What do we mean by a stochastic process with continuous paths on R^d with characteristics $\{a_{i,j}(t,\omega)\}$ and $\{b_j(t,\omega)\}$ or solution to the martingale problem corresponding to $(\{a_{i,j}(t,\omega); \{b_j(t,\omega)\})$?

$\Omega = C[[0,T]; R^d]$ is the space of continuous R^d valued function on $[0,T]$. $\mathcal{F}_t$ is the σ-field generated by $\{x(s)\}$, $0 \le s \le t$. $T can be finite or \infty$ in which case we have $[0,\infty)$ instead of $[0,T]$. A function $u : \Omega \times [0,T] \to R^k$ is progessively measurable if, for each $t \ge 0$, u is a (jointly) mesurable map from $(\Omega \times [0,t], \mathcal{F}_t \times \mathcal{B}(]0,T])$ to R^k. $\{a_{i,j}(t,x)\}$ is a symmetric positive semidefinite matrix, assumed to be uniformly bounded (for simplicity) and progressively measurable. $\{b_j(t,x)\}$ are similarly bounded progressively mesurable with values in R^d.

We say that P is a process with characteristics a, b with initial distribution μ if $P[x(0) \in A] = \mu(A)$ and any one of the following which is equivalent are true.

1. For any smooth function f with compact support on R^d

$$f(x(t,\omega)) - f(x(0,\omega)) - \int_0^t (L_{s,\omega} f)(s, x(s,\omega)) ds \tag{1}$$

is a martingale with respect to $(\Omega, \mathcal{F}_t, P)$ Here

$$(L_{s,\omega} f)(s,x) = \frac{1}{2} \sum_{i,j} a_{i,j}(s,\omega) \frac{\partial^2 f}{\partial x_i \partial x_j}(s,x) + \sum_j b_j(s,\omega) \frac{\partial f}{\partial x_j}(s,x)$$

2. For any function $f(t,x)$ in $C^{1,2}([0,T] \times R^d)$

$$f(t, x(t,\omega)) - f(0, x(0,\omega))) - \int_0^t [\frac{\partial f}{\partial s}(s, x(s,\omega)) + (L_{s,\omega} f)(s, x(s,\omega))] ds \tag{2}$$

is a martingale.

3. For any function f in $C^{1,2}([0,T] \times R^d)$

$$\exp[f(x(t,\omega) - f(x(0,\omega)) - \int_0^t [e^{-f}(\frac{\partial}{\partial s} + L_{s,\omega}) e^f](s, x(s,\omega)) ds] \tag{3}$$

is a martingale with respect to $(\Omega, \mathcal{F}_t, P)$

Rematk: f and its derivatives can have growth $o(|x|^2)$ at infinity. In particular

$$P[\sup_{0 \le t \le T} \|x(t,\omega)\| \ge \ell] \le C(T) \exp[-c_0(T)\ell^2]$$

4. For any $\theta \in R^d$

$$\exp[< \theta, x(t,\omega) - x(0,\omega) > - \frac{1}{2} \int_0^t < a(s,\omega)\theta, \theta > ds - \int_0^t < b(s,\omega), \theta > ds]$$

is a martingale.

Proofs. We can assume without loss of generality that $f(t,x)$ is $C^\infty[[0,T]\times R^d]$.

$$\begin{aligned}
&E[f(t,x(t))-f(s,x(s))|\mathcal{F}_s]\\
&\quad=E[f(t,x(t))-f(s,x(t))+f(s,x(t))-f(s,x(s))|\mathcal{F}_s]\\
&\quad=E[\int_s^t f_v(v,x(t))dv+\int_s^t(L_{v,\omega}f)(s,x(v))dv|\mathcal{F}_s]\\
&\quad=E[\int_s^t f_v(v,x(v))dv+\int_s^t dv\int_v^t L_{u,\omega}f_v(v,x(u))du\\
&\qquad\qquad+\int_s^t(L_{v,\omega}f)(v,x(v))dv-\int_s^v du\int_s^t(L_{v,\omega}f)_u(u,x(v))dv|\mathcal{F}_s]\\
&\quad=E[\int_s^t f_v(v,x(v))dv+\int_s^t(L_{v,\omega}f)(v,x(v))dv]
\end{aligned}$$

Lemma. Let $M(t)$ be a continuous martingale on $(\Omega,\mathcal{F}_t,P)$ and $A(t)$ a progressively measurable continuous function of bounded variation with $A(0)=0$. Assume for any finite T, $M(T)$ is square integrable and the total variation $|A|(T)$ of $A(t)$ on $[0,T]$ is square intgrable, then

$$A(t)M(t)-\int_0^t M(s)dA(s)$$

is a martingale with respect to $(\Omega,\mathcal{F}_t,P)$.

Proof.

$$\begin{aligned}
&E[A(t)M(t)-A(s)M(s)-\int_s^t M(u)dA(u)|\mathcal{F}_s]\\
&\quad=\lim_{\pi\downarrow 0}\sum_j E[A(t_j)M(t_j)-A(t_{j-1})M(t_{j-1})-\int_{t_{j-1}}^{t_j}M(u)dA(u)|\mathcal{F}_s]\\
&\quad=\lim_{\pi\downarrow 0}\sum_j E[A(t_j)M(t_j)-A(t_{j-1})M(t_j)-\int_{t_{j-1}}^{t_j}M(u)dA(u)|\mathcal{F}_s]\\
&\quad=0
\end{aligned}$$

To go from **2.** to **3.**

$$M(t)=e^{f(t,x(t,\omega))}-\int_0^t[(\frac{\partial}{\partial s}+L_{s,\omega})e^f](s,x(s,\omega))ds$$

$$A(t)=\exp[-f(0,x(0,\omega))-\int_0^t[(e^{-f}(\frac{\partial}{\partial s}+L_{s,\omega})e^f)(s,x(s,\omega))]ds]$$

$M(t)A(t) - \int_0^t M(s)dA(s)$ simplifies to (3) because

$$A(t)\int_0^t [(\frac{\partial}{\partial s} + L_{s,\omega})e^f](s, x(s,\omega))ds + \int_0^t M(s)dA(s) = 0$$

To verify this let us differentiate with respect to t.

$$A'(t)\int_0^t [(\frac{\partial}{\partial s} + L_{s,\omega})e^f](s, x(s,\omega))ds + A(t)[(\frac{\partial}{\partial s} + L_{s,\omega})e^f](s, x(s,\omega))] + A'(t)M(t) = 0 \ ?$$

$$A'(t) = -A(t)[(e^{-f}(\frac{\partial}{\partial s} + L_{s,\omega})e^f)(s, x(s,\omega))]$$

$$M(t) = e^{f(t,x(t,\omega))} - \int_0^t [(\frac{\partial}{\partial s} + L_{s,\omega})e^f](s, x(s,\omega))ds$$

We see that after dividing by $A(t)$

$$\begin{aligned}
&- [(e^{-f}(\frac{\partial}{\partial s} + L_{s,\omega})e^f)(s, x(s,\omega))]\int_0^t [(\frac{\partial}{\partial s} + L_{s,\omega})e^f](s, x(s,\omega))ds \\
&+ [(\frac{\partial}{\partial s} + L_{s,\omega})e^f](s, x(s,\omega))] \\
&- [(e^{-f}(\frac{\partial}{\partial s} + L_{s,\omega})e^f)(s, x(s,\omega))]e^{f(t,x(t,\omega))} \\
&+ (e^{-f}(\frac{\partial}{\partial s} + L_{s,\omega})e^f)(s, x(s,\omega))]\int_0^t [(\frac{\partial}{\partial s} + L_{s,\omega})e^f](s, x(s,\omega))]ds
\end{aligned}$$

First and last terms cancel each other as do the second and third.

3 implies **4**.

Limits of nonnegative martingales is a supermartingale. Let $X_n(t)$ be a sequence of non negative martingales with $E[X_n(t)] = 1$ and let $X(t) = \lim_{n\to\infty} X_n(t)$ a.e. Then $M(t)$ is a supermartingale.

Proof. Let $E_k(s) = \{\omega : \sup_n X_n(s) \le k\}$. $E_k(s) \in \mathcal{F}_s$ and $E_k(s) \uparrow \Omega$

$$\int_{A\cap E_k(s)} X_n(s)dP = \int_{A\cap E_k(s)} X_n(t)dP$$

Let $n \to \infty$ and use Fatou on the right and bounded convergence theorem on the left.

$$\int_{A\cap E_k(s)} X(s)dP \ge \int_{A\cap E_k(s)} X(t)dP$$

Let $k \to \infty$.

$$\int_A X(s)dP \ge \int_A X(t)dP$$

or $E[X(t)|\mathcal{F}_s] \leq X(s)$ a.e.

The function $< \theta, x >$ is not bounded but can be approximated by smooth bounded functions and

$$\exp[< \theta, x(t) - x(0) > - \int_0^t < \theta, b(s,\omega) > ds - \frac{1}{2}\int_0^t < \theta, a(s,\omega)\theta > ds]$$

is a supemartingale.

$$E^P[\exp[< \theta, x(t) - x(0) > - \int_0^t < \theta, b(s,\omega) > ds - \frac{1}{2}\int_0^t < \theta, a(s,\omega)\theta > ds]] \leq 1$$

$$E^P[\exp[< \theta, x(t) - x(0) >]] \leq \exp[t(c_1\|\theta\| + c_2\|\theta\|^2)]$$

It is clear that $E[\exp[\lambda\|x(t)\|]] < \infty$ for all $\lambda > 0$. The approximations can be constructed with uniform linear bounds.

Hence

$$X_\theta(t) = \exp[< \theta, x(t,\omega) - x(0,\omega) > -\frac{1}{2}\int_0^t < a(s,\omega)\theta, \theta > ds - \int_0^t < b(s,\omega), \theta > ds]$$

are martingales. If $y(t) = x(t) - x(0) - \int_0^t b(s,\omega)ds$, then

$$P[\sup_t \|y(t)\| \geq \ell] \leq c_1 \exp[-\frac{c_2\ell^2}{t}]$$

4 implies **1**.

Continue analytically. Replace θ by $i\theta$.

$$Y_\theta(t) = \exp[< i\theta, x(t,\omega) - x(0,\omega) > +\frac{1}{2}\int_0^t < a(s,\omega)\theta, \theta > ds - i\int_0^t < b(s,\omega), \theta > ds]$$

are martingales. Take

$$A(t) = \exp[-\frac{1}{2}\int_0^t < a(s,\omega)\theta, \theta > ds + i\int_0^t < b(s,\omega), \theta > ds]$$

Then $Y_\theta(t)A(t) - \int_0^T Y_\theta(s)dA(s)$ reduces to **1** with $f = e^{i<\theta,x>}$. Note that $y(t) - \int_0^t b(s,\omega)ds$ and $y_i(t)y_j(t) - \int_0^t a_{i,j}(s,\omega)ds$ are martingales.

Stochastic Integrals. Given $(\Omega, \mathcal{F}_s, x(s,\omega), P, \{a(s,\omega), b(s,\omega)\})$. A progressively measurable function $e(s,\omega)$ with values in R^d we want to define

$$z(t,\omega) = \int_0^t < e(s,\omega), dx(s) >= \int_0^t < e(s,\omega), dy(s) > + \int_0^t < e(s,\omega), b(s,\omega) > ds$$

It is only the dy integral that is a problem. Let us take for simplicity $b = 0$. Take a subdivision $t_j = \frac{j}{N}$

Step 1. Assume e is uniformly bounded, is piecewise (in time) constant $e = e_j(\omega)$ on $[t_{j-1}, t_j]$ which is $\mathcal{F}_{t_{j-1}}$ measurable. Then for $t_j \leq t \leq t_{j+1}$

$$z_N(t) = \sum_{1 \leq i \leq j} < e(t_{i-1}), y(t_i) - y(t_{i-1}) > + < e(t_j), y(t) - y(t_j) >$$

$z(\cdot)$ is linear in e, almost surely continuous and for any such e it is a martingale and so is

$$z^2(t) - \int_0^t < e(s,\omega), a(s,\omega)e(s,\omega) > ds$$

and by Doob's inequality

$$E[[\sup_{0 \leq s \leq T} |z(s)|]^2] \leq 4E^P[\int_0^T < e(s,\omega), a(s,\omega)e(s,\omega) > ds]$$

and

$$\exp[z(t) - \int_0^t < e(s,\omega), b(s,\omega) > ds - \frac{1}{2}\int_0^t < e(s,\omega), a(s,\omega)e(s,\omega) > ds]$$

are martingales.

Step 2. If $e(s,\omega)$ is uniformly bounded and continuous we can approximate $e(s,\omega)$ by $(e, \frac{[ns]}{n})$ which is agiain progressively measurable. We can pass to the limit. The limit exists and satisfy the smae properties as before.

Step 2. Given a bounded progressively measurable e we define e_n for $s \geq \frac{1}{n}$ by

$$e_n(s) = n \int_{s-\frac{1}{n}}^s e(v,\omega) dv$$

$\int_0^T \|e_n(s) - e(s)\|^2 ds \to 0$ and therefore $z_n(t)$ has a limit.

Step 3. If $E^P[\int_0^T \|e(s,\omega)\|^2 ds] < \infty$ we can truncate by

$$e_\ell(s,\omega) = e(s,\omega)\mathbf{1}_{\|e(s,\omega)\| \leq \ell}$$

and let $\ell \to \infty$.

In conclusion we can define

$$z(t) = \int_0^t < e(s,\omega), dx(s) >$$

provided

$$E^P[\int_0^T \|e(s,\omega)\|^2 ds] < \infty$$

Then $z(t) - \int_0^t < e(s,\omega, b(s,\omega)ds >$ is a square ntegrable martingale. If e is uniformly bounded then

$$\exp[z(t) - \int_0^t < e(s,\omega, b(s,\omega)ds > -\frac{1}{2}\int_0^t < e(s,\omega, a(s,\omega)e(s,\omega)ds > ds >]$$

is martingale.

The linear algebra of Stochastic Integrals.

$[\Omega, \mathcal{F}_s, P, x(s,\omega), a(s,\omega), b(s,\omega)]$

$x \in R^d, b \in R^d, a \in S_d^+$ S_d^+ is positive semidefinite $d \times d$ matrices. Let $y(t) = \int_0^t c(s,\omega)ds + \int_0^t e(s,\omega)dx(s)$ where $c \in R^n, e \in W_{n,d}$ where $W_{n,d}$ is the set of $n \times d$ matrices. Then $[\Omega, \mathcal{F}_s, P, y(s,\omega), \hat{a}(s,\omega), \hat{b}(s,\omega)]$ and $y \in R^n, \hat{b} \in R^n, \hat{a} \in S_n^+$, $\hat{b} = c + eb$ and $\hat{a} = eae^*$

If X is Gaussian with mean μ and covariance A, $Y = eX + c$ is Gaussian with mean $e\mu + c$ and covariance eAe^*.

Itô's Formula. Let $f(t,x)$ be a smooth bounded function function. Let $g_{\lambda,\theta}(t,x) = \lambda f(t,x) + < \theta, x >$.

$$\exp[(g(t,x(t)) - g(0,x(0))) - \int_0^t H(s,\omega)ds]$$

is a martingale, where

$$H(s,\omega) = \frac{\partial g}{\partial s}(s, x(s,\omega)) + \frac{1}{2}\sum_{i,j} a_{i,j}(s,\omega)\frac{\partial^2 g}{\partial x_i \partial x_j}(s, x(s,\omega))$$
$$+ \sum_j b_j(s,\omega)\frac{\partial g}{\partial x_j}(s, x(s,\omega)) + \frac{1}{2} < a(s,\omega)(\nabla g)(s,\omega), (\nabla g)(s,\omega) >$$

Let $y(t) = f(t,x(t)) - f(0,x(0))$. Then

$$[\Omega, \mathcal{F}_s, P, y(s,\omega), x(s,\omega), \hat{a}(s,\omega), \hat{b}(s,\omega)]$$

where

$$\hat{b} = [\frac{\partial f}{\partial s}(s, x(s,\omega)) + \frac{1}{2}\sum_{i,j} a_{i,j}(s,\omega)\frac{\partial^2 f}{\partial x_i \partial x_j}(s, x(s,\omega))$$
$$+ \sum_j b_j(s,\omega)\frac{\partial f}{\partial x_j}(s, x(s,\omega)), b(s,\omega)]$$

$$\hat{a} = \begin{pmatrix} < a(s,\omega)(\nabla f)(s, x(s,\omega), (\nabla f(s,\omega) > & a(s,\omega)(\nabla f)(s, x(s,\omega) \\ a(s,\omega)(\nabla f)(s, x(s,\omega) & a(s,\omega) \end{pmatrix}$$

Let us define a new process

$$w(t) = f(t, x(t)) - f(0, x(0)) - \int_0^t f_s(s, x(s))ds - \int_0^t \langle (\nabla f)(s, x(s)), dx(s) \rangle$$

$$dw = dy - f_s(s, x(s))ds - \langle (\nabla f)(s, x(s,\omega)), dx(s) \rangle$$

$$[\Omega, \mathcal{F}_s, P, w(s,\omega), \tilde{a}(s,\omega), \tilde{b}(s,\omega)]$$

$$\tilde{b} = \frac{1}{2} \sum_{i,j} a_{i,j}(s,\omega) \frac{\partial^2 f}{\partial x_i \partial x_j}(s, x(s,\omega))$$

$$\tilde{a} = \langle (1, -(\nabla f)(s, x(s))), \hat{a}(s,\omega)(1, -(\nabla f)(s, x(s))) \rangle = 0$$

$$df(t, x(t)) = f_t dt + \langle (\nabla f), dx \rangle + \frac{1}{2} \sum_{i,j} a_{i,j}(t,\omega) \frac{\partial^2 f}{\partial x_i \partial x_j}(t, x(t,\omega))dt$$