Chapter 4

Sobolev Spaces.

4.1 Generalized derivatives

Let $f(x)$ and $g(x)$ be two functions on R. What does it mean to say $g(x)$ is the derivative of f? Clearly the different quotient

$$g_h(x) = \frac{f(x+h) - f(x)}{h}$$

must converge to g. The sense in which the convergence takes place is to be specified. Here are some possibilities. Uniform convergence on finite intervals.

$$\sup_{a \le x \le b} |g_h(x) - g(x)| = 0$$

for every finite interval $[a, b]$. Require that for every x or for almost all x.

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = g(x)$$

point wise. This is not easy to work with unless f is known a priori to be absolutely continuous. Another possibility is to require that for some $1 \le p < \infty$

$$\lim_{h \to 0} \| \frac{f(x+h) - f(x)}{h} - g(x) \|_p = 0$$

or locally for every finite interval $[a, b]$,

$$\lim_{h \to 0} \int_a^b | \frac{f(x+h) - f(x)}{h} - g(x)|^p dx = 0$$

One could avoid all limits and require that

$$f(b) - f(a) = \int_a^b g(x)dx$$

which can be a problem if f is only known a priori to be a function in L_p. What makes sense always is to demand that for any smooth function ϕ with compact support

$$\int \phi'(x)f(x)dx + \int \phi(x)g(x)dx = 0$$

We only need to assume only that f and g are locally in $L_1[a,b]$ on any finite interval. Another possibility is to consider a smoothened version

$$f_\epsilon(x) = \int f(x-y)\psi_\epsilon(y)dy$$

where $\psi_\epsilon(x) = \frac{1}{\epsilon}\psi(\frac{x}{\epsilon})$ and $\psi(\cdot)$ is a nonnegative compactly supported infinitely differentiable function with $\int_{-\infty}^{\infty}\psi(y)dy = 1$. Ask now that $g_\epsilon = f'_\epsilon$ have a limit g as $\epsilon \to 0$, either uniformly on bounded sets, or in $L_p(R)$ or $L_p[a,b]$.

The Sobolev spaces $W_{k,p}(R^d)$ are defined as the space of functions u on R^d such that u and all its partial derivatives $D_{x_1}^{n_1}\cdots D_{x_d}^{n_d}u$ of order $n = n_1 + \cdots + n_d \le k$ are in L_p. We could start with C^∞ functions with compact support on R^d and complete it in the norm $\|u\|_{k,p}$ defined by

$$\|u\|_{k,p}^p = \sum_{\substack{n_1,\ldots n_d \\ 0\le n=n_1+\cdots+n_d\le k}} \|D_{x_1}^{n_1}\cdots D_{x_d}^{n_d}u\|_p^p \tag{4.1}$$

If $p = 2$, in terms of Fourier Transforms,

$$\|u\|_{k,2}^2 = \int_{R^d} |\widehat{u}|^2(\xi)[\sum_{\substack{n_1,\ldots,n_d \\ 0\le n_1+\cdots+n_d\le n}} |\xi_1|^{2n_1}\cdots|\xi_d|^{2n_d}]\Pi_i d\xi_i$$

4.2 Embedding Theorems.

If $u \in L_p$ and $D_i u = \frac{\partial u}{\partial x_i} \in L_p$ for $i = 1, 2\ldots, d$, one should expect u to be more regular than a function in L_p. If $d = 1$,

$$|u(b) - u(a)| \le \int_a^b |u'(x)|dx \le |a-b|^{\frac{1}{q}}\|u'\|_p$$

Showing that u is Hölder continuous of order $1-\frac{1}{p}$. What is the situation if $d \geq 2$?

Let us consider the operator $\widehat{A}$ which in terms of Fourier transforms is multiplication by $(1+|\xi|^2)^{-\frac{1}{2}}$.

$$(\widehat{A}u)(\xi) = \frac{1}{(1+|\xi|^2)^{\frac{1}{2}}}\,\widehat{u}(\xi)$$

and consider its represenation by the kernel

$$(Au)(x) = \int_{R^d} u(x+y)a(y)dy$$

where

$$\begin{aligned} a(x) &= c_d \int_{R_d} \frac{e^{-i<x,\xi>}}{(1+|\xi|^2)^{\frac{1}{2}}} d\xi = \frac{c_d}{\sqrt{\pi}} \int_{R^d} \int_0^\infty e^{-i<x,\xi>} e^{-t(1+|\xi|^2)} \frac{1}{\sqrt{t}} d\xi dt \\ &= k_d \int_0^\infty \frac{e^{-t}}{t^{\frac{d+1}{2}}} e^{-\frac{|x|^2}{4t}} dt = \frac{k_d}{|x|^{d-1}} \int_0^\infty e^{-t\,|x|^2} e^{-\frac{1}{4t}} \frac{dt}{t^{\frac{d+1}{2}}} \end{aligned}$$

decays very rapidly at ∞, is smooth for $x \neq 0$ and has a singularity of order $|x|^{1-d}$ near the origin for $d \geq 2$ and a logarithmic singularity at 0 when $d = 1$. In particular $a(\cdot) \in L_q$ for $q < \frac{d}{d-1}$. By Hölder's inequality, A will map L_p into L_∞ for $p > d$. If $p = d$ and $d > 1$ the result is false. Let us take $d = 2$ and a nonnegative function f with compact support such that $f \in L_2$ but $\int_{R^2} \frac{f(x)}{|x|} dx = \infty$. We saw that Af has a singularity at 0. Let us consider $u = D_1(Af)$. Clearly

$$\|u\|_2^2 = \|\widehat{u}\|_2^2 = \int_{R^2} \frac{\xi_1^2}{1+|\xi|^2} |\widehat{f}(\xi)|^2 d\xi \leq \|\widehat{f}\|_2^2 = \|f\|_2^2$$

By Young's inequality any $K \in L_q$ maps $L_p \to L_{p'}$ provided $\frac{1}{p} - \frac{1}{p'} = 1 - \frac{1}{q}$. Therefore $f \in W_{1,p}$ implies $f \in L_{p'}$ so long as $\frac{1}{p} - \frac{1}{p'} < \frac{1}{d}$. By induction $f \in W_{k,p}$ implies that $f \in W_{1,p}$ implies $f \in L_{p'}$ so long as $\frac{1}{p} - \frac{1}{p'} < \frac{k}{d}$). Therefore on R^d, $f \in W_{k,p}$ implies the continuity of f if $k > \frac{d}{p}$.

Actually one can prove a stronger result to the effect that if $\frac{1}{p} - \frac{1}{p'} = \frac{1}{d}$. then $W_{1,p} \subset L_{p'}$ as long as $1 < p' < \infty$. This requires the following theorem.

Theorem 4.1. *Let T_a be the operator of convolution by the kernel $|x|^{a-d}$ on R^d.*

$$(T_a f)(x) = \int_{R^d} |y|^{a-d} f(x+y)dy \tag{4.2}$$

Then T_a is bounded from L_p to $L_{p'}$ provided $1 < p < \frac{d}{a}$ and $\frac{1}{p'} = \frac{1}{p} - \frac{a}{d}$.

Proof. First, we note that for $a > 0$, T_a is well defined on bounded functions with compact support. We start by proving a weak type inequlity of the form

$$\mu[x : |(T_a f)(x)| \geq \ell] \leq C \frac{\|f\|_p^q}{\ell^q}$$

For any choice of $1 < p < \frac{d}{a}$ let $f \in L_p$. We can assume without loss of generality that $f \geq 0$. We write

$$\begin{aligned}(T_a f)(x) &= \int_{|y| \leq \rho} |y|^{a-d} f(x+y)dy + \int_{|y| \geq \rho} |y|^{a-d} f(x+y)dy \\ &\leq u_1 + u_2\end{aligned}$$

and estimate u_1, u_2 by

$$\begin{aligned}\|u_1\|_p &\leq C_1 \rho^a \|f\|_p \\ \|u_2\|_\infty &\leq \left(\int_{|y| \geq \rho} |y|^{p^*(a-d)} dy \right)^{\frac{1}{p^*}} \|f\|_p = C_2 \rho^{a-d+\frac{d}{p^*}} \|f\|_p\end{aligned}$$

We can now pick $\rho = (\frac{2C_2\|f\|_p}{\ell})^{\frac{p}{d-ap}}$ and estimate $\sup_x u_2(x) \leq \frac{\ell}{2}$ as well as

$$\mu[x : u_1(x) \geq \frac{\ell}{2}] \leq 2^p C_1^p \rho^{a\,p} \frac{\|f\|_p^p}{\ell^{\,p}} = C_3 \left(\frac{\|f\|_p}{\ell} \right)^{\frac{ap^2}{d-ap}+p} = C_3 \left(\frac{\|f\|_p}{\ell} \right)^q$$

where $q = \frac{pd}{d-ap}$ or $\frac{1}{q} = \frac{1}{p} - \frac{a}{d}$. □

Now, an application of Marcinkiewicz interpolation gives boundedness from L_p to L_q in the same range and with the same relation between p and q.

What about the trace of a function on a lower dimensional set? For example if $u \in R^n \in W_{m,2}$ what can one say about the function

$$v(x_1, \cdots, x_k) = u(x_1, \ldots, x_k, 0, \ldots, 0)$$

the restriction of u to a k dimensional hyperplane.

Theorem 4.2.

$$\|v\|_{m-\frac{n-k}{2},2} \leq C\|u\|_{m,2}$$

Lose $\frac{1}{2}$ derivative for each restriction to co-dimension 1.

Proof. Assume $k = n - 1$. Let $\widehat{u}(y_1, \ldots, y_n)$ be the Fourier transform.

$$\widehat{v}(y_1, \ldots, y_{n-1}) = \int \widehat{u}(y_1, \ldots, y_n) dy_n$$

$$\int_{R^{n-1}} \left[| \int_R \widehat{u}(y_1, \ldots, y_n) dy_n|^2 \right] \left[1 + |y_1|^2 + \cdots + |y_{n-1}|^2 \right]^{m-\frac{1}{2}}] dy_1 \cdots dy_{n-1}$$

Write

$$\widehat{u}(y_1, \ldots, y_n) = \widehat{u}(y_1, \ldots, y_n)[1 + |y_1|^2 + \cdots + |y_n|^2]^{\frac{m}{2}} [1 + |y_1|^2 + \cdots + |y_n|^2]^{-\frac{m}{2}}$$

By Schwartz inequality

$$\begin{aligned} &\int_{R^{n-1}} | \int_R \widehat{u}(y_1, \ldots, y_n) dy_n|^2 dy_1 \cdots dy_{n-1} \\ &\qquad \leq |\widehat{u}|_{m,2}|^2 \left[\sup_{y_1, \cdots, y_{n-1}} \int [1 + |y_1|^2 + \cdots + |y_n|^2]^{-\frac{m}{2}} dy_n \right] \end{aligned}$$

$$\int [1 + |y_1|^2 + \cdots + |y_n|^2]^{-\frac{m}{2}} dy_n \leq C[1 + |y_1|^2 + \cdots + |y_{n-1}|^2]^{\frac{m-1}{2}}$$

provided $m > \frac{1}{2}$. □

4.3 Fractional Derivatives.

We can also define the fractional derivative operarors

$$(|D|^a f)(x) = \int_{R^d} \frac{f(x+y) - f(x)}{|y|^{d+a}} dy \tag{4.3}$$

for $0 < a < 2$. A calculation shows that in terms of Foirier transforms it is multiplication by

$$\int_{R^d} \frac{e^{i<\xi\,,y>} - 1}{|y|^{d+a}} dy = c_{d,a}|\xi|^a$$

Therefore $|D|^a$ and T_a are essentially (upto a constant) inverses of each other. If $r > 0$ is written as $k + a$, where k is a nonnegative integer and $0 \le a < 1$, then one defines the norm corresponding to r^{th} derivative by

$$\|u\|_{r,p} = \sum_{\sum_i n_i \le k} \|D_1^{n_1} \cdots D_d^{n_d} u\|_p + \sum_{\sum_i n_i = k} \|D_1^{n_1} \cdots D_d^{n_d} u\|_{a,p} \tag{4.4}$$

where for $0 \le a < 1$, $\|u\|_{a.p} = \||D|^a u\|_p$. This way the Sobolev spaces $W_{r,p}$ are defined for fractional derivatives as well.

Theorem 4.3. *The inclusion map is well defined and bounded from $W_{r,p}$ into $W_{s,q}$ provided $s < r$, $1 < p < q < \infty$, and $\frac{1}{q} \ge \frac{1}{p} - \frac{r-s}{d}$. The extreme value of $q = \infty$ is allowed if $\frac{1}{q} > \frac{1}{p} - \frac{r-s}{d}$.*

Proof. We can assume without loss of generality that $0 < r - s < 1$. We can go from $W_{r,p}$ to $W_{s,q}$ in a finite number of steps, with $0 < r - s < 1$ at each step. We write $\mathcal{I} = c_{d,a} T_a |D|^a$ where $a = r - s$. By definition $|D|^a$ maps $W_{r,p}$ boundedly into $W_{s,p}$. By the earlier theorem T_a maps $W_{s,p}$ boundedly into $W_{s,q}$. Although we proved it for $s = 0$, it is true for every s because T_a commutes with $|D|^a$. The cae $q = \infty$ is covered as well by this argument. □

4.4 Generalized Functions.

Let us begin with the space $W_{1,2}$. This is a Hilbert Space with the inner product

$$< f, g >_1 = \int_{R^d} [f\bar{g} + \sum_1^d f_{x_i} \bar{g}_{x_i}] dx = \int_{R^d} f\bar{h} dx$$

where $h = g - \sum_1^d g_{x_i x_i}$. Since $g \in W_{1,2}$, $g_{x_i} \in L_2$ and $g_{x_i x_i}$ is the derivative of an L_2 function. In fact since we can write $\int f g_{x_i} dx$ as $-\int f_{x_i} g dx$, Any derivative of an L_2 function can be thought of as a bounded linear functional on the space $W_{1,2}$. A simlar reasoning applies to all the spaces $W_{r,p}$. The dual space of $W_{r,p}$ is $W_{-r,q}$ where $\frac{1}{p} + \frac{1}{q} = 1$.

For a function to be in L_p its singularities as well as decay at ∞ must be controlled. We can get rid of the condition at ∞ by demaniding that f be in $L_p(K)$ for every bounded set K or equivalently by insisting that $\phi f \in L_p$ for every C^∞ function ϕ with compact support. This definition makes sense for $W_{r,p}$ as well. We say that $f \in W_{r,p}^{loc}$ if $\phi f \in W_{r,p}$ for every C^∞ function ϕ with compact support. One needs to check that on $W_{r,p}$ mutiplication by a smooth function is a bounded linear map. One can use Leibnitz's rule if r is an integer. For $0 < r < 1$ we need the following lemma.

Lemma 4.4. *If $f \in W_{r,p}$ and $\phi \in C^{r'}$ with $r < r' \le 1$ i.e. ϕ is a bounded function satisfying $|\phi(x) - \phi(y)| \le C|x - y|^{r'}$, for all x, y, then $\phi f \in W_{r,p}$.*

Proof. We need to prove

$$g(x) = \int_{R^d} \frac{\phi(y)f(y) - \phi(x)f(x)}{|y - x|^{d+r}} dy$$

is in L_p. We can write

$$\phi(y)f(y) - \phi(x)f(x) = \phi(x)[f(y) - f(x)] + [\phi(y) - \phi(x)]f(y).$$

The contribution of first term is easy to control. To control the second term it is sufficient to show that

$$\sup_x \int_{R^d} \frac{|\phi(y) - \phi(x)|}{|y - x|^{d+r}} dy < \infty$$

because

$$\| \int_{R^d} K(x, y)f(y)dy\|_p \le (\sup_x \int_{R^d} |K(x, y)|dy)\|f\|_p$$

To this end we split the integral into two regions $|x - y| \le 1$ and $|x - y| > 1$, use the Hölder property of ϕ to obtain an estimate on the integral over $|x - y| \le 1$ and the boundedness of ϕ to get an estimate over $|x - y| > 1$, both of which are uniform in x. ☐