We consider a probability space $(\Omega, \mathcal{F}, P)$ on which R^d acts as a group $\{\tau_x : x \in R^d\}$ of measure preserving transformations. P is assumed to be ergodic under this action. Let the function $H(p,\omega) : R^d \times \Omega \to R$ be convex in p for each ω. L is the conjugate function

$$L(y,\omega) = \sup_p [< p, y > -H(p,\omega)]$$

We will assume some growth and regularity conditions on H or equivalently on L.

For any given $\epsilon > 0$ and $\omega \in \Omega$, we consider the solution $u_\epsilon = u_\epsilon(t,x,\omega)$ of equation

$$\frac{\partial u_\epsilon}{\partial t} = \frac{\epsilon}{2}\Delta u_\epsilon + H(\nabla u_\epsilon, \tau_{\frac{x}{\epsilon}}\omega), \quad (t,x) \in [0,\infty) \times R^d, \tag{1}$$

with the initial condition $u_\epsilon(0,x) = f(x)$.

We wish to show that as $\epsilon \to 0$ the solutions u_ϵ of 1 converge to the solution of an effective equation

$$u_t = \overline{H}(\nabla u) \tag{2}$$

with the same initial condition $u_\epsilon(0,x) = f(x)$

We note that the solution $u_\epsilon(t,x,\omega)$ of (1) is equal to $\epsilon v_\epsilon(\frac{t}{\epsilon}, \frac{x}{\epsilon}, \omega)$, the rescaled version of v_ϵ that solves

$$\frac{\partial v_\epsilon}{\partial t} = \frac{1}{2}\Delta v_\epsilon + H(\nabla v_\epsilon, \tau_x\omega), \quad (t,x) \in [0,\infty) \times R^d,$$

with $v_\epsilon(0,x) = \epsilon^{-1} f(\epsilon x)$.

We now construct the convex function $\overline{H}(p)$ that appears in (2). The translation group $\{\tau_x : x \in R^d\}$ acting on $L^2(\Omega, \mathcal{F}, P)$ will have infinitesimal generators $\{\nabla_i : 1 \leq i \leq d\}$ in the coordinate directions and the corresponding Laplace operator $\Delta = \sum_i \nabla_i^2$. For reasonable choices of $b(\omega) : \Omega \to R^d$, the operator

$$\mathcal{A}_b = \frac{1}{2}\Delta + < b(\omega), \nabla >$$

will define a Markov process on Ω. Construction of this Markov Process is not difficult. Given a starting point $\omega \in \Omega$, we define $b(x,\omega) : R^d \to R^d$ by $b(x,\omega) = b(\tau_x\omega)$. This allows us to define the diffusion $Q^{b,\omega}_{0,0}$, starting from 0 at time 0, in the random environment that corresponds to the generator

$$\frac{1}{2}\Delta + < b(x,\omega), \nabla >$$

The diffusion is then lifted to Ω by evolving ω randomly in time by the rule $\omega(t) = \tau_{x(t)}\omega$. The induced measure $P^{b,\omega}$ defines the Markov process on Ω that corresponds to $\mathcal{A}_b$. The problem of finding the invariant measures for the process $P^{b,\omega}$ with generator $\mathcal{A}_b$ on Ω is very hard and nearly impossible to solve. However, if we can find a density $\phi > 0$ such that $\phi \, dP$ is an invariant ergodic probability measure for $\mathcal{A}_b$, then one has by the ergodic theorem,

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t F(\omega(s))ds = \int_\Omega F(\omega)\, \phi(\omega)\, dP$$

a.e $P^{b,\omega}$ or in $L^1(P^{b,\omega})$ for almost all ω with respect to P. Let us denote by $\mathcal{B}$ the space of essentially bounded maps from $\Omega \to R^d$ and by $\mathcal{D}$ the space of probability densities $\phi : \Omega \to R$ relative to P, with $\phi, \nabla\phi, \nabla^2\phi$ essentially bounded and ϕ in addition having a positive essential lower bound. Let us denote by $\mathcal{E}$ the following subset of $\mathcal{B} \times \mathcal{D}$

$$\mathcal{E} = \left\{ (b, \phi) : \frac{1}{2}\Delta\phi = \nabla \cdot (b\, \phi) \right\}.$$

Here we assume that the equation $\frac{1}{2}\Delta\phi = \nabla \cdot (b\, \phi)$ is satisfied in the weak sense. We define the convex function $\overline{H}$ on R^d by

$$\overline{H}(p) = \sup_{(b,\phi)\in\mathcal{E}} [< p, E^P[b(\omega)\, \phi(\omega)] > -E^P[L(b(\omega), \omega)\phi(\omega)]] \tag{3}$$

The corresponding variational solution of (2) is given by

$$u(t, x) = \sup_y [f(y) - t\mathcal{I}(\frac{y - x}{t})],$$

where $\mathcal{I}$ is related to $\overline{H}$ by the duality relation

$$I(x) = \sup_p [< p, x > -\overline{H}(p)]$$

We will show that

$$\lim_{\epsilon\to 0} u_\epsilon(t, x) = u(t, x)$$

The first step in establishing the lower bound is the variational representation of solutions of Hamilton-Jacobi-Bellman equations. Let $\mathcal{C}$ be the set of all bounded maps $c(s, x)$ from $[0, T] \times R^d$ to R^d such that $\sup_{s,x} \|c(s, x)\| < \infty$. Consider the diffusion $Q^c_{0,x}$ on R^d starting from $x \in R^d$ at time 0 with time dependent generator

$$\frac{1}{2}\Delta + c(s, x) \cdot \nabla$$

in the time interval $[0, t]$. For each $c \in \mathcal{C}$ and $\omega \in \Omega$ we consider

$$v_c(t, x, \omega) = E^{Q^c_{0,x}} \left(f(x(t)) - \int_0^t L(c(s, x(s)), \tau_{x(s)}\omega)ds \right),$$

If

$$v(t, x, \omega) = \sup_{c\in\mathcal{C}} v_c(t, x, \omega)$$

then v is the solution of
$$\frac{\partial v}{\partial t} = \frac{1}{2}\Delta v + H(\nabla v, \tau_x \omega)$$
with $v(0,x) = f(x)$.

There is a simple relation between $v(t,y,\cdot)$ and $v(t,0,\cdot)$. If we define $f^y(x) = f(x+y)$, then the solution of (1) with initial data $v(0,x) = f^y(x)$ and $\omega' = \tau_y \omega$ is given by

$$v^y(t,x,\omega') = v^y(t,x,\tau_y\omega) = v(t,x+y,\omega).$$

In particular,

$$v(t,y,\omega) = v^y(t,0,\tau_y\omega). \tag{3}$$

The solution u_ϵ of (1) with initial data $f(x)$ is related to the solution v_ϵ of (2) with initial data $\epsilon^{-1}f(\epsilon x)$ by
$$u_\epsilon(t,x,\omega) = \epsilon v_\epsilon\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}, \omega\right).$$

We, therefore, obtain the following variational expression for $u_\epsilon(t,x)$.

$$\begin{aligned} u_\epsilon(t,x,\omega) &= \sup_{c\in\mathcal{C}} E^{Q^c_{0,x/\epsilon}}\left(f\left(\epsilon x\left(t/\epsilon\right)\right) - \epsilon\int_0^{t/\epsilon} L(c(s,x(s)),\tau_{x(s)}\omega)ds\right) \\ &= \sup_{c\in\mathcal{C}} E^{Q^{c,\epsilon}_{0,x}}\left[f(x(t)) - \xi_\epsilon(t)\right] \end{aligned}$$

where $Q^{c,\epsilon}_{0,x}$ is the diffusion on R^d starting from x corresponding to the generator

$$\frac{\epsilon}{2}\Delta + c(s,x)\cdot\nabla$$

i.e. almost surely with respect to $Q^{c,\epsilon}_{0,x}$, $x(t)$ satisfies

$$x(t) = x + \int_0^t c(s,x(s))ds + \sqrt{\epsilon}\beta(t)$$

and
$$\xi_\epsilon(t) = \int_0^t L(c(s,x(s)),\tau_{\epsilon^{-1}x(s)}\omega)ds$$

Since the supremum over $c \in \mathcal{C}$ is taken for each ω one can choose c to depend on ω. A special choice for $c(t,x)$, one that depends on $\omega \in \Omega$ but not on t, is the choice $c(t,x) = c(t,x,\omega) = c(x,\omega) = b(\tau_x\omega)$ with $(b,\phi) \in \mathcal{E}$. With that choice we can consider either the process $\{Q^{b,\omega}_{0,x}\}$ on R^d or the process $\{P^{b,\omega}\}$ with values in Ω. It is easy to see that for any $y \in R^d$, the translation map $\hat{\tau}_y$ on $C([0,T];R^d)$ defined by $x(\cdot) \to x(\cdot) + y$ has the property
$$Q^{b,\omega}_{0,y} = Q^{b,\tau_y\omega}_{0,0}\hat{\tau}_y^{-1},$$

which is essentially a restatement of (3). Since $(b,\phi)\in\mathcal{E}$, by the ergodic theorem we have

$$\lim_{\epsilon\to 0}\epsilon\int_0^{\frac{t}{\epsilon}} b(\omega(s))ds = t\int b(\omega)\phi(\omega)dP = t\,m(b,\phi)$$

and

$$\lim_{\epsilon\to 0}\epsilon\int_0^{\frac{t}{\epsilon}} L(b(\omega(s)),\omega(s))ds = t\int L(b(\omega),\omega)\phi(\omega)dP = t\,h(b,\phi)$$

Both limits are valid in $L^1(P^{b,\omega})$ for P almost all ω. If we define $\mathbf{A}\subset R^d\times R$ as

$$\mathbf{A} = \{(m(b,\phi),h(b,\phi)) : (b,\phi)\in\mathcal{E}\}$$

then

$$\liminf_{\epsilon\to 0} u_\epsilon(t,0,\omega) \geq [f(t\,m) - t\,h]$$

for every $(m,h)\in\mathbf{A}$. Therefore, for almost all ω with respect to P

$$\begin{aligned}\liminf_{\epsilon\to 0} u_\epsilon(t,0,\omega) &\geq \sup_{(m,h)\in\mathbf{A}}[f(t\,m) - t\,h]\\ &= \sup_{y\in R^d}(f(y) - t\mathcal{I}(\frac{y}{t}))\\ &= u(t,0)\end{aligned}$$

This is a very weak form of convergence and work has to be done in order to strengthen it to locally uniform convergence.

The upper bound is first obtained for linear f and then extended to general f. By using the convex duality and the minimax theorem the right hand side of (3) is rewritten in terms of the dual problem. If we take $f(x) =< p,x >$, we have established an asymptotic lower bound for u_ϵ, which is the solution

$$u(t,x) =< p,x > +t\overline{H}(p)$$

of (2) with $u(0,x) =< p,x >$. Here

$$\begin{aligned}\overline{H}(p) &= \sup_{(b,\phi)\in\mathcal{E}} E^P[[< p, b(\omega) > -L(b(\omega),\omega)]\phi(\omega)]\\ &= \sup_\phi\sup_b\inf_\psi E^P[[< p, b(\omega) > -L(b(\omega),\omega) + \mathcal{A}_b\psi]\phi(\omega)]\\ &= \sup_\phi\inf_\psi\sup_b E^P[[< p, b(\omega) > -L(b(\omega),\omega) + \mathcal{A}_b\psi]\phi(\omega)]\\ &= \sup_\phi\inf_\psi\sup_b E^P[[< p+\nabla\psi, b(\omega) > -L(b(\omega),\omega) + \frac{1}{2}\Delta\psi]\phi(\omega)]\\ &= \sup_\phi\inf_\psi[H(p+(\nabla\psi)(\omega),\omega) + \frac{1}{2}\Delta\psi]\\ &= \inf_\psi\sup_\phi[H(p+(\nabla\psi)(\omega),\omega) + \frac{1}{2}\Delta\psi]\\ &= \inf_{\psi(\cdot)} \mathit{ess\ sup}_\omega[H(p+(\nabla\psi)(\omega),\omega) + \frac{1}{2}(\Delta\psi)(\omega)].\end{aligned}$$

We have used the fact that

$$\inf_{\psi} E^P[\mathcal{A}_b\psi\,\phi] = -\infty$$

unless $\phi\, dP$ is an invariant measure for $\mathcal{A}_b$, in which case it is 0. It follows that for any $\delta > 0$, there exists a "ψ" such that

$$\frac{1}{2}(\Delta\psi)(\omega) + H(\theta + (\nabla\psi)(\omega), \omega) \leq \overline{H}(\theta) + \delta.$$

The "ψ" is a weak object and one has to do some work before we can use it as a test function and obtain the upper bound by comparison. The interchange of inf and sup that we have done freely needs justification.

We start with the formula

$$\overline{H}(p) = \sup_{(b,\phi)\in\mathcal{E}} E^P[[< p, b(\omega) > -L(b(\omega),\omega)]\phi(\omega)]$$

If $L(b,\omega)$ grows faster than linear in b, say like a power $|b|^\alpha$ (uniformly in ω), then $\overline{H}(p)$ is finite and grows at most like the conjugate power $|p|^{\alpha'}$. Since $\mathcal{E}$ is an inconvenient set to work with, we rewrite this as

$$\overline{H}(p) = \sup_{\phi}\sup_{b}\inf_{\psi} E^P[[< p, b(\omega) > -L(b(\omega),\omega) + \mathcal{A}_b\psi]\phi(\omega)]$$

The inf over ψ is $-\infty$ unless $(b,\phi) \in \mathcal{E}$ in which case it is 0. We limit the sup over b to a bounded set $\mathcal{B}_k = \{b : \|b\|_\infty \leq k\}$ and ϕ to a set $\mathcal{D}_r$ with $\frac{1}{r} \leq \phi \leq r$ and $|\nabla\phi| \leq r^2$. We would then have

$$\overline{H}(p) \geq \sup_{\phi\in\mathcal{D}_r}\sup_{b\in\mathcal{B}_k}\inf_{\psi} E^P[[< p, b(\omega) > -L(b(\omega),\omega) + \mathcal{A}_b\psi]\phi(\omega)]$$

We now are in a position to interchange the sup and inf in order to rewrite

$$\overline{H}(p) \geq \sup_{\phi\in\mathcal{D}_r}\inf_{\psi}\sup_{b\in\mathcal{B}_k} E^P[[< p, b(\omega) > -L(b(\omega),\omega) + \mathcal{A}_b\psi]\phi(\omega)]$$

We can carry out the sup over b for each ω to obtain

$$\overline{H}(p) \geq \sup_{\phi\in\mathcal{D}_r}\inf_{\psi} E^P[[\frac{1}{2}(\Delta\psi)(\omega) + H_k((p + \nabla\psi)(\omega),\omega)]\phi(\omega)]$$

where

$$H_k(p,\omega) = \sup_{b:|b|\leq k}[< p, b > -L(b,\omega)]$$

After integration by parts

$$\overline{H}(p) \geq \sup_{\phi\in\mathcal{D}_r}\inf_{\psi} E^P[[-\frac{1}{2} < (\nabla\psi)(\omega), \frac{\nabla\phi}{\phi}(\omega) > + H_k((p + \nabla\psi)(\omega),\omega)]\phi(\omega)]$$

We can again interchange the sup and inf to get

$$\overline{H}(p) \geq \inf_{\psi} \sup_{\phi \in \mathcal{D}_r} E^P[[-\frac{1}{2} < (\nabla\psi)(\omega), \frac{\nabla\phi}{\phi}(\omega) > +H_k((p + \nabla\psi)(\omega), \omega)]\phi(\omega)]$$

In other words we have $\psi_{k,r}$ such that for all $\phi \in \mathcal{D}_r$

$$E^P[[-\frac{1}{2} < (\nabla\psi_k)(\omega), \frac{\nabla\phi}{\phi}(\omega) > +H_k((p + \nabla\psi_k)(\omega), \omega)]\phi(\omega)] \leq \overline{H}(p) + \delta$$

While H_k will only grow linearly but the rate grows with k and it is not hard to see that because $H_k \uparrow H$, $\nabla\psi_{k,r}$ is an uniformly integrable sequence in k. We can take a weak limit to get W_r that satisfies $E^P[W_r] = 0$, $\nabla \times W_r = 0$, and for all $\phi \in \mathcal{D}_r$

$$E^P[-\frac{1}{2} < W_r(\omega), \nabla\phi(\omega) > +H((p + W_r)(\omega), \omega)\phi(\omega)] \leq \overline{H}(p)$$

It is easy to see that, under suitable growth conditions on H, W_r is bounded in $L_\beta(P)$ for some $\beta > 1$, and if W is a weak limit, we have $W \in L_\beta(P)$, $E^P W = 0$, $\nabla \times W = 0$ and

$$E^P[-\frac{1}{2} < W(\omega), \nabla\phi(\omega) > +H((p + W)(\omega), \omega)\phi(\omega)] \leq \overline{H}(p)$$

for all $\phi \in \cup_r \mathcal{D}_r$, and therefore

$$ess\ sup_\omega[H(p + W(\omega), \omega) + \frac{1}{2}(\nabla \cdot W)(\omega)] \leq \overline{H}(p)$$

If we define $W(x, \omega) = W(\tau_x\omega)$, then we can integrate it on R^d to get $U(x, \omega)$ with $U(0, \omega) = 0$ and $\nabla U = W$ on R^d. Then for almost all ω

$$H(p + (\nabla U)(x, \omega)) + \frac{1}{2}(\Delta U)(x, \omega) \leq \overline{H}(p)$$

on R^d as a distribution. Morally, by the ergodic theorem U will be sublinear and and $< p, x > +U(x, \omega)$ can be used as test function to bound u_ϵ with $f(x) =< p, x >$ with the help of the maximum principle. To actually do it one may have to mollify U and that can be done provided H is regular enough to transfer the convolution inside.