The 'quenched' large deviations.

Let a stationary and ergodic random environment $\{\pi(\omega, x, y)\}$ be given. In other words we have a probability space (Ω, Σ, P) on which $\{\tau_z : z \in Z^d\}$ acts ergodically as measure preserving transformations and $\pi(\omega, x, y) = p(\tau_x \omega, y - x)$.

Hypothesis A. $p(\omega, \pm \mathbf{e}_i) \geq \rho$ for some nonrandom $\rho > 0$, where $\{\pm \mathbf{e}_i\}$ are the nearest neighbors of 0 along the coordinate axes.

Let $\{\pi^{(n)}(\omega, x, y)\}$ be the n step transition probability derived from $\pi(\omega, x, y)$. Let $c > 0$ be given. Define for each ω, t, x and y,

$$q_c(\omega, t, x, y) = \sup_{n \geq 0} \left[\pi^{(n)}(\omega, x, y) e^{-c|n-t|} \right]$$

For each c and t, $q_c(\omega, t, x, y) = q_c(\tau_x \omega, t, 0, y - x)$ is a stationary process with respect to translations $\{\tau_z : z \in Z^d\}$. From the lower bound on $\pi(\omega, x, x \pm \mathbf{e}_i)$ it follows that $q_c(\omega, t, x, y) > 0$ and

$$|\log q_c(\omega, t, x, y) - \log q_c(\omega, s, x, y)| \leq c|t - s|$$

Since we can go from $x + z_1$ to x in $|z_1|$ steps and from y to $y + z_2$ in $|z_2|$ steps with probabilities at least $\rho^{|z_1|}$ and $\rho^{|z_2|}$ respectively, we obtain

$$\begin{aligned}
q_c(\omega, t, x + z_1, y + z_2) &= \sup_{n \geq 0} \left[\pi^{(n)}(\omega, x + z_1, y + z_2) e^{-c|n-t|} \right] \\
&= \sup_{n, n_1, n_2 \geq 0} \left[\pi^{(n+n_1+n_2)}(\omega, x + z_1, y + z_2) e^{-c|n+n_1+n_2-t|} \right] \\
&\geq \rho^{|z_1|+|z_2|} e^{-c[|z_1|+|z_2|]} \sup_{n \geq 0} \left[\pi^{(n)}(\omega, x, y) e^{-c|n-t|} \right] \\
&= (\rho e^{-c})^{|z_1|+|z_2|} q_c(\omega, t, x, y)
\end{aligned}$$

Therefore for each $c < \infty$, there is a constant c' such that the function $g_c(\omega, t, x, y) = -\log q_c(\omega, t, x, y)$ satisfies

$$|g_c(\omega, t, x, y) - g_c(\omega, t + s, x + z_1, y + z_2)| \leq c'[s + |z_1| + |z_2|]$$

In addition

$$\begin{aligned}
q_c(\omega, t + s, 0, x + y) &= \sup_{n \geq 0} \left[\pi^{(n)}(\omega, 0, x + y) e^{-c|n-(t+s)|} \right] \\
&= \sup_{n, m \geq 0} \left[\pi^{(n+m)}(\omega, 0, x + y) e^{-c|(n+m)-(t+s)|} \right] \\
&= \sup_{n, m \geq 0} \left[e^{-c|(n+m)-(t+s)|} \sum_z \pi^{(n)}(\omega, 0, z) \pi^{(m)}(\omega, z, x + y) \right] \\
&\geq \sup_{n, m \geq 0} \left[e^{-c|n-t|-c|m-s|} \pi^{(n)}(\omega, 0, x) \pi^{(m)}(\omega, x, x + y) \right] \\
&= q_c(\omega, t, 0, x)\, q_c(\omega, s, x, x + y)
\end{aligned}$$

i.e.

$$g_c(\omega, t+s, 0, x+y) \le g_c(t, 0, x) + g_c(s, x, x+y)$$

and g_c is a stationary and ergodic relative to $x \to x+z$ $y \to y+z$.

We can use the subadditive ergodic theorem to prove

Theorem 1. Let $(\Omega, \mathcal{F}, P)$ be a probability space on which $\{\tau_z : z \in Z^d\}$ acts ergodically as measure preserving transformations. Let $V(\omega, t, z_1, z_2)$ be a function $\Omega \times [0,\infty) \times Z^d \times Z^d \to [0,\infty)$, with the properties

1. $|V(\omega, t, z_1, z_2) - V(\omega, t'.z\prime_1, z_2')| \le C[|t-t'| + |z_1 - z_1'| + |z_2 - z_2'|]$

2. $V(\omega, t+s, z_1, z_3) \le V(\omega, t, z_1, z_2) + V(\omega, s, z_2, z_3)$ for all $z_1, z_2, z_3 \in Z^d$ and $t, s \ge 0$.

3. $V(\omega, t, z+z_1, z+z_2) = V(\tau_z\omega, t, z_1, z_2)$ for all ω, t, z_1 and z_2.

Then there exists a deterministic convex function $\bar{V}(\xi)$ on R^d such that for almost all ω

$$\lim_{\substack{n\to\infty \\ \frac{y_n}{n} \to \eta}} \frac{1}{n} V(\omega, nt, 0, y_n) = t\, v\Big(\frac{\eta}{t}\Big) \tag{1}$$

Proof. The proof can be found in a paper of Rezakhanlou although the result is not quite stated in the form we need. We will give a sketch of the proof. The random functions

$$V^{(n)}(t, \xi, \eta) = \frac{1}{n} V(\omega, nt, n\xi, n\eta)$$

are uniformly Lipschitz in t, ξ and η and are defined on $[0,\infty) \times \frac{1}{n}Z^d \times \frac{1}{n}Z^d$. They have subsequences that converge in distribution to an almost surely continuous stochastic process $V^\infty(t, \xi, \eta)$ defined on $[0,\infty) \times R^d \times R^d$, that is stationary under the transformations $\{\tau_\zeta : \zeta \in R^d\}$, $\xi \to \xi + \zeta, \eta \to \eta + \zeta$ and satisfies the subadditivity property

$$V^\infty(t+s, \xi, \zeta) \le V^\infty(t, \xi, \eta) + V^\infty(s, \eta, \zeta)$$

with probability 1.

If $\xi \in Z^d$ then

$$\lim_{n\to\infty} V^{(n)}(t, 0, \xi) = \widehat{v}(t, \xi) \tag{2}$$

exists almost surely and is a deterministic function of t and ξ by the standard subadditive ergodic theorem and $\widehat{v}(kt, k\xi) = k\widehat{v}(t, \xi)$ for all $k \in Z^+$, $t \in R$ and $\xi \in Z^d$. One can therefore extend $\widehat{v}(t, \xi)$ for ξ with rational coordinates by the relation $\widehat{v}(t, \xi) = \frac{1}{k}\widehat{v}(kt, k\xi)$ and $\widehat{v}$ will satisfy $\widehat{v}(rt, r\xi) = r\,\widehat{v}(t, \xi)$ for all positive rational r and $\xi \in R^d$ with rational coordinates. Since $\widehat{v}$ is Lipschitz in t and ξ it has a natural extension as a function of the form $t\, v(\frac{\xi}{t})$.

The next step is to deduce (1) from (2). Let $\xi \in R^d$ have rational coordinates. Then $q\,\xi \in Z^d$ for some integer q. Clearly

$$\lim_{n\to\infty} \frac{1}{n} V(\omega, nqt, 0, nq\xi) = \widehat{v}(qt, q\xi) = q\,\widehat{v}(t,\xi)$$

By writing $n = n'q + r$, with a remainder r, $0 \le r \le q-1$, and using the Lipschitz property we can control the error and

$$\frac{1}{n}|V(\omega, nt, 0, y_n) - V(\omega, n'qt, 0, n'q\,\xi)| \le \frac{C}{n}[|n - n'q|t + |y_n - n'q\,\xi|] \to 0$$

Therefore

$$\frac{1}{n} V(\omega, nt, 0, y_n) \to \widehat{v}(t,\xi) = t\,v(\frac{\xi}{t})$$

It is routine now to approximate arbitrary $\xi \in R^d$ by rational ξ and conclude that (1) holds for all $\xi \in R^d$. In particular $V^\infty(t, 0, \xi)$ is almost surely equal to $t\,v(\frac{\xi}{t})$. From stationarity

$$V^\infty(t, \xi, \eta) = t\,v(\frac{\eta - \xi}{t})$$

almost surely. The subadditivity of V^∞ tells us that

$$(t+s)\,v(\frac{\eta}{t+s}) \le t\,v(\frac{\xi}{t}) + s\,v(\frac{\eta-\xi}{s})$$

which is exactly the convexity of v.

In order to prove the large deviation principle we apply Theorem 1 to g_c to get a limit h_c and then let $c \to \infty$. The functions h_c are nondecreasing and the limit h, which is also convex, is easily seen to be the rate function.

Theorem 2. There exists a nonrandom convex rate function $h(\xi)$ such that, for almost all ω with respect to P,

$$\limsup_{n\to\infty} \frac{1}{n} \log P^\omega\big[\frac{x_n}{n} \in C\big] \le -\inf_{\xi \in C} h(\xi) \quad \text{for closed} \quad C \subset R^d$$
$$\liminf_{n\to\infty} \frac{1}{n} \log P^\omega\big[\frac{x_n}{n} \in G\big] \ge -\inf_{\xi \in G} h(\xi) \quad \text{for open} \quad G \subset R^d$$

Proof. For every c,

$$\pi^{(n)}(\omega, 0, y_n) \le q_c(\omega, n, 0, y_n)$$

Since the number of lattice points such that $|\frac{y_n}{n} - \xi| \le \epsilon$ is of order n^d, the upper bound holds with any c and so we can let $c \to \infty$. For the lower bound if $h(\xi) = \infty$ there is nothing to prove. Assume $h(\xi) = \ell < \infty$. Let $\epsilon > 0$ be given. Choose c is large enough such that $(\ell+1)C < c\,\epsilon$, where C is un upper bound for the size of the jumps. Clearly $h_c(\xi) \le \ell$. Then, if $y_n = n\xi + o(n)$, for almost all ω,

$$\sup_{k\ge 0}\left[\pi^{(k)}(\omega, x, y)e^{-c|k-n|}\right] \ge e^{-nh(\xi)+o(n)} \ge e^{-n\ell+o(n)}$$

For sufficiently large n, the supremum is clearly attained in the set $\{k : |k-n| \le \frac{(\ell+1)n}{c}\}$. Hence

$$\sup_{k\,:\,c\,|n-n|\le(\ell+1)n} \pi^{(k)}(\omega,0,y_n) \ge e^{-n\ell+o(n)}$$

Since the chain moves with a maximum speed of C, if it is at y_n at some time within $\frac{\ell+1}{c}n$ of n, then at time n, it is within $\frac{\ell+1}{c}nC$ or ϵn of y_n and we are done. While we cannot calculate $h(\cdot)$, we will be able, later on, to say some thing about where $h(\cdot)$ is 0.

Although this is a quick proof of the Large Deviation Principle for the quenched case it says very little about the rate function. We shall use another approach to the problem, and will illustrate it in the context of diffusions in a random medium, more specifically Brownian motion with a random drift.

We can have R^d acting on (Ω, Σ, P) ergodically and consider a diffusion on R^d with a random infinitesimal generator

$$(\mathcal{L}^\omega u)(x) = \frac{1}{2}(\Delta u)(x) + < b(\omega, x), (\nabla u)(x) >$$

acting on smooth function on R^d. Here $b(\omega, x)$ is generated from a map $b(\omega) : \Omega \to R^d$ by the action $\{\tau_x\}$ of R^d

$$b(\omega, x) = b(\tau_x \omega)$$

Again there is the quenched measure Q^ω that corresponds to the diffusion with generator $\mathcal{L}^\omega$ that starts from 0 at time 0. This model is referred to as diffusion with a random drift. We can also define a diffusion on Ω with generator

$$\mathcal{L} = \frac{1}{2}\Delta + < b(\omega), \nabla >$$

where $\nabla = \{D_i\}$ are the generators of the translation group $\{\tau_x : x \in R^d\}$. This is essentially the image of lifting the paths $x(t)$ of the diffusion on R^d corresponding to $\mathcal{L}^\omega$ to Ω by

$$\omega(t) = \tau_{x(t)}\omega$$

While there is no possibility of having an invariant measure on R^d, on Ω one can hope to find an invariant density $\phi(\omega)$ provided we can find $\phi(\omega) \ge 0$ in $L_1(P)$, that solves

$$\frac{1}{2}\Delta\phi = \nabla \cdot (b\phi)$$

If such a ϕ exists, then we have an ergodic theorem for the diffusion process Q^ω corresponding $\mathcal{L}$ on Ω

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t f(\omega(s))ds = \int f(\omega)\phi(\omega)dP \qquad \text{a.e. } Q^\omega \quad \text{a.e. } P$$

This translates to an ergodic theorem on R^d as well

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t f(\omega, x(s))ds = \int f(\omega)\phi(\omega)dP \qquad \text{a.e. } Q^\omega \quad \text{a.e. } P$$

where now Q^ω is the quenched process in the random environment. Since

$$x(t) = \int_0^t b(\omega, x(s)ds + \beta(t)$$

it is clear that

$$\lim_{t\to\infty} \frac{x(t)}{t} = \int b(\omega)\phi(\omega)dP \qquad \text{a.e. } Q^\omega \quad \text{a.e. } P$$

providing a law of large numbers for $x(t)$. While we can not be sure of finding ϕ for given b it is easy to find a b for given ϕ. For instance we could take $b = \frac{\nabla\phi}{2\phi}$. Or more generally $b = \frac{\nabla\phi}{2\phi} + \frac{c}{\phi}$ with $\nabla \cdot c = 0$. If we change b to $b' = \frac{\nabla\phi}{2\phi} + c$ with $\nabla \cdot c = 0$, the new process will have relative entropy

$$E^{Q^{b',\omega}}\left[\frac{1}{2}\int_0^t \|b(\omega(s)) - \frac{\nabla\phi(\omega(s))}{2\phi(\omega(s))} - \frac{c(\omega(s))}{\phi(\omega(s))}\|^2 ds\right]$$

More over, for almost all ω with respect to P, almost surely with respect to $Q^{b',\omega}$

$$\lim_{t\to\infty} \frac{x(t)}{t} = \int [\frac{\nabla\phi}{2\phi} + \frac{c}{\phi}]\phi dp = \int cdP$$

If we fix $\int cdP = a$, the bound

$$\liminf_{t\to\infty} \frac{1}{t} \log Q^\omega[\frac{x(t)}{t} \simeq a] \geq -\frac{1}{2}\int \|b - \frac{\nabla\phi}{2\phi} - \frac{c}{\phi}\|^2 \phi dP$$

is easily obtained. If we define

$$I(a) = \inf_{\substack{\nabla\cdot c=0 \\ \int cdP=a}} \frac{1}{2}\int \|b - \frac{\nabla\phi}{2\phi} - \frac{c}{\phi}\|^2 \phi dP$$

then

$$\liminf_{t\to\infty} \frac{1}{t} \log Q^\omega[\frac{x(t)}{t} \simeq a] \geq -I(a)$$

Of course these statements are valid a.e Q^ω a.e P. One can check that I is convex and the upper bound amounts to proving the dual estimate

$$\lim_{t\to\infty} \frac{1}{t} \log E^{Q^\omega}[e^{<\theta, x(t)>}] \leq \Psi(\theta)$$

where

$$\psi(\theta) = \sup_a [< a, \theta > -I(a)]$$

We need a bound on the solution of

$$u_t = \frac{1}{2}\Delta u + < b, \nabla u >$$

with $u(0) = \exp[<\theta, x>]$. By Hopf-Cole transformation $v = \log u$ this reduces to estimating

$$v_t = \frac{1}{2}\Delta v + \frac{1}{2}\|\nabla v\|^2 + <b, \nabla v>$$

with $v(0) = <\theta, x>$. This can be done if we can construct a subsolution

$$\frac{1}{2}\nabla \cdot w + \frac{1}{2}\|\nabla w\|^2 + <b, w> \leq \psi(\theta)$$

on Ω, where $w : \Omega \to R^d$ satisfies $\int w dP = \theta$ and w is closed in the sense that $D_i w_j = D_j w_i$. The existence of the subsolution comes from convex analysis.

$$\begin{aligned}
\psi(\theta) &= \sup_{\substack{\phi \\ \nabla\cdot c = 0}} \Big[\int <c, \theta> dP - \frac{1}{2}\int \|b - \frac{\nabla\phi}{2\phi} - \frac{c}{\phi}\|^2 \phi dP \Big] \\
&= \sup_{\phi}\sup_{c}\inf_{u} \Big[\int <c, \theta + \nabla u> dP - \frac{1}{2}\int \|b - \frac{\nabla\phi}{2\phi} - \frac{c}{\phi}\|^2 \phi dP \Big] \\
&= \sup_{\phi}\inf_{u}\sup_{c} \Big[\int <c, \theta + \nabla u> dP - \frac{1}{2}\int \|b - \frac{\nabla\phi}{2\phi} - \frac{c}{\phi}\|^2 \phi dP \Big] \\
&= \sup_{\phi}\inf_{u}\int \sup_{c} \Big[<c, \theta + \nabla u> - \frac{1}{2}\|b - \frac{\nabla\phi}{2\phi} - \frac{c}{\phi}\|^2 \phi \Big] dP \\
&= \sup_{\phi}\inf_{u}\int [<b - \frac{\nabla\phi}{2\phi}, \theta + \nabla u> + \frac{1}{2}\|\theta + \nabla u\|^2]\phi dP \\
&= \sup_{\phi}\inf_{u}\int \Big[[<b, \theta + \nabla u> + \frac{1}{2}\|\theta + \nabla u\|^2]\phi - \frac{1}{2} <\nabla u, \nabla\phi> \Big] dP \\
&= \sup_{\phi}\inf_{u}\int [\frac{1}{2}\Delta u + <b, \theta + \nabla u> + \frac{1}{2}\|\theta + \nabla u\|^2]\phi dP \\
&= \sup_{\phi} \inf_{\substack{w \text{ closed} \\ \int w dP = \theta}} \int [\frac{1}{2}\nabla\cdot w + <b, w> + \frac{1}{2}\|w\|^2]\phi dP \\
&= \inf_{\substack{w \text{ closed} \\ \int w dP = \theta}} \sup_{\phi} \int [\frac{1}{2}\nabla\cdot w + <b, w> + \frac{1}{2}\|w\|^2]\phi dP \\
&= \inf_{\substack{w \text{ closed} \\ \int w dP = \theta}} \sup_{\omega} [\frac{1}{2}\nabla\cdot w + <b, w> + \frac{1}{2}\|w\|^2]
\end{aligned}$$

which proves the existence of the subsolution.

This can be viewed as showing the existence of a limit as $\epsilon \to 0$ (homogenization) of the solution of

$$u^\epsilon_t = \frac{\epsilon}{2}\Delta u^\epsilon + \frac{1}{2}\|\nabla u^\epsilon\|^2 + <b(\frac{x}{\epsilon}, \omega), \nabla u^\epsilon>$$

with $u^\epsilon(0,x) = f(x)$. The limit satisfies

$$u_t = \Psi(\nabla u)$$

with $u(0,x) = f(x)$.

This can be generalized to equations of the form

$$u^\epsilon_t = \frac{\epsilon}{2}\Delta u^\epsilon + H(\frac{x}{\epsilon}, \nabla u^\epsilon, \omega)$$