Homework 3

1. Show that $\nu_p(n!) < n$.

2. Let $\mathbb{Q}_p$ be the completion of $\mathbb{Q}$ with respect to the $p$-adic distance $| \cdot |_p$. Compute $\sqrt{-1}$ in $\mathbb{Q}_{29}$ to 5 digits.

3. For $c \in \mathbb{N}$ show that the sequence $c_n := c^{p^n}$ converges in $\mathbb{Q}_p$. Let $\gamma = \lim c_n$. Then $\gamma = c \mod p$ and $\gamma^{p-1} = 1$.

4. Prove that every sequence of integers has a subsequence which is Cauchy with respect to $| \cdot |_p$.

5. Let

$$G(x_1, x_2, x_3) := x_1^4 + x_2^4 + x_3^4 - (x_1^2 x_2^2 + x_1^2 x_3^2 + x_2^2 x_3^2) - x_1 x_2 x_3 (x_1 + x_2 + x_3)$$

and

$$F(x_1, \ldots, x_{18}) := G(x_1, x_2, x_3) + G(x_4, x_5, x_6) + G(x_7, x_8, x_9) + 4G(x_{10}, x_{11}, x_{12}) + 4G(x_{13}, x_{14}, x_{15}) + 4G(x_{16}, x_{17}, x_{18}).$$

Show that $F(x) = 0$ has only the trivial zero in $\mathbb{Q}_2$. 