Chapter 12

Random Time Change

If $x(t)$ is a Markov process with generator $\mathcal{L}$, i.e. on some space of trajectories with filtration $(\Omega, \mathcal{F}_t, x(t), P)$ this is captured by the requirement that for a large class of test functions f,

$$f(x(t)) - f(x(0)) - \int_0^t (\mathcal{L}f)(x(s))ds$$

is a martingale. If we speed up time and write $y(t) = x(2t)$ then the generator for the process $y(t)$ is just $2\mathcal{L}$. This extends to random time changes of a certain type. Let $a(x)$ be a positive measurable function bounded above and below i.e. $0 < c \leq a(x) \leq C < \infty$. Consider the stopping times $\{\tau(t)\}$ defined for $t > 0$ by

$$\int_0^{\tau(t)} \frac{ds}{a(x(s))} = t; \quad d\tau'(t) = a(y(t))dt \tag{12.1}$$

These are bounded stopping times, and τ_t is strictly increasing. On the space of functions $\Omega = D[[0,\infty), X]$ they define a map $x(t) \to y(t) = x(\tau(t))$. We denote this map of $\Omega \to \Omega$ by $T_{a(\cdot)}$. It is easy to verify that $T_{a(\cdot)}T_{b(\cdot)} = T_{b(\cdot)}T_{a(\cdot)} = T_{ab(\cdot)}$. If in addition to (11.1) we also have

$$\int_0^{\sigma(t)} \frac{ds}{b(y(s))} = t; \quad \sigma'(t) = b(z(t))dt$$

Then $z(t) = y(\sigma(t)) = x(\theta(t))$, with $\theta(t) = \tau(\sigma(t))$, and we have

$$d\theta(t) = \tau'(\sigma(t))\sigma'(t)dt = a(y(\sigma(t))b(z(t))dt = a(z(t))b(z(t))dt$$

proving $T_{ab} = T_a T_b = T_b T_a$. In particular $T_a^{-1} = T_{a^{-1}}$. The map T_a is one to one and on to map between solutions of $\mathcal{L}$ and $a(x)\mathcal{L}$. To see this we observe that

$$f(x(\tau(t)) - f(x(0)) - \int_0^{\tau(t)} (\mathcal{L}f)(x(s))ds$$

is a martingale with respect to $(\Omega, \mathcal{F}_{\tau(t)}, P)$. We can rewrite the above as

$$f(y(t)) - f(y(0)) - \int_0^t a(y(s))(\mathcal{L}f)(y(s))ds$$

is a martingale. If $x(t)$ is a solution for $\mathcal{L}$, then $y(t)$ is a solution for $a(x)\mathcal{L}$. In particular in one dimension any process for $[a(x), b(x)]$ with $0 < c \leq a(x) \leq C < \infty$ can be obtained easily from Brownian motion. A random time change will get us to $[a(x), 0]$ from $[1, 0]$ and a Girsanov transformation with the suitable Radon Nikodym derivative will bring us to $[a, b]$. These are reversible steps and therefore uniqueness for $[a, b]$ follows from the uniqueness for $[1, 0]$ which is the characterization of Brownian motion.

Chapter 13

Local time

Formally the local time of Brownian motion is

$$l(t,y) = \int_0^t \delta(x(s)-y)ds = \lim_{h\to 0} \frac{1}{2h} \int_0^t \mathbf{1}_{[y-h,y+h]}(x(s))ds$$

Theorem 13.1. *The limit $l(t,y)$ exists, with probability 1 as a jointly continuous function of t and y, and is uniquely defined by the property*

$$\int_R f(y)l(t,y)dy = \int_0^t f(x(s))ds$$

for any bounded measurable f.

Remark 13.1. If we define $L(t,A) = \int_0^t \mathbf{1}_A(x(s))ds$, i.e. the amount of time spent by the Brownian path in the set A during $[0,t]$, then $L(t,\cdot,\omega)$ is a random measure on R with total mass t. The theorem says it is almost surely absolutely continuos with a continuous density $l(t,y)$. We will show that with probability 1, it is continuous in y for each t and continuous in t for each y. In fact with a little more work one can prove that with probability 1, it is jointly continuous in t and y.

Proof. If we try to apply Itô's formula for $f(x) = |x-a|$ we could say $f'(x) = \sigma(x-a)$ and $f''(x) = 2\delta(x-a)$, where $\sigma(x) = \frac{x}{|x|} = \pm 1$ for $x \neq 0$. Therefore

$$|x(t)-a| - |a| = \int_0^t \sigma(x(s))dx(s) + \int_0^t \delta(x(s)-a)ds$$

or

$$l(t,a) = |x(t)-a| - |a| - \int_0^t \sigma(x(s))dx(s)$$

If we define $u_h(x)$ as the solution of $u''(x) = \frac{1}{h}\mathbf{1}_{[-h,h]}(x)$ with $u(x) = u'(x) = 0$, it can be explicitly solved and we can verify that $u_h(x) \to |x|$ and $u_h'(x) \to \sigma(x)$. ∎

For each $h > 0$ we have

$$u_h(x(t)-a) - u_h(-a) - \int_0^t u_h(x(s)-a)dx(s) = \int_0^t \frac{1}{2h}\mathbf{1}_{[a-h,a+h]}(x(s))ds$$

It is easy to check that limit exists on the left hand side and so it does on the right hand side as well. If we multiply both sides by $f(a)$ and integrate we get

$$F(x(t)) - F(x(0)) - \int_0^t F'(x(s)ds = \int l(t,y)f(y)dy$$

and $F''(x) = 2f(x)$. Comparing with Itô's formula proves that with probability 1,

$$\int_0^t f(x(s))ds = \int l(t,y)f(y)dy$$

The questions of continuity are easily established. Clearly the stochastic integral

$$\int_0^t \sigma(x(s))dx(s)$$

is continuous almost surely. As for continuity in y for fixed t we estimate for $y < z$

$$\begin{aligned}
&E[|\int_0^t [\sigma(x(s)-y) - \sigma(x(s)-z)]dx(s)|^4] \\
&\quad \le CE\big[[\int_0^t [\sigma(x(s)-y) - \sigma(x(s)-z)]^2 ds]^2\big] \\
&\quad = CE[|\int_0^t \mathbf{1}_{[y,z]}(x(s))ds|^2] \\
&\quad = 2C\int_{0\le s_1\le s_2\le t}\int_y^z\int_y^z \frac{1}{\sqrt{2\pi s_1}}e^{-\frac{x_1^2}{2s_1}}\frac{1}{\sqrt{2\pi(s_2-s_1)}}e^{-\frac{(x_2-x_1)^2}{2(s_2-s_1)}}dx_1dx_2ds_1ds_2 \\
&\quad \le C|y-z|^2\int_{0\le s_1\le s_2\le t}\frac{1}{\sqrt{2\pi s_1}}\frac{1}{\sqrt{2\pi(s_2-s_1)}}ds_1ds_2 \\
&\quad \le C(t)|y-z|^2
\end{aligned}$$

which is enough to prove continuity in y for fixed t. One can estimate

$$E[l(t_1,x_1) - l(t_2,x_2)|^6] \le C|x_1-x_2|^3 + C|t_1-t_2|^3$$

and this would do it. (Two dimensional version of Kolmogorov's theorem). The multidimensional version of Kolmogorov's theorem says that a sufficient condition for a process $\xi(x) : x \in R^d$ to be almost surely continuos is the estimate

$$E[|\xi(x) - \xi(y)|^p] \le C|x-y|^{d+\alpha}$$

for some $p > 0$ and $\alpha > 0$.