Given $a(s,\omega), b(s,\omega)$ progressively measurable on $(\Omega, \mathcal{F}_s)$ and an initial distribution μ is there a probability distribution P on $\Omega = C[[0,\infty) : R^d$ such that $P[x(0) \in A] = \mu(A)$ for all Borel sets A and for any smooth f with compact support in R

$$X_f(t) = f(x(t)) - f(x(0)) - \int_0^t \langle b(s,\omega), (\nabla f)(x(s))\rangle ds - \frac{1}{2}\int_0^t \sum_{i,j} a_{i,j}(s,\omega)\frac{\partial^2 f}{\partial x_i \partial x_j}(x(s))ds$$

is a martingale relative to $(\Omega, \mathcal{F}_s, P)$? Is it unique?

Let $\|a(t,\omega)\|, \|b(t,\omega)\|$ be uniformly bounded. The family Itô process corresponding to any $[a,b]$ with the distribution μ of $x(0)$ satisfying a uniform tightness condition is a compact family on $C[0,T]$.

Some Estimates.

1

$$P[\sup_{0\le s\le t}\|x(s)\| \ge \ell] \le C\exp[-c_1\frac{(\ell - c_3 t)_+^2}{t}]$$

If $y(t) = x(t) - x(0) - \int_0^t b(s,\omega)ds$ then

$$\exp[\langle\theta, y(t)\rangle - \frac{1}{2}\int \langle\theta, a(s,\omega)\theta\rangle ds]$$

is a martingale and with $\theta = \pm e_i$ and $\lambda > 0$,

$$P[\sup_{0\le s\le t}\langle\theta, y(s)\rangle \ge \ell] \le P[\sup_{0\le s\le t}\exp[\lambda\langle\theta, y(t)\rangle - \frac{\lambda^2}{2}\int_0^t \langle\theta, a(s,\omega)\theta\rangle ds] \ge e^{\lambda\ell - c\lambda^2}]$$
$$\le e^{-\lambda\ell + ct\lambda^2}$$

Optimizing over λ proves the inequality.

$$P[\sup_{0\le s\le t}\|y(s)\| \ \ge \ell] \le Ce^{-\frac{c_1\ell^2}{t}}$$

This is enough to provide the estimates

$$E[|y(t) - y(s)|^4] \le C|t-s|^2$$

establishing tightness. If a, b are constants we have BM, with covariance a and drift b. Run BM with $[a(0,\omega), b(0,\omega)]$ for time h then update to $[a(h,\omega), b(h,\omega)]$ for the next period of length h and go on. We have a process P_h for which

$$Z(f,h,t) = f(x(t)) - f(x(0)) - \int_0^t \langle b(h[\frac{s}{h}],\omega), (\nabla f)\rangle(x(s))ds$$
$$- \frac{1}{2}\int_0^t \text{ Tr } a(h[\frac{s}{h}],\omega)\cdot(\nabla^2 f)(x(s))ds$$

are martingales for smooth f. Assuming that a and b are continuous, and P is the limit of P_h along a subsequence then

$$Z(f,t) = f(x(t)) - f(x(0)) - \int_0^t \langle b(s)\omega), (\nabla f)\rangle(x(s))ds - \frac{1}{2}\int_0^t \text{ Tr } a(s,\omega)\cdot(\nabla^2 f)(x(s))ds$$

is a martingale with respect to $(\Omega, \mathcal{F}_t, P)$.

$$\lim_{n\to\infty} \int F_n(\omega)dP_n \to \int F(\omega)dP$$

provided, $F_n(\omega_n) \to F(\omega)$ if $\omega_n \to \omega$ and $\sup_n \sup_\omega |F_n(\omega)| \le C$

$$\int G(\omega)Z(f,t,\omega)dP = \int G(\omega)Z(f,s,\omega)dP$$

for all bounded $\mathcal{F}_s$ measurable G implies $E^P[Z(f,t,\omega)|\mathcal{F}_s] = Z(f,s,\omega)$

In particular if $a(t,x), b(t,x)$ are bounded and continuous as functions of t, x with values in S_d^+ and R^d respectively, for every $(s_0, x_0) \in [0,T] \times R^d$, there is atleast on solution P a probability measure on $C[[s,T]; R^d$ such that $P[x(s_0) = x_0] = 1$ and with respect to P for $s \ge s_0$

$$\begin{aligned} z(f,t,\omega) = Z(f,t) = f(x(t)) - f(x(s_0)) - \int_{s_0}^t \langle b(x(s)), (\nabla f)\rangle(x(s))ds \\ - \frac{1}{2}\int_0^t \text{ Tr } a(s,x(s))\cdot(\nabla^2 f)(x(s))ds \end{aligned}$$

are martingales.

Theorem.. Let $(C[[0,T]; R^d, \mathcal{F}_t, P)$ be a solution to the martingale problem corresponding to $[a,b]$ with $P[x(0) \in A] = \mu(A)$ for $A \in \mathcal{B}(R^d)$ and $a(t,\omega) = \sigma(t,\omega)\sigma^*(t,\omega)$. $\sigma \in d \times k$ matrices. Let $((C[0,T]; R^k), \mathcal{G}_t, Q)$ be Brownian motion. Then on $\Omega = ((C[0,T]; R^d \times R^k), \mathcal{F}_t \otimes \mathcal{G}_t, P \otimes Q)$ there is a Brownian motion $\hat{\beta}$ on Ω such that

$$x(t) - x(0) = \int_0^t \sigma(s,\omega)d\hat{\beta}(s) + \int_0^t b(s,\omega)ds$$

In particular, if $a(t,\omega) = a(t,x(t))$ and $b(t,\omega) = b(t,x(t))$, the above equation takes the form

$$x(t) - x(0) = \int_0^t \sigma(s,x(s))d\hat{\beta}(s) + \int_0^t b(s,x(s))ds$$

Let $\tau(s,\omega)$ be a pseudo inverse of $\sigma(S,\omega)$ and $P_1(s,\omega) = \sigma(s,\omega)\tau(s,\omega)$ and $P_2(s,\omega) = \tau(s,\omega)\sigma(s,\omega)$ or orthogonal projections that depend on s, ω.

Then if $y(t) = x(t) - x(0) - \int_0^t b(s,\omega)ds$ then

$$\hat{\beta}(t) = \int_0^t \tau(s,\omega)dy(s) + \int_0^t (I - P_2(s,\omega))d\beta(s)$$

is a martingale,

$$\tau a\tau^* + (I - P_2) = \tau\sigma\sigma^*\tau^* + (I - P_2) = P_2 + (I - P_2) = I$$

and

$$\int_0^t dy(s) - \int_0^t \sigma(s,\omega)d\hat{\beta}(s) = \int_0^t (I - \sigma(s,\omega)\tau(s,\omega))dy(s) - \int_0^t \sigma(s,\omega)(I - P_2(s,\omega))d\beta(s)$$

$$(I - P_1)a(I - P_1) + \sigma(I - P_2)(I - P_2)\sigma^* = 0$$

because P_1 is the projection onto the range of a and

$$\sigma(I - P_2)(I - P_2)\sigma^* = \sigma(I - P_2)\sigma^* = \sigma\sigma^* - \sigma P_2\sigma^* = \sigma\sigma^* - \sigma\tau\sigma\sigma^* = a - P_1 a = 0$$

Girsanov's formula. If P corresponds to $[a(t,\omega), b(t,\omega)]$ and $c(t,\omega)$ is bounded then

$$Y_0(t) = \exp[\int_0^t [\langle c(s,\omega), (dx(s) - b(s,\omega)ds)\rangle] - \frac{1}{2}\int_0^t [\langle c(s,\omega), a(s,\omega)c(s,\omega)\rangle]ds]$$

is a martingale with respect to $(\Omega, \mathcal{F}_t, P)$. $dQ = Y(t)dP$ is well defined. Moreover if $\xi(t)$ is a martingale w.r.t. Q, if and only if $\xi(t)Y(t)$ is a martingale w.r.t.P. The following are martingales w.r.t. P

$$\begin{aligned} Y_\theta(t) = &\exp[\int_0^t [\langle(\theta + c(s,\omega)), (dx(s) - b(s,\omega)ds)\rangle] \\ &- \frac{1}{2}\int_0^t [\langle(\theta + c(s,\omega)), a(s,\omega)(\theta + c(s,\omega))\rangle]ds] \\ = \quad & Y_0(t)Z_\theta(t) \end{aligned}$$

where

$$Z_\theta(t) = \exp[\langle\theta, x(t) - x(0)\rangle - \int_0^t [\langle\theta, b(s,\omega) + a(s,\omega)c(s,\omega)\rangle ds - \frac{1}{2}\int_0^t \langle\theta, a(s,\omega)\theta\rangle ds]$$

is a martingale w.r.t. Q defined by $dQ = Y_0(t)dP$.

Q corresponds to $[a, b + ac]$. c need not be bounded . Enough $< c, ac >$ is bounded. $c = \tau c^*$ where τ is the pseudo inverse of σ with $\sigma\sigma^* = a$ and c^* is bounded.

Martingales and conditioning. r.c.p.d or disintegration. If $M(t)$ is martingale w.r.t. $(\Omega, \mathcal{F}_t, P)$ and $Q_{t_0,\omega}$ is r.c.p.d given $\mathcal{F}_{t_0}$, $M(t)$ for $t \geq 0$ is a martingale w.r.t. $Q_{t_0,\omega}$ for almost all ω.

$$\int_A M(t_1) dQ_{t_0,\omega} = \int_A M(t_2) dQ_{t_0,\omega}$$

$t_0 \le t_1 < t_2$ and $A \in \mathcal{F}_{t_1}$.

$$\int_B [\int_A M(t_1) dQ_{t_0,\omega}] dP = \int_B [\int_A M(t_2) dQ_{t_0,\omega}] dP$$

for all $B \in \mathcal{F}_{t_0}$. Need

$$\int_{A\cap B} M(t_1) dP = \int_{A\cap B} M(t_2) dP$$

valid since $A \cap B \in \mathcal{F}_{t_1}$.

Corollary. If $a(s,\omega) = a(s,x(s))$ and $b(s,\omega) = b(s,x(s))$ then uniqueness for all staring points (s,x) implies the processes are all Markov with transition probability

$$p(s,x,t,A) = P_{s,x}[x(t) \in A]$$

If it is $a(x(s)), b(x(s))$ then $p(s,x,t,A) = p(t-s,x,A)$. Extends to stopping times. Uniqueness implies the strong Markov property.

Uniqueness and Stability. If P_n is a solution for $[a_n, b_n]$ and if $[a_n, b_n] \to [a, b]$, if P is the unique solution for $[a, b]$ then $P_n \to P$ in the weak topology.