We now turn our attention to the case where $\tau_{t,x} : (t,x) \in R \times R^d$ acts on (Ω, Σ, P). We still have the same problem of homogenizing, i.e. determining the limit as $\epsilon \to 0$ of

$$u_t^\epsilon + \frac{\epsilon}{2}\Delta + H(\nabla u^\epsilon, \tau_{\frac{t}{\epsilon},\frac{x}{\epsilon}}\omega) = 0$$

with $u^\epsilon(T,x) = f(x)$ The formal method is still the same as the time homogeneous case. Instead of looking at $\mathcal{E}$ consisting of $(b(\omega), \phi(\omega))$ satisfying $\frac{1}{2}\Delta\phi = \nabla(b\phi)$ we now look at pairs (b,ϕ) that satisfy $\frac{1}{2}\Delta\phi = D_t\phi + \nabla(b\phi)$. We denote as before the space derivatives by ∇ and the time derivative on Ω is denoted by D_t. Denoting by $L(b,\omega)$ the conjugate of $H(p,\omega)$, we have the same formula

$$\overline{H}(p) = \sup_{(b,\phi)\in\mathcal{E}} E^P[[< p, b(\omega) > -L(b(\omega),\omega)]\phi(\omega)]dP$$

The lower bound offers no new difficulties. But interchanging sup and inf becomes a lot harder, mainly due to the lack of control on $D_t\psi$. In addition to $\mathcal{D}_r$ will consider $\mathcal{C}_r$ defined by $\phi : \frac{1}{r} \le \phi \le r$. If we convolve $\phi \in \mathcal{C}_r$ by a mollifier ρ then $\phi * \rho \in \mathcal{D}_r$ for large r. It is easy to construct by the same method as before a sequence ψ_r such that

$$E^P[(D_t\psi_r + H_r(p + \nabla\psi_r, \omega))\phi - \frac{1}{2} < \nabla\psi_r, \nabla\phi >] \le \overline{H}(p) + \frac{1}{r}$$

for all $\phi \in \mathcal{D}_r$. Our goal is to construct a subsolution

$$D_t\psi + \frac{1}{2}\Delta\psi + H(p + \nabla\psi, \omega) \le \overline{H}(p)$$

Taking expectations we can control $E^P[H(p+\nabla\psi.\omega)]$ and hence $E^P[|\nabla\psi|^\beta]$ for some $\beta > 1$ with growth conditions on H. If we mollify ψ, in space, we can control $E^P[[D_t(\psi * \rho)]^+]$ and since $E^P[D_t(\psi * \rho)] = 0$ we can at best control $E^P[|D_t(\psi * \rho)|]$. We can at best produce, for each mollifier ρ functions f_ρ, g_ρ, potential D_t and ∇ of some functions that may not exist such that

$$f_\rho(\omega) + \frac{1}{2}\nabla \cdot g_\rho(\omega) + \int H(p + g(\tau_{0,x}\omega), \tau_{0,x}\omega)\rho(x)dx \le \overline{H}(p)$$

g will be in $L_\beta(P)$ but f_ρ will be only in $L_1(P)$ and as $\rho \to \delta_0$ we will lose control of $\int |f_\rho|dP$. But this is enough to provide a comparison and derive the upper bound. Even after convoluting with ρ the existence of the limiting object is quite technical. The main steps are the following.

1. We start with

$$E^P[(D_t\psi_r + H_r(p + \nabla\psi_r, \omega))\phi - \frac{1}{2} < \nabla\psi_r, \nabla\phi >] \le \overline{H}(p) + \frac{1}{r}$$

valid for $\phi \in \mathcal{D}_r$. Taking $\phi = 1$ we get the bound

$$E^P[H_r(p + \nabla\psi_r, \omega))] \le \overline{H}(p) + \frac{1}{r}$$

This will show that $\nabla\psi_r$ is bounded in $L_1(P)$ and uniformly integrable. If g is a weak limit $E^P[g] = 0$, $\nabla \times g = 0$ and

$$E^P[H_r(p + g(\omega), \omega))] \leq \overline{H}(p)$$

and by monotone convergence theorem

$$E^P[H(p + g(\omega), \omega))] \leq \overline{H}(p)$$

and consequently $g \in L_\beta(P)$ for some $\beta > 1$.

2. Denoting $D_t\psi_r$ and $\nabla\psi_r$ by f_r and g_r we have for $\phi \in \mathcal{D}_r$,

$$E^P[(f_r + H_r(p + g_r, \omega))\phi - \frac{1}{2} < g_r, \nabla\phi >] \leq \overline{H}(p) + \frac{1}{r}$$

For $f_n * \rho = f_n^\rho$ we can now get the estimates

$$E^P[f_n^\rho \phi] \leq C(\rho)$$

and

$$E^P[f_n^\rho : f_n^\rho > \ell] \leq \frac{C(\rho)}{\ell}$$

for $\phi \in \cup_r \mathcal{C}_r$. This implies a bound

$$E^P[|f_n^\rho|] \leq C(\rho)$$

as well as the uniform integrability of $(f_n^\rho)^+$ in $L_1(P)$.

3. If we denote by f_ρ a weak limit in L_1 with perhaps a defect in the negative part, we still have

$$E^P[(f_\rho + \int H(p + g(\tau_{0,x}\omega), \tau_{0,x}\omega))\rho(x)dx) + \frac{1}{2}\nabla \cdot g_\rho)\phi] \leq \overline{H}(p)$$

4. One checks that f_ρ are consistent. That is $f_\rho * \rho' = f_{\rho * \rho'}$

5. This is sufficient to construct a parabolic supsolution, that can be used to obtain an upper bound.

Further problems. What is the Martin Boundary of a Random Walk in a random environment or diffusion with a random drift. Are there bounded harmonic functions? You can consider them on Ω or R^d. It is clear that any solution W of

$$\frac{1}{2}\nabla \cdot W + \frac{1}{2}\|W\|^2 + < b(\omega), W >= 0$$

with $\nabla \times W = 0$ will give rise to Harmonic functions on R^d. Are there others?