Sigma'), \mathbb{Z}). \end{aligned}$$

Proof. The first isomorphism follows from Theorem 15, Proposition 12, and Corollary 10, with β being the class of a vertical line. Indeed, we have

$$\begin{aligned} \mathrm{H}_i(\mathrm{Sect}(\mathcal{X}/B, d, \Sigma), \mathbb{Z}) &\cong \mathrm{H}_i(S^{\mathrm{stop}}_{d,\Sigma}, \mathbb{Z}) \\ &\cong \mathrm{H}_i(\mathrm{Sect}^{\mathrm{stop}}(\mathcal{X}/B, d, \Sigma) \times \mathrm{Pic}^d(B), \mathbb{Z}) \\ &\cong \mathrm{H}_i(\mathrm{Sect}^{\mathrm{stop}}(\mathcal{X}/B, d+1, \Sigma) \times \mathrm{Pic}^d(B), \mathbb{Z}) \\ &\cong \mathrm{H}_i(S^{\mathrm{stop}}_{d+1,\Sigma}, \mathbb{Z}) \\ &\cong \mathrm{H}_i(\mathrm{Sect}(\mathcal{X}/B, d+1, \Sigma), \mathbb{Z}). \end{aligned}$$

The second isomorphism follows from Theorem 15, Proposition 12, Proposition 6, and Proposition 8. We have

$$\begin{aligned} \mathrm{H}_i(\mathrm{Sect}(\mathcal{X}/B, d, \Sigma), \mathbb{Z}) &\cong \mathrm{H}_i(S^{\mathrm{stop}}_{d,\Sigma}, \mathbb{Z}) \\ &\cong \mathrm{H}_i(\mathrm{Sect}^{\mathrm{stop}}(\mathcal{X}/B, d, \Sigma) \times \mathrm{Pic}^d(B), \mathbb{Z}) \\ &\cong \mathrm{H}_i(\mathrm{Sect}^{\mathrm{stop}}(\mathcal{X}/B, d, \Sigma') \times \mathrm{Pic}^d(B), \mathbb{Z}) \\ &\cong \mathrm{H}_i(S^{\mathrm{stop}}_{d,\Sigma'}, \mathbb{Z}) \\ &\cong \mathrm{H}_i(\mathrm{Sect}(\mathcal{X}/B, d, \Sigma'), \mathbb{Z}). \end{aligned}$$

□

Conic bundles over curves. Let

$$\pi : \mathcal{S} \to B,$$

be a smooth conic bundle over B, i.e., $\mathcal{S}$ is a smooth projective surface such that ω_π^{-1} is π-relatively ample. Any singular fiber is the union of two lines meeting at a point. Let $\mathfrak{d}$ be the number of singular fibers. There are $2^{\mathfrak{d}}$ birational morphisms

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{\varphi_r} & \mathcal{S}_r = \mathbb{P}(\mathcal{E}_r) \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi_r} \\ B & = & B \end{array}$$

contracting vertical (-1)-curves; here $\mathcal{E}_r$ is normalized so that

$$\omega_{\pi_r}^{-1} \cong \mathcal{O}_{\mathbb{P}(\mathcal{E}_r)/B}(2) \otimes \mathcal{O}(\mathcal{S}_b)^{\otimes \varepsilon(r)},$$

where $\mathcal{S}_b$ is a general fiber of π and $\varepsilon(r) = 0$, or 1. Given a section σ of π class α we let $\varphi_{r(\alpha)}$ be the morphism contracting curves meeting σ. Put

$$h = \deg(\sigma^*\omega_\pi^{-1}), \quad d = \deg(\sigma^*\varphi_{r(\alpha)}^*\mathcal{O}_{\mathbb{P}(\mathcal{E}_{r(\alpha)})/B}(1)) \tag{6.1}$$

In particular,

$$h = 2d + \epsilon(r) - \mathfrak{d}.$$

Let Σ be a set of admissible jets for π and $\Sigma_{r(\alpha)}$ the set of admissible jets for $\pi_{r(\alpha)}$ such that Σ is the strict transform of $\Sigma_{r(\alpha)}$. We have

$$\deg(\Sigma_{r(\alpha)}) = \deg(\Sigma) + \mathfrak{d}, \quad |\Sigma_{r(\alpha)}| \le |\Sigma| + \mathfrak{d}.$$

Proposition 5 implies that there is an isomorphism

$$\mathrm{Sect}(\mathcal{S}/B, h, \Sigma) \simeq \mathrm{Sect}(\mathcal{S}_{r(\alpha)}/B, h + \mathfrak{d}, \Sigma_{r(\alpha)}). \tag{6.2}$$

Theorem 17. *Let*

$$\pi : \mathcal{S} \to B$$

be a smooth conic bundle. Let Σ, Σ' be non-empty sets of admissible jets for π, such that

$$\pi(\Sigma) = \pi(\Sigma') \quad \text{and} \quad \deg(\Sigma) \ge \deg(\Sigma').$$

Let α be the class of a section of π, h its degree as in (6.1)*, and*

$$\ell(h) := \frac{h}{2} - \frac{1}{2} - |\Sigma| - \frac{\mathfrak{d}}{2} - A(\mathcal{E}_{r(\alpha)}, \deg(\Sigma) + \mathfrak{d}) - 2.$$

Then, for all $i \leq \ell(h)$, we have isomorphisms

$$\begin{aligned}\mathrm{H}_i(\mathrm{Sect}(\mathcal{S}/B, h, \Sigma), \mathbb{Z}) &\cong \mathrm{H}_i(\mathrm{Sect}(\mathcal{S}/B, h+2, \Sigma), \mathbb{Z}) \\ &\cong \mathrm{H}_i(\mathrm{Sect}(\mathcal{S}/B, h, \Sigma'), \mathbb{Z}).\end{aligned}$$

Proof. This follows from (6.2) and Theorem 16. □

Quadric surface bundles. Let

$$\pi : \mathcal{X} \to B$$

be a smooth quadric surface bundle with relative Picard rank one and singular fibers with at most one singularity, of type A_1. Let

$$F_1(\mathcal{X}) \to D \xrightarrow{\iota} B$$

be the Stein factorization of the map from the space of lines in the fibers of π to B (see, e.g., [HT12, Section 3] for more details). The covering involution ι is branched along the discriminant divisor $\mathfrak{d}$ of π; the map $F_1(\mathcal{X}) \to D$ is a *smooth* $\mathbb{P}^1$-bundle, i.e.,

$$\pi_{\mathcal{Y}} : \mathcal{Y} := F_1(\mathcal{X}) = \mathbb{P}(\mathcal{E}) \to D,$$

for a rank-2 vector bundle $\mathcal{E}$ on D. Every point $x \in \mathcal{X}$, in a smooth fiber of π, gives rise to points $y, y' \in \mathcal{Y}$, in distinct fibers of $\pi_{\mathcal{Y}}$. In detail, let

$$\mathcal{D} \subset \mathcal{X} \times_B F_1(\mathcal{X}).$$

be the universal family of lines. Given a section $\sigma : B \to \mathcal{X}$, define $\mathcal{D}_\sigma$ as the fiber product

$$\begin{array}{ccc} \mathcal{D}_\sigma & \hookrightarrow & B \times_B F_1(\mathcal{X}) \\ \downarrow & & \downarrow{\scriptstyle \sigma\times\mathrm{id}} \\ \mathcal{D} & \hookrightarrow & \mathcal{X} \times_B F_1(\mathcal{X}). \end{array}$$

Then

$$\mathcal{D}_\sigma \to B \times_B F_1(\mathcal{X}) \to B$$

is a double cover ramified along $\mathfrak{d}$, so that $\mathcal{D}_\sigma$ is isomorphic to D. We obtain a section

$$\tau : D \to \mathcal{F}_1(\mathcal{X}).$$

This construction applies to jets as well: let

$$\hat{\sigma} : \mathrm{Spec}(\mathbb{C}[z]/(z^{N+1})) \to \mathcal{X}$$

be an admissible Nth-jet. This yields an admissible jet

$$\hat{\tau} : \mathcal{D}_{\hat{\sigma}} \to F_1(\mathcal{X}).$$

When $\hat{\sigma}$ is supported in a smooth fiber,

$$\mathcal{D}_{\hat{\sigma}} \to \mathrm{Spec}(\mathbb{C}[z]/(z^{N+1}))$$

is étale, so it consists of two copies of $\mathrm{Spec}(\mathbb{C}[z]/(z^{N+1}))$. When $\hat{\sigma}$ is supported in a singular fiber, $\mathcal{D}_{\hat{\sigma}}$ is isomorphic to $\mathrm{Spec}(\mathbb{C}[z]/(z^{2N+2}))$. In this way, a set of admissible jets Σ_B induces a set of admissible jets Σ_D, with

$$\deg(\Sigma_D) = 2\deg(\Sigma_B), \quad |\Sigma_D| \le 2|\Sigma_B|.$$

Lemma 18. *Let Σ_B be a set of admissible jets for $\pi : \mathcal{X} \to B$ and Σ_D the induced set of admissible jets of $\pi_{\mathcal{Y}} : \mathcal{Y} \to D$. There is an isomorphism*

$$\mathrm{Sect}(\mathcal{X}/B, h, \Sigma_B) \simeq \mathrm{Sect}(\mathcal{Y}/D, d, \Sigma_D),$$

for

$$h = 2d + \deg(\mathcal{E}) - \frac{|\mathfrak{d}|}{2}.$$

Proof. The last formula follows from the normalization of $\mathcal{E}$ in [HT12, Section 3]. □

Theorem 19. *Let*

$$\pi : \mathcal{X} \to B$$

be a smooth quadric surface bundle with relative Picard rank one and singular fibers with at most one A_1-singularity. Let Σ_B, Σ'_B be non-empty sets of admissible jets for π, such that

$$\pi(\Sigma_B) = \pi(\Sigma'_B) \quad \text{and} \quad \deg(\Sigma_B) \ge \deg(\Sigma'_B).$$

Let

$$\ell(h) = \frac{h}{2} - \frac{\deg(\mathcal{E})}{2} + \frac{|\mathfrak{d}|}{4} - 2|\Sigma_B| - A(\mathcal{E}, 2\deg(\Sigma_B)) - 2.$$

Then, for all $i \le \ell(h)$, we have isomorphisms

$$\begin{aligned} \mathrm{H}_i(\mathrm{Sect}(\mathcal{X}/B, h, \Sigma_B), \mathbb{Z}) &\cong \mathrm{H}_i(\mathrm{Sect}(\mathcal{X}/B, h+2, \Sigma_B), \mathbb{Z}) \\ &\cong \mathrm{H}_i(\mathrm{Sect}(\mathcal{X}/B, h, \Sigma'_B), \mathbb{Z}). \end{aligned}$$

Proof. This follows from Lemma 18 and Theorem 16. □

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