INVARIANTS IN EQUIVARIANT BIRATIONAL GEOMETRY

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Abstract. We discuss invariants in equivariant birational geometry.

1. Introduction

In this survey, we consider problems in classical birational geometry and their extensions to the setting of equivariant geometry. The focus is on new invariants, introduced in [84], building on [79], [78], [82], and [59]. In our applications, we work over an algebraically closed ground field k of characteristic zero.

The term *birational geometry* refers to the characterization of isomorphism classes of function fields of algebraic varieties over k, i.e., fields K which are finitely generated over k. Of particular interest is the *(stable) rationality problem*, i.e., the characterization of *(stably) rational* function fields. Recall that an algebraic variety X over k is called *rational*, respectively, *stably rational*, if its function field $K = k(X)$, respectively, the field $K(x_1, \dots, x_n)$ for suitable n, is isomorphic to a purely transcendental extension of k. An interesting related notion is that of *unirationality*, i.e., the property that $K \subseteq k(x_1, \dots, x_n)$. The (stable) rationality problem is settled in dimension 2, but remains elusive in dimensions ≥ 3.

One can ask about birationality of varieties with group actions. Concretely, let G be a finite group, acting on $K = k(X)$, and trivially on k. If the action is generically free, then G is identified with a subgroup of

$$\operatorname{Bir}\operatorname{Aut}(X),$$

the group of birational automorphisms of X over k. The main question in this context is:

When are two such subgroups conjugate in $\operatorname{Bir}\operatorname{Aut}(X)$*?*

There is always a smooth projective model for which the action is *regular* (see Section 3). Taking X to be such a model, we can view G as a subgroup of

$$\operatorname{Aut}(X),$$

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the group of biregular automorphisms of X. For example, when $X = \mathbb{P}^2$, the projective plane, we have $\mathrm{Aut}(X) \simeq \mathrm{PGL}_3$, and it is known classically how to distinguish finite subgroups up to conjugation in PGL_3. But the problem of distinguishing finite subgroups of PGL_3 up to conjugation in the *Cremona group*

$$\mathrm{Cr}_2 := \mathrm{Bir}\,\mathrm{Aut}(\mathbb{P}^2),$$

the group of birational automorphism of $\mathbb{P}^2$, is a recent advance [38].

In this paper, we discuss new invariants in G-equivariant birational geometry, and in particular, the *equivariant Burnside group*

$$\mathrm{Burn}_n(G),$$

introduced in [84]. Needless to say, there is an enormous literature on equivariant birational geometry; we will not be able to give a complete account of all important ideas and techniques. We will point to sources that should allow an interested reader to learn more about this fascinating subject.

One of the motivations to study G-birational geometry lies in strongly suggestive analogies with birational geometry over nonclosed fields, a booming subject. These analogies have been emphasized in [94] and [95]: the role of G is played by Galois symmetries, which act on special loci, geometric invariants, etc. There are also appreciable differences between these theories. We will not pursue this theme here, but refer to [66], [67], [83] for recent developments inspired by this dual point of view.

Here is a roadmap of this paper. In Sections 2 and 3 we discuss the basics of equivariant birational geometry and the evolution of ideas and constructions that led us to the definition of $\mathrm{Burn}_n(G)$:

- *analytic tools*: motivic integration and (specialization of) rationality,
- *geometric tools*: equivariant weak factorization, birational rigidity, stacks.

We give a short summary of known obstructions in equivariant birational geometry in Section 4. In Section 5, we discuss the Amitsur group and introduce a higher version of the classical Amitsur invariant, based on the Leray spectral sequence for motivic complexes of stacks. In Section 6, we explain how to choose suitable birational models for G-actions. In Section 7, we recall the definition of equivariant Burnside groups from [84] and explain how to compute the class of the action. We comment on first structural properties of these new invariants in Section 8:

- *filtrations*,
- *incompressible symbols*,
- *specialization*.

We present several applications in Section 9 and discuss limitations of these invariants towards the end of the paper, in Section 10.

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2. Key ideas

Throughout, we work over a field k of characteristic zero.

Motivic integration. Motivic integration was introduced by Kontsevich in his generalization of Batyrev's theorem [7]: birational Calabi-Yau varieties have equal *Hodge numbers*. It is now a major research direction, with many applications, e.g., to the study of singularities, McKay correspondence, etc.; see [34].

Let

$$\mathrm{Var}_{n,k}$$

be the set of isomorphism classes of algebraic varieties over k of dimension n, and put

$$\mathrm{Var}_k = \bigsqcup_{n\geq 0} \mathrm{Var}_{n,k}, \qquad \mathrm{Var}_k^{\leq d} = \bigsqcup_{0\leq n\leq d} \mathrm{Var}_{n,k}\,.$$

We will write $[X]$ for the class of an algebraic variety (i.e., a reduced, separated, finite-type k-scheme) in Var_k. The *Grothendieck ring*

$$\mathrm{K}_0(\mathrm{Var}_k)$$

is defined as the quotient of the $\mathbb{Z}$-module

$$\mathbb{Z}[\mathrm{Var}_k]$$

by the relations:

- *Excision*: $[X] = [Z] + [X \setminus Z]$, for closed $Z \subset X$,
- *Product*: $[X \times Y] = [X] \times [Y]$, for all X, Y.

The ring $\mathrm{K}_0(\mathrm{Var}_k)$ carries a natural filtration, by the images of $\mathbb{Z}[\mathrm{Var}_k^{\leq d}]$.

A *motivic measure* with values in a commutative ring R is a ring homomorphism

$$\mu\colon \mathrm{K}_0(\mathrm{Var}_k) \to R.$$

For example, for $k = \mathbb{C}$ we take $R = \mathbb{Z}[u, v]$ and define, for smooth projective X,

$$\mu([X]) := \sum_{p,q} (-1)^{p+q}\, h^{p,q}(X) u^p v^q,$$

the *Hodge-Deligne polynomial*. Kontsevich's refinement of Batyrev's theorem shows that birational Calabi-Yau varieties have the same class in $\mathrm{K}_0(\mathrm{Var}_k)$, thus the same Hodge-Deligne polynomials, and thus the same Hodge numbers.

Stable rationality and $\mathrm{K}_0(\mathrm{Var}_k)$. The beautiful paper [93] revealed an unexpected connection with birational geometry: smooth projective varieties X and Y are *stably birational* if and only if

$$[X] \equiv [Y] \pmod{[\mathbb{A}^1]},$$

We have a surjective homomorphism

$$\mathrm{K}_0(\mathrm{Var}_k) \to \mathbb{Z}[\mathrm{SBir}_k] \tag{2.1}$$

to the free abelian group on *stable birationality* classes of algebraic varieties over k, with kernel the ideal generated by $[\mathbb{A}^1]$. In particular, every motivic measure μ which is trivial on $\mathbb{A}^1$ factors through the homomorphism (2.1).

The proof is based on

- *Weak factorization*: every birational map can be factored into a sequence of blow-ups and blow-downs with smooth centers,
- *Bittner's presentation*: the defining *Excision* relation in $\mathrm{K}_0(\mathrm{Var}_k)$ can be replaced by the *Blow-up* relation: If $\widetilde{X}$ is the blow-up of a smooth projective variety X in smooth $Z \subsetneq X$ and E is the exceptional divisor then

$$[X] - [Z] = [\widetilde{X}] - [E].$$

The paper [79] introduced the set

$$\mathrm{Bir}_{n,k}$$

of isomorphism classes of function fields of algebraic varieties of dimension n over k, i.e., the set of *birationality* classes of n-dimensional algebraic varieties. The free abelian group on $\mathrm{Bir}_{n,k}$ is

$$\mathrm{Burn}_{n,k} := \mathbb{Z}[\mathrm{Bir}_{n,k}].$$

The group

$$\mathrm{Burn}_k := \bigoplus_{n \geq 0} \mathrm{Burn}_{n,k}$$

has a natural ring structure, induced by the product operation on algebraic varieties. There is a surjection

$$\mathrm{Burn}_k \to \mathrm{gr}(\mathrm{K}_0(\mathrm{Var}_k)),$$

with nontrivial kernel [27, Thm. 2.13].

Specialization of (stable) rationality. Applications of *specialization* to rationality problems go back (at least) to [10], which established failure of rationality of certain Fano threefolds by degenerating them to conic bundles; for these it is easier to compute the intermediate Jacobian, and if it fails to be the Jacobian of a curve, then the same holds for the generic fiber.

The next conceptual leap was due to Voisin [123] and Colliot-Thélène–Pirutka [47] who realized that specialization holds for cycle-theoretic (stable) birational invariants such as *integral decomposition of the diagonal*, respectively, *universal* CH_0*-triviality*. Again, the computation of these invariants can be easier in the special fiber. Immediately, this led to a wealth of new results on failure of stable rationality, see, e.g., [123], [47], [77], [115], [58], [63], [61], [62], [60], [81], [111], [4], [112], [100], as well as the surveys [102] and [46].

It turned out that both $\mathbb{Z}[\mathrm{SBir}_k]$ and Burn_k admit specialization homomorphisms as well [101], [79]. The key construction was the *motivic volume* homomorphism

$$\mathrm{K}_0(\mathrm{Var}_K) \to \mathrm{K}_0(\mathrm{Var}_k), \quad K = k((t)),$$

respectively, the *Burnside volume* homomorphism

$$\mathrm{Burn}_K \to \mathrm{Burn}_k\,.$$

A common refinement

$$\mathrm{K}_0(\mathrm{Var}_K^{\dim}) \to \mathrm{K}_0(\mathrm{Var}_k^{\dim})$$

of these two volume homomorphisms, with the *Grothendieck ring graded by dimension* $\mathrm{K}_0(\mathrm{Var}_k^{\dim}) = \bigoplus_{d\geq 0} \mathrm{K}_0(\mathrm{Var}_k^{\leq d})$, was given in [99].

3. Equivariant birational geometry

Given these recent developments it is natural to seek *equivariant* analogs of the main constructions outlined in Section 2. Here, we do this in the context of actions of a *finite* group G; a version for varieties with logarithmic volume forms has been proposed in [33].

3.1. **Background.** We follow the conventions of [85, Sect. 2]: a G-variety is a reduced, separated, finite-type scheme over k, with regular action of G that is transitive on the set of irreducible components. The G-action is generically free if it is free on a nonempty invariant open subvariety.

Let X and Y be G-varieties. We call a G-equivariant rational map $X \dashrightarrow Y$ a *G-rational map*; if birational, then we call it *G-birational*. A G-rational map $X \dashrightarrow Y$ is *proper* if there exist a G-variety Z and proper G-equivariant morphisms $Z \to X$ and $Z \to Y$, with $Z \to X$ birational

(cf. [72, Sect. 2.12], [57, App. A]). Fundamental results in this setting are:

- **Equivariant desingularization**, which holds, e.g., by [18], and lets us replace any model by a smooth model.
- **Equivariant weak factorization** [2]: Any proper G-birational map $X \dashrightarrow Y$, where X and Y are smooth, can be factored as a composition of finitely many G-birational maps given by blow-ups of invariant smooth G-subvarieties, and their inverses.
- **Regular models**. Any rational G-action is regular on some projective model, which by equivariant desingularization we may take to be nonsingular; see [31].

By convention, we work with regular G-actions. We say that X and Y are *G-birational*, written,

$$X \sim_G Y$$

when there exists a proper G-birational map $X \dashrightarrow Y$. We say that two G-varieties X and Y are *stably G-birational* if

$$X \times \mathbb{P}(W) \sim_G Y \times \mathbb{P}(V),$$

for some G-representations V, W. In particular, we do not assume that X and Y have the same dimension.

A projective G-variety X is called *linearizable*, respectively *stably linearizable*, if there exist G-representations V, W such that

$$X \sim_G \mathbb{P}(V), \quad \text{resp.} \quad X \times \mathbb{P}(W) \sim_G \mathbb{P}(V).$$

If X is only quasi-projective, then we apply the same terminology for the respective condition, applied to an equivariant projective model.

By a result recalled in Section 3.5, the No-name lemma, in the case of *generically free* actions, the condition of stable G-birationality, respectively, stable linearizability, can be equivalently stated as

$$X \times \mathbb{P}^n \sim_G Y \times \mathbb{P}^m, \quad \text{resp.} \quad X \times \mathbb{P}^n \sim_G \mathbb{P}(V).$$

for some n, m, with trivial action on the second factors.

A G-action on X is called *unirational* if there exists a G-equivariant dominant rational map

$$\mathbb{P}(V) \dashrightarrow X.$$

By convention, we let G act on X on the right, and we write $X \leftarrow G$. Correspondingly, the G-action on $K = k(X)$ is on the left, $G \curvearrowright K$, where we adopt the further convention that $k(X)$ is the product of the function fields of the irreducible components of X.

3.2. **Cremona group.** Of particular interest are varieties X with *large* birational automorphism groups

$$\operatorname{Bir}\operatorname{Aut}(X).$$

There are currently no general methods to determine this group, given a variety X. Of course, this is easy for curves, but is a formidable problem in dimensions ≥ 2.

There are important classes of varieties, where $\operatorname{Bir}\operatorname{Aut}(X)$ is accessible, in principle:

- varieties of *general type*, via canonical models supplied by the Minimal Model Program [56],
- K3 surfaces: every birational automorphism is a regular automorphism, and those are classified, see, e.g., [71, Chapter 15] or [30],
- *birationally rigid varieties*, e.g., smooth quartic threefolds [74]; see [108].

However, one of the most important examples,

$$\mathrm{Cr}_n := \operatorname{Bir}\operatorname{Aut}(\mathbb{P}^n),$$

the *Cremona group* in dimension $n \geq 2$, remains mysterious. Taking k to be algebraically closed, the classification of finite subgroups of Cr_2 was a culmination of decades of efforts [52]. There is a classification of finite nonabelian *simple* and *quasi-simple* subgroups of Cr_3 [105], [21], and there are fascinating results about the overall structure of Cr_n [107], [22], but there are also many concrete unanswered questions, that fall under the general, overarching problems:

- When are two (necessarily linear) G-actions on $\mathbb{P}^n$ conjugate in Cr_n?
- Is a given G-action on a rational variety (stably) linearizable?

3.3. **Equivariant birational rigidity.** The study of G-equivariant birational rigidity is a thriving area, with many beautiful constructions, see, e.g., [106], [37]. In principle, one can approach the problem of (equivariant) birationality via classification of all possible blow-ups and blow-downs. In practice, this is feasible only in small dimensions, and under the assumption that G is large. It entails a detailed analysis of singularities, explicit invariant theory, geometric inequalities, etc. In dimension 2, the main technical tool is the theory of equivariant *Sarkisov links*; it allowed to classify finite subgroups of Cr_2 in [52]. In dimension 3, it is the equivariant *Noether-Fano inequality*. Here is a sample application:

Example: The permutation $\mathfrak{A}_5$-action (and thus also the $\mathfrak{S}_5$-action) on the diagonal quadric

$$X = \Big\{\sum_{j=1}^{5} x_j^2 = 0\Big\} \subset \mathbb{P}^4$$

is not linearizable [36], [104]. By [41, Thm. 4.1], the $\mathfrak{S}_5$-action is stably linearizable.

3.4. **Stacks.** Let X be a G-variety and $[X/G]$ the associated Deligne-Mumford stack. When X is a point (with trivial G-action), this is the classifying stack

$$BG = [\mathrm{Spec}(k)/G].$$

A systematic introduction to the birational geometry of Deligne-Mumford stacks can be found in [86].

While there is loss of information in the passage

$$X \circlearrowleft G \quad \rightsquigarrow \quad [X/G] \quad \rightsquigarrow \quad k(X)^G,$$

there is still a rich source of obstructions to G-birationality to be explored in the setting of stacks; we will see such examples below in Section 5.

An equivariant sheaf (of abelian groups) on X (for the G-action) determines a sheaf on $[X/G]$, by which we mean a sheaf on the étale site of $[X/G]$ [50, Defn. 4.10]. As on any site, there are cohomology groups of sheaves of abelian groups. For BG, when the base field k is algebraically closed, sheaf cohomology recovers group cohomology

$$\mathrm{H}^p(BG, \mathcal{F}) = \mathrm{H}^p(G, \mathcal{F}(\mathrm{Spec}(k))), \tag{3.1}$$

by the Čech spectral sequence for the covering of BG by $\mathrm{Spec}(k)$. Still supposing k to be algebraically closed, the Leray spectral sequence, applied to the morphism of stacks

$$[X/G] \to BG,$$

yields

$$\mathrm{H}^p(G, \mathrm{H}^q(X, \mathcal{F})) \Rightarrow \mathrm{H}^{p+q}([X/G], \mathcal{F}). \tag{3.2}$$

One can also consider an arithmetic version, where instead of a G-action we consider the action of the absolute Galois group $G_k = \mathrm{Gal}(\bar{k}/k)$. In this case, the spectral sequence takes the form

$$\mathrm{H}^p(G_k, \mathrm{H}^q(X_{\bar{k}}, \mathcal{F})) \Rightarrow \mathrm{H}^{p+q}(X_k, \mathcal{F}). \tag{3.3}$$

This spectral sequence has been studied in great detail; see, e.g., [48, Sect. 1.5].

3.5. **No-name lemma.** This surprisingly useful result in equivariant geometry is applicable to a G-vector bundle, whenever the G-action on the base is *generically free*. In one form, it asserts the existence of a G-birational map, over the base, from the total space of the vector bundle to a product with an affine space (with trivial G-action on the affine space); cf. [43, Lemma 4.4]. Here we state a projective version.

Lemma 3.1 (No-name lemma, projective bundle version)**.** *Let X be a G-variety, where the G-action on X is generically free. Let $E \to X$ be a G-vector bundle of rank $n \geq 1$. Then, letting G act trivially on $\mathbb{P}^{n-1}$ we have an equivariant birational equivalence*

$$\mathbb{P}(E) \sim_G X \times \mathbb{P}^{n-1},$$

which is compatible with the two projection maps to X.

Proof. By assumption, the G-action is free on some dense invariant open subscheme $W \subset X$. If $V \subset W$ is an affine dense open subscheme, then the intersection

$$U = \bigcap_{g \in G} V \cdot g$$

of the translates of V is an affine dense invariant open subscheme of X. Writing $U = \operatorname{Spec}(R)$, we have quotient $Z = \operatorname{Spec}(R^G)$ such that $U \to Z$ has a structure of a G-torsor. Then, by [98, Prop. 0.9, Ampfl. 1.3] and standard faithfully flat descent, we may identify G-vector bundles on U with vector bundles on Z. In particular, when we shrink U to a suitable invariant dense affine open subscheme and correspondingly shrink Z, we may suppose that the restriction of E to U is G-equivariantly trivial. With this we obtain a proper G-birational map $\mathbb{P}(E) \dashrightarrow X \times \mathbb{P}^{n-1}$. □

As an immediate consequence, we have

Corollary 3.2. *Let $G \to \mathrm{GL}(V^\vee)$ and $G \to \mathrm{GL}(W^\vee)$ be representations, that determine faithful projective representations $G \to \mathrm{PGL}(V^\vee)$ and $G \to \mathrm{PGL}(W^\vee)$. Then $\mathbb{P}(V)$ and $\mathbb{P}(W)$ are stably G-birational.*

Another useful observation concerns the canonical compactification $\mathbb{P}(E \oplus 1)$ of a G-vector bundle $E \to X$. The G-rational map $E \dashrightarrow \mathbb{P}(E)$ becomes a morphism after blowing up the zero-section 0_E. This extends to the compactification, and we have an equivariant isomorphism

$$B\ell_{0_E}\mathbb{P}(E \oplus 1) \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}(E)}(-1) \oplus 1)$$

over $\mathbb{P}(E)$.

Corollary 3.3. *Let X be a G-variety and $E \to X$ a G-vector bundle of rank ≥ 1. If the G-action on $\mathbb{P}(E)$ is generically free, then*

$$\mathbb{P}(E \oplus 1) \sim_G \mathbb{P}(E) \times \mathbb{P}^1,$$

where G acts trivially on $\mathbb{P}^1$.

4. Obstructions in G-equivariant geometry

Here, we assume that k (the base field, of characteristic zero) is algebraically closed. A regular action of a group G on a smooth projective n-dimensional variety X naturally induces an action on its points $X(k)$ and on various geometric invariants associated with X. Among those geometric invariants are:

- spaces of differential forms,
- Chow and Hilbert schemes,
- Picard and Brauer groups,
- étale cohomology and Chow groups,
- Intermediate Jacobians, etc.

Frequently, the information extracted from these data can be turned into a birational invariant of the G-action. In this section, we recall some of these extensively studied invariants and resulting obstructions to (stable) linearizability, and introduce new invariants.

4.1. **Dimension.** Let X be a smooth projective rational variety with a generically free action of a finite group G. The simplest obstruction to linearizability of the action is the absence of faithful G-representations of dimension $\dim(X) + 1$.

Examples: Quadrics with actions of large groups. E.g., the smallest faithful representation of the symmetric group $\mathfrak{S}_n$ has dimension $n-1$, and the same holds for the alternating group $\mathfrak{A}_n$, for $n \geq 6$. It follows that the $(n-3)$-dimensional quadric X, defined in $\mathbb{P}^{n-1}$ by

$$\sum_{j=1}^{n} x_j^2 = \sum_{j=1}^{n} x_j = 0$$

with the obvious permutation action of $\mathfrak{A}_n$, is not linearizable. For $n = 6$, the action is stably linearizable, by [41, Thm. 4.1].

Another example is supplied by the Mathieu group $G = \mathfrak{M}_{11}$. Its smallest faithful representations have dimension 10. One of these is a self-dual representation V admitting a nontrivial invariant quadratic form, thus a nonlinearizable action on an 8-dimensional quadric $X \subset \mathbb{P}(V)$. By [67, Prop. 5.3], stable linearizability of the G-action on X is implied by stable linearizability of the action of the 2-Sylow subgroup $G_2 \subset G$, which

in this case is the semi-dihedral group of order 16. The restriction of V to G_2 admits a decomposition $W \oplus 1 \oplus 1$, from which we may conclude $X \sim_{G_2} \mathbb{P}(W \oplus 1)$. Thus the G-action on X is stably linearizable.

Yet another example is the Pfaffian, and thus rational, cubic fourfold $X \subset \mathbb{P}^5$ considered in [26, Rmk. 15]:

$$\sum_{j=1}^{6} x_j^3 = \sum_{j=1}^{6} x_3 = 0,$$

with the action of the Frobienius group $\mathrm{AGL}_1(\mathbb{F}_7)$ of order 42. Its smallest faithful representation has dimension 6, and the group cannot act generically freely on $\mathbb{P}^4$. As shown in [26, Thm. 16], $X \times \mathbb{P}^1$ is linearizable.

4.2. **Fixed points.** Actions of cyclic groups on smooth projective rational varieties always have fixed points. However, this may fail for noncyclic groups (or for actions of cyclic groups on irrational varieties).

Examples: The faithful action of the Klein four-group $\mathfrak{K}_4 = C_2^2$ on $\mathbb{P}^1$ has no fixed points. Also, there are no fixed points for the action of the unique $C_2 \oplus C_4 \subset \mathrm{Cr}_2$ not fixing a curve of genus ≥ 1 and not birational to an action on $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$, see [19, Thm. 5].

Clearly,

$$X^G \neq \emptyset \quad \Leftrightarrow \quad (X \times \mathbb{P}^n)^G \neq \emptyset,$$

with G acting trivially on the second factor.

Proposition 4.1 ([109])**.** *Let A be an abelian group and let X and Y be smooth projective A-birational varieties. Then*

$$X^A \neq \emptyset \quad \Leftrightarrow \quad Y^A \neq \emptyset.$$

Example 4.2. Consider a blow-up of a fixed point $x \in X^A$ on a smooth projective A-variety. The exceptional divisor $E = \mathbb{P}(\mathcal{T}_{X,x})$ is the projectivization of the tangent bundle $\mathcal{T}_{X,x}$, a representation of A. The projectivization necessarily has fixed points.

However, for *nonabelian* G, the existence of fixed points is *not* a G-birational invariant of smooth projective G-varieties. Consider $G = \mathfrak{S}_3$, acting on $\mathbb{P}^2$, canonical compactification of the standard 2-dimensional representation. When we blow up the origin, the unique fixed point, we obtain exceptional divisor $\mathbb{P}^1$, which has has no fixed points.

(A): Existence of fixed points upon restriction to abelian subgroups

$$A \subseteq G.$$

Condition **(A)** is a G-equivariant stable birational invariant of smooth projective varieties. It holds when the G-action is stably linearizable, and more generally, when the G-action is unirational.

4.3. **Determinant.** For the action of a finite abelian group A, we get a finer invariant by considering the *weights* of the A-action in the tangent space $\mathcal{T}_{X,x}$ to a fixed point $x \in X^A$. Let

$$\beta(x) := [b_1, \dots, b_n], \quad b_j \in A^\vee := \mathrm{Hom}(A, k^\times),$$

be the collection of these weights, i.e., characters of A. Let

$$\det(\beta(x)) \in \wedge^n(A^\vee)$$

be the *determinant*, defined as the wedge product $b_1 \wedge \cdots \wedge b_n$ and well-defined up to sign.

Proposition 4.3 ([110])**.** *Let*

$$\pi\colon \widetilde{X} \to X$$

be an equivariant birational morphism of smooth projective A-varieties. Then for all $x \in X^A$ and $\tilde{x} \in \widetilde{X}^A$ with $\pi(\tilde{x}) = x$ we have

$$\det(\beta(\tilde{x})) = \pm \det(\beta(x)) \in \wedge^n(A^\vee).$$

Example: The action of the cyclic group C_5 on $\mathbb{P}^1$, canonical compactification of a nontrivial one-dimensional representation, falls into one C_5-birational equivalence class for weight ± 1 and another C_5-birational equivalence class for weight ± 2 (and we have the same description for C_5-isomorphism classes, because of dimension 1). Using Proposition 4.3 we see that for any $n \geq 1$ and odd prime p the canonical compactifications of n-dimensional faithful linear representations of C_p^n fall into precisely $(p-1)/2$ equivariant birational equivalence classes.

(Det): Existence of A-fixed points, for some abelian subgroup $A \subseteq G$, with a given determinant class

$$[\det(\beta)] \in \wedge^n(A^\vee)/\pm$$

is a G-equivariant birational invariant for smooth projective G-varieties.

Remark 4.4. The invariant **(Det)** can only deliver finer information than the mere existence of A-fixed points in the case of G-varieties of dimension equal to the rank of A.

4.4. **Cohomology.** One can investigate the G-action on the Picard group $\mathrm{Pic}(X)$. A key observation is that a blow-up of a smooth G-stable subvariety adds a permutation module to $\mathrm{Pic}(X)$, and that the multiplication with projective space with trivial G-action adds $\mathbb{Z}$ to $\mathrm{Pic}(X)$, a trivial permutation module. It follows that $\mathrm{Pic}(X)$, modulo permutation modules is a stable birational invariant. This yields an invariant and obstruction to stable linearizability:

Let X be a smooth projective rational variety with a generically free G-action. The similarity class

$$[\mathrm{Pic}(X)]$$

in the set of isomorphism classes of G-lattices, modulo the equivalence relation given by direct sums with permutation lattices, is a stable birational invariant.

(SP): The property that the G-module $\mathrm{Pic}(X)$ is stably permutation:

$$[\mathrm{Pic}(X)] = [0].$$

Condition **(SP)** holds for stably linearizable actions.

There exist stably permutation modules that are not permutation modules; some examples are recalled in [68, Sect. 1]. We are not aware of algorithms to check whether a given G-module is stably permutation. However, the next, related, invariant can be computed effectively.

The first group cohomology

$$\mathrm{H}^1(G, \mathrm{Pic}(X))$$

is a stable birational invariant.

(H1): The vanishing of $\mathrm{H}^1(H, \mathrm{Pic}(X))$, for all subgroups $H \subseteq G$.

If the G-action is stably linearizable then Condition **(H1)** holds. In some cases, the invariant $\mathrm{H}^1(G, \mathrm{Pic}(X))$ can be determined from the stabilizer stratification of X [23], [83].

Example: Let $G = C_p$ act on a smooth projective rational surface X. Assume that X^G contains a curve of genus $g \geq 1$; such a curve is necessarily smooth and unique. Then by [23],

$$\mathrm{H}^1(G, \mathrm{Pic}(X)) \cong (\mathbb{Z}/p\mathbb{Z})^{2g}.$$

4.5. **Coniveau filtrations.** Let X be a smooth projective G-variety of dimension n and A a G-module. As in [89, Sect. 3] there is the cohomology

$$\mathrm{H}^i_G(X, A) := \mathrm{H}^i([X/G], A).$$

Let us suppose that A is m-torsion, for a positive integer m, and consider the subgroups:

- $\mathrm{N}^1\mathrm{H}^i_G(X,A)$, the group of all $\alpha \in \mathrm{H}^i_G(X,A)$, such that α vanishes in $\mathrm{H}^i_G(U,A)$ for some nonempty G-invariant open $U \subset X$,
- $\widetilde{\mathrm{N}}^1\mathrm{H}^i_G(X,A)$, consisting of classes induced via equivariant morphisms $Y \to X$, with Y a smooth projective G-variety of dimension $n-1$.

In the definition of $\widetilde{\mathrm{N}}^1\mathrm{H}^i_G(X,A)$, classes are induced via the Gysin map

$$\mathrm{H}^{i-2}([Y/G], A \otimes \mu_m^{\otimes -1}) \to \mathrm{H}^i([X/G], A).$$

We have

$$\widetilde{\mathrm{N}}^1\mathrm{H}^i_G(X,A) \subseteq \mathrm{N}^c\mathrm{H}^i_G(X,A); \tag{4.1}$$

see [13] for background. Exactly as in [13, Prop. 2.4] we obtain:

Proposition 4.5. *Suppose the action of G on X is generically free. Then the quotient*

$$\mathrm{N}^1\mathrm{H}^i_G(X,A)/\widetilde{\mathrm{N}}^1\mathrm{H}^i_G(X,A) \tag{4.2}$$

is a stable G-birational invariant of X.

This invariant has been studied in the non-equivariant setting: the quotient (4.2) (with $G = 1$) can be nontrivial, e.g., with $i = 3$, $A = \mathbb{Z}/2\mathbb{Z}$ [76].

Remark 4.6. Benoist and Ottem [13, Thm. 4.3] consider a variant of this construction, where for particular G and A there are smooth projective varieties X which approximate in a certain sense the cohomology of BG, for which the quotient (4.2) is nontrivial.

4.6. **Intermediate Jacobians.** An important notion in birational geometry of threefolds over closed and nonclosed ground fields is the *intermediate Jacobian*. It allowed to show that a smooth cubic threefold over $\mathbb{C}$ is irrational [45], showed irrationality of many conic bundles [10], gave rise to examples of irrational but stably rational threefolds [11]. Recent developments in [14], [64], [65], [15], [91], [92] concerning *torsors* under intermediate Jacobians led to rationality criteria for certain geometrically rational threefolds over nonclosed fields. Equivariant versions of intermediate Jacobians and cycle invariants appeared in [67].

The following gives an analog of a birational invariant introduced in [32, Thm. 3.6]; our definition does not depend on the theory of *atoms* from that paper.

Theorem 4.7 ([80, Prop. 5.2])**.** *Let X be a smooth projective rational threefold with a regular action of a cyclic group $G = C_p$ of prime order p. We have a decomposition of the fixed locus $X^G = \bigsqcup_\alpha F_\alpha$ into a disjoint*

union of smooth irreducible components. Let C be a smooth projective curve of genus $g \geq 2$. Consider

$$I := -I_1 - 2I_2 + I_3, \tag{4.3}$$

where

- *I_1 is the number of F_α isomorphic to C,*
- *I_2 is the number of F_α birational to $C \times \mathbb{P}^1$, and*
- *I_3 is the number of factors of the intermediate Jacobian $\mathrm{IJ}(X)$ isomorphic to $\mathrm{J}(C)$, with trivial G-action.*

Then I is an equivariant birational invariant, which vanishes when the G-action is linearizable.

Proof. The proof is parallel to the proof in [32]. By equivariant weak factorization, it suffices to consider blow-ups of smooth G-orbits; only the following blow-ups can change the shape of the invariant:

(1) blow-up of a G-fixed curve isomorphic to C not lying on a G-fixed surface,
(2) blow-up of a G-fixed curve isomorphic to C on a G-fixed surface,
(3) blow-up of a G-stable but not G-fixed curve isomorphic to C,
(4) blowing up a G-orbit of curves isomorphic to C.

Case (1) admits subcases, depending on whether or not the G-weights in the normal bundle of C are equal. In the first subcase,

$$\Delta(I_1) = 2 - 1, \quad \Delta(I_2) = 0, \quad \Delta(I_3) = 1,$$

where Δ denotes the discrepancy. In the second subcase,

$$\Delta(I_1) = -1, \quad \Delta(I_2) = 1, \quad \Delta(I_3) = 1.$$

In Case (2),

$$\Delta(I_1) = 1; \quad \Delta(I_2) = 0, \quad \Delta(I_3) = 1.$$

In Case (3), there are no changes in I_j. In Case (4), the intermediate Jacobian changes by a G-orbit of $\mathrm{J}(C)$; all I_j remain unchanged.

Finally, for a linear action on $\mathbb{P}^3$, all terms in (4.3) vanish. □

5. Amitsur groups

Classical Amitsur invariant. It is well-known that the canonical line bundle on a nonsingular G-variety is automatically G-linearized. On the other hand, not every element of the invariant Picard group $\mathrm{Pic}(X)^G$ can be represented by a G-linearized line bundle. These observations can be formalized, via an introduction of a cohomological invariant, the Amitsur group; see, e.g., [21, App. A]:

$$\mathrm{Am}^2(X, G) := \mathrm{Pic}(X)^G / \{[L] \,|\, G\text{-linearized line bundle } L\}.$$

Then, for instance, $\mathrm{Am}^2(X,G) = 0$ when $X = \mathbb{P}(V)$ for a linear representation V of G. If $\mathrm{Am}^2(X,G) = 0$, then every element of $\mathrm{Pic}(X)^G$ admits a lift to the group

$$\mathrm{Pic}(X,G) = \mathrm{Pic}([X/G])$$

of G-linearized line bundle L on X.

To see that $\mathrm{Am}^2(X,G)$ is a stable G-birational invariant, let X be a smooth projective G-variety and $Z \subset X$ a smooth G-subvariety in codimension ≥ 2. When we blow up Z in X to produce $\widetilde{X}$ we obtain $\mathrm{Pic}(\widetilde{X})$, differing from $\mathrm{Pic}(X)$ by a permutation module, generated by components of the exceptional divisor. The corresponding G-invariant class is automatically G-linearized. It follows that

$$\mathrm{Am}^2(\widetilde{X},G) = \mathrm{Am}^2(X,G).$$

A similar argument takes care of invariance under product with a projective space.

Higher Amitsur invariants. Additional cohomological invariants can be extracted from the Leray spectral sequence (3.2):

$$\mathrm{H}^p(G, \mathrm{H}^q(X,\mathcal{F})) \Rightarrow \mathrm{H}^{p+q}([X/G],\mathcal{F}),$$

for a G-linearized sheaf $\mathcal{F}$ on a G-variety X.

We start by recording some basic results:

- For $i = 1$ and $\mathcal{F} = \mathbb{G}_m$, we have
$$\mathrm{H}^1([X/G],\mathbb{G}_m) = \mathrm{Pic}(X,G);$$
see, e.g., [83, Sect. 3].
- If k is algebraically closed, then by (3.1) we have
$$\mathrm{H}^i(BG,\mathbb{G}_m) = \mathrm{H}^i(G,k^\times),$$
for all i; when $i > 0$ this recovers classical group cohomology $\mathrm{H}^i(G,k^\times) \cong \mathrm{H}^{i+1}(G,\mathbb{Z})$, cf. [83, §2.1].

Combining these results, setting $\mathcal{F} = \mathbb{G}_m$, and taking X to be a smooth projective variety with regular G-action and k algebraically closed, the spectral sequence (3.2) yields (see, e.g., [83, Sect. 3]):

$$\begin{aligned} 0 \to \mathrm{H}^1(G,k^\times) \to \mathrm{Pic}(X,G) \to \mathrm{Pic}(X)^G \xrightarrow{\delta_2} \mathrm{H}^2(G,k^\times) \\ \longrightarrow \ker(\mathrm{Br}([X/G]) \to \mathrm{Br}(X)) \to \mathrm{H}^1(G,\mathrm{Pic}(X)) \xrightarrow{\delta_3} \mathrm{H}^3(G,k^\times), \end{aligned} \tag{5.1}$$

where

$$\mathrm{Br}(-) := \mathrm{H}^2(-,\mathbb{G}_m)_{\mathrm{tors}}$$

is the *Brauer group*; since X is smooth, the full group $\mathrm{H}^2(X, \mathbb{G}_m)$ is torsion and thus is $\mathrm{Br}(X)$; by [3, Prop. 2.5 (iii)] the same holds for $[X/G]$. The obstruction to linearizing line bundles with classes in $\mathrm{Pic}(X)^G$ is cohomological:

$$\mathrm{Am}^2(X, G) = \mathrm{Im}(\delta_2).$$

We can also consider

$$\mathrm{Am}^3(X, G) := \mathrm{Im}(\delta_3) \subset \mathrm{H}^3(G, k^\times);$$

this is also a stable birational invariant, by the stable birational invariance of $\mathrm{H}^1(G, \mathrm{Pic}(X))$ and functoriality of the spectral sequence [83, Sect. 3]. In each of the following cases we have the vanishing of $\mathrm{Am}^2(X, G)$ and $\mathrm{Am}^3(X, G)$:

- When X has a G-fixed point [83, Sect. 3].
- When the G-action is stably linearizable, or more generally, when the G-action is unirational [89, Sect. 2].

All possibilities for Am^2 for rational surfaces have been determined in [21, Prop. 6.7]; for Am^3 this has been done in [118].

Universal torsor obstruction. One can extend the analysis of the spectral sequence further, using different coefficients. We suppose that k is algebraically closed. We can consider

$$\mathcal{F} = T_{\mathrm{NS}},$$

the Néron-Severi torus, viewed as a sheaf. For X smooth, projective, and rational, this is the torus with character group $\mathrm{NS}(X) = \mathrm{Pic}(X)$. Then, the low-degree sequence reads:

$$0 \to \mathrm{H}^1(G, T_{\mathrm{NS}}(k)) \to \mathrm{H}^1([X/G], T_{\mathrm{NS}}) \to \mathrm{H}^1(X, T_{\mathrm{NS}})^G \xrightarrow{\delta_2} \mathrm{H}^2(G, T_{\mathrm{NS}}(k))$$

We can identify

$$\mathrm{H}^1(X, T_{\mathrm{NS}}) = \mathrm{End}(\mathrm{Pic}(X)). \tag{5.2}$$

This has a canonical, G-invariant, element $1_{\mathrm{Pic}(X)}$. As explained in [66] and [89, Sect. 5],

$$\beta(X, G) := \delta_2(1_{\mathrm{Pic}(X)})$$

is the obstruction to lifting the G-action from X to a *universal* torsor

$$\begin{array}{ccc} & & \mathcal{T}_X \\ & & \big\downarrow{\scriptstyle T_{\mathrm{NS}}} \\ G \times X & \longrightarrow & X \end{array}$$

This gives rise to:

Condition (T): $\beta(X, G) = 0$.

This condition is a stable birational invariant of smooth projective irreducible rational G-varieties. Its failure is an obstruction to unirationality of the G-action on X [89, Prop. 5.1].

By [89, Thm. 6.1], for toric varieties, where the action is via automorphisms of the torus (as an algebraic variety), the vanishing is also sufficient for the unirationality of the action; indeed, up to strata in codimension ≥ 2, $\mathcal{T}_X$ is an affine space, and the lifted action on it is linear. A classification of such liftable actions on 3-dimensional toric varieties can be found in [119].

The formalism of torsors for varieties over BG in [66] was inspired by the corresponding theory of torsors over nonclosed fields, developed by Colliot-Thélène–Sansuc, and others. Some, but not all, features of the theory carry over. The most important difference is the absence of an analog of Hilbert's Theorem 90, used extensively in [48]. On the other hand, over BG there are additional geometric applications, including a wealth of stably linearizable, non-linearizable actions (via birational rigidity or Burnside invariants).

The motivation for the introduction and detailed study of universal torsors is rooted in the expectation that the birational geometry of $\mathcal{T}_X$ is in some sense "easier" than that of the base variety X. One instance of this is the following analog of [48, Thm. 2.1.2].

Proposition 5.1. *Let X be a smooth projective rational variety with a regular action of a finite group G. Assume that $\beta(X,G)=0$, i.e., the G-action lifts to a universal torsor $\mathcal{T}_X \to X$. Let $\overline{\mathcal{T}}_X$ be a smooth projective G-equivariant compactification of $\mathcal{T}_X$. Then $\mathrm{Pic}(\overline{\mathcal{T}}_X)$ is a permutation module and hence*

$$\mathrm{H}^1(G, \mathrm{Pic}(\overline{\mathcal{T}}_X)) = 0.$$

Proof. We follow the proof of [48, Thm. 2.1.2]. Since, non-equivariantly, $\mathcal{T}_X$ is the complement of the zero-sections in a sum of line bundles, whose classes form a basis of $\mathrm{Pic}(X)$, we have $k[\mathcal{T}_X]^\times = k^\times$. Thus $\mathrm{Pic}(\overline{\mathcal{T}}_X)$ is freely generated by the components of the boundary $\overline{\mathcal{T}}_X \setminus \mathcal{T}_X$. The G-action is a permutation action. □

Example 5.2. A natural geometric construction of new actions goes as follows: given a regular G-action on a smooth projective rational X, we can consider the action of $\widetilde{G} := G^n \wr \mathfrak{S}_n$ on $\widetilde{X} := X^n$, combining the given action with the permutation action. Then

$$\beta(X,G)=0 \quad \Rightarrow \quad \beta(\widetilde{X},\widetilde{G})=0.$$

Indeed, we have

$$\mathrm{Pic}(\widetilde{X}) = \mathrm{Pic}(X)^n;$$

Putting $\mathcal{T}_{\widetilde{X}} := (\mathcal{T}_X)^n$, where $\mathcal{T}_X$ be a universal torsor of X, we obtain a universal torsor for $\widetilde{X}$. The G-action lifts to $\mathcal{T}_X$, by assumption. Then $\widetilde{G}$-action lifts to $\mathcal{T}_{\widetilde{X}}$.

Condition (T), torsors, and Amitsur invariants. We continue to suppose k algebraically closed and X smooth, projective, and rational. We connect the $\beta(X,G)$-obstruction with the problem of lifting G-actions to general torsors and the Amitsur group $\mathrm{Am}^2(X,G)$.

Consider a G-torus S, i.e., non-equivariantly S is isomorphic to $\mathbb{G}_m^n$ for some $n \in \mathbb{N}$, and the G-action is given by a representation $G \to \mathrm{GL}(M)$, where M denotes the character lattice of S. We fix a G-equivariant homomorphism

$$\lambda\colon M \to \mathrm{Pic}(X).$$

For this torus as well there is the low-degree sequence of the spectral sequence (3.2):

$$0 \to \mathrm{H}^1(G, S(k)) \to \mathrm{H}^1([X/G], S) \to \mathrm{Hom}_G(M, \mathrm{Pic}(X)) \to \mathrm{H}^2(G, S(k)),$$

where we use the equivariant identification

$$\mathrm{H}^1(X, S) = \mathrm{Hom}(M, \mathrm{Pic}(X)).$$

We interpret $\mathrm{H}^1([X/G], S)$ as isomorphism classes of G-equivariant S-torsors on X, and for such, mapping to $\lambda \in \mathrm{Hom}_G(M, \mathrm{Pic}(X))$, we call λ the *equivariant type* of the corresponding G-equivariant S-torsor.

The low-degree sequence is related to the sequence (3.2) by functoriality, with $1_{\mathrm{Pic}(X)} \in \mathrm{End}(\mathrm{Pic}(X))$ mapping to λ. Consequently, the image of $\beta(X,G)$ in $\mathrm{H}^2(G, S(k))$ is the obstruction to realizing λ as the G-equivariant type of a G-equivariant S-torsor on X.

The case $M = \mathbb{Z}$, with trivial action, corresponds to a G-invariant class $[L] \in \mathrm{Pic}(X)^G$. Here, $\mathrm{H}^2(G, S(k)) = \mathrm{H}^2(G, k^\times)$, and the functoriality relation supplies the commutative diagram

$$\begin{array}{ccc} \mathrm{End}_G(\mathrm{Pic}(X)) & \xrightarrow{\delta_2} & \mathrm{H}^2(G, T_{\mathrm{NS}}(k)) \\ \downarrow & & \downarrow \\ \mathrm{Pic}(X)^G & \xrightarrow{\delta_2} & \mathrm{H}^2(G, k^\times) \end{array}$$

where the vertical maps are given by evaluation on $[L]$. Consequently,

$$\delta_2([L]) = \mathrm{ev}_{[L]}(\beta(X,G)),$$

and

$$\mathrm{Am}^2(X,G) = \{\mathrm{ev}_{[L]}(\beta(X,G)) \mid [L] \in \mathrm{Pic}(X)^G\}.$$

Proposition 5.3. *Let X be a smooth projective rational variety with regular G-action. If $\beta(X,G)=0$, then every G-equivariant homomorphism $\lambda\colon M\to \mathrm{Pic}(X)$ arises as the equivariant type of some G-equivariant S-torsor on X. In particular,*

$$\beta(X,G)=0 \quad \Rightarrow \quad \mathrm{Am}^2(X,G)=0.$$

Examples with *nonvanishing* $\beta(X,G)$ and vanishing $\mathrm{Am}^2(X,G)$ appear in [89] (one with vanishing and one with nonvanishing $\mathrm{Am}^3(X,G)$).

Motivic cohomology. Let q be an integer. The qth *motivic complex* is the bounded above cochain complex of étale sheaves with transfers

$$\mathbb{Z}(q):=\begin{cases}\mathrm{Sing}^{\mathbb{A}^1}(\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge q}))[-q], & \text{if } q\geq 0,\\ 0, & \text{if } q<0,\end{cases}$$

on smooth separated finite-type k-schemes; see, e.g., [96, Defn. 3.1]. Here,

- $\mathrm{Sing}^{\mathbb{A}^1}$ denotes the Suslin-Voevodsky construction [96, Defn. 2.14], which associates to a given presheaf with transfers the complex, formed by its values on products with algebraic simplices,
- $\mathbb{Z}_{tr}(X)$ is the presheaf with transfers sending U to the free abelian group on the set of correspondences (finite over U) to X, and
- $\mathbb{G}_m$ denotes the pointed scheme $(\mathbb{A}^1\setminus\{0\},1)$, with qth iterated smash product $\mathbb{G}_m^{\wedge q}$.

For $q=0$, we have $\mathbb{Z}(0)=\mathrm{Sing}^{\mathbb{A}^1}(\mathbb{Z})$, quasi-isomorphic to $\mathbb{Z}$. For $q=1$, we have $\mathbb{Z}(1)[1]=\mathrm{Sing}^{\mathbb{A}^1}(\mathbb{Z}_{tr}(\mathbb{G}_m))$, quasi-isomorphic to the étale sheaf $\mathbb{G}_m$.

We are interested in étale motivic cohomology groups

$$\mathrm{H}^p(X,\mathbb{Z}(q)):=\mathbb{H}^p_{\acute{e}t}(X,\mathbb{Z}(q)|_{X_{\acute{e}t}}). \tag{5.3}$$

For instance, when k is algebraically closed and X is a smooth projective variety, we have

$$\begin{aligned}
&\mathrm{H}^0(X,\mathbb{Z}(0))=\mathbb{Z}, && \mathrm{H}^0(X,\mathbb{Z}(1))=0,\\
&\mathrm{H}^1(X,\mathbb{Z}(0))=0, && \mathrm{H}^1(X,\mathbb{Z}(1))=k^\times,\\
&\mathrm{H}^2(X,\mathbb{Z}(0))=\mathrm{Hom}(\pi_1(X),\mathbb{Q}/\mathbb{Z}), && \mathrm{H}^2(X,\mathbb{Z}(1))=\mathrm{Pic}(X),\\
& && \mathrm{H}^3(X,\mathbb{Z}(1))=\mathrm{Br}(X).
\end{aligned}$$

(Normal schemes have vanishing $\mathrm{H}^1(-,\mathbb{Z})$, with $\mathrm{H}^i(-,\mathbb{Q})$ for all $i\geq 1$ [51, (2.1)], so we get the assertions in the left-hand column from the short exact sequence $0\to\mathbb{Z}\to\mathbb{Q}\to\mathbb{Q}/\mathbb{Z}\to 0$, and the right-hand column shows the étale cohomology of $\mathbb{G}_m$ in low degrees.)

In (5.3), the hypercohomology on the right is $\mathrm{H}^p(R\Gamma(X,\mathbb{Z}(q)|_{X_{\acute{e}t}}))$. Another approach is to work in the category of étale motives. This arises

in a two-step construction: *localization* of the derived category $\mathbf{D} = \mathbf{D}(\mathrm{Sh}_{\acute{e}t}(Cor_k))$ of étale sheaves with transfers at $\mathbb{A}^1$-weak equivalences to produce the category of effective étale motives $\mathbf{DM}^{\mathrm{eff}}_{\acute{e}t}(k)$ [96, Defn. 9.2], and *stabilization*, which formally inverts tensoring with $\mathbb{Z}(1)$ to yield the category of étale motives $\mathbf{DM}_{\acute{e}t}(k)$. The natural functor

$$\mathbf{DM}^{\mathrm{eff}}_{\acute{e}t}(k) \to \mathbf{DM}_{\acute{e}t}(k)$$

is fully faithful, by [70, Prop. A.3] and [6, Lemme 4.2].

Theorem 5.4. *Let k be a field of characteristic 0 and X a smooth separated k-scheme of finite type. Then*

$$\mathbb{H}^p_{\acute{e}t}(X, \mathbb{Z}(q)|_{X_{\acute{e}t}}) \cong \mathrm{Hom}_{\mathbf{DM}_{\acute{e}t}(k)}(\mathbb{Z}_{tr}(X), \mathbb{Z}(q)[p]).$$

For the convenience of the reader we record a proof of this standard result; cf. [5, Thm. 4.12]. As described in [114], for unbounded complexes, notions such as K-injective are relevant. For example, among unbounded complexes the injective objects are those that not only are exact (by factoring $1_{A^\bullet}$ through $\mathrm{cone}(1_{A^\bullet})$) and termwise injective (exact left adjoint to the nth term functor), but are also K-injective; cf. [69, Exa. 3.2]. Over a Grothendieck category there is the *injective model structure* on cochain complexes (loc. cit., see also [12]): fibrations are epimorphisms with termwise injective K-injective kernel, cofibrations are monomorphisms, and weak equivalences are quasi-isomorphisms. In particular, for complexes of sheaves of abelian groups on an essentially small site, $R\Gamma$ is obtained by applying Γ to a fibrant replacement, i.e., a quasi-isomorphic termwise injective K-injective complex.

Proof. We use the identification of étale hypercohomology with hyperext:

$$\mathbb{H}^p_{\acute{e}t}(X, \mathcal{L}^\bullet|_{X_{\acute{e}t}}) \cong \mathrm{Hom}_{\mathbf{D}}(\mathbb{Z}_{tr}(X), \mathcal{L}^\bullet[p]), \tag{5.4}$$

for any complex $\mathcal{L}^\bullet$ of étale sheaves with transfers. This fact [96, Exer. 6.25] is conveniently addressed using the local projective model structure on complexes of presheaves on Cor_k, respectively Sm_k (smooth separated finite-type k-schemes), respectively $X_{\acute{e}t}$, described in [44, §5.2], and the corresponding right-lifted model structures on respective complexes of sheaves (cf. [54, Cor. 2.7]). In each case, the fibrant objects are those that satisfy étale hyperdescent; exact forgetful functors, that forget the transfers and restrict to $X_{\acute{e}t}$, send fibrant objects to fibrant objects. It suffices to show that the object $\mathbb{Z}(q)$ of $\mathbf{D}$ is $\mathbb{A}^1$-local, since by [96, Lemma 9.19] the Hom on the right in the statement of the theorem is then identified with $\mathrm{Hom}_{\mathbf{D}}(\mathbb{Z}_{tr}(X), \mathbb{Z}(q)[p])$, and we conclude by (5.4).

We show, more generally, that for any étale sheaf with transfers $\mathcal{K}$, the Suslin-Voevodsky construction $\mathrm{Sing}^{\mathbb{A}^1}(\mathcal{K})$ gives the $\mathbb{A}^1$-localization of $\mathcal{K}$, meaning:

- the natural map $\mathcal{K} \to \mathrm{Sing}^{\mathbb{A}^1}(\mathcal{K})$ is an $\mathbb{A}^1$-weak equivalence,
- $\mathrm{Sing}^{\mathbb{A}^1}(\mathcal{K})$ is an $\mathbb{A}^1$-local complex of étale sheaves with transfers.

The first assertion holds by [96, Lemma 9.15]. The second is more subtle and relies on $\mathrm{char}(k) = 0$, but quickly reduces to the case that k is algebraically closed. Indeed, writing $\bar{k}$ for an algebraic closure and

$$p\colon \mathbb{A}^1_k \to \mathrm{Spec}(k), \quad \bar{p}\colon \mathbb{A}^1_{\bar{k}} \to \mathrm{Spec}(\bar{k}), \quad \alpha\colon \mathrm{Spec}(\bar{k}) \to \mathrm{Spec}(k),$$

we have generally for a complex $\mathcal{L}^\bullet$ over $\mathrm{Sh}_{\acute{e}t}(Cor_k)$, that $\mathcal{L}^\bullet$ is $\mathbb{A}^1$-local if and only if the natural map $\mathcal{L}^\bullet \to Rp_*p^*\mathcal{L}^\bullet$ is a quasi-isomorphism. Being a quasi-isomorphism may be detected after applying α^*. By [6, Lemme 4.2] we get an identification of $\alpha^*Rp_*p^*\mathcal{L}^\bullet$ with $R\bar{p}_*\bar{p}^*\alpha^*\mathcal{L}^\bullet$. Now the reduction is clear, since essentially by definition, we have an identification $\alpha^*\mathrm{Sing}^{\mathbb{A}^1}(\mathcal{K}) \cong \mathrm{Sing}^{\mathbb{A}^1}(\alpha^*\mathcal{K})$.

For the rest of the proof, we suppose k is algebraically closed. Then we may apply [96, Prop. 9.30], which tells us that a complex over $\mathrm{Sh}_{\acute{e}t}(Cor_k)$, which has strictly $\mathbb{A}^1$-homotopy invariant cohomology sheaves, is $\mathbb{A}^1$-local. For $n \in \mathbb{Z}$, the cohomology presheaf $\mathrm{H}^n(\mathrm{Sing}^{\mathbb{A}^1}(\mathcal{K}))$ is $\mathbb{A}^1$-homotopy invariant by [96, Cor. 2.19]. This leads to strict $\mathbb{A}^1$-homotopy invariance of the sheafification $\mathcal{F} := a_{\acute{e}t}\mathrm{H}^n(\mathrm{Sing}^{\mathbb{A}^1}(\mathcal{K}))$, i.e., the property that

$$R\Gamma(U, \mathcal{F}) \to R\Gamma(U \times \mathbb{A}^1, \mathcal{F})$$

is a quasi-isomorphism for U in Sm_k. Indeed, this holds if and only if the same is true with $\mathcal{F}$ replaced by $\mathcal{F} \otimes^{\mathbf{L}} \mathbb{Z}/\ell\mathbb{Z}$ for ℓ prime and by $\mathcal{F} \otimes \mathbb{Q}$, since cohomology commutes with $\otimes^{\mathbf{L}}\mathbb{Z}/\ell\mathbb{Z}$, and with $\otimes\mathbb{Q}$, the latter an instance of commuting with filtered colimits [97, Rmk. III.3.6]. The case of $\mathcal{F} \otimes^{\mathbf{L}} \mathbb{Z}/\ell\mathbb{Z}$ is taken care of by the Suslin rigidity theorem [96, Thm. 7.20], and $\mathcal{F} \otimes \mathbb{Q}$, by [96, Cor. 14.22] and [121, Thm. 5.6, Prop. 5.27]. □

Remark 5.5. By the same argument, for an arbitrary complex $\mathcal{K}^\bullet$ of étale sheaves with transfers, $\mathrm{Sing}^{\mathbb{A}^1}(\mathcal{K}^\bullet)$ gives the $\mathbb{A}^1$-localization of $\mathcal{K}^\bullet$ [5, Cor. 4.11].

By [122, Prop. 3.3.1], we have the exact sheafification functor

$$\mathrm{Sh}_{Nis}(Cor_k) \to \mathrm{Sh}_{\acute{e}t}(Cor_k),$$

thus a triangulated functor

$$\mathbf{DM}^{\mathrm{eff}}_{Nis}(k) \to \mathbf{DM}^{\mathrm{eff}}_{\acute{e}t}(k).$$

We have the projective bundle and blow-up formulas in $\mathbf{DM}^{\mathrm{eff}}_{Nis}(k)$ [96, Thm. 15.12, Cor. 15.13]:

$$\mathbb{Z}_{tr}(\mathbb{P}(E)) \cong \bigoplus_{r=0}^{d-1} \mathbb{Z}_{tr}(X)(r)[2r] \qquad (\mathrm{rk}(E) = d),$$

$$\mathbb{Z}_{tr}(B\ell_Z(X)) \cong \mathbb{Z}_{tr}(X) \oplus \bigoplus_{r=1}^{d-1} \mathbb{Z}_{tr}(Z)(r)[2r] \qquad (\mathrm{codim}(Z) = d).$$

So the same formulas are valid in $\mathbf{DM}^{\mathrm{eff}}_{\acute{e}t}(k)$ and we conclude:

Proposition 5.6. *For a vector bundle $E \to X$ of rank d on a smooth separated finite-type scheme X over k, respectively, a smooth subscheme $Z \subset X$ of codimension d, we have*

$$\mathrm{H}^p(\mathbb{P}(E), \mathbb{Z}(q)) \cong \bigoplus_{r=0}^{d-1} \mathrm{H}^{p-2r}(\mathbb{P}(E), \mathbb{Z}(q-r)),$$

$$\mathrm{H}^p(B\ell_Z(X), \mathbb{Z}(q)) \cong \mathrm{H}^p(X, \mathbb{Z}(q)) \oplus \bigoplus_{r=1}^{d-1} \mathrm{H}^{p-2r}(Z, \mathbb{Z}(q-r)).$$

Proof. We apply Theorem 5.4 and the projective bundle and blow-up formulas in $\mathbf{DM}^{\mathrm{eff}}_{\acute{e}t}(k)$. □

Applications of motivic cohomology. We suppose that k is algebraically closed and X is a smooth projective rational variety. This implies, in particular, that the birational invariant $\mathrm{Br}(X)$ of X vanishes. Let a regular action of G on X be given.

We use scheme approximations in order to apply a result such as Proposition 5.6 to $[X/G]$. This is the construction [53, Sect. 6.3], used to define equivariant (higher) Chow groups. For each $i > 0$, a representation V of G is chosen, with invariant open $U \subset V$ and $V \setminus U$ of codimension $\geq i$, such that a G-principal bundle $U \to U/G$ exists in the category of smooth separated schemes of finite type over k. By [98, Prop. 7.1], the same holds for

$$X \times U \to (X \times U)/G.$$

The scheme approximation $(X \times U)/G$ has the property, that

$$\mathrm{H}^p([X/G], \mathbb{Z}(q)) \to \mathrm{H}^p((X \times U)/G, \mathbb{Z}(q))$$

is an isomorphism, provided $i \geq \min((p+1)/2, q)$, as we see by comparing the Leray spectral sequence (3.2) for X and $X \times U$ and applying homotopy invariance and purity [75, Prop. 3.13].

In the Leray spectral sequence (3.2), we put $\mathcal{F} := \mathbb{Z}(1)$ and analyze the outcome. In particular, we are interested in the kernel $\ker(\varphi)$ of the natural homomorphism

$$\varphi\colon \mathrm{H}^4([X/G],\mathbb{Z}(1)) \to \mathrm{H}^4(X,\mathbb{Z}(1))^G.$$

Considering deeper terms in the spectral sequence, we obtain an exact sequence

$$\mathrm{H}^3(G,k^\times) \to \ker(\varphi) \to \mathrm{H}^2(G,\mathrm{Pic}(X)) \xrightarrow{\delta_4} \mathrm{H}^4(G,k^\times), \tag{5.5}$$

where $\delta_4 := d_2^{22}$ is the differential of the spectral sequence. This allows to define the degree 4 Amitsur group

$$\mathrm{Am}^4(X,G) := \mathrm{Im}(\delta_4) \subseteq \mathrm{H}^4(G,k^\times).$$

Theorem 5.7. *Let X be a smooth projective rational variety, equipped with a regular action of a finite group G. Then $\mathrm{Am}^4(X,G)$ satisfies the following properties:*

1. *It is a stable birational invariant of the G-action.*
2. *It vanishes if $X^G \neq \emptyset$.*
3. *Given a G-equivariant morphism $Y \to X$ of smooth projective rational G-varieties one has*

$$\mathrm{Am}^4(X,G) \subseteq \mathrm{Am}^4(Y,G).$$

If this induces an isomorphism $\mathrm{Pic}(X) \cong \mathrm{Pic}(Y)$ then

$$\mathrm{Am}^4(X,G) = \mathrm{Am}^4(Y,G).$$

4. *If V is a linear representation of G then*

$$\mathrm{Am}^4(\mathbb{P}(V),G) = 0.$$

Proof. First we address the birational invariance. By equivariant weak factorization, we only need to consider blow-ups of smooth subvarieties.

Let X be a smooth projective variety and $Z \subset X$ a smooth subvariety of codimension d. Let $\widetilde{X} \to X$ be the blow-up of X in Z. By Proposition 5.6 and the vanishing of $\mathbb{Z}(q)$ when $q < 0$, we have

$$\mathrm{H}^p(\widetilde{X},\mathbb{Z}(1)) = \mathrm{H}^p(X,\mathbb{Z}(1)) \oplus \mathrm{H}^{p-2}(Z,\mathbb{Z}(0)). \tag{5.6}$$

For $p = 4$, the additional term takes the form

$$\mathrm{H}^2(Z,\mathbb{Z}(0)) = \mathrm{Hom}(\pi_1(Z),\mathbb{Q}/\mathbb{Z}). \tag{5.7}$$

Same formulas hold for stacks, as can be seen via scheme approximations.

Assume that Z is a smooth G-subvariety of X, and let Z_0 be a component of Z. We let H be the subgroup of G that stabilizes the component Z_0. Then, $[Z/G] \cong [Z_0/H]$. We apply the blow-up formula:

$$\begin{array}{ccc} \mathrm{H}^4([\widetilde{X}/G], \mathbb{Z}(1)) & \xrightarrow{\quad\tilde{\varphi}\quad} & \mathrm{H}^4(\widetilde{X}, \mathbb{Z}(1))^G \\ \| & & \| \\ \mathrm{H}^4([X/G], \mathbb{Z}(1)) \oplus \mathrm{H}^2([Z_0/H], \mathbb{Z}(0)) & \longrightarrow & \mathrm{H}^4(X, \mathbb{Z}(1))^G \oplus \mathrm{H}^2(Z_0, \mathbb{Z}(0))^H \end{array}$$

Using the low-order terms of the Leray spectral sequence for $\mathcal{F} = \mathbb{Z}(0)$, we have

$$0 \to \mathrm{H}^2(H, \mathbb{Z}) \to \mathrm{H}^2([Z_0/H], \mathbb{Z}(0)) \to \mathrm{H}^2(Z_0, \mathbb{Z}(0))^H,$$

thus

$$\ker(\tilde{\varphi}) \cong \ker(\varphi) \oplus \mathrm{H}^2(H, \mathbb{Z}).$$

One the other hand,

$$\mathrm{H}^2(G, \mathrm{Pic}(\widetilde{X})) \cong \mathrm{H}^2(G, \mathrm{Pic}(X)) \oplus \mathrm{H}^2(H, \mathbb{Z}).$$

Comparing the exact sequence (5.5) for X and $\widetilde{X}$, we see that

$$\mathrm{Am}^4(X, G) = \mathrm{Am}^4(\widetilde{X}, G).$$

We now address invariance under upon passage to the product with a projective space with trivial G-action. This uses the following formula for motivic cohomology, a trivial case of the projective bundle formula:

$$\mathrm{H}^p(X \times \mathbb{P}^n, \mathbb{Z}(1)) = \mathrm{H}^p(X, \mathbb{Z}(1)) \oplus \mathrm{H}^{p-2}(X, \mathbb{Z}(0)).$$

Then we can argue as above and obtain the invariance.

Properties (2) and (3) are immediate from the functoriality of the Leray spectral sequence. Property (4) follows by considering the G-rational map

$$\mathbb{P}(V \oplus 1) \dashrightarrow \mathbb{P}(V).$$

By Property (2), $\mathrm{Am}^4(\mathbb{P}(V \oplus 1), G) = 0$. After blowing up the origin this becomes a morphism

$$B\ell_0 \mathbb{P}(V \oplus 1) \to \mathbb{P}(V).$$

By Property (1), we still have $\mathrm{Am}^4(B\ell_0 \mathbb{P}(V \oplus 1), G) = 0$; now it suffices to apply Property (3). □

Corollary 5.8. *Let X be a smooth projective rational variety with regular action of a finite group G. If X is G-unirational, then* $\mathrm{Am}^4(X, G) = 0$.

Example 5.9. One can compute $\mathrm{Am}^4(X,G)$, following [90, p. 84]. We carry this out, taking G to be the Klein 4-group, acting on $\mathbb{P}^1$ by $x \mapsto -x$ and $x \mapsto x^{-1}$. The result is

$$\mathrm{Am}^4(\mathbb{P}^1, G) = \mathrm{H}^4(G, k^\times) \cong (\mathbb{Z}/2\mathbb{Z})^2.$$

For the computation we use the invariant set of divisors $\{0, \infty\}$, and we get

$$\mathrm{H}^2(G, \mathrm{Pic}(\mathbb{P}^1)) = \mathrm{H}^2(G, \mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^2,$$

mapping isomorphically to $\mathrm{H}^3(G, \mathbb{Z} \cdot (0-\infty)) \cong (\mathbb{Z}/2\mathbb{Z})^2$, mapping isomorphically to $\mathrm{H}^4(G, k^\times)$.

Bogomolov multipliers. Given that many cohomological invariants vanish upon existence of fixed points, and taking into account Condition **(A)**, which is necessary for equivariant unirationality and (stable) linearizability of the action, it is natural to consider generalized *Bogomolov multipliers*

$$\mathrm{B}^j(G, M) := \mathrm{Ker}\left(\mathrm{H}^j(G, M) \to \bigoplus_A \mathrm{H}^j(A, M)\right),$$

where M is a G-module and the sum is over all abelian subgroups $A \subseteq G$, see [118, Sect. 2]. In search of interesting examples, we focus on G and M with nontrivial $\mathrm{B}^j(G, M)$, for some j.

We suppose that k is algebraically closed. For $j = 2$ and $M = k^\times$, with trivial G-action, we obtain the classical Bogomolov multiplier

$$\mathrm{B}^2(G, k^\times) = \mathrm{Br}_{\mathrm{nr}}(k(V)^G),$$

the unramified Brauer group of the field of invariants of a faithful representation V of G [24] over an algebraically closed field k of characteristic zero.

We can extend [118, Prop. 2] to the degree 4 Amitsur group.

Proposition 5.10. *Let X be a smooth projective rational variety with regular action of G. If X satisfies Condition* **(A)**, *then*

$$\mathrm{Am}^4(X, G) \subseteq \mathrm{B}^4(G).$$

6. Equivariant birational geometry – choice of a model

In this section, we fix a finite group G and assume that the base field k contains a primitive eth root of unity, where e is the least common multiple of the orders of the elements of G.

6.1. **Stabilizer stratification.** Let X be a smooth G-variety, and suppose that the G-action is generically free. One can introduce the *stabilizer poset*

$$\mathcal{P}(X,G) := \{\mathrm{Stab}(\bar{x}) \,|\, \bar{x} \in X(\bar{k})\}.$$

the set of subgroups of G (ordered by inclusion) which occur as stabilizer groups of geometric points of X.

We are interested in the fixed locus X^G of the action, i.e., the maximal closed subscheme of X on which G acts trivially, and more generally in the fixed loci X^H for subgroups $H \subseteq G$. These are smooth (see, e.g., [49, Prop. A.8.10]), and may have several components, of various dimensions. They are collected in the *stabilizer stratification* [28]:

$$\mathcal{S}(X,G) := \{\text{components of } X^H \,|\, H \subseteq G\}.$$

Any $F \in \mathcal{S}(X,G)$ gives rise to two associated subgroups of G:

- The *generic stabilizer group*

$$\mathrm{I}(F) \in \mathcal{P}(X,G).$$

- The *component stabilizer group*

$$\mathrm{D}(F) := \{g \in G \,|\, F \cdot g = F\}.$$

The generic stabilizer group is a normal subgroup of the component stabilizer group:

$$\mathrm{I}(F) \lhd \mathrm{D}(F).$$

In case X is affine, $X = \mathrm{Spec}(A)$, with quotient variety $\mathrm{Spec}(B)$, $B = A^G$, and $F = \mathrm{Spec}(A/\mathfrak{P})$, with prime ideal $\mathfrak{P}$ over a prime ideal $\mathfrak{p}$ of B, these reproduce the classical inertia and decomposition groups:

- $\mathrm{I}(F) = \mathrm{I}(\mathfrak{P}/\mathfrak{p})$, the inertia group of $\mathfrak{P}$ over $\mathfrak{p}$.
- $\mathrm{D}(F) = \mathrm{D}(\mathfrak{P}/\mathfrak{p})$, the decomposition group of $\mathfrak{P}$ over $\mathfrak{p}$.

The extension of residue fields $A_{\mathfrak{P}}/\mathfrak{P}A_{\mathfrak{P}}$ of $B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}$ is Galois, and the Galois group is canonically identified with $\mathrm{D}(F)/\mathrm{I}(F)$.

Example 6.1. Let the dihedral group $\mathfrak{D}_4$ of order 8 act on affine space $\mathbb{A}^3 = \mathrm{Spec}(k[x,y,z])$, where generators ρ and σ act by

$$(x,y,z) \mapsto (-y,x,z) \qquad \text{and} \qquad (x,y,z) \mapsto (x,-y,-z),$$

respectively. Then the subvariety F, defined by $x = y = 0$, belongs to $\mathcal{S}(\mathbb{A}^3, \mathfrak{D}_8)$, with $\mathrm{I}(F) = \langle\rho\rangle$ and $\mathrm{D}(F) = \mathfrak{D}_4$.

6.2. **Models.** One could try to develop a theory of birational invariants of G-actions by analyzing the arrangement $\mathcal{S}(X,G)$ on a given model X. It turns out that the theory admits significant simplifications, under some assumptions on the model, which can be achieved by performing suitable equivariant blow-ups. We comment on the nature of the simplification in Section 10.

Our setting is, still, a smooth G-variety, where the G-action is generically free. One of the first simplifications is to obtain a model X for which $\mathcal{P}(X,G)$ consists of *abelian* subgroups of G. Abelianization of stabilizers has been discovered and rediscovered several times; the earliest reference we found is [25]. Subsequently, the result appeared in [109], [8], and [28]. In a more modern formulation, it appears in the framework of *destackification* in the theory of algebraic stacks [16], [17]. The procedure, described below in Corollary 6.10, achieves several conditions:

- **Abelian stabilizers** – every $H \in \mathcal{P}(X,G)$ is abelian.
- **Divisorial form** – the action has abelian stabilizers, and for every $H \in \mathcal{P}(X,G)$ and $F \in \mathcal{S}(X,G)$ with $\mathrm{I}(F) = H$, putting $Y := \mathrm{D}(F)/H$, the composite

  $$\mathrm{Pic}(X,G) \to \mathrm{H}^1(\mathrm{D}(F), k(F)^\times) \to \mathrm{H}^1(H, k(F)^\times)^Y \to H^\vee$$

  is surjective. The leftmost map is given by restriction to the generic point of F, the middle map is the restriction map of the inflation-restriction exact sequence, and the rightmost map is

  $$\mathrm{H}^1(H, k(F)^\times)^Y \subseteq \mathrm{H}^1(H, k(F)^\times) = \mathrm{Hom}(H, k(F)^\times) \cong H^\vee.$$

- **Standard form** – there exists G-invariant open $U \subset X$, with simple normal crossing boundary, such that
  - G acts freely on U,
  - for every irreducible component Z of the boundary $X \setminus U$, every $g \in G$ satisfies

    $$Z \cdot g \cap Z = \emptyset \qquad \text{or} \qquad Z \cdot g = Z.$$

In [84], "divisorial form" was called "Assumption 2"; a similar divisoriality condition, with surjection from a given group of classes of orbifold line bundles of an algebraic orbifold, appeared in [82]. The condition "standard form" was introduced in [109].

Example 6.2. If G is abelian, then the action of G on X is automatically in divisorial form. Indeed, we have $G^\vee \to \mathrm{Pic}(X,G)$, and the composite with the map from the definition of divisorial form is $G^\vee \to H^\vee$ is the restriction map, Cartier dual to the inclusion $H \to G$. This is surjective.

All three conditions are preserved under equivariant blow-ups. Among the advantages of divisorial form are: its formulation does not require a generically free G-action, so for instance we have Example 6.2 also without the requirement of a generically free G-action; divisorial form is stable under general equivariant morphisms, as we may observe by the following characterization.

Proposition 6.3. *Let X be a smooth G-variety, where the G-action on X is generically free. Then the action is in divisorial form if and only if the action has abelian stabilizers and at every $\bar{x} \in X(\bar{k})$ the characters of $\mathrm{Stab}(\bar{x})$, determined by the elements of $\mathrm{Pic}(X, G)$, generate $\mathrm{Stab}(\bar{x})^\vee$.*

Proof. Given $\bar{x} \in X(\bar{k})$, over a closed point $x \in X$ we let $H = \mathrm{Stab}(\bar{x})$ and $F \in \mathcal{S}(X, G)$ be the unique component of X^H containing x. The action of G on X restricts to the trivial action of H on F, and in particular, on the local ring $\mathcal{O}_{F,x}$. The composite map in the definition of divisorial form factors through $\mathrm{H}^1(H, \mathcal{O}_{F,x}^\times)$ and agrees with the composite with

$$\mathrm{H}^1(H, \mathcal{O}_{F,x}^\times) \to \mathrm{H}^1(H, \bar{k}^\times) \cong H^\vee.$$

So the condition from the definition of divisorial form, for F as above, is equivalent to generation of $\mathrm{Stab}(\bar{x})$ by elements coming from $\mathrm{Pic}(X, G)$. □

Remark 6.4. Given a collection of linearized line bundles $\{L_i\}_{i \in I}$ on X, we can replace $\mathrm{Pic}(X, G)$ by the subgroup generated by the classes of the L_i in the definition of divisorial form, to get a variant, *divisorial form with respect to* $\{L_i\}_{i \in I}$. This notion appears, e.g., in [88, Sect. 5], and Proposition 6.3 remains valid for this notion and subgroup. If X is in divisorial form, then X is in divisorial form with respect to some $\{L_i\}_{i \in I}$ with I *finite*. This is observed in [84, Remark 3.2], which also points out the following stack-theoretic formulation (I finite): X is in divisorial form with respect to $\{L_i\}_{i \in I}$ if and only if the associated morphism of stacks

$$[X/G] \to \prod_{i \in I} B\mathbb{G}_m, \tag{6.1}$$

is *representable* (i.e., induces monomorphisms on geometric stabilizers).

Example 6.5. Divisorial form has abelian stabilizers, but not every action with abelian stabilizers is in divisorial form: consider the projectivization X of the 3-dimensional irreducible representation of the alternating group $G := \mathfrak{A}_4$. There are points with stabilizer H of order 4. We compute $\mathrm{Pic}(X, G)$ using exact sequence (5.1): the torsion is $\mathrm{Hom}(\mathfrak{A}_4, k^\times) \cong \mathbb{Z}/3\mathbb{Z}$, the free part is spanned by the class of $\mathcal{O}_X(1)$

with its natural linearization. There is no surjective homomorphism from $\mathrm{Pic}([X/G]) \cong \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ to $H^\vee \cong (\mathbb{Z}/2\mathbb{Z})^2$.

Example 6.6. Standard form is divisorial, but not every divisorial action is in standard form: consider $X := \mathbb{P}^3$ with action of $G := (\mathbb{Z}/2\mathbb{Z})^4$, where respective generators send $(x : y : z : w)$ to

$$(x : y : -z : -w), \quad (z : w : x : y), \quad (x : -y : z : -w), \quad (y : x : w : z).$$

Every subgroup of G of order 2 appears as generic stabilizer along curves in X. Since G is abelian, the action is in divisorial form. With

$$Q : \ x^2 + y^2 + z^2 + w^2 = 0,$$

we get $\mathrm{Pic}(X, G) = \langle [Q] \rangle \oplus G^\vee$, by (5.1). For a subgroup $H \subseteq G$, $|H| = 2$, the homomorphism $\mathrm{Pic}(X, G) \to H^\vee$ (definition of divisorial form) maps $[Q]$ to $1 \in H^\vee$ if $C \subset Q$, otherwise to 0, for a curve $C \in \mathcal{S}(X, G)$ with $\mathrm{I}(C) = H$, and on $G^\vee$ is given by restriction. Such C can be contained in at most two boundary components of a standard form. But exhaustive checking reveals: any subset $T \subseteq \mathrm{Pic}(X, G)$, generating $H^\vee$ for every subgroup $H \subseteq G$, $|H| = 2$, has to have at least 3 elements mapping to $1 \in H^\vee$ for some subgroup $H \subseteq G$, $|H| = 2$.

Proposition 6.7. *Let G be a finite group with generically free action on a smooth variety X. If the action is in divisorial form, then $\mathrm{I}(F)$ lies in the center of $\mathrm{D}(F)$, for every $F \in \mathcal{S}(X, G)$.*

Proof. With the notation of the definition of divisorial form, the action of Y on $\mathrm{H}^1(H, k(F)^\times)$ has to be trivial. Since the action is given by conjugation, $\mathrm{I}(F)$ is a central subgroup of $\mathrm{D}(F)$. □

Example 6.8. In Example 6.1, $\mathrm{I}(F)$ does not lie in the center of $\mathrm{D}(F)$. With this linear action of $\mathfrak{D}_4$ on $\mathbb{A}^3$, or its projective compactification $\mathbb{P}^3$, we have additional examples of actions with abelian stabilizers, not in divisorial form.

Proposition 6.9. *Let a generically free action of a finite group G on a smooth algebraic variety X be given, together with a simple normal crossing divisor*

$$D = D_1 \cup \cdots \cup D_\ell,$$

on X such that each D_i is smooth and G-invariant. Given $\bar{x} \in X(\bar{k})$ over $x \in X$, we define

$$H' \subseteq H := \mathrm{Stab}(\bar{x})$$

to be the intersection of the kernels of the characters determined by the G-linearized line bundles $\mathcal{O}_X(D_i)$ for all i with $x \in D_i$, and let $d(\bar{x})$ denote the dimension of the nontrivial part of the representation of H'

on the tangent space $\mathcal{T}_{X,\bar{x}}$. Then the function d is identically zero if and only if the action is in standard form with boundary D. If the maximal value m of d is positive, then

$$\{\bar{x} \in X(\bar{k}) \,|\, d(\bar{x}) = m\}$$

is the set of $\bar{k}$-points of a smooth G-invariant closed subscheme W of pure codimension m, meeting D transversally, such that the blow-up $\widetilde{X}$ of X along W has a simple normal crossing divisor formed by the exceptional divisor and the pre-images of the D_i, and $d(\bar{y}) < m$ for all $\bar{y} \in \widetilde{X}(\bar{k})$.

Proof. We start by proving the first assertion. For an action in standard form with boundary D, the generation condition of Proposition 6.3 is satisfied at $\bar{x} \in X(\bar{k})$ over $x \in X$ with just the G-linearized line bundles $\mathcal{O}_X(D_i)$ with $x \in D_i$, thus $d(\bar{x}) = 0$. The converse follows from the observation, that for x in the complement of D, the vanishing of $d(\bar{x})$ implies the triviality of H.

For the remaining assertions, the condition to have geometric stabilizer group contained in H defines H-invariant open $X^\circ \subset X$, with $x \in X^\circ$. At any $\bar{k}$-point of X° the value of the function d stays the same when we replace G by H. The function d also remains unchanged under passage to an equivariant étale cover that induces bijections on geometric stabilizer groups. So we are reduced to checking the remaining assertions under the assumption that $G = H$ and x is the origin in a linear representation $X = V$ for some $G \to \mathrm{GL}(V^\vee)$,

$$V = \chi_1 \oplus \cdots \oplus \chi_\ell \oplus V',$$

direct sum of ℓ one-dimensional representations of G and a final factor V', with the D_i given by the first ℓ factors.

The subgroup H' is the intersection of the kernels of $\chi_1, \ldots, \chi_\ell$. We write

$$V' = (V')^{H'} \oplus V''$$

for some subrepresentation V''. Now $m = \dim(V'')$, with $d(\bar{v}) \leq m$ for all $\bar{v} \in V(\bar{k})$, and equality holds if and only if $\bar{v}$ projects to $0 \in V''$. So

$$W = \chi_1 \oplus \cdots \oplus \chi_\ell \oplus (V')^{H'}.$$

Blowing up W in V replaces V'' by $B\ell_0(V'')$. At a $\bar{k}$-point $\bar{y}$ of the exceptional divisor, the nontrivial part of the representation on $\mathcal{T}_{B\ell_W V,\bar{y}}$ of the intersection of kernels of characters of $\mathrm{Stab}(\bar{y})$ is contained in the tangent space to the image of $\bar{y}$ in $\mathbb{P}(V'')$. This has dimension $m-1$. □

Corollary 6.10 ([17])**.** *A smooth algebraic G-variety, where the G-action is generically free, can be brought into standard form via a sequence of blow-ups at G-invariant smooth centers.*

Proof. Starting with the empty divisor, we repeatedly blow up the locus indicated in Proposition 6.9, until the function d is identically zero. □

Remark 6.11. The function d of Proposition 6.9 depends only on the closed point $x \in X$ and may be extended to arbitrary points of X, by replacing the tangent space at $\bar{x}$ by the dual of the fiber at x of the conormal sheaf of the closure of $\{x\}$. Then we recover the *divisorial index* of [17]. Corollary 6.10 records the outcome of the *divisorialification* algorithm of [17, Thm. A].

7. Equivariant Burnside groups – definitions

Consider a smooth projective G-variety X of dimension n, where the base field k is assumed to contain enough roots of unity; the precise requirement for "enough" is stated at the beginning of Section 6. We assume that the G-action is generically free. Let $\bar{x}$ be a $\bar{k}$-point of X, with abelian geometric stabilizer group $H \subseteq G$. The equivalence class of the induced representation of H on the tangent space $\mathcal{T}_{X,\bar{x}}$ may be encoded by a sequence

$$\beta(\bar{x}) := (b_1, \dots, b_n)$$

of characters of H. If $\bar{x}$ is an isolated point of X^H, then all of the characters will be nontrivial. In general, the multiplicity of the zero character will be the codimension in X of the component of X^H containing $\bar{x}$. The characters $b_1, \dots, b_n$ will generate the character group $H^\vee$.

We formulate, first, the groups of symbols that encode these actions. At a $\bar{k}$-point $\bar{x} \in X$ with abelian geometric stabilizer group H, we get a sequence of characters of H, that is defined up to order. At another point in the G-orbit of $\bar{x}$ the geometric stabilizer group $H \subseteq G$ gets replaced by a conjugate subgroup $H' := gHg^{-1}$, and the characters get replaced accordingly by their g-conjugates. The corresponding symbol groups are subject to order and conjugation relations. In the next step, blow-up relations are introduced; only after introducing these relations do we obtain groups, where the equivariant birational invariants take their values.

7.1. **Symbols.** Let H be a finite abelian group, with character group

$$A := H^\vee.$$

For $n \in \mathbb{N}$, we introduce

$$\mathcal{S}_n(H),$$

the abelian group generated by *symbols*, which take the form of an n-tuple of characters

$$\beta = (b_1, \dots, b_n), \qquad b_1, \dots, b_n \in A,$$

such that $\langle b_1, \dots, b_n \rangle = A$. These are subject to the relation
(O) (order) $\beta = (b_1, \dots, b_n)$ is equal to $\beta' = (b'_1, \dots, b'_n)$ if there exists a permutation $\sigma \in \mathfrak{S}_n$, with $b'_i = b_{\sigma(i)}$ for $i = 1, \dots, n$.

Notice that we allow 0 to appear as a character in β.

Let G be a finite group. The *combinatorial symbols group*

$$\mathcal{SC}_n(G)$$

is generated by *symbols*

$$(H, Y, \beta)$$

where

- $H \subseteq G$ is an abelian subgroup,
- Y is a subgroup of $Z_G(H)/H$ (with $Z_G(H)$ the centralizer of H in G), and
- β is a sequence of characters in $H^\vee$, generating $H^\vee$, of length at most n.

These are subject to relations
(O) $(H, Y, \beta) = (H, Y, \beta')$ if β' is a reordering of β,
(C) (conjugation) For $g \in G$ we have $(H, Y, \beta) = (H', Y', \beta')$, where $H' = gHg^{-1}$, $Y' = gYg^{-1}$, and by conjugation by g we obtain the characters in β' from those in β.

Lastly, we introduce the abelian group

$$\mathrm{Symb}_n(G),$$

where the generators are *symbols*

$$(H, Y \circlearrowright K, \beta).$$

Here,

- $H \subseteq G$ is an abelian subgroup,
- $Y \subseteq Z_G(H)/H$ is a subgroup,
- β is a sequence of characters in $H^\vee$, generating $H^\vee$,
- K is a finitely generated extension field of k, with faithful action over k by Y, such that, if we denote by Y_0 the pre-image of Y in $Z_G(H)$ (by the canonical homomorphism), the restriction map

$$\mathrm{H}^1(Y_0, K^\times) \to \mathrm{H}^1(H, K^\times) \cong H^\vee \tag{7.1}$$

  is surjective, and
- denoting by d the transcendence degree of K over k, the length of the sequence of characters β is $n - d$.

These symbols are subject to relations
(O) $(H, Y \subset K, \beta) = (H, Y \subset K, \beta')$ if β' is a reordering of β,
(C) For $g \in G$ we have $(H, Y \subset K, \beta) = (H', Y' \subset K', \beta')$, where

$$H' = gHg^{-1}, \quad Y' = gYg^{-1},$$

K' is isomorphic as a k-algebra to K in a manner that is compatible with the respective actions, and β' obtained from β by conjugation by g.

Remark 7.1. The surjectivity condition (7.1) here was called "Assumption 1" in [84]. The map (7.1) is extraced from the inflation-restriction exact sequence of group cohomology. This has, further to the left, the term $\mathrm{H}^1(Y, K^\times)$, which vanishes by Hilbert's Theorem 90. Consequently, surjectivity of (7.1) is equivalent to having here an isomorphism.

7.2. **Relations.** We define relations, mirroring the impact on the weights when a smooth G-variety is blown up along a smooth G-subvariety. It suffices to consider the case of a blow-up in codimension 2 (this will be explained later, in the proof of Theorem 7.4); this is reflected in the form of the relations.

We consider the quotient

$$\mathcal{S}_n(H) \to \mathcal{B}_n(H)$$

by the relation:

(B) (blow-up) For $\beta = (b_1, \dots, b_n)$, $n \geq 2$,

$$\beta = \begin{cases} (0, b_2, \dots, b_n), & \text{if } b_1 = b_2, \\ \beta_1 + \beta_2, & \text{if } b_1 \neq b_2, \end{cases}$$

where

$$\beta_1 := (b_1 - b_2, b_2, b_3, \dots, b_n), \qquad \beta_2 := (b_1, b_2 - b_1, b_3, \dots, b_n).$$

Similarly, we consider the quotient

$$\mathcal{SC}_n(G) \to \mathcal{BC}_n(G)$$

by the relations
(V) (vanishing) $(H, Y, \beta) = 0$ whenever $b_i = 0$ for some i,
(B) if the sequence of characters $\beta = (b_1, b_2, \dots)$ has length ≥ 2, then

$$(H, Y, \beta) = (H, Y, \beta_1) + (H, Y, \beta_2) + (\overline{H}, \overline{Y}, \bar{\beta}),$$

where $\beta_1 := (b_1 - b_2, b_2, \dots)$, $\beta_2 := (b_1, b_2 - b_1, \dots)$, $\overline{H} := \ker(b_1 - b_2)$, $\bar{\beta} := (\bar{b}_2, \bar{b}_3, \dots)$ (restrictions of characters), and $\overline{Y} := Y_0/\overline{H}$, with Y_0 the pre-image of Y in $Z_G(H)$, as before.

Finally, we consider the quotient

$$\mathrm{Symb}_n(G) \to \mathrm{Burn}_n(G)$$

by the relations
(V) $(H, Y \circlearrowright K, \beta) = 0$ whenever $b_i = 0$ for some i,
(B) for a symbol $(H, Y \circlearrowright K, \beta)$ with $n - d \geq 2$ (where d denotes the transcendence degree of K over k),

$$(H, Y \circlearrowright K, \beta) = \\ (H, Y \circlearrowright K, \beta_1) + (H, Y \circlearrowright K, \beta_2) + (\overline{H}, \overline{Y} \circlearrowright \overline{K}, \bar{\beta}), \quad (7.2)$$

with notation as above and additionally

$$\overline{K} := K(t),$$

where the following recipe is carried out to produce an action of $\overline{Y}$ on $\overline{K}$. Consider $b := b_1 - b_2$ as a primitive character of $H/\overline{H}$. By the surjectivity of (7.1) and Remark 7.1, b admits a unique lift to $\mathrm{H}^1(Y_0, K^\times)$. A choice of cocycle representation leads to a Y_0-action on $K(t)$, where $\overline{H}$ acts trivially, and we get an induced action of $\overline{Y}$; see [84, Sect. 2].

Remark 7.2. The formulation of relations **(V)** and **(B)** follows [88, Sect. 2]. Earlier papers such as [84] and [85] allowed only symbols with sequences of *nonzero* characters, generating $H^\vee$, with relations **(B1)**–**(B2)**, that are however equivalent to the formulation with more general symbols and relations **(V)** and **(B)**. An additional, conventional difference in the present formulation is the requirement for K to be a *field*, rather than a finite product of fields, as in [84] (formulation with Galois algebras for the group $N_G(H)/H$), or [88] (with Galois algebras for subgroups of $N_G(H)/H$); here, $N_G(H)$ denotes the normalizer of H in G.

7.3. **Class of the action.** Given a good model for the G-action, i.e., one that is in *divisorial form*, as in Section 6.2, we define its class in $\mathcal{B}_n$, $\mathcal{BC}_n$, respectively, Burn_n as follows:

- For G *abelian*, we let $\mathcal{F}(X, G)$ be the set of components of the fixed locus
$$X^G = \bigsqcup_{F \in \mathcal{F}(X,G)} F$$
and, for each, record the characters of the G-action
$$\beta_F(X) = (b_1, \ldots, b_n), \quad b_j \in G^\vee,$$
in the tangent bundle at a geometric point of F. The class
$$[X \circlearrowleft G] := \sum_{F \in \mathcal{F}(X,G)} [k' : k] \beta_F(X) \in \mathcal{B}_n(G)$$

with k' the algebraic closure of k in $k(F)$, yields a well-defined G-equivariant birational invariant [78, Thm. 3].

- For arbitrary G, we consider the stabilizer stratification $\mathcal{S}(X,G)$ introduced in Section 6.1; recall that all stabilizers are abelian. For every component $F \in \mathcal{S}(X,G)$ of dimension d we record a symbol

$$(H, Y, \beta) \qquad \text{resp.} \qquad (H, Y \circlearrowright k(F), \beta),$$

where $H = \mathrm{I}(F)$ (an abelian subgroup of G), and the symbol records $Y = \mathrm{D}(F)/H \subseteq Z_G(H)/H$ (containment by Proposition 6.7), respectively Y with residual action, and $\beta = \beta_F(X) = (b_1, \dots, b_{n-d})$ is the sequence of characters of H, for the action of H in the normal bundle to F in X at the generic point of F. We record only one such symbol for the G-orbit of F. In particular, H, Y, and β are defined only up to conjugation (reflecting the passage to a different component in the G-orbit of F); furthermore, the sequence β is only defined up to order. The class is

$$[X \circlearrowleft G] := \sum_{F \in \mathcal{S}(X,G)/G} [k' : k](H, Y, \beta_F(X)) \in \mathcal{BC}_n(G),$$

with k' as before the algebraic closure of k in $k(F)$, respectively,

$$[X \circlearrowleft G] := \sum_{F \in \mathcal{S}(X,G)/G} (H, Y \circlearrowright k(F), \beta_F(X)) \in \mathrm{Burn}_n(G),$$

where the sum runs over G-orbit representatives of the stabilizer stratification.

Remark 7.3. In parallel to the conventional differences in the definition of symbols, mentioned in Remark 7.2, there are alternative expressions for $[X \circlearrowleft G]$. In [84], $[X \circlearrowleft G] \in \mathrm{Burn}_n(G)$ is expressed as a sum over conjugacy class representatives H of abelian subgroups of G, of sums of symbols $\sum_F \mathfrak{s}_F$, where F in the inner sum runs over $N_G(H)$-orbits of elements of $\mathcal{S}(X,G)$ with generic stabilizer H. In the analogous expression in [85], the inner sum is over $Z_G(H)$-orbits, but this comes at the cost of a more complicated outer sum, requiring the notion of $\mathrm{Pic}(X,G)$-pairing on an abelian subgroup of G; an advantage of the shift to $Z_G(H)$-orbits was a simpler formulation of the blow-up relations. The closest to the present formulation is [87, p. 3027], an expression as a sum over points $x_0 \in X/G$ (in the scheme sense, i.e., not just closed points). When we have the generic point of some $F \in \mathcal{S}(X,G)/G$ over x_0, the contribution is $(H, Y \circlearrowright k(F), \beta_F(X))$. Otherwise, the contribution is a

symbol with 0 in the sequence of characters, and this vanishes by relation **(V)**. More details are provided in [88, Sect. 2].

Theorem 7.4. *The recipe, for a projective n-dimensional G-variety with generically free G-action, to choose a smooth projective model X in divisorial form and associate the element*

$$[X \leftarrow G],$$

defines a G-equivariant birational invariant class in $\mathcal{BC}_n(G)$, respectively in $\mathrm{Burn}_n(G)$.

We recall, a smooth projective model in divisorial form may be obtained by performing equivariant resolution of singularities, followed by the sequence of blow-ups given by Corollary 6.10.

Proof. It needs to be checked that two different smooth projective models in divisorial form give rise to the same class in $\mathcal{BC}_n(G)$, respectively in $\mathrm{Burn}_n(G)$. This is established in [84, Thm. 5.1] by using equivariant weak factorization to reduce to the case of a blow-up of a smooth G-subvariety of codimension $j \geq 2$, and relations in $\mathrm{Burn}_n(G)$ of the form

$$(H, Y \curvearrowright K, \beta) = \sum_{\emptyset \neq I \subset \{1,\dots,j\}} (H_I, Y_I \curvearrowright K_I, \beta_I),$$

provided $\mathrm{trdeg}_k(K) \leq n - j$ (or analogous relations in $\mathcal{BC}_n(G)$); a key point is that these relations are shown, in [84, Prop. 4.7(ii)], to follow from relations **(B)** and **(V)**. Here,

$$H_I := \bigcap_{i,i' \in I} \ker(b_i - b_{i'})$$

and, writing $Y = Y_0/H$ with $Y_0 \subseteq Z_G(H)$,

$$Y_I := Y_0/H_I.$$

We construct an action of Y_I on the purely transcendental extension K_I of K, $\mathrm{trdeg}_K(K_I) = |I| - 1$, in an analogous fashion to $\overline{Y} \curvearrowright \overline{K}$ in (7.2); this is the *action construction* from [84, Sect. 2], see also [85, Sect. 2]. All b_i for $i \in I$ have a common class $\bar{b} \in (H_I)^\vee$; then,

$$\bar{\beta} := (\bar{b}, \bar{b}_{i_1}, \dots, \bar{b}_{i_r}),$$

with $\{1, \dots, n-d\} \setminus I = \{i_1, \dots, i_r\}$. □

Algorithms for the computation of the class $[X \leftarrow G] \in \mathrm{Burn}_n(G)$ for actions arising from (projectivizations of) linear representations and actions on smooth projective toric varieties can be found in [85] and [87].

7.4. **Class of an open subvariety.** In applications, we also need to consider a G-invariant open U in a smooth projective G-variety X. We suppose that the G-action on X is generically free, and X is in divisorial form. There are *two* ways to define the class of $U \circlearrowleft G$ (in any flavor of Burnside group). The most naive way to do this is to copy the formula from Section 7.3, replacing X by U:

$$[U \circlearrowleft G]^{\mathrm{naive}} := \sum_{F \in \mathcal{F}(U,G)} [k' : k]\beta_F(U) \in \mathcal{B}_n(G),$$

$$[U \circlearrowleft G]^{\mathrm{naive}} := \sum_{F \in \mathcal{S}(U,G)/G} [k' : k](H, Y, \beta_F(U)) \in \mathcal{BC}_n(G),$$

$$[U \circlearrowleft G]^{\mathrm{naive}} := \sum_{F \in \mathcal{S}(U,G)/G} (H, Y \curvearrowright k(F), \beta_F(U)) \in \mathrm{Burn}_n(G).$$

These classes are birational invariants of $U \circlearrowleft G$, by [84, Lemma 5.3].

However, motivated by the form of the specialization map, even in its original, nonequivariant form [79], we also need a more sophisticated formula with an alternating sum over boundary strata. For this we make the further assumption, that $X \setminus U$ is a simple normal crossing divisor

$$D = D_1 \cup \cdots \cup D_\ell,$$

such that each D_i is G-invariant. For $\emptyset \neq I \subseteq \mathcal{I} := \{1, \dots, \ell\}$ we have $D_I ; = \bigcap_{i \in I} D_i$, with normal bundle $\mathcal{N}_{D_I/X}$ and *punctured normal bundle* $\mathcal{N}^\circ_{D_I/X}$. The latter is defined by removing, from $\mathcal{N}_{D_I/X} \cong \bigoplus_{i \in I} \mathcal{N}_{D_i/X}|_{D_I}$, the fiber over $D_{I \cup \{j\}}$, for all $j \in \mathcal{I} \setminus I$, as well as the zero-section of $\mathcal{N}_{D_i/X}|_{D_I}$, for all $i \in I$. Then in any flavor of Burnside group we have

$$\begin{aligned}[U \circlearrowleft G] &:= [X \circlearrowleft G] + \sum_{\emptyset \neq I \subseteq \mathcal{I}} (-1)^{|I|} [\mathcal{N}_{D_I/X} \circlearrowleft G]^{\mathrm{naive}} \\ &= [U \circlearrowleft G]^{\mathrm{naive}} + \sum_{\emptyset \neq I \subseteq \mathcal{I}} (-1)^{|I|} [\mathcal{N}^\circ_{D_I/X} \circlearrowleft G]^{\mathrm{naive}}.\end{aligned}$$

The definition is [84, Defn. 5.4], and the equality is [84, Lemma 5.7]. This class is a birational invariant of $U \circlearrowleft G$, by [84, Thm. 5.15].

7.5. **Comparisons.** The different versions of Burnside groups are related to each other, via forgetful homomorphisms. The homomorphism

$$\mathrm{Burn}_n(G) \to \mathcal{BC}_n(G)$$

forgets the birational type of the stratum F in the symbol [88, Prop. 8.2]:

$$(H, Y \curvearrowright k(F), \beta) \mapsto [k' : k](H, Y, \beta).$$

For G abelian, we have a further homomorphism [88, Exa. 8.8]

$$\mathcal{BC}_n(G) \to \mathcal{B}_n(G),$$

where we quotient by the subgroup generated by symbols with $H \subsetneq G$. A more refined formalism of intermediate quotients, with respect to natural filtrations on $\mathrm{Burn}_n(G)$ is described in Section 8.2.

8. Equivariant Burnside groups – properties

We continue to assume that k has enough roots of unity, as in Sections 6 and 7.

8.1. **Vanishing in stable range.** In the analysis of relations in the various Burnside groups, we often use the following combinatorial consequence of the blow-up relation [84, Prop. 4.7(i)]: a symbol

$$(H, Y \circlearrowright K, (b_1, \dots, b_{n-d}))$$

vanishes in $\mathrm{Burn}_n(G)$, if $\sum_{i\in I} b_i = 0$ for some $\emptyset \neq I \subseteq \{1, \dots, n-d\}$. This implies the vanishing of any symbol of the form

$$(H, Y \circlearrowright K(t_1, \dots, t_m), (b_1, \dots, b_{n-d}))$$

where m is at least one less than the minimum of the orders of the b_i [87, Prop. 4.1]. As a consequence, if X is a smooth projective n-dimensional variety with generically free G-action, and we consider $X \times \mathbb{P}^m$, with trivial G-action on $\mathbb{P}^m$, then

$$[X \times \mathbb{P}^m \looparrowleft G] = (1, G \circlearrowright k(X)(t_1, \dots, t_m), ()) \in \mathrm{Burn}_{n+m}(G),$$

provided $m \geq -1 + \max_{g\in G} |\langle g\rangle|$ [87, Rmk. 4.2]. (More generally, if X is a smooth projective G-variety and X_0 an irreducible component of X, then this holds with $(1, \mathrm{D}(X_0) \circlearrowright k(X_0)(t_1, \dots, t_m), ())$ on the right.)

8.2. **Filtrations.** There is the possibility to restrict attention just to certain stabilizer groups. This can be done systematically using the notion of filter, introduced in [88, §3]. A *G-filter* is a subset $\mathbf{H}$ of the set of pairs (H, Y), consisting of an abelian subgroup $H \subseteq G$ and a subgroup $Y \subseteq Z_G(H)/H$, that is closed under conjugation and satisfies the following additional property. For $(H, Y) \in \mathbf{H}$, $H \neq 1$, and $g \in Z_G(H)$, with $\bar{g} \in Y$ and $Y \subseteq Z_G(g)/H$, we require $(\langle H, g\rangle, Y/\langle \bar{g}\rangle) \in \mathbf{H}$. Then $\mathrm{Burn}_n^{\mathbf{H}}(G)$ is defined to be the quotient of $\mathrm{Burn}_n(G)$ by the subgroup generated by all symbols $(H, Y \circlearrowright K, \beta)$ with $(H, Y) \notin \mathbf{H}$. A similar definition is made for $\mathcal{BC}_n(G)$. The properties of a filter guarantee [88, Prop. 3.3, Prop. 8.7] that $\mathrm{Burn}_n^{\mathbf{H}}(G)$ is the abelian group, generated by symbols $(H, Y \circlearrowright K, \beta)$ with $(H, Y) \in \mathbf{H}$, subject to relations **(O)**, **(C)**,

(V), **(B)**, applied just to these symbols, and the analogous fact holds for $\mathcal{BC}_n^{\mathbf{H}}(G)$.

There is also a comparison homomorphism between the Burnside and combinatorial Burnside groups (§7.5) in the presence of a filter $\mathbf{H}$, and this is compatible with the quotient maps $\mathrm{Burn}_n(G) \to \mathrm{Burn}_n^{\mathbf{H}}(G)$ and $\mathcal{BC}_n(G) \to \mathcal{BC}_n^{\mathbf{H}}(G)$.

For instance, when G is abelian we define (cf. [84, §8])

$$\mathrm{Burn}_n^G(G) := \mathrm{Burn}_n^{\{(G,1)\}}(G), \qquad \mathcal{BC}_n^G(G) := \mathcal{BC}_n^{\{(G,1)\}}(G).$$

By [88, Exa. 8.8], we have

$$\mathcal{BC}_n^G(G) = \mathcal{B}_n(G).$$

8.3. **Incompressibles.** The analysis of the blow-up relation, and in particular, the determination of the vanishing of a symbol in $\mathrm{Burn}_n(G)$ can be tricky. However, in many applications, G-actions can be distinguished upon projection to the subgroup, freely generated by *incompressible* symbols. These are divisor symbols, that is, symbols

$$(H, Y \supset K, (b))$$

with $\mathrm{trdeg}_k(K) = n-1$ and nontrivial cyclic H, $H^\vee = \langle b \rangle$, that cannot be obtained, via the action construction, as rightmost term in a relation (7.2), coming from relation **(B)**. The notion appears in [85, Defn. 3.3].

Among classical precursors of this notion are fixed curves of birational involutions on rational surfaces, leading to the classification of involutions as Bertini, Geisser, and de Jonquières; cf. [9]. A more recent version of this is the *normalized fixed curve with action (NFCA)* invariant, introduced by Blanc [20]. This takes into account the stabilizer and the residual action, and gives a finer invariant for cyclic groups of nonprime order.

8.4. **Specialization.** Let $\mathfrak{o}$ be a complete DVR with residue field k, and let K be the fraction field of $\mathfrak{o}$. Fix a uniformizer $t \in \mathfrak{o}$; so, $\mathfrak{o} \cong k[[t]]$. We proceed to describe the specialization map

$$\rho_t^G \colon \mathrm{Burn}_{n,K}(G) \to \mathrm{Burn}_n(G).$$

The definition works with symbols in $\mathrm{Burn}_{n,K}(G)$, but the motivation is the specialization of $[X \leftarrow G] \in \mathrm{Burn}_{n,K}(G)$, where X is a smooth projective G-variety over K with generically free G-action.

We take $\mathcal{X}$ to be a regular model, projective over $\mathfrak{o}$, such that the special fiber is a simple normal crossing divisor $D = D_1 \cup \cdots \cup D_\ell$, where the G-action extends to $\mathcal{X}$, each D_i is G-invariant, and there is $d_i \in \mathbb{N}$, common multiplicity of the components of D_i in the special fiber. The set-up reminds us of Section 7.4, and we have similarly the normal bundle

$\mathcal{N}_{D_I/\mathcal{X}}$ and punctured normal bundle $\mathcal{N}^\circ_{D_I/\mathcal{X}}$ for $\emptyset \neq I \subseteq \mathcal{I} := \{1, \dots, \ell\}$. We have a map, given by the canonical section of a twist of $\mathcal{O}_{D_I}$, and isomorphism, supplied by t:

$$\bigotimes_{i\in I} \mathcal{N}^{\otimes d_i}_{D_i/\mathcal{X}}|_{D_I} \to \Big(\bigotimes_{i\in I} \mathcal{N}^{\otimes d_i}_{D_i/\mathcal{X}}|_{D_I}\Big) \otimes \mathcal{O}_{D_I}\Big(\sum_{j\in\mathcal{I}\setminus I} d_j D_{I\cup\{j\}}\Big) \cong \mathcal{O}_{D_I}.$$

We define the regular function ω_I on $\mathcal{N}_{D_I/\mathcal{X}}$ via the natural identification with $\bigoplus_{i\in I} \mathcal{N}_{D_i/X}|_{D_I}$, to be the composite with the morphism

$$\bigoplus_{i\in I} \mathcal{N}_{D_i/X}|_{D_I} \to \bigotimes_{i\in I} \mathcal{N}^{\otimes d_i}_{D_i/\mathcal{X}}|_{D_I}, \qquad (s_i)_{i\in I} \mapsto \prod_{i\in I} s_i^{d_i}.$$

The locus of vanishing of ω_I is the union of the fibers over $D_{I\cup\{j\}}$ for $j \in \mathcal{I} \setminus I$ and the zero-sections of the $\mathcal{N}_{D_i/X}|_{D_I}$. Thus

$$\omega_I^{-1}(\mathbb{A}^1 \setminus \{0\}) = \mathcal{N}^\circ_{D_I/\mathcal{X}}.$$

The specialization map is defined as follows. Given a symbol

$$(H, Y \circlearrowright K(W), \beta)$$

with W a smooth projective G-variety over K, we take $\mathcal{W}$ to be a regular model of W as above. So the G-action extends to $\mathcal{W}$, projective over $\mathfrak{o}$, with simple normal crossing special fiber $D_1 \cup \cdots \cup D_\ell$, such that each D_i is G-invariant with a multiplicity d_i. There is the regular function ω_I on $\mathcal{N}_{D_I/\mathcal{W}}$, as above. Then

$$\rho^G_\pi((H, Y \circlearrowright K(W), \beta)) := \sum_{\emptyset\neq I\subseteq\mathcal{I}} (-1)^{|I|-1}(H, Y \circlearrowright k(\omega_I^{-1}(1)), \beta),$$

which is well-defined, i.e., independent of the choice of $\mathcal{W}$ [84, Sect. 6].

We return to our motivation, to understand the image of $[X \circlearrowleft G]$ under ρ^G_t. This was addressed in [84, Thm. 6.6], with a formula as an explicit sum of symbols. We state this in a conceptually simpler form.

Theorem 8.1. *Let X be a smooth projective G-variety over K with generically free G-action, and let $\mathcal{X}$ be a regular model to which the action extends, such that the special fiber is a simple normal crossing divisor $D = D_1 \cup \cdots \cup D_\ell$, where each D_i is G-invariant with a multiplicity d_i. Then with ω_I as above we have*

$$\rho^G_t([X \circlearrowleft G]) = \sum_{\emptyset\neq I\subseteq\mathcal{I}} (-1)^{|I|-1}[\omega_I^{-1}(1) \circlearrowleft G]^{\mathrm{naive}}.$$

A technical point is that we wish to employ geometric arguments over k, but X is defined over K. Néron desingularization [29, §3.6] lets us replace $\mathfrak{o}$ by a suitable smooth $k[t]$-subalgebra A, and K by the fraction field K_A of A, thereby enabling direct geometric arguments. First, A may

be chosen with projective A-scheme X_A, locally a complete intersection with $\mathcal{X} \cong X_A \times_{\mathrm{Spec}(A)} \mathrm{Spec}(\mathfrak{o})$. The simple normal crossing boundary translates into a local equation form for $\mathcal{X}$, which we may assume valid for X_A. Then X_A is smooth over k with simple normal crossing boundary $X_{A/tA} := X_A \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A/tA)$, and the same holds when A is replaced by a larger smooth $k[t]$-subalgebra of $\mathfrak{o}$. By means of monoidal transform we may suppose that $X_{A/tA}$ is equivariantly isomorphic to $D \times \mathrm{Spec}(A/tA)$ (where G acts trivially on the second factor).

The normal crossing boundary of X_A defines a morphism

$$[X_A/G] \to [\mathbb{A}^\ell/\mathbb{G}_m^\ell], \tag{8.1}$$

which by [17, Rmk. 3.4] is smooth. The stack $[\mathbb{A}^\ell/\mathbb{G}_m^\ell]$ has a dense open substack $[(\mathbb{A}^1\backslash\{0\})^\ell/\mathbb{G}_m^\ell] \cong \mathrm{Spec}(k)$, with pre-image $X_{A[t^{-1}]} = X_A\backslash X_{A/tA}$ by (8.1). The pre-image of $X_{A[t^{-1}]}$ under $\mathcal{X} \to X_A$ is X.

We observe that in Theorem 8.1 there is no loss of generality in supposing X to be in divisorial form. For (8.1) there is a relative notion of divisoriality and a functorial procedure to achieve this by iterated blow-up; see [17, Thm. 6.6]. Each center of blow-up is smooth over $[\mathbb{A}^\ell/\mathbb{G}_m^\ell]$, i.e., smooth over k and transverse to the normal crossing boundary. We claim, also, transverse intersection on $\mathcal{N}^\circ_{D_I/\mathcal{X}}$ with $\omega_I^{-1}(1)$, for every I. This claim does not see the group action; with respect to a local trivialization of $\mathcal{N}_{D_I^\circ/\mathcal{X}}$, we have ω_I given as unit times monomial, so this is clear. Thus, in Theorem 8.1, relative divisorialification preserves the hypothesis, as well as both sides of the equality (by birational invariance of the class in the Burnside group).

The notion of divisoriality makes sense also for $\mathcal{X}$, as explained in [84, Sect. 6]: $\mathcal{X}$, to be divisorial, should have abelian stabilizers with X divisorial, and for every $F \in \mathcal{S}(\mathcal{X}, G)$ supported in the special fiber, with generic stabilizer group H, we should have surjective composite

$$\mathrm{Pic}(\mathcal{X}, G) \to \mathrm{H}^1(\mathrm{D}(F), k(F)^\times) \to \mathrm{H}^1(H, k(F)^\times)^{\mathrm{D}(F)/H} \to H^\vee.$$

But if X is divisorial, then $\mathcal{X}$ is divisorial. The corresponding assertion for orbifolds is [86, Lemma 9.1(ii)]; we give a proof in the present setting. Say X is divisorial with respect to linearized line bundles $L_1, \ldots, L_r$. We suppose A, as above, is taken so that the linearized line bundles are restrictions of linearized line bundles $M_1, \ldots, M_r$ on X_A. We apply relative divisorialification to the map

$$[X_A/G] \to B\mathbb{G}_m^r \times [\mathbb{A}^\ell/\mathbb{G}_m^\ell]$$

that combines (6.1) and (8.1). A numerical quantity that measures the relative stabilizers, if not identically zero, takes its maximal value on a locus W, smooth over $B\mathbb{G}_m^r \times [\mathbb{A}^\ell/\mathbb{G}_m^\ell]$, in particular, transverse to the

normal crossing boundary. By the equivariant isomorphism of $X_{A/tA}$ with $D \times \mathrm{Spec}(A/tA)$ and the analysis of the numerical quantity in terms of representation theory (cf. proof of Proposition 6.9), every component of W is either contained in $X_{A[t^{-1}]}$ or meets every fiber of $X_{A/tA}$ nontrivially. The image in $\mathrm{Spec}(A)$ of a component of W cannot contain the generic point, but by transversality cannot be equal to $\mathrm{Spec}(A/tA)$. Thus W is contained in $X_{A[t^{-1}]}$, with $W \times_{X_{A[t^{-1}]}} X = \emptyset$.

The proof of Theorem 8.1 will be helped by a more convenient description of ω_I. The monomial map

$$[\mathbb{A}^\ell/\mathbb{G}_m^\ell] \to [\mathbb{A}^1/\mathbb{G}_m]$$

with exponents $d_1, \ldots, d_\ell$, when composed with (8.1), is the map given by the boundary divisor $X_{A/tA} \subset X_A$. The chosen uniformizer t supplies a lift of the composite map to $\mathbb{A}^1$ and a corresponding lift of (8.1) to

$$[X_A/G] \to [\mathbb{A}^\ell/\mathcal{K}], \qquad \mathcal{K} := \ker(\mathbb{G}_m^\ell \to \mathbb{G}_m). \tag{8.2}$$

Away from the boundary, (8.2) restricts to

$$[X_{A[t^{-1}]}/G] \to [(\mathbb{A}^1 \setminus \{0\})^\ell/\mathcal{K}] \cong \mathbb{A}^1 \setminus \{0\},$$

given by t. The morphism (8.2) is smooth. Since formation of the normal bundle commutes with flat base change, we may describe $\mathcal{N}_{D_I/\mathcal{X}}$ by base change from $[\mathbb{A}^\ell/\mathcal{K}]$, where we have D_i defined by $x_i = 0$, and D_I, by $x_i = 0$ for all $i \in I$. Then $\mathcal{N}_{D_I/\mathbb{A}^\ell} \cong \mathbb{A}^\ell$, but now x_i, for $i \in I$, is a normal bundle coordinate; ω_I is given by $x_1^{d_1} \ldots x_\ell^{d_\ell}$. Summarizing, with $0_I \colon \mathbb{A}^\ell \to \mathbb{A}^\ell$ given by $(x_i)_{i\in\mathcal{I}} \mapsto (\mathbb{1}_{\mathcal{I}\setminus I}(i)x_i)_{i\in\mathcal{I}}$, we have the diagram with fiber square

$$\begin{array}{ccccc} \mathcal{N}_{D_I/\mathcal{X}} & \longrightarrow & [\mathbb{A}^\ell/\mathcal{K}] & \xrightarrow{x_1^{d_1}\cdots x_\ell^{d_\ell}} & \mathbb{A}^1 \\ \downarrow & & \downarrow{\scriptstyle 0_I} & & \\ \mathcal{X} & \longrightarrow & [\mathbb{A}^\ell/\mathcal{K}] & \xrightarrow{x_1^{d_1}\cdots x_\ell^{d_\ell}} & \mathbb{A}^1 \end{array} \tag{8.3}$$

where the composite top map is ω_I, and the composite bottom map is t.

In the proof of Theorem 8.1 we suppress the explicit mention of X_A, but its use is implicit in application of facts, valid for smooth morphisms, to a morphism such as $[\mathcal{X}/G] \to [\mathbb{A}^\ell/\mathcal{K}]$, where the justification will use the smooth morphism (8.2).

Proof of Theorem 8.1. We use the description of $\mathcal{N}_{D_I/\mathcal{X}}$, and ω_I, from the diagram (8.3). Restriction to $\mathbb{A}^1 \setminus \{0\}$ in the top row has the effect of replacing $\mathcal{N}_{D_I/\mathcal{X}}$ by $\mathcal{N}^\circ_{D_I/\mathcal{X}}$ and making the right-hand map an

isomorphism. Stabilizer components in a smooth family are smooth, so

$$\begin{aligned}\sum_{E\in\mathcal{S}(\mathcal{N}^\circ_{D_I/\mathcal{X}},G)/G} &(H, Y \subset k(E\cap\omega_I^{-1}(1)), \beta_E(\mathcal{N}^\circ_{D_I/\mathcal{X}}))\\ &= \sum_{F\in\mathcal{S}(\omega_I^{-1}(1),G)/G} (H, Y \subset k(F), \beta_F(\omega_I^{-1}(1))).\end{aligned} \tag{8.4}$$

We bridge the gap between $\rho_t^G([X \leftarrow G])$, a sum over $\mathcal{S}(X,G)/G$, and the sum over $\mathcal{S}(\mathcal{N}^\circ_{D_I/\mathcal{X}},G)/G$ in (8.4), by means of deformation to the normal bundle. Classically, for a smooth scheme X with a smooth closed subscheme V, the construction

$$\mathcal{M}^\circ_{V/X} := B\ell_{V\times\{0\}}(X\times\mathbb{A}^1) \setminus B\ell_{V\times\{0\}}(X\times\{0\})$$

yields a scheme with morphisms to X and to $\mathbb{A}^1$, such that:

- The morphism to $\mathbb{A}^1$ is smooth.
- The fiber over 0 is $\mathcal{N}_{V/X}$, projecting to $V\subset X$.
- The fiber over $\mathbb{A}^1\setminus\{0\}$ is $X\times(\mathbb{A}^1\setminus\{0\})$.

(Usually V and X are not assumed smooth; the general case is called deformation to the normal cone. In the first property "smooth" is replaced by "flat", and in the second, instead of the normal bundle we have the normal cone.) We are interested in D_I in $\mathcal{X}$, but by flatness $\mathcal{M}^\circ_{D_I/\mathcal{X}}$ is obtained by base change from D_I in $[\mathbb{A}^\ell/\mathcal{K}]$. In the latter case the deformation to the normal bundle yields $[\mathbb{A}^\ell/\mathcal{K}]\times\mathbb{A}^1$. Denoting by u the coordinate on the factor $\mathbb{A}^1$, we have morphisms $((x_i)_{i\in\mathcal{I}},u)\mapsto(u^{\mathbb{1}_I(i)}x_i)_{i\in\mathcal{I}}$ to $[\mathbb{A}^\ell/\mathcal{K}]$ and projection to $\mathbb{A}^1$.

There is the additional morphism $((x_i)_{i\in\mathcal{I}},u)\mapsto x_1^{d_1}\cdots x_\ell^{d_\ell}$ to $\mathbb{A}^1$. The additional morphism restricts to ω_I on the fiber over 0; we denote it by $\tilde{\omega}_I$. The morphism $\tilde{\omega}_I$ is flat, and the restriction over $\mathbb{A}^1\setminus\{0\}$ is smooth. Upon restriction we get smooth $(\tilde{\omega}_I,\mathrm{pr}_2)$ to $(\mathbb{A}^1\setminus\{0\})\times\mathbb{A}^1$, and subject to the caveat mentioned just before the start of the proof, the same can be asserted when we map the corresponding open subscheme of $\mathcal{M}^\circ_{D_I/\mathcal{X}}$ to $(\mathbb{A}^1\setminus\{0\})\times\mathbb{A}^1$. The fiber over $(\mathbb{A}^1\setminus\{0\})\times(\mathbb{A}^1\setminus\{0\})$ is isomorphic to $X\times(\mathbb{A}^1\setminus\{0\})$, and the fiber over $(\mathbb{A}^1\setminus\{0\})\times\{0\}$ is $\mathcal{N}^\circ_{D_I/\mathcal{X}}$.

Thus, we have a G-equivariant map $\mathcal{S}(\mathcal{N}^\circ_{D_I/\mathcal{X}},G)\to\mathcal{S}(X,G)$ and for $E\in\mathcal{S}(\mathcal{N}^\circ_{D_I/\mathcal{X}},G)$ mapping to $W\in\mathcal{S}(X,G)$ we have $\beta_E(\mathcal{N}^\circ_{D_I/\mathcal{X}}) = \beta_W(X)$. For $W\in\mathcal{S}(X,G)$ the closure $\mathcal{W}$ in $\mathcal{X}$ is a regular model of W, with normal crossing special fiber $(D_1\cap\mathcal{W})\cup\cdots\cup(D_\ell\cap\mathcal{W})$ and the

same multiplicities d_i as for the special fiber $D_1 \cup \cdots \cup D_\ell$ of $\mathcal{X}$. So

$$\begin{aligned}
&\rho_t^G([X \circlearrowleft G]) \\
&\quad = \sum_{\emptyset \neq I \subseteq \mathcal{I}} (-1)^{|I|-1} \sum_{W \in \mathcal{S}(X,G)/G} (H, Y \circlearrowright k(\mathcal{W} \times_{\mathcal{X}} \omega_I^{-1}(1)), \beta_W(X)) \\
&\quad = \sum_{\emptyset \neq I \subseteq \mathcal{I}} (-1)^{|I|-1} \sum_{E \in \mathcal{S}(\mathcal{N}^\circ_{D_I/\mathcal{X}},G)/G} (H, Y \circlearrowright k(E \cap \omega_I^{-1}(1)), \beta_E(\mathcal{N}^\circ_{D_I/\mathcal{X}})),
\end{aligned}$$

and we conclude by (8.4). □

9. Applications

We suppose that k is algebraically closed. Our main interest is the study of finite subgroups of the Cremona group Cr_n, i.e., automorphisms of rational varieties, see Section 3.1. Representative examples of rational varieties include:

- quadric hypersurfaces and quadric bundles over $\mathbb{P}^1$,
- del Pezzo surfaces and rational Fano threefolds,
- rational conic bundles over rational surfaces,
- singular cubic hypersurfaces (other than cones),
- toric varieties,
- equivariant compactifications of linear algebraic groups,
- generalized flag varieties.

One of the main problems is the *linearization problem*, i.e., the determination whether or not a given action is equivariantly birational to the projectivization of a representation. This is settled in dimension 2 [103], but is largely open in dimensions ≥ 3.

Another important problem is to prove that a given G-action on a rational variety is *stably linearizable*. The main tool for this is Lemma 3.1 (No-name lemma), applied to various vector bundle constructions. This is particularly interesting when the action is not linearizable.

In this section, we discuss several geometric applications of Burnside groups, quotient stacks, and cohomological invariants, with special regard to

- stabilizer stratification, Amitsur and Brauer groups,
- filtrations, incompressibles,
- specialization.

Fields of invariants. An application that we have already seen, of Lemma 3.1 (No-name lemma), is the stable G-birationality of projectivized representations from Corollary 3.2. By essentially the same argument, the quotients $\mathbb{P}(V)/G$ and $\mathbb{P}(W)/G$ are stably birational. However, the corresponding questions for (G-)birationality are more subtle.

First examples of nonbirational linear G-actions appeared in [110]; these were based on Condition **(Det)** in Section 4.3. Further examples, via the Burnside group formalism, can be found in [38] and [116].

We are not aware of examples of nonbirational *quotients* $\mathbb{P}(V)/G$, $\mathbb{P}(W)/G$, for projectivized representations of the same dimension, where the G-actions are generically free. The same questions can be asked for nonlinear actions, e.g., actions on quadrics.

Translation actions. Let G be a connected linear algebraic group over k; as a variety, G is rational. Let $G \subset \mathsf{G}$ be a finite subgroup. When is the natural translation action of G on G (stably) linearizable?

The action is stably linearizable, when G is *special*, i.e.,

$$\mathrm{H}^1(K, \mathsf{G}) = 1, \quad \forall\, K/k;$$

see, e.g., [67, Prop. 4.1].

Here we show how to address the stable linearizability problem for the *nonspecial* group

$$\mathsf{G} = \mathrm{PGL}_n, \quad n \geq 2.$$

Let V be a faithful representation of G, and $K := k(V)^G$ the field of invariants. We have, by functoriality:

$$\mathrm{H}^1(BG, \mathsf{G}) \to \mathrm{H}^1([V/G], \mathsf{G}) \to \mathrm{H}^1(K, \mathsf{G}). \tag{9.1}$$

The inclusion of G in G supplies an element $\alpha \in \mathrm{H}^1(BG, \mathsf{G})$.

Lemma 9.1. *If α has trivial image in $\mathrm{H}^1(K, \mathsf{G})$, then α also has trivial image in $\mathrm{H}^1(k(W)^G, \mathsf{G})$ for any other faithful representation W of G.*

Proof. Let $L := k(W)^G$, and let β denote the image of α in $\mathrm{H}^1(L, \mathsf{G})$. By the No-name lemma, we have on a suitable invariant open $W^\circ \subset W$ an equivariant identification $V \times W^\circ \cong \mathbb{A}^d \times W^\circ$, $d = \dim(V)$. Since α is trivial in $\mathrm{H}^1(K, \mathsf{G})$ it is also trivial on a dense open subset of $[V/G]$, and this determines a dense open subset of $\mathbb{A}^d_L$ on which β becomes trivial. Since L-points in $\mathbb{A}^d_L$ are Zariski dense, we obtain the vanishing of β. □

We define

$$\alpha^{\mathsf{G}}(G, V) \in \mathrm{H}^1(k(V)^G, \mathsf{G})$$

to be the image of α under (9.1). By Lemma 9.1, the triviality of $\alpha^{\mathsf{G}}(G, V)$ is independent of the choice of faithful representation V.

The argument in the proof of [67, Prop. 4.1] implies:

Proposition 9.2. *If $\alpha^{\mathsf{G}}(G,V)$ is trivial, then G is G-equivariantly stably linearizable.*

In fact, the triviality of $\alpha^{\mathsf{G}}(G,V)$ is both necessary and sufficient for the stable linearizability of the translation action, and indeed is encoded in the classical obstruction to lifting G to a *linear* representation

$$G \to \mathrm{GL}_n.$$

There is a natural PGL_n-equivariant compactification

$$\mathrm{PGL}_n \subset \mathbb{P}^{n^2-1},$$

the projectivization of the coordinates of an $n \times n$ matrix. Those with a single nonzero column form an invariant linear subspace $\mathbb{P}^{n-1} \subset \mathbb{P}^{n^2-1}$. Since the inclusion induces an isomorphism on Picard groups, by [90, Lemma 2.1] we have $\mathrm{Am}^2(\mathbb{P}^{n-1},G) = \mathrm{Am}^2(\mathbb{P}^{n^2-1},G)$. But $\mathrm{Am}^2(\mathbb{P}^{n-1},G)$ is cyclic, generated by a class $\gamma \in \mathrm{H}^2(G,k^\times)$ that represents the obstruction to lifting $G \to \mathrm{PGL}_n$ to a linear representation. If $\gamma = 0$, then such a lift exists, and by functoriality $\alpha^{\mathsf{G}}(G,V)$ lies in the image of

$$\mathrm{H}^1(K,\mathrm{GL}_n) \to \mathrm{H}^1(K,\mathrm{PGL}_n),$$

hence is trivial, and by Proposition 9.2, PGL_n is G-equivariantly stably linear. If $\gamma \neq 0$, then we have nontrivial $\mathrm{Am}^2(\mathbb{P}^{n^2-1},G)$, and this obstructs not only the G-equivariant stable linearizability of PGL_n but also the G-equivariant unirationality.

Condition (A). Actions of abelian groups on del Pezzo surfaces have been classified in [19], in particular, there is a classification of all actions satisfying Condition **(A)**. Such a classification for smooth Fano threefolds can be found in [1].

Amitsur and Brauer groups. The Amitsur invariant $\mathrm{Am}^2(X,G)$ from [21, Sect. 6] has been computed for rational surfaces [21, Prop. 6.7] and can be extracted for smooth Fano threefolds from the analysis in [113]. Computations of $\mathrm{Am}^3(X,G)$ for del Pezzo surfaces can be found in [118]. The related Condition **(T)** (Section 5) has been investigated for Fano threefolds and toric varieties in [89], [119]. It would be desirable to undertake a systematic study of $\mathrm{Am}^4(X,G)$.

An algorithm for computing Brauer groups of stacks $[X/G]$ and smooth projective models of quotient spaces X/G has been presented in [90].

Incompressibles. The description of incompressible symbols is easy in $\mathrm{Burn}_2(G)$. There, the incompressible symbols are the divisor symbols $(H, Y \subset k(C), (b))$, where C is either a curve of positive genus, or $C \cong \mathbb{P}^1$ is acted on by a *noncyclic* group Y. Already the simplest example

$$(C_2, \mathfrak{S}_3 \subset k(\mathbb{P}^1), (1)),$$

is useful, e.g., reproving a result of Iskovskikh [73] about nonbirationality of a linear action and a toric action of $G = C_2 \times \mathfrak{S}_3$, see [59, §7.6].

For $\mathrm{Burn}_3(G)$ the situation is more involved. There are many examples of incompressible symbols, but a full classification is unavailable; it would require the detailed analysis of [52]. Proofs of nonlinerizability of G-actions on rational varieties of dimension ≥ 3, based on incompressibles, can be found for

- actions on singular cubic threefolds, e.g., [39, Exa. 2.7], or
- the Burkhardt quartic, [40, Prop. 7.2].

Among further applications of incompressible symbols to proofs of nonbirationality of stably birational actions are:

- on $\mathbb{P}^3$, see [85, Thm. 11.2], or [117, Exa. 8.1],
- cubic threefolds and degree 14 Fano threefolds [120].

Specialization. An immediate application of Theorem 8.1 is the fact very general members of families of G-varieties equivariantly specializing to (mildly singular) nonlinearizable G-varieties are also not linearizable [84, Cor. 6.8, Exa. 6.9].

This motivated a search for such families. For example, a systematic study of specialization patterns for singular cubic threefolds to cubics with cohomological obstructions to stable linearizability was offered in [39], [35]. In particular, there is now a classification of actions on singular cubics $X \subset \mathbb{P}^4$ failing Condition **(H1)**; these come from specific C_2 or C_3-actions, see [42, Sect. 2]. A representative example of such a specialization is given in [39, Prop. 4.3].

10. What if ...

Nonabelian stabilizers. *A priori*, one could try to encode a given regular action of a finite group G by the stabilizer stratification and the induced action in the normal bundles, which would decompose into irreducible representations of the stabilizer groups. There are at least two reasons for bypassing this, via passage to divisorial forms:

- one would have to keep track of these representations, and it is easier to deal with sums of one-dimensional representations,

- encoding the interaction of the action of the stabilizers with the residual action on the stratum might involve nonabelian second cohomology [55], which we wanted to avoid.

This explains the importance of achieving abelian stabilizers.

Not in divisorial form. The definition of divisorial form ensures that the formula for the class in the equivariant Burnside group yields a sum of symbols.

Example 10.1. Consider the action of $\mathfrak{D}_4$ on $\mathbb{P}^3_{\mathbb{C}}$, the projective compactification of the action of Example 6.1. The subvariety F defined by $x = y = 0$ has generic stabilizer H, cyclic of order 4, and nontrivial residual action of $\mathfrak{D}_4/H$ on $\mathbb{C}(F)$. As observed in [84, Exa. 3.5], primitive characters of H do not lift to $\mathrm{H}^1(\mathfrak{D}_4, \mathbb{C}(F)^\times)$.

Such lifts of characters are needed in the definition of relation **(B)**, which is crucial for the equivariant Burnside group. The definition of symbols (§7.1) restricts to actions $Y \circlearrowleft K$, where Y is quotient by H of a subgroup Y_0 of G, *in which H is central*. Only then is there the possibility for all characters of H to lift to $\mathrm{H}^1(Y_0, K^\times)$ (see Proposition 6.7 and [88, Lemma 2.1]).

Tori. We suppose that k is algebraically closed. In principle, the theory developed above can be extended to the case of an algebraic torus $T = \mathbb{G}_m^d$. We can imitate the construction of the equivariant Burnside group and define analogous groups, where the symbols involve subgroups, quotient groups, and sequences of characters.

For example, there would be an analogous group

$$\mathcal{B}_n(T),$$

generated by unordered sequences of characters

$$\beta = (b_1, \dots, b_n), \quad b_j \in M = \mathfrak{X}^*(T),$$

and subject to the blow-up relation as in Section 7.1.

The first issue is that $\mathcal{B}_n(T)$ has infinitely many generators and relations, so that it is not obvious that one can effectively check whether or not two such symbols are equal in $\mathcal{B}_n(T)$.

However, there are geometric reasons for not pursuing the formalism for $\mathrm{Burn}_n(T)$. Indeed, T-torsors are Zariski locally trivial, and thus a T-action on X is equivariantly birational to a T-action on $T \times Y$, with $Y = X/T$, standard action of T on itself, and trivial action of T on Y. Moreover, actions of T on X and X' are equivariantly birational if and only if the quotients Y and Y' are (non-equivariantly) birational.

Extensions of finite groups by tori. Another direction would be to consider algebraic groups fitting into an exact sequence

$$1 \to T \to N \to G \to 1,$$

where G is a finite group and $T = \mathbb{G}_m^d$.

Example 10.2. Consider the singular cubic threefold $X \subset \mathbb{P}^4$ given by

$$x_1x_2x_3 + x_3^3 + x_3x_4x_5 + x_4^3 + x_5^3 = 0,$$

with singularities at $[1:0:0:0:0]$ and $[0:1:0:0:0]$, and the action of

$$1 \to \mathbb{G}_m \to N \to C_2 \to 1,$$

given by rescaling

$$\eta_a \colon x_1 \mapsto ax_1, \quad x_2 \mapsto a^{-1}x_2$$

and $\tau \colon x_1 \leftrightarrow x_2$, switching the coordinates. The fixed locus is the smooth curve given by $x_1 = x_2 = 0$, of genus 1. We can consider $\langle \eta_a, \tau \rangle$, $a \in k^\times$, defining a family of actions of (infinite, for general a) dihedral groups on X. When a is a primitive even root of unity, the action is not linearizable, by [35, Prop. 5.17].

Reductive groups. One might inquire, whether the theory developed above could be extended to linear algebraic groups, more general than tori. Reichstein and Youssin have established the existence of a standard form, after suitable equivariant blow-up [109]. However, several issues arise:

- The stabilizer groups are extensions by unipotent groups of diagonalizable groups, so their representation theory is not determined by characters.
- The stabilizer groups can *vary* in a stratum.

Example 10.3. Consider the action of GL_2 by right multiplication on the affine space of 2×2-matrices. The stabilizer groups of rank 1 matrices consist of a whole conjugacy class of $\mathbb{G}_a$-extensions of $\mathbb{G}_m$. As this stratum has codimension 1, there is no possibility to improve the situation by blowing up.

References

[1] H. Abban, I. Cheltsov, T. Kishimoto, and F. Mangolte. Smooth Fano 3-folds satisfying Condition (A), 2025. arXiv:2505.13684.

[2] D. Abramovich and M. Temkin. Functorial factorization of birational maps for qe schemes in characteristic 0. *Algebra Number Theory*, 13(2):379–424, 2019.

[3] B. Antieau and L. Meier. The Brauer group of the moduli stack of elliptic curves. *Algebra Number Theory*, 14(9):2295–2333, 2020.

[4] A. Auel, Chr. Böhning, and A. Pirutka. Stable rationality of quadric and cubic surface bundle fourfolds. *Eur. J. Math.*, 4(3):732–760, 2018.
[5] J. Ayoub. A guide to (étale) motivic sheaves. In *Proceedings of the International Congress of Mathematicians—Seoul 2014. Vol. II*, pages 1101–1124. Kyung Moon Sa, Seoul, 2014.
[6] J. Ayoub. La réalisation étale et les opérations de Grothendieck. *Ann. Sci. Éc. Norm. Supér. (4)*, 47(1):1–145, 2014.
[7] V. V. Batyrev. Birational Calabi-Yau n-folds have equal Betti numbers. In *New trends in algebraic geometry (Warwick, 1996)*, volume 264 of *London Math. Soc. Lecture Note Ser.*, pages 1–11. Cambridge Univ. Press, Cambridge, 1999.
[8] V. V. Batyrev. Canonical abelianization of finite group actions, 2000. `arXiv:math/0009043`.
[9] L. Bayle and A. Beauville. Birational involutions of $\mathbf{P}^2$. *Asian J. Math.*, 4(1):11–17, 2000. Kodaira's issue.
[10] A. Beauville. Variétés de Prym et jacobiennes intermédiaires. *Ann. Sci. École Norm. Sup. (4)*, 10(3):309–391, 1977.
[11] A. Beauville, J.-L. Colliot-Thélène, J.-J. Sansuc, and P. Swinnerton-Dyer. Variétés stablement rationnelles non rationnelles. *Ann. of Math. (2)*, 121(2):283–318, 1985.
[12] T. Beke. Sheafifiable homotopy model categories. *Math. Proc. Cambridge Philos. Soc.*, 129(3):447–475, 2000.
[13] O. Benoist and J. C. Ottem. Two coniveau filtrations. *Duke Math. J.*, 170(12):2719–2753, 2021.
[14] O. Benoist and O. Wittenberg. The Clemens-Griffiths method over non-closed fields. *Algebr. Geom.*, 7(6):696–721, 2020.
[15] O. Benoist and O. Wittenberg. Intermediate Jacobians and rationality over arbitrary fields. *Ann. Sci. Éc. Norm. Supér. (4)*, 56(4):1029–1084, 2023.
[16] D. Bergh. Functorial destackification of tame stacks with abelian stabilisers. *Compos. Math.*, 153(6):1257–1315, 2017.
[17] D. Bergh and D. Rydh. Functorial destackification and weak factorization of orbifolds, 2019. `arXiv:1905.00872`.
[18] E. Bierstone and P. D. Milman. Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant. *Invent. Math.*, 128(2):207–302, 1997.
[19] J. Blanc. Finite abelian subgroups of the Cremona group of the plane, 2006. Ph.D. Thesis, Université de Genève, `arXiv:math/0610368`.
[20] J. Blanc. Elements and cyclic subgroups of finite order of the Cremona group. *Comment. Math. Helv.*, 86(2):469–497, 2011.
[21] J. Blanc, I. Cheltsov, A. Duncan, and Yu. Prokhorov. Finite quasisimple groups acting on rationally connected threefolds. *Math. Proc. Cambridge Philos. Soc.*, 174(3):531–568, 2023.
[22] J. Blanc, S. Lamy, and S. Zimmermann. Quotients of higher-dimensional Cremona groups. *Acta Math.*, 226(2):211–318, 2021.
[23] F. Bogomolov and Yu. Prokhorov. On stable conjugacy of finite subgroups of the plane Cremona group, I. *Cent. Eur. J. Math.*, 11(12):2099–2105, 2013.
[24] F. A. Bogomolov. The Brauer group of quotient spaces of linear representations. *Izv. Akad. Nauk SSSR Ser. Mat.*, 51(3):485–516, 688, 1987.

[25] F. A. Bogomolov. Stable cohomology of groups and algebraic varieties. *Mat. Sb.*, 183(5):3–28, 1992.
[26] Chr. Böhning, H.-Chr. Graf v. Bothmer, and Yu. Tschinkel. Equivariant birational geometry of cubic fourfolds and derived categories. *Adv. Math.*, 469:Paper No. 110249, 32, 2025. With an appendix by Brendan Hassett.
[27] L. A. Borisov. The class of the affine line is a zero divisor in the Grothendieck ring. *J. Algebraic Geom.*, 27(2):203–209, 2018.
[28] L. A. Borisov and P. E. Gunnells. Wonderful blowups associated to group actions. *Selecta Math. (N.S.)*, 8(3):373–379, 2002.
[29] S. Bosch, W. Lütkebohmert, and M. Raynaud. *Néron models*, volume 21 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1990.
[30] S. Brandhorst and T. Hofmann. Finite subgroups of automorphisms of K3 surfaces. *Forum Math. Sigma*, 11:Paper No. e54, 57, 2023.
[31] M. Brion. On models of algebraic group actions. *Proc. Indian Acad. Sci. Math. Sci.*, 132(2):Paper No. 61, 17, 2022.
[32] L. Cavenaghi, L. Katzarkov, and M. Kontsevich. Atoms meet symbols, 2025. `arXiv:2509.15831`.
[33] A. Chambert-Loir, M. Kontsevich, and Yu. Tschinkel. Burnside rings and volume forms with logarithmic poles, 2023. `arXiv:2301.02899`, to appear in *PAMQ*.
[34] A. Chambert-Loir, J. Nicaise, and J. Sebag. *Motivic integration*, volume 325 of *Progress in Mathematics*. Birkhäuser/Springer, New York, 2018.
[35] I. Cheltsov, L. Marquand, Yu. Tschinkel, and Zh. Zhang. Equivariant geometry of singular cubic threefolds, II. *J. Lond. Math. Soc. (2)*, 112(1):Paper No. e70224, 46, 2025.
[36] I. Cheltsov, A. Sarikyan, and Z. Zhuang. Birational rigidity and alpha invariants of Fano varieties. In *Higher dimensional algebraic geometry—a volume in honor of V. V. Shokurov*, volume 489 of *London Math. Soc. Lecture Note Ser.*, pages 286–318. Cambridge Univ. Press, Cambridge, 2025.
[37] I. Cheltsov and C. Shramov. *Cremona groups and the icosahedron*. Monographs and Research Notes in Mathematics. CRC Press, Boca Raton, FL, 2016.
[38] I. Cheltsov, Yu. Tschinkel, and Zh. Zhang. Conjugacy classes of linear actions in the plane Cremona group, 2025. `arXiv:2508.09929`.
[39] I. Cheltsov, Yu. Tschinkel, and Zh. Zhang. Equivariant geometry of singular cubic threefolds. *Forum Math. Sigma*, 13:Paper No. e9, 52, 2025.
[40] I. Cheltsov, Yu. Tschinkel, and Zh. Zhang. Equivariant geometry of the Segre cubic and the Burkhardt quartic. *Selecta Math. (N.S.)*, 31(1):Paper No. 7, 36, 2025.
[41] I. Cheltsov, Yu. Tschinkel, and Zh. Zhang. Equivariant unirationality of Fano threefolds, 2025. `2502.19598`.
[42] I. Cheltsov, Yu. Tschinkel, and Zh. Zhang. Rationality of singular cubic threefolds over $\mathbb{R}$. *Adv. Math.*, 487:Paper No. 110756, 2026.
[43] V. Chernousov, P. Gille, and Z. Reichstein. Resolving G-torsors by abelian base extensions. *J. Algebra*, 296(2):561–581, 2006.
[44] U. Choudhury and M. Gallauer Alves de Souza. Homotopy theory of dg sheaves. *Comm. Algebra*, 47(8):3202–3228, 2019.

[45] C. H. Clemens and P. A. Griffiths. The intermediate Jacobian of the cubic threefold. *Ann. of Math. (2)*, 95:281–356, 1972.
[46] J.-L. Colliot-Thélène. Introduction to work of Hassett-Pirutka-Tschinkel and Schreieder. In *Birational geometry of hypersurfaces*, volume 26 of *Lect. Notes Unione Mat. Ital.*, pages 111–125. Springer, Cham, 2019.
[47] J.-L. Colliot-Thélène and A. Pirutka. Hypersurfaces quartiques de dimension 3: non-rationalité stable. *Ann. Sci. Éc. Norm. Supér. (4)*, 49(2):371–397, 2016.
[48] J.-L. Colliot-Thélène and J.-J. Sansuc. La descente sur les variétés rationnelles. II. *Duke Math. J.*, 54(2):375–492, 1987.
[49] B. Conrad, O. Gabber, and G. Prasad. *Pseudo-reductive groups*, volume 26 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, second edition, 2015.
[50] P. Deligne and D. Mumford. The irreducibility of the space of curves of given genus. *Inst. Hautes Études Sci. Publ. Math.*, (36):75–109, 1969.
[51] Ch. Deninger. A proper base change theorem for nontorsion sheaves in étale cohomology. *J. Pure Appl. Algebra*, 50(3):231–235, 1988.
[52] I. V. Dolgachev and V. A. Iskovskikh. Finite subgroups of the plane Cremona group. In *Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. I*, volume 269 of *Progr. Math.*, pages 443–548. Birkhäuser Boston, Boston, MA, 2009.
[53] D. Edidin and W. Graham. Equivariant intersection theory. *Invent. Math.*, 131(3):595–634, 1998.
[54] R. Garner, M. Kędziorek, and E. Riehl. Lifting accessible model structures. *J. Topol.*, 13(1):59–76, 2020.
[55] J. Giraud. *Cohomologie non abélienne*. Die Grundlehren der mathematischen Wissenschaften, Band 179. Springer-Verlag, Berlin-New York, 1971.
[56] Ch. D. Hacon, J. McKernan, and Ch. Xu. On the birational automorphisms of varieties of general type. *Ann. of Math. (2)*, 177(3):1077–1111, 2013.
[57] B. Hassett and D. Hyeon. Log canonical models for the moduli space of curves: the first divisorial contraction. *Trans. Amer. Math. Soc.*, 361(8):4471–4489, 2009.
[58] B. Hassett, A. Kresch, and Yu. Tschinkel. Stable rationality and conic bundles. *Math. Ann.*, 365(3-4):1201–1217, 2016.
[59] B. Hassett, A. Kresch, and Yu. Tschinkel. Symbols and equivariant birational geometry in small dimensions. In *Rationality of varieties*, volume 342 of *Progr. Math.*, pages 201–236. Birkhäuser, Cham, 2021.
[60] B. Hassett, A. Pirutka, and Yu. Tschinkel. Intersections of three quadrics in $\mathbb{P}^7$. In *Surveys in differential geometry 2017. Celebrating the 50th anniversary of the Journal of Differential Geometry*, volume 22 of *Surv. Differ. Geom.*, pages 259–274. Int. Press, Somerville, MA, 2018.
[61] B. Hassett, A. Pirutka, and Yu. Tschinkel. Stable rationality of quadric surface bundles over surfaces. *Acta Math.*, 220(2):341–365, 2018.
[62] B. Hassett, A. Pirutka, and Yu. Tschinkel. A very general quartic double fourfold is not stably rational. *Algebr. Geom.*, 6(1):64–75, 2019.
[63] B. Hassett and Yu. Tschinkel. On stable rationality of Fano threefolds and del Pezzo fibrations. *J. Reine Angew. Math.*, 751:275–287, 2019.
[64] B. Hassett and Yu. Tschinkel. Cycle class maps and birational invariants. *Comm. Pure Appl. Math.*, 74(12):2675–2698, 2021.

[65] B. Hassett and Yu. Tschinkel. Rationality of complete intersections of two quadrics over nonclosed fields. *Enseign. Math.*, 67(1-2):1–44, 2021. With an appendix by Jean-Louis Colliot-Thélène.
[66] B. Hassett and Yu. Tschinkel. Torsors and stable equivariant birational geometry. *Nagoya Math. J.*, 250:275–297, 2023.
[67] B. Hassett and Yu. Tschinkel. Equivariant geometry of low-dimensional quadrics. *Taiwanese J. Math.*, 29(6):1381–1402, 2025.
[68] A. Hoshi and A. Yamasaki. Rationality problem for algebraic tori. *Mem. Amer. Math. Soc.*, 248(1176):v+215, 2017.
[69] M. Hovey. Cotorsion pairs, model category structures, and representation theory. *Math. Z.*, 241(3):553–592, 2002.
[70] A. Huber and B. Kahn. The slice filtration and mixed Tate motives. *Compos. Math.*, 142(4):907–936, 2006.
[71] D. Huybrechts. *Lectures on K3 surfaces*, volume 158 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2016.
[72] S. Iitaka. *Algebraic geometry*, volume 24 of *North-Holland Mathematical Library*. Springer-Verlag, New York-Berlin, 1982. An introduction to birational geometry of algebraic varieties, Graduate Texts in Mathematics, 76.
[73] V. A. Iskovskikh. Two non-conjugate embeddings of $S_3 \times Z_2$ into the Cremona group. II. In *Algebraic geometry in East Asia—Hanoi 2005*, volume 50 of *Adv. Stud. Pure Math.*, pages 251–267. Math. Soc. Japan, Tokyo, 2008.
[74] V. A. Iskovskikh and Yu. I. Manin. Three-dimensional quartics and counterexamples to the Lüroth problem. *Mat. Sb. (N.S.)*, 86(128):140–166, 1971.
[75] B. Kahn and Nguyen T. K. Ngan. Sur l'espace classifiant d'un groupe algébrique linéaire, I. *J. Math. Pures Appl. (9)*, 102(5):972–1013, 2014.
[76] M. Kameko. Coniveau filtrations with $\mathbb{Z}/2$-coefficients, 2025. `arXiv:2504.19388`.
[77] Zh.-L. Kol′ë-Telèn and E. V. Piryutko. Cyclic covers that are not stably rational. *Izv. Ross. Akad. Nauk Ser. Mat.*, 80(4):35–48, 2016.
[78] M. Kontsevich, V. Pestun, and Yu. Tschinkel. Equivariant birational geometry and modular symbols. *J. Eur. Math. Soc. (JEMS)*, 25(1):153–202, 2023.
[79] M. Kontsevich and Yu. Tschinkel. Specialization of birational types. *Invent. Math.*, 217(2):415–432, 2019.
[80] A. Kresch, S. Tanimoto, and Yu. Tschinkel. Intermediate Jacobians and Burnside invariants, 2025. `arXiv:2511.07101`, to appear in J. Math. Soc. Japan.
[81] A. Kresch and Yu. Tschinkel. Stable rationality of Brauer-Severi surface bundles. *Manuscripta Math.*, 161(1-2):1–14, 2020.
[82] A. Kresch and Yu. Tschinkel. Birational types of algebraic orbifolds. *Mat. Sb.*, 212(3):54–67, 2021.
[83] A. Kresch and Yu. Tschinkel. Cohomology of finite subgroups of the plane Cremona group, 2022. `arXiv:2203.01876`, to appear in Algebraic Geom. and Physics.
[84] A. Kresch and Yu. Tschinkel. Equivariant birational types and Burnside volume. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 23(2):1013–1052, 2022.
[85] A. Kresch and Yu. Tschinkel. Equivariant Burnside groups and representation theory. *Selecta Math. (N.S.)*, 28(4):Paper No. 81, 39, 2022.
[86] A. Kresch and Yu. Tschinkel. Birational geometry of Deligne-Mumford stacks, 2023. `arXiv:2312.14061`, to appear in Ann. Sc. Norm. Super. Pisa Cl. Sci.

[87] A. Kresch and Yu. Tschinkel. Equivariant Burnside groups and toric varieties. *Rend. Circ. Mat. Palermo (2)*, 72(5):3013–3039, 2023.
[88] A. Kresch and Yu. Tschinkel. Equivariant Burnside groups: structure and operations. *Taiwanese J. Math.*, 29(6):1507–1529, 2025.
[89] A. Kresch and Yu. Tschinkel. Equivariant unirationality of toric varieties, 2025. arXiv:2506.07152.
[90] A. Kresch and Yu. Tschinkel. Unramified Brauer group of quotient spaces by finite groups. *J. Algebra*, 664:75–100, 2025.
[91] A. Kuznetsov and Yu. Prokhorov. Rationality of Fano threefolds over non-closed fields. *Amer. J. Math.*, 145(2):335–411, 2023.
[92] A. Kuznetsov and Yu. Prokhorov. Rationality over nonclosed fields of Fano threefolds with higher geometric Picard rank. *J. Inst. Math. Jussieu*, 23(1):207–247, 2024.
[93] M. Larsen and V. A. Lunts. Motivic measures and stable birational geometry. *Mosc. Math. J.*, 3(1):85–95, 259, 2003.
[94] Yu. I. Manin. Rational surfaces over perfect fields. *Inst. Hautes Études Sci. Publ. Math.*, (30):55–113, 1966.
[95] Yu. I. Manin. Rational surfaces over perfect fields. II. *Mat. Sb. (N.S.)*, 72 (114):161–192, 1967.
[96] C. Mazza, V. Voevodsky, and C. Weibel. *Lecture notes on motivic cohomology*, volume 2 of *Clay Mathematics Monographs*. American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2006.
[97] J. S. Milne. *Étale cohomology*. Princeton Mathematical Series, No. 33. Princeton University Press, Princeton, N.J., 1980.
[98] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric invariant theory*, volume 34 of *Ergeb. Math. Grenzgeb.* Berlin: Springer-Verlag, 3rd enl. edition, 1994.
[99] J. Nicaise and J. C. Ottem. A refinement of the motivic volume, and specialization of birational types. In *Rationality of varieties*, volume 342 of *Progr. Math.*, pages 291–322. Birkhäuser/Springer, Cham, 2021.
[100] J. Nicaise and J. C. Ottem. Tropical degenerations and stable rationality. *Duke Math. J.*, 171(15):3023–3075, 2022.
[101] J. Nicaise and E. Shinder. The motivic nearby fiber and degeneration of stable rationality. *Invent. Math.*, 217(2):377–413, 2019.
[102] E. Peyre. Progrès en irrationalité [d'après C. Voisin, J.-L. Colliot-Thélène, B. Hassett, A. Kresch, A. Pirutka, B. Totaro, Y. Tschinkel et al.]. Number 407, pages Exp. No. 1123, 91–116. 2019. Séminaire Bourbaki. Vol. 2016/2017. Exposés 1120–1135.
[103] A. Pinardin, A. Sarikyan, and E. Yasinsky. Linearization problem for finite subgroups of the plane Cremona group, 2024. arXiv:2412.12022.
[104] A. Pinardin and Zh. Zhang. $\mathfrak{A}_5$-equivariant geometry of quadric threefolds, 2025. arXiv:2508.11496.
[105] Yu. Prokhorov. Simple finite subgroups of the Cremona group of rank 3. *J. Algebraic Geom.*, 21(3):563–600, 2012.
[106] Yu. Prokhorov. Finite groups of birational transformations. In *European Congress of Mathematics (Portorož, 2021)*, pages 413–437. EMS Press, Berlin, 2023.
[107] Yu. Prokhorov and C. Shramov. Jordan property for Cremona groups. *Amer. J. Math.*, 138(2):403–418, 2016.

[108] A. Pukhlikov. *Birationally rigid varieties*, volume 190 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2013.
[109] Z. Reichstein and B. Youssin. Essential dimensions of algebraic groups and a resolution theorem for G-varieties. *Canad. J. Math.*, 52(5):1018–1056, 2000. With an appendix by János Kollár and Endre Szabó.
[110] Z. Reichstein and B. Youssin. A birational invariant for algebraic group actions. *Pacific J. Math.*, 204(1):223–246, 2002.
[111] S. Schreieder. On the rationality problem for quadric bundles. *Duke Math. J.*, 168(2):187–223, 2019.
[112] S. Schreieder. Stably irrational hypersurfaces of small slopes. *J. Amer. Math. Soc.*, 32(4):1171–1199, 2019.
[113] Sh. Sharma. Actions on the Picard group of smooth Fano threefolds, 2025. `arXiv:2511.12447`.
[114] N. Spaltenstein. Resolutions of unbounded complexes. *Compositio Math.*, 65(2):121–154, 1988.
[115] B. Totaro. Hypersurfaces that are not stably rational. *J. Amer. Math. Soc.*, 29(3):883–891, 2016.
[116] Yu. Tschinkel, K. Yang, and Zh. Zhang. Combinatorial Burnside groups. *Res. Number Theory*, 8(2):Paper No. 33, 17, 2022.
[117] Yu. Tschinkel, K. Yang, and Zh. Zhang. Equivariant birational geometry of linear actions. *EMS Surv. Math. Sci.*, 11(2):235–276, 2024.
[118] Yu. Tschinkel and Zh. Zhang. Cohomological obstructions to equivariant unirationality, 2025. `arXiv:2504.10204`.
[119] Yu. Tschinkel and Zh. Zhang. Equivariant unirationality of tori in small dimensions, 2025. `arXiv:2509.17008`.
[120] Yu. Tschinkel and Zh. Zhang. Stable equivariant birationalities of cubic and degree 14 Fano threefolds, 2025. `arXiv:2409.08392`, to appear in Algebraic Geom. and Physics.
[121] V. Voevodsky. Cohomological theory of presheaves with transfers. In *Cycles, transfers, and motivic homology theories*, volume 143 of *Ann. of Math. Stud.*, pages 87–137. Princeton Univ. Press, Princeton, NJ, 2000.
[122] V. Voevodsky. Triangulated categories of motives over a field. In *Cycles, transfers, and motivic homology theories*, volume 143 of *Ann. of Math. Stud.*, pages 188–238. Princeton Univ. Press, Princeton, NJ, 2000.
[123] C. Voisin. Unirational threefolds with no universal codimension 2 cycle. *Invent. Math.*, 201(1):207–237, 2015.

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