Assignment 3 (revised).

Due Oct 2.

We have a coin that has probability p for coming up head and $q = 1 - p$ for coming up tail when tossed. It is tossed n times and the number of times X that head appeared is counted and $t = \frac{X}{n}$ is offered as an unbiased "estimate" of p, in the sense that $E[t] \equiv p$ or

$$\sum_{r=0}^{n} \frac{r}{n} \binom{n}{r} p^r (1-p)^{n-r} \equiv p$$

Its variance is

$$\sum_{r=0}^{n} (\frac{r}{n} - p)^2 \binom{n}{r} p^r (1-p)^{n-r} = \frac{p(1-p)}{n}$$

Can we do better? Is there some other function $f(r)$ such that

$$\sum_{r=0}^{n} f(r) \binom{n}{r} p^r (1-p)^{n-r} \equiv p$$

but

$$\sum_{r=0}^{n} (f(r) - p)^2 \binom{n}{r} p^r (1-p)^{n-r} < \frac{p(1-p)}{n}$$

for some p?

Hint. Prove that the $n+1$ polynomials $g_r(p) = p^r(1-p)^{n-r}$ for $r = 0, 1, \ldots, n$ are linearly independent in the $(n+1)$ dimensional vector space of polynomials of degree n and use it.