Chapter 2

Singular Integrals.

2.1 Marcinkiewicz Interpolation Theorem.

Interpolation theorems play a very important role in Harmonic Analysis. An example is the following theorem. Let (X, Σ, μ) be a measure space. μ need not be a finite measure. A bounded map $T : L_p \to L_p$ satisfies $\|Tf\|_p \le C\|f\|_p$ for some $C < \infty$. By Tchebychev's inequality

$$\mu[x : |Tf| \ge \ell] \le \frac{\|Tf\|_P^p}{\ell^p} \le \frac{C^p\|f\|_p^p}{\ell^p}$$

This type of inequality, known as weak type inequality can hold even when T is not bounded.

Theorem 2.1 (Marcinkiewicz)**.** *Let T be a sublinear map defiened on $L_p \cap L_q$ that satisfies weak type inequlities*

$$\mu[x : (Tf)(x)| \ge \ell] \le \frac{C_i\|f\|_{p_i}^{p_i}}{\ell^{p_i}} \tag{2.1}$$

for $i = 1, 2$ where $1 \le p_1 < p_2 < \infty$. Then for $p_1 < p < p_2$, there are constants C_p such that

$$\|Tf\|_p \le C_p\|f\|_p \tag{2.2}$$

Note that T need not be linear. It need only satisfy for each x

$$|(T(f + g))(x)| \le |(Tf)(x)| + |(Tg)(x)| \tag{2.3}$$

Proof. Let $p \in (p_1, p_2)$ be fixed. For any function $f \in L_p$ and for any positive number a we deine $f_a = f\chi_{\{|f|\le a\}}$ and $f^a = f\chi_{\{|f|>a\}}$. Clearly $f_a \in L_{p_2}$ and $f^a \in L_{p_1}$

$$\begin{aligned}
&\mu[x : |(Tf)(x)| \ge \ell] \\
&\le \mu[x : |(Tf_a)(x)| \ge \frac{\ell}{2}] + \mu[x : |(Tf^a)(x)| \ge \frac{\ell}{2}] \\
&\le \frac{C_2 2^{p_2}}{\ell^{p_2}} \int_{|f(x)|\le a} |f(x)|^{p_2} d\mu + \frac{C_1 2^{p_1}}{\ell^{p_1}} \int_{|f(x)|>a} |f(x)|^{p_1} d\mu
\end{aligned}$$

Choose $a = \ell$. Multiply by $p\ell^{p-1}$ and integrate over $[0, \infty)$. Denote by $\sigma(d\tau)$ the distribution of $\tau = |f(z)|$.

$$\begin{aligned}
\|Tf\|_p^p &= \int_0^\infty p\,\ell^{p-1}\mu[z : |(Tf)(z)| \ge \ell]d\ell \\
&\le \int_0^\infty \frac{C_2 2^{p_2} p\,\ell^{p-1}}{\ell^{p_2}} \int_{|f(z)|\le\ell} |f(x)|^{p_2} d\mu\, d\ell \\
&\quad + \int_0^\infty \frac{C_1 2^{p_1} p\,\ell^{p-1}}{\ell^{p_1}} \int_{|f(z)|>\ell} |f(x)|^{p_1} d\mu\, d\ell \\
&= \int_0^\infty \frac{C_2 2^{p_2} p\,\ell^{p-1}}{\ell^{p_2}} \int_{\tau\le\ell} \tau^{p_2}\, \sigma(d\tau)\, d\ell \\
&\quad + \int_0^\infty \frac{C_1 2^{p_1} p\,\ell^{p-1}}{\ell^{p_1}} \int_{\tau>\ell} \tau^{p_1} \sigma(d\tau)\, d\ell \\
&= \int \tau^{p_2} \int_\tau^\infty \frac{C_2 2^{p_2} p\,\ell^{p-1}}{\ell^{p_2}} d\ell\, \sigma(d\tau) \\
&\quad + \int \tau^{p_1} \int_0^\tau \frac{C_1 2^{p_1} p\,\ell^{p-1}}{\ell^{p_1}} d\ell\, \sigma(d\tau) \\
&= C(p_1, p_2, p, C_1, C_2) \int \tau^p \sigma(d\tau)
\end{aligned}$$

□

There is a slight variation of the argument that allows p_2 to be infinite provided T is bounded on L_∞. If we assume the bound $\|(Tf)\|_\infty \le C_2\|f\|_\infty$

we obtain the estimate

$$\begin{aligned}\mu[x : |Tf(x)| \geq (1+C_2)\ell] &\leq \mu[x : |Tf^\ell(x)| \geq C_2\ell] + \mu[x : |Tf_\ell(x)| \geq \ell]\\ &= \mu[x : |Tf^\ell(x)| \geq \ell]\\ &\leq \frac{C_1}{\ell^{p_1}} \int_{|f(x)|\geq \ell} |f^\ell(x)|^{p_1} d\mu\\ &= \frac{C_1}{\ell^{p_1}} \int_\ell^\infty \tau^{p_1}\sigma(d\tau)\end{aligned}$$

multiply by $p\ell^{p-1}$ and integrate as before.

A different interpolation theorem for **linear** maps T is the following

Theorem 2.2 (Riesz-Thorin). *If a linear map T is bounded from L_{p_i} into L_{p_i} with a bound C_i for $i = 1, 2$ then for $p_1 \leq p \leq p_2$ it is bounded from L_p into L_p with a bound C_p that can be taken to be*

$$C_p = C_1^t C_2^{1-t} \tag{2.4}$$

where t is determined by

$$\frac{1}{p} = \frac{t}{p_1} + \frac{1-t}{p_2} \tag{2.5}$$

Proof. The proof uses methods from the theory of functions of a complex variable. The starting point is the maximum modulus principle. Let us assume that $u(z)$ is analytic in the open strip $a < Re\, z < b$ and bounded and continuous in the closed strip $a \leq Re\, z \leq b$. Let $M(x)$ be the maximum modulus of the function on the line $Re\, z = x$. Then $\log M(x)$ is a convex function of x. This is not hard to see. Clearly the maximum principle dictates that

$$M(x) \leq \max[M(a), M(b)]$$

If one is worried about the maximum being attained, one can always mutiply by $e^{\epsilon z^2}$ and let ϵ go to 0. Replacing $u(z)$ by $u(z)e^{tz}$ yields the inequality

$$M(x)e^{tx} \leq \max[M(a)e^{at}, M(b)e^{bt}]$$

Pick t so that $M(a)e^{at} = M(b)e^{bt}$, i.e $t = \frac{1}{b-a}\log\frac{M(a)}{M(b)}$. We get

$$M(x) \leq M(a)^{\frac{b-x}{b-a}} M(b)^{\frac{x-a}{b-a}}$$

Since $a\frac{b-x}{b-a} + b\frac{x-a}{b-x} = x$ this proves the required convexity.

We note that the maximum of any collection of convex functions is again convex. The proof is completed by representing $\log F(p)$, where $F(p)$ is the norm of T from L_p to L_p, as the supremum of a bunch of functions that are convex in $x = \frac{1}{p}$.

$$\begin{aligned}
\|T\|_{p,p} &= \sup_{\substack{\|f\|_p \le 1 \\ \|g\|_q \le 1}} |\int g(Tf) d\mu| \\
&= \sup_{\substack{\|f\|_p \le 1, f \ge 0, |\phi|=1 \\ \|g\|_q \le 1, g \ge 0, |\psi|=1}} |\int (g\psi)(T(f\phi)) d\mu| \\
&= \sup_{\substack{\|f\|_1 \le 1, f > 0, |\phi|=1 \\ \|g\|_1 \le 1, g > 0, |\psi|=1}} |\int (g^x\psi)(T(f^{1-x}\phi)) d\mu| \\
&= \sup_{\substack{\|f\|_1 \le 1, f > 0, |\phi|=1 \\ \|g\|_1 \le 1, g > 0, |\psi|=1 \\ Rez=x}} |\int (g^z\psi)(T(f^{1-z}\phi)) d\mu| \\
&= \sup_{\substack{\|f\|_1 \le 1, f > 0, |\phi|=1 \\ \|g\|_1 \le 1, g > 0, |\psi|=1}} \sup_{Rez=x} |u(f, g, \phi, \psi, z)| \\
&= \sup_{\substack{\|f\|_1 \le 1, f > 0, |\phi|=1 \\ \|g\|_1 \le 1, g > 0, |\psi|=1}} M_{f,g,\phi,\psi}(x)
\end{aligned}$$

□

In particular for the Hardy-Littlewood or Poisson maximal function the L_∞ bound is trivial and we now have a bound for the L_p norm of the maximal function in terms of the L_p norm of the original function provided $p > 1$.

2.2 Weak type inequality.

We saw that for a convolution operator of the form

$$(Tf)(x) = \int_{\mathbf{T}} f(y)k(x-y)dy \tag{2.6}$$

to be bounded as an operator from L_1 into itself we need k to be in L_1. However for $1 < p < \infty$ the operator can some times be bounded even if k is not in L_1. This is proved by establishing a bound from L_2 to L_2 and a weak type inequality in L_1. We can then use a combination of Marcinkiewicz interpolation, Riesz-Thorin interpolation and duality to prove the boundedness of T form $L_p \to L_p$ for $1 < p < \infty$.

Theorem 2.3. *If*

$$\hat{k}(n) = \int e^{inz} k(z) dz$$

satisfies $\sup_n |\widehat{k}(n)| \leq C$*, then the convolution operator given by equation (2.6) is bounded by* C *as an operator from* L_2 *to* L_2*.*

Proof. Use the the orthonormal basis $e_n(x) = \frac{1}{\sqrt{2\pi}} e^{-inx}$ to diagonalize T

$$Te_n(x) = \hat{k}(n) e_n(x) \tag{2.7}$$

□

We now proceed to establish weak type $(1,1)$ estimate. We shall assume that we have a kernel k in L_1 that satisfies

1. $$\sup_n |\int k(y) e^{iny} dy| = C_1 < \infty \tag{2.8}$$

2. $$\sup_y \int_{x:|x-y|>2|y|} |k(x-y) - k(x)| dx = C_2 < \infty \tag{2.9}$$

   Here $|x-y|$ in $\mathbf{T}$ is the length of the shorter arc connecting x and y in $\mathbf{T}$. In particular $|x-y| \leq \pi$ for all $x, y \in \mathbf{T}$.

Although we have assumed that k is in L_1 we will prove a weak type $(1,1)$ bound that depends only on C_1 and C_2.

Theorem 2.4. *The operator of convolution by* k

$$(T_k f)(x) = \int_{-\pi}^{\pi} k(x-y) f(y) dy \tag{2.10}$$

satisfies the weak type inequality (1,1)

$$\mu[x : |(T_k f)(x)| \geq \ell] \leq \frac{C}{\ell} \|f\|_1 \tag{2.11}$$

with a constant C *that depends only on* C_1 *and* C_2*.*

Proof. Proof involves several steps.

- First we observe that the Hardy-Littlewood maximal function given by (1.16) satisfies the estimate (1.17). The set $G = [x : M_f(x) > \ell]$ is an open set in $\mathbf{T}$ and has Lebesgue measure at most $\frac{3\|f\|_1}{\ell}$. We assume that $\ell > \frac{3\|f\|_1}{2\pi}$ so that $B = G^c$ is nonempty. We write the open set G as a possible countable union of **disjoint** open intervals I_j of length r_j centered at x_j. Note that the end points are not in G and that implies that at all the end points $x_j \pm \frac{1}{2}r_j$, $M_f(x_j \pm r_j) \le \ell$. The maximal inequality assures us that

$$\sum_j r_j \le \frac{3\|f\|_1}{\ell}$$

- Let us define the averages

$$m_j = \frac{1}{r_j}\int_{I_j} f(y)dy$$

and write f in the form

$$\begin{aligned} f(x) &= [f(x)1_B(x) + \sum_j m_j 1_{I_j}(x)] + \sum_j [f(x) - m_j]1_{I_j}(x) \\ &= g(x) + \sum_j h_j(x) \end{aligned}$$

- We have the bounds

$$\begin{aligned} |m_j| &\le \frac{1}{r_j}\int_{I_j} |f(y)|dy \le \frac{1}{r_j}\int_{\tilde{I}_j} |f(y)|dy \\ &\le 2\frac{1}{2r_j}\int_{\tilde{I}_j} |f(y)|dy \le 2M_f(x_j \pm \frac{r_j}{2}) \le 2\ell \end{aligned}$$

Here $\tilde{I}_j$ is the interval centered around $x_j \pm \frac{r_j}{2}$ of length $2r_j$ which covers I_j. In particular $\|g\|_\infty \le 2\ell$. On the other hand since $\{I_j\}$ are disjoint

$$\sum_j \|h_j\|_1 = \sum_j \int_{I_j} |f(y) - m_j| dy \le 2 \sum_j \int_{I_j} |f(y)| dy \le 2\|f\|_1$$

We therefore have

$$\|g\|_1 \le 3\|f\|_1$$

Note that the decomposition depends on ℓ. Let us write the corresponding sum

$$u = T_k f = T_k g + \sum_j T_k h_j = v + \sum_j w_j = v + w$$

- We estimate the L_2 norm of v and the L_1 norm of w on large enough set. Then use Tchebychev's inequality.

$$\mu[x : |v(x)| \ge \frac{\ell}{2}] \le \frac{\|v\|_2^2}{\ell^2} \le \frac{C_1\|g\|_2^2}{\ell^2} \le \frac{2\ell C_1 \|g\|_1}{\ell^2} = \frac{6C_1\|f\|_1}{\ell}$$

Let us denote by $\hat{I}_j$ the interval centered around x_j of length $3r_j$ and by $U = \cup_j \hat{I}_j$. We begin by estimating $\|w.1_{U^c}\|_1$.

$$\begin{aligned}
\|w.1_{U^c}\|_1 &\le \int_{U^c} \sum_j |\int_{I_j} k(x-y)[f(y) - m_j] dy| dx \\
&= \int_{U^c} \sum_j |\int_{I_j} [k(x-y) - k(x-x_j)][f(y) - m_j] dy| dx \\
&\le \int_{U^c} \sum_j \int_{I_j} |k(x-y) - k(x-x_j)||f(y) - m_j| dy dx \\
&= \sum_j \int_{I_j} |f(y) - m_j| dy \int_{U^c} |k(x-y) - k(x-x_j)| dx \\
&\le \sum_j \int_{I_j} |f(y) - m_j| dy \int_{\hat{I}_j^c} |k(x-y) - k(x-x_j)| dx \\
&\le \sum_j \int_{I_j} |f(y) - m_j| dy \int_{x:|x-y| \ge 2|y-x_j|} |k(x-y) - k(x-x_j)| dx \\
&\le C_2 \sum_j \int_{I_j} |f(y) - m_j| dy \\
&\le 2C_2 \|f\|_1
\end{aligned}$$

We have used here two facts. $f(y)-m_j$ has mean zero on I_j . If $y \in I_j$ and $x \in \tilde{I}_j^c$, then $|y-x| \geq r_j \geq 2|y-x_j|$. On the other hand

$$\mu(U) \leq \sum \mu(\tilde{I}_j) \leq 3\sum \mu(I_j) = 3\sum_j r_j \leq \frac{9\|f\|_1}{\ell}$$

- Finally we can put the pieces together.

$$\begin{aligned}\mu(x : |u(x)| \geq 2\ell) &\leq \mu(x : |v(x)| \geq \ell) + \mu(x : |w(x)| \geq \ell)\\ &\leq \frac{6C_1\|f\|_1}{\ell} + \frac{9\|f\|_1}{\ell} + \frac{2C_2\|f\|_1}{\ell}\end{aligned}$$

or

$$\mu(x : |u(x)| \geq \ell) \leq \frac{(12C_1 + 18 + 4C_2)\|f\|_1}{\ell} = \frac{C\|f\|_1}{\ell}$$

□

There is one point that we should note. For the interval doubling construction on the circle we should be sure that we do not see for instance any interval of lenghth larger than $\frac{\pi}{2}$ in G. This can be ensured if we take $\ell > \frac{6\|f\|_1}{\pi}$. The inequality is however satisfied for all ℓ because we can assume $C \geq 12$.

We want to look at the special kernel $k(y) = \frac{1}{y}$ which is not in L_1. We consider its truncation

$$k_\delta(y) = \frac{1}{y}\mathbf{1}_{\{|y|\geq\delta\}}(y)$$

Theorem 2.5. *Convolution by the kernel* $\frac{1}{x}$ *is a bounded operator from* $Lp \to L_p$ *for* $1 < p < \infty$.

We truncate it and consider

$$k_\delta(x) = \begin{cases} \frac{1}{x} \text{ if } |x| \geq \delta \\ 0 \text{ if } |x| > \delta \end{cases}$$

First we estimate the Fourier transform

$$\begin{aligned}|\int_{|y|\geq\delta}\frac{e^{iny}}{y}dy| &= 2|\int_{\delta}^{\pi}\frac{\sin ny}{y}dy| \\ &= 2|\int_{n\delta}^{n\pi}\frac{\sin y}{y}dy| \leq 4\sup_{0<a<\infty}|\int_0^a\frac{\sin y}{y}dy| \leq C_1\end{aligned}$$

Next in order to verify the condition (2.9) we need to estimate the following quantity uniformly in y and δ.

$$\int_{x:|x-y|>2|y|}|k_\delta(x-y)-k_\delta(x)|dx$$

There are three sets over which the integral does not vanish.

$$\begin{aligned}F_1 &= \{x:|x-y|>2|y|,|x-y|\geq\delta,|x|\geq\delta\} \\ F_2 &= \{x:|x-y|>2|y|,|x-y|\leq\delta,|x|\geq\delta\} \\ F_3 &= \{x:|x-y|>2|y|,|x-y|\geq\delta,|x|\leq\delta\}\end{aligned}$$

We consider

$$\begin{aligned}\int_{F_1}|\frac{1}{x-y}-\frac{1}{x}|dx &\leq \int_{x:|x-y|\geq2|y|}|\frac{1}{x-y}-\frac{1}{x}|dx \\ &\leq \int_{|z-1|\geq2}|\frac{1}{z-1}-\frac{1}{z}|dz \\ &= C_3\end{aligned}$$

It is clear that $F_2\subset[-2\delta,2\delta]$. Therefore

$$\int_{F_2}\frac{1}{|x|}dx\leq 2\int_{\delta}^{2\delta}\frac{dx}{x}=C_4$$

Finally $F_3\subset[x:|x-y|\leq2\delta]$ and works similarly. With $C_2=C_3+2C_4$ we have an estimate that is uniform in δ and we are are done.

We are now ready to prove

Theorem 2.6. *For any $f\in L_p$ the partial sums $s_N(f,x)$ converge to f in L_p provided* $1<p<\infty$*.*

Proof. We need only prove, for $1 < p < \infty$, a bound from L_p to L_p, for the partial sum operators

$$(T_N f)(x) = \int f(x-y) k_N(y) dy$$

with

$$k_N(z) = \frac{1}{2\pi} \frac{\sin(N+\frac{1}{2})z}{\sin\frac{z}{2}}$$

that is uniform in N. We are looking for a uniform L_p bound for the operators defined by convolution with a kernel whose Fourier transform is

$$\hat{k}_N(n) = \mathbf{1}_{\{|n| \le N\}}(n)$$

This can be reduced to proving the boundedness of a single operator the Hilbert transform S which in terms of Fourier transform multiplication by *signum* n given by

$$h(n) = \begin{cases} 1 & \text{if } n > 0 \\ -1 & \text{if } n < 0 \\ 0 & \text{if } n = 0 \end{cases}$$

We need the projection operator $Pf = \frac{1}{2\pi} \int f(x) dx$ onto constants acting on Fourier transforms as multiplication by

$$\chi_{\mathbf{0}}(n) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases}$$

Finally M_k is the operator of multiplication of a function by e^{ikx} or acting on Fourier transforms as shift operator $(M_k a)(n) = a(n+k)$. And the operator T_N is multiplication by

$$\tau_N(n) = \begin{cases} 1 \text{ if } |n| \le N \\ 0 \text{ if } |n| > N \end{cases}$$

$$\frac{1}{2}[M_{-N}(h + \chi_0)](n) - \frac{1}{2}[M_{-N-1}(h + \chi_0)](n) = \tau_N(n)$$

The operator M_N are uniformly bounded by 1 in every L_p space. It is therefore sufficient to show thatthe Hilbert transform is bounded from$L_p \to L_p$ for $1 < p < \infty$. Its kernel is

$$s(z) = \frac{1}{2\pi} \cot \frac{z}{2}$$

This can be replaced by the modified kernel

$$k(z) = \frac{1}{\pi z}$$

and we are done. □

2.3 Exercises.

1. In theorem 2.1 instead of taking $a = \ell$ take $a = k\ell$obtain the constant C explicitly and optimize over k
2. Consider multiplication of the Fourier transform by a sequence $a(n)$ that is real monotone and satisfies $\lim_{n\to-\infty} = 0, \lim_{n\to\infty} = 1$. Does there exist a kernel $A(x)$ that corresponds to it? Does it define a bounded operator from $L_p \to L_p$?