We are all aware of what a random walk is. A random walk on Z^d is a stochastic process $X_n = \xi_1 + \xi_2 + \cdots + \xi_n$ defined for $n \geq 0$ with $X_0 = 0$, where $\{\xi_i\}$ are a sequence of independent identically distributed random variables with a common distribution $P[\xi_i = x] = \alpha(x)$. For simplicity one may assume that α has finite support and that $\alpha(\pm e_i) > 0$ where $\{e_i\}$ are the coordinate directions. It is viewed as a Markov chain on Z^d with transition probabilities

$$\pi(x, y) = \alpha(y - x)$$

Various properties of the random walks are easy to establish. For instance in three or more dimensions all random walks are transient. If the dimension is one or two then the random walk is recurrent if and only if the mean is 0.

One has the law of large numbers that states that with probability 1

$$\lim_{n\to\infty} \frac{X_n}{n} = m = E[\xi_1] = \sum_x x\,\alpha(x)$$

and the central limit theorem that states

$$\frac{X_n - n\,m}{\sqrt{n}} \simeq N[0, C]$$

i.e. $\frac{X_n - n\,m}{\sqrt{n}}$ has an asymptotic distribution that is normal with mean 0 and covariance

$$\langle Cu, v\rangle = \sum_x \langle x - m, u\rangle\langle x - m, v\rangle\alpha(x)$$

where $u, v \in R^d$. Finally there is the large deviation result of Cramér, that shows that for reasonable sets $A \in R^d$,

$$P[\frac{S_n}{n} \in A] = \exp[-n \inf_{x\in A} I(x) + o(n)]$$

where

$$I(x) = \sup_\theta[< \theta, x > - \log M(\theta)]$$

and

$$M(\theta) = E[e^{<\theta,\xi>}] = \sum_x e^{<\theta,x>}\alpha(x)$$

Our goal initially is to see if the analogous results are valid for Random Walks in Random Environments.

We begin with a definition of what a random walk in a random environment is. Instead of translation invariance of the type $\pi(x,y) = \alpha(y-x)$, we assume that $\pi(x,y) = \pi(x,y,\omega)$ is random but is translation invariant in a statistical sense, i.e the processes $\pi(x,y,\omega)$ and $\pi(x+z,y+z,\omega)$ have the same distribution. This is usually modeled by a measure preserving (ergodic) action τ_z of $z \in Z^d$ on some probability space (Ω, Σ, P), a random probability measure $\hat{\pi}(z,\omega)$ on Z^d, i.e a map $\hat{\pi} : \Omega \to \mathcal{M}(Z^d)$, where $\mathcal{M}(Z^d)$ is the space of all probability distributions on Z^d, We shall assume for simplicity that $\hat{\pi}(\cdot,\omega)$ has a fixed finite support, i.e. there exists ℓ such that $\hat{\pi}(z,\omega) = 0$ for all ω if $|z| \geq \ell$ and $\hat{\pi}(\pm e_i,\omega) \geq c > 0$ for some positive constant c, a condition that is often referred to as the ellipticity condition. The random environment on Z^d is defined by

$$\pi(x,y,\omega) = \hat{\pi}(y-x, \tau_x\omega)$$

If the probability distribution P is such that $\pi(\tau_x,\omega)$ are all statistically independent then the environment is called a product environment and the distribution β on $\mathcal{M}(Z^d)$ of $\pi(\omega,\cdot)$ essentially determines P, provided we take $\Omega = \Pi_{x\in z^d}\mathcal{M}(Z^d)$.

For each ω one has a Markov Chain on Z^d with transition probability $\pi(x,y,\omega)$, starting from 0 at time 0 and we denote this measure by Q^ω. Properties valid for Q_ω for almost all ω with respect to P are said to be valid for the quenched case. We can also define the averaged measure

$$\bar{Q} = \int Q_\omega P(d\omega)$$

and see which results are valid for it. Laws of large numbers are about properties valid with probability 1 and so validity for Q_ω for almost all ω is the same as validity for $\bar{P}$. On the other hand central limit theorem could be valid for $\bar{Q}$ with out being valid for Q_ω. The large deviation results could be different except for the restriction imposed by Jensen's inequlity.

. The special case of $d = 1$, $\pi(x,y,\omega) = 0$ unless $y = x \pm 1$ is interesting because a certain amount of exact computations are possible. The Markov Chain is determined by $\pi(x,x+1,\omega) = p(x)$ and $\pi(x,x-1,\omega) = 1-p(x)$. Here $\{p(x)\}$ is a stationary process in $x \in Z$, and is bounded away from 0 and 1. Let us try to compute, for $x \leq 0$,

$$u(x) = u(x,\omega) = Q^\omega[\sigma_1 < \infty | X_0 = x]$$

where σ_1 is the first time X_j equals 1. This is the solution of the linear equation

$$u(x) = p(x)u(x+1) + (1-p(x))u(x-1)$$

with $u(1) = 1$. Either $u(x) \equiv 1$ for all $x \leq 0$ or $u(x) < 1$ and $u(x) \to 0$ as $x \to -\infty$. The latter is the necessary and sufficient condition for $X_n \to -\infty$ as $n \to \infty$. If we denote by $v(x) = u(x+1) - u(x)$, then

$$v(x) = \frac{p(x)}{1-p(x)} v(x+1)$$

In particular either $v(x) = 0$ for all $x \leq 0$, i.e $u(x) \equiv 1$ or $v(x) > 0$ for every $x \leq 0$ and $\sum_x v(x) < \infty$, the latter condition being the necessary and sufficient condition for $X_n \to -\infty$ as $n \to \infty$. But

$$v(x) = v(0) \prod_{y=x}^{0} \frac{p(y)}{1-p(y)}$$

It is not hard to see that if $E[\log \frac{p(y)}{1-p(y)}] < 0$, then $\sum_{x<0} \prod_{y=x}^{0} \frac{p(y)}{1-p(y)} < \infty$ and we have $X_n \to -\infty$ as $n \to \infty$ and transience. Similarly if $E[\log \frac{p(y)}{1-p(y)}] > 0$, then we have $X_n \to \infty$ as $n \to \infty$ and again transience. One can check that if $E[\log \frac{1-p(y)}{p(y)}] = 0$ we have recurrence. This does not require any assumption about the product nature of the environment. No matter how dependent the recurrence or transience depends only on the distribution of $p(y)$ at one site.

We next look at the law of large numbers or the speed with which $X_n \to \pm\infty$. Let us compute

$$E^{Q_\omega}[\sigma_1 | X_0 = 0]$$

If $E^P[g(\omega)] = \theta < \infty$, then

$$\lim_{n\to\infty} \frac{X_n}{n} = m = \theta^{-1}$$

We have the following simple identity.

$$g(\omega) = p(0) + (1-p(0))[1 + g(\tau^{-1}\omega) + g(\omega)]$$

or

$$p(0)g(\omega) = p(0) + (1-p(0)) + (1-p(0))g(\tau^{-1}\omega)$$

i.e.

$$g(\omega) = 1 + \frac{1-p(0)}{p(0)} + \frac{1-p(0)}{p(0)} g(\tau^{-1}\omega)$$

In a product environment $p(0)$ and $g(\tau^{-1}\omega)$ are independent. Therefore

$$\theta = 1 + E^P[\frac{1-p(0)}{p(0)}](1+\theta)$$

yielding

$$\theta[1 - E^P[\frac{1-p(0)}{p(0)}]] = [1 + E^P[\frac{1-p(0)}{p(0)}]]$$

or

$$m = \frac{[1 - E^P[\frac{1-p(0)}{p(0)}]]}{[1 + E^P[\frac{1-p(0)}{p(0)}]]}$$

provided $E^P[\frac{1-p(0)}{p(0)}] < 1$. For S_n to tend to ∞ we need $E^P[\log \frac{p(0)}{1-p(0)}] > 0$ or equivalently $E^P[\log \frac{1-p(0)}{p(0)}] < 0$. One can have that and still have $E^P[\frac{1-p(0)}{p(0)}] > 1$. In this case $\frac{S_n}{n}$ will tend to 0 as $n \to \infty$. This cannot happen for ordinary random walks. The explanation is that you come across environments that can slow you down. You have to get through them. One cannot avoid them if $d = 1$.

Let us look at an invariant measure for the chain. This should satisfy

$$q(x-1)p(x-1) + q(x+1)(1 - p(x+1)) = q(x)$$

or

$$q(x-1)p(x-1) - q(x+1)p(x+1) = q(x) - q(x+1)$$

or equivalently

$$q(x-1)p(x-1) + q(x)p(x) = q(x) + c$$

i.e.

$$q(x-1) = \frac{1-p(x)}{p(x-1)}q(x) + \frac{c}{p(x-1)}$$

admitting a solution

$$q(x,\omega) = c\left[\frac{1}{p(x,\omega)} + \sum_{y=x+1}^{\infty}\left[\prod_{z=x}^{y-1}\frac{1-p(z+1,\omega)}{p(z,\omega)}\right]\frac{1}{p(y,\omega)}\right]$$

Notice the covariant nature of the solution, i.e.

$$q(x,\omega) = q(0,\tau_x\omega) = \hat{q}(\omega)$$

In order to normalize with $E^P[\hat{q}(\omega)] = 1$, we need $E[\frac{1-p(0)}{p(0)}] < 1$. Then we can consider the Markov Chain on Ω with transition probability $\omega \to \tau_1\omega$ or $\tau_{-1}\omega$ with probability $p(\omega)$ and $1 - p(\omega)$ respectively. $\hat{q}(\omega)$ is an invariant density (with respect to P) for the chain and is ergodic. Therefore if ω_n is the chain and $\hat{Q}_\omega$ is the measure of the ω_n-process, which is the environment as seen by the random walk, then

$$X_n - \sum_{j=0}^{n-1}[2p(\omega_j) - 1]$$

is a Martingale and therefore

$$\lim_{n\to\infty}\frac{X_n}{n} = \int_\Omega [2p(\omega) - 1]\hat{q}(\omega)dP = \frac{2c}{1 - E^P[\frac{1-p(0)}{p(0)}]} - 1$$

Normalization constant c can be evaluated by

$$c + 1 = \frac{2c}{1 - E^P[\frac{1-p(0)}{p(0)}]}$$

which matches with the earlier calculation.

The situation in Z^d with $d \geq 2$ is much more complicated. There is no simple criterion for determining when a random walk in a random environment is recurrent or transient, even in the case of a product environment. One trivial sufficient condition for transience is

$$\sum_y < \ell, y - x > \pi(x, y, \omega) \geq c > 0 \tag{1}$$

for almost all ω. A uniform positive drift in some direction will make the random walk go off to ∞!. Even for a product environment, under this assumption, it is not easy to show the existence of the limit

$$\lim_{n \to \infty} \frac{x_n}{n} = m$$

a.e. There is no simple formula for m. The density $\hat{q}(\omega)$ does not seem to exist, because the chain on Ω may not have an invariant measure that is absolutely continuous with respect to P.

We will approach this problem from a different perspective. We look at the process $\bar{Q} = \int Q^\omega dP$ where P is a product measure.

We need to look at the process $\bar{Q}$ in its own right. Let β be the marginal distribution of $\pi(x, x + \cdot)$ viewed as a probability distribution on $\mathcal{M}(Z^d)$. Then $\bar{Q}$ is a process with long memory which is not Markov. If we start from 0, the first step ξ_1 under $\bar{Q}$ has distribution

$$\bar{\pi}(z) = \int_{\mathcal{M}(Z^d)} \pi(z) d\beta(\pi)$$

which is the prior distribution of jumps at any site. If we arrive at a site that has never been visited before, the conditional distribution of the next jump is still the same, since the environment is a product measure. However if the current site of the random walk has already been visited k times and produced $k(z)$ jumps of size z, with $k = \sum_z k(z)$, then *a posteriori* probability of the jump is given by

$$\hat{\pi}(z|w) = \frac{\int \pi(z) \prod [\pi(z')]^{k(z')} d\beta(\pi)}{\int \prod [\pi(z')]^{k(z')} d\beta(\pi)} \tag{2}$$

where w is the past information. It is more convenient to slide the walk back so it ends at the origin rather starting from it. Let us denote by $\mathcal{W}_n$ the space of walks that end at 0, starting from time $-n$ at some point which is really $-X_n$, if X_n is where the original walk ended up at time n. We denote by $\mathcal{W}_\infty = \cup_n \mathcal{W}_n$ and we have a Markov Chain on $\mathcal{W}$. The transition probability is $w \to T_z w$ with probabilty $\hat{\pi}(z|w)$. Here $T_z w$ is obtained by appending a jump of size z at the front and shifting back by $-z$. Of course T_z maps $\mathcal{W}_n \to \mathcal{W}_{n+1}$. One can denote by $\overline{\mathcal{W}} = \mathcal{W} \cup \mathcal{W}_{tr}$ and include transient paths with infinite past. Then $\hat{\pi}(z|w)$ can be extended to $\overline{\mathcal{W}}$ and a Markov Chain defined on it with transition probability given by $\hat{\pi}(z|w)$.

Suppose ν is an invariant measure for this Markov Chain. Suppose it is unique. Then this will define a process with stationary increments. It is not hard to see that we can expect

$$\frac{X_n}{n} \to m(\nu) = \int_{\overline{\mathcal{W}}} \big[\sum_{z} z\hat{\pi}(z|w)\big] d\nu(w)$$

a.e $\bar{Q}$ and therefore a.e Q^{ω} a.e P.

Under the assumption (1), we will prove the existence and uniqueness of the invariant measure ν.

The transition probabilities from w to $T_z w$ are given by $\{\hat{\pi}(z|w)\}$ given by formula (2). We note the fact that $\{\hat{\pi}(z|w)\}$ depends on local information, i.e on $\{k(z') : z' \in F\}$ for some finite F. We assume bounded step size as well as ellipticity, i.e. for some fixed function $\bar{Q}(\cdot) \geq 0$ with $\bar{Q}(z) = 0$ for $|z| > C$ and $\bar{Q}(\pm\mathbf{e}_i) > 0$, (where $\pm\mathbf{e}_i$ are the nearest neighbors of 0) and a constant $c > 0$, we have

$$c\bar{Q}(z) \geq \pi(z) \geq \bar{Q}(z), a.e\ \beta$$

Existence. The problem is to keep the process uniformly transient. A uniform exponential estimate for $Q^w[z_1 + z_2 + \cdots z_n = 0]$ will do the trick. Since the z_i have a uniform bound and $E[< s, z_{j+1} > |z_1, z_2, \ldots, z_j] \geq c > 0$ this is elementary with the help of

$$\psi_j = \exp[-\lambda < s, z_1 + z_2 + \cdots + z_j > +f(\lambda)j]$$

which is a supermartingale with $f(\lambda) > 0$ provided λ is sufficiently small.

Some mixing properties.

Given a starting point w, we have the measure Q^w corresponding to the Markov chain starting with this initial walk w. The measure in particular will generate a random walk $\{X_j : j \geq 1\}$ in the future with steps $\mathbf{z} = \{z_j\}$ where $z_j = z_0(w_j)$ and $X_k = z_1+z_2+\cdots+z_k$. Remember that $z_0(w)$ is the last jump of w and z_j will be the last jump after j steps. Given a set $A \subset Z^d$, let us denote by

$$H(n, A, \mathbf{z}) = \#\Big\{k : 0 \leq k \leq n \text{ and } X_k \in A\Big\}$$

the number of visits to A by $X_k = z_1 + z_2 + \cdots + z_k$ during $0 \leq k \leq n$. Clearly, by our ellipticity condition,

$$|\log \frac{dQ^{w_1}}{dQ^{w_2}}(\mathbf{z})|_{\mathcal{F}_n} \leq CH(n, S(w_1) \cup S(w_2), \mathbf{z})$$

where $\mathcal{F}_n$ is the σ-filed generated by $z_1, z_2, \ldots, z_n$ and $S(w) = \{X_j(w) : j \leq 0\}$ is the range of the past. Let us fix a direction $s \in S^{d-1}$ and define the events $E_s \subset F_s$

$$E_s = \{\mathbf{z} : \lim_{n\to\infty} \langle s, X_n\rangle = \infty\}$$
$$F_s = \{\mathbf{z} : \limsup_{n\to\infty} \langle s, X_n\rangle = \infty\}$$

and their probabilities $U(w) = Q^w[E_s]$ and $V(w) = Q^w[F_s]$ with $U(w) \leq V(w)$. Let

$$s^*(w) = \sup_{y \in S(w)} < s, y >$$

Lemma. If for some $\bar{w}$ with $s^*(\bar{w}) < \infty$, we have $U(\bar{w}) > 0$, then for every w with $s^*(w) < \infty$ we have $U(w) = V(w) > 0$ and

$$1 - U(w) = Q^w\left[\lim_{n\to\infty} \langle s, X_n\rangle = -\infty\right]$$

Proof: We will first prove an estimate of the form

$$U(w) \geq u(s^*(w))$$

for some function $u(r)$ that is non-increasing in r and positive for each $r > 0$. Since $U(\bar{w}) > 0$, for every k there is a $r = r(k)$ such that

$$Q^{\bar{w}}\left[\langle s, X_n\rangle \geq \max\{k, s^*(\bar{w})\} + 1 \text{ for all } n \geq r(k)\right] \geq \frac{1}{2}U(\bar{w})$$

Then for any w and $\mathbf{z}$ satisfying $\langle s, X_n\rangle \ \geq\ \max\{s^*(w), s^*(\bar{w})\} + 1$ for all $n \geq r(s^*(w))$

$$\sup_n H(n, S(w) \cup S(\bar{w}), \mathbf{z}) \leq r(s^*(w))$$

providing a lower bound

$$U(w) \geq \frac{1}{2}U(\bar{w})e^{-Cr(s^*(w))}$$

Since

$$U(w_n) = Q^w[E|\mathcal{F}_n]$$

is a Q^w martingale it must converge to 0 almost surely on E_s^c. Which means that

$$\lim_{n\to\infty} s^*(w_n) = \infty \text{ a.e. } Q^w \text{ on } E_s^c$$

But clearly $s^*(w_n) = 0$ infinitely often on F_s, proving that F_s and E_s^c are almost surely disjoint. In fact because of ellipticity $\limsup_{n\to\infty}\langle s, X_n\rangle$ can only be $\pm\infty$, proving the lemma.

Theorem: Let $\{\nu_i : i = 1, 2\}$ be two invariant ergodic measures for $\hat{\pi}(z|w)$ such that $m(\nu_i) \neq 0$ for $i = 1, 2$. Then $m(\nu_1) = c_0 m(\nu_2)$ for some $c_0 < 0$.

Proof: Assume otherwise. Then there is an $s' \in S^{d-1}$ such that $\langle s', m(\nu_i)\rangle > 0$ for $i = 1$ and $i = 2$. From our assumptions

$$\sup_n H(n, S(w), \mathbf{z}) < \infty$$

a.e. $\alpha_i(dw)Q^w(d\mathbf{z})$ for both $i = 1$ and $i = 2$. This proves the mutual absolute continuity of $Q^{w_1}(d\mathbf{z})$ and $Q^{w_2}(d\mathbf{z})$ for almost all w_1, w_2, with respect to $\nu_1 \times \nu_2$. This implies, by the ergodic theorem, that $\nu_1 = \nu_2$. Since under assumption (1), $< s, m(\nu) >\geq c > 0$ there is at most one invariant measure.

Remark: A zero-one law of Merkl and Zerner proves in $d = 2$ ($d = 1$ is trivial) that for any $w \in \overline{\mathcal{W}}$, in particular for $w = \phi$, $U(w) = 0$ or 1.