Solutions for Quiz 2

1. All quadratic residues modulo 31 are:

$$\begin{array}{llll} 1^2 \equiv 1 \mod 31, & 2^2 \equiv 4 \mod 31, & 3^2 \equiv 9 \mod 31, & 4^2 \equiv 16 \mod 31 \\ 5^2 \equiv 25 \mod 31 & 6^2 \equiv 5 \mod 31, & 7^2 \equiv 18 \mod 31, & 8^2 \equiv 2 \mod 31 \\ 9^2 \equiv 19 \mod 31, & 10^2 \equiv 7 \mod 31, & 11^2 \equiv 28 \mod 31, & 12^2 \equiv 20 \mod 31 \\ 13^2 \equiv 14 \mod 31, & 14^2 \equiv 10 \mod 31, & 15^2 \equiv 8 \mod 31 & \end{array}$$

2. By quadratic reciprocity,

$$\begin{aligned} \left(\frac{68}{113}\right) &= \left(\frac{4}{113}\right)\left(\frac{17}{113}\right) = \left(\frac{17}{113}\right) \\ &= \left(\frac{113}{17}\right) = \left(\frac{11}{17}\right) = \left(\frac{17}{11}\right) \\ &= \left(\frac{6}{11}\right) = \left(\frac{2}{11}\right)\left(\frac{3}{11}\right) = -\left(\frac{3}{11}\right) \\ &= \left(\frac{11}{3}\right) = \left(\frac{2}{3}\right) = -1 \end{aligned}$$

Thus, 68 is a quadratic non residue modulo 113, and $x^2 \equiv 68 \mod 113$ is not solvable.

3. The prime factorization of 15! is given by:

$$15! = 2^{11} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11^1 \cdot 13^1$$

Thus,

$$\begin{aligned} \varphi(15!) &= \varphi(2^{11})\varphi(3^6)\varphi(5^3)\varphi(7^2)\varphi(11)\varphi(13) \\ &= (2^{11} - 2^{10})(3^6 - 3^5)(5^3 - 5^2)(7^2 - 7)(11 - 1)(13 - 1) \\ &= 1024 \cdot 486 \cdot 100 \cdot 42 \cdot 10 \cdot 12 \\ &= 250822656000 \\ &= 2^{17} \cdot 3^7 \cdot 5^3 \cdot 7^1 \end{aligned}$$