Invariant Distributions.

Theorem. Let

$$L = \frac{1}{2}\sum_{i,j=1}^{d} a_{i,j}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^{d} b_j(x)\frac{\partial}{\partial x_j}$$

where $a_{i,j}, b_j$ are continuous functions on R^d perhaps unbounded. Assume that the solution to the martingale problem exists (i.e.does not blow up and is unique for every x. So there is a family of measures $\{P_x\}$ on $C[[0,\infty); R^d]$. Suppose μ is a probability distribution on R^d such that $\int_{R^d}(Lu)(x)d\mu(x) = 0$ for every smooth function u with compact support on R^d. Then μ is an invariant distribution for L and $\int P_x d\mu$ is a stationary Markov process with marginal μ

Proof.

For $h > 0$ we will construct a family $\pi_h(x, dy)$ of transition probability functions for Markov Chains such that

1. μ is invariant measure for each π_h. i.e.

$$\int \pi(x, A)\mu(dx) = \mu(A)$$

2. The Markov chain with time step h and transition probability π_h converges to the process $\{P_x\}$ as $h \to 0$.

Our candidate for π_h is $(I - hL)^{-1}$. We have the domain $\mathcal{D}$ of C^2 functions that are constant outside a compact set and $(I - hL)$ maps it into continuous functions that are constant outside a compact set. Let $u - Lu = f$ and $f \geq 0$. The minimum of u is attained at some point z and $Lu(z) \geq 0$ implying tha $u(z) = f(z) + Lu(z) \geq 0$. Thus $f \geq 0$ implies $u \geq 0$. This implies that $(I - hL)$ is invertible and the inverse π_h maps the range $\mathcal{R}_h$ of $(I_h L)$ into $\mathcal{D}$. It maps nonnegative functions to nonnegative ones and $\pi_h \mathbf{1} = \mathbf{1}$. This implies that $\|\pi_h f\|_\infty \leq \|f\|_\infty$. $\int Lf d\mu = 0$ implies $\int \pi_h f d\mu = \int f d\mu$. The problem is $\mathcal{R}_h$ is not necessarily dense and we may have to extend π from $\mathcal{R}_h$ to $C(R^d)$ preserving nonnegativity and the identity $\int \pi_h f d\mu = \int f d\mu$. We construct a distribution $\lambda_h(dx, dy)$ on $R^d \times R^d$ with both marginals equal to μ and $E^\lambda[f(y)|x] = (\pi_h f)(x)$ for $f \in \mathcal{R}_h$. To this end we define a linear functional Λ_h on (not necessarily closed) subspace of functions in $C[R^d \times R^d]$ of the form

$$w(x, y) = \sum_{r=1}^{n} u_r(x) f_r(y) + u_0(y)$$

where u_0, u_r are continuous functions with a limit at ∞ and $f_r = g_r - hLg_r \in \mathcal{R}_h$. We define

$$\Lambda_h(w) = \sum_{r=1}^{n} \int_{R^d} u_r(x) g_r(x)\mu(dx) + \int_{R^d} u_0(x)\mu(dx)$$

We need to show that if $w \geq 0$ then $\Lambda_h(w) \geq 0$. We can then extend Λ_h to all of $C_0(R^d \times R^d)$ functions with limit at infinity. Riesz theorem will give a bivariate distribution λ_h with both marginals equal to μ and the conditional probability distribution $\hat{\pi}_h$ will agree with π_h on $\mathcal{R}_h$. Let us define

$$\psi(z_1, \ldots, z_n) = \inf_x [\sum_{r=1}^n u_r(x) z_r]$$

It is concave. Let us pretend it is smooth. Then $\psi(f_1(x), \cdots, f_n(x)) + u_0(x) \geq 0$.

$$\int \psi(f_1(x) - hLf_1(x), \cdots, f_n(x) - hLf_n(x)) d\mu$$

is a concave function of h. Derivative with respect to h at $h = 0$ is given by

$$-\int \sum_j \frac{\partial \psi}{\partial z_j}(f_1(x), \cdots, f_n(x))(Lf_j)(x) d\mu \leq -\int L\psi(f_1(x), \cdots, f_n(x)) d\mu = 0$$

The maximum principle implies that for a concave function $\psi(z_1, \ldots, z_n)$,

$$L\psi(f_1, \ldots, f_n) \leq \sum_j \frac{\partial \psi}{\partial z_j} Lf_j$$

As a function of h it is concave and has negative slope at $h = 0$. So it is decreasing for $h \geq 0$. Hence

$$\int \psi(f_1(x), \ldots, f_n(x)) d\mu \geq \int \psi(f_1(x) - hLf_1(x), \ldots, f_n(x) - hLf_n(x)) d\mu$$

Now,

$$\begin{aligned}
&\int \sum_{r=1}^n u_r(x) g_r(x) d\mu + \int u_0(x) d\mu \\
&\geq \int \psi(g_1(x), \ldots, g_r(x)) d\mu + \int u_0(x) d\mu \\
&\geq \int \psi(g_1(x) - hLg_1(x), \ldots, g_n(x) - hLg_n(x)) d\mu + \int u_0(x) d\mu \\
&= \int \psi(f_1(x), \ldots, f_n(x)) d\mu + \int u_0(x) d\mu \\
&= \int [\psi(f_1(x), \ldots, f_n(x)) + u_0(x)] d\mu \\
&\geq 0
\end{aligned}$$

Hahn-Banch Theorem. Let B be the Banach space of real valued bounded continuous functions on a compact space X and B_0 a linear subspace containing constants. $\Lambda(f)$ is a linear functional that is nonnegative, i.e. $f \geq 0$ implies $\Lambda(f) \geq 0$. Let $\Lambda(\mathbf{1}) = 1$. Then Λ

can be extended as a nonnegative linear functional on B and represented by Riesz theorem as integral with resprct to a probability measure on X.

Let $g \notin B_0$. Let $c^+(g) = \inf_{f \in B_0; f \geq g} \Lambda(f)$ and $c^- = \sup_{f \in B_0; f \leq g} \Lambda(f)$. It is easy to check that $c^+(g) \geq C^-(g)$ and let us define $\Lambda(g) = 1$ where $c^+(g) \geq a \geq C^-(g)$. Then we need to check that if $f + cg \geq 0$ then $\Lambda(f) + ca \geq 0$. Then we would have extended Λ from B_0 to $B_1 = \text{span } \{B_0, g\}$ If $c = 0$, there is nothing to prove. If $c > 0$ then $g \geq -\frac{f}{c}$. By choice $\Lambda(g) = a \geq \Lambda(-\frac{f}{c}) = -\frac{\Lambda(f)}{c}$. Thus $\Lambda(f) + ca \geq 0$. The case when $c < 0$ is similar. The rest is routine.

It is clear that both the marginals are fixed at μ. If we denote by $\hat{\pi}_h(x, dy)$ the r.c.p.d then the Markov Chain has μ as marginal and since $\pi_h((I - hL)f = f$, $\frac{1}{h}(\pi_h u_h - u_h) = f$ where $u_h = f - hLf \to u$. This is enough by martingale arguments to show tightness and convergence to $\{P_x\}$.