Chapter 13

The two dimensional case

If we have a diffusion in $2-d$ with bounded b and a which has its eigenvalues uniformly bounded between λ_1 and λ_2, by Girsanov transformation we can get rid of b and by random time change normalize so that $Tra = 2$. Then if we want to solve

$$\lambda u - \mathcal{L}u = f$$

i.e invert $\lambda I - \mathcal{L}$ we treat it as perturbation of $I - \frac{1}{2}\Delta$. We need to show that with $\mathcal{L} = \frac{1}{2}a_{i,j}(x)D_iD_j$ and $E = \mathcal{L} - \frac{1}{2}\Delta$,

$$[\lambda I - \mathcal{L}]^{-1} = [\lambda I - \frac{1}{2}\Delta - E]^{-1} = [\lambda I - \frac{1}{2}\Delta]^{-1}[\lambda I - E[I - \frac{1}{2}\Delta]^{-1}]^{-1}$$

is well defined in some function space. We will show that it maps $L_2 \to H_2$. This will be done in two steps. $[\lambda I - \frac{1}{2}\Delta]^{-1}$ maps $L_2 \to H_2$ for each positive λ and $\|E[\lambda I - \frac{1}{2}\Delta]^{-1}\| \leq \rho < 1$ uniformly for all $\lambda > 0$. The first step is accomplished by Fourier transform. The operation is multiplication by $e_\lambda(\xi) = (\lambda + \frac{1}{2}|\xi|^2)^{-1}$. For positive λ and $i = 1, 2$, $e_\lambda(\xi), \xi_i e_\lambda(\xi)$ are bounded while $\xi_i\xi_j e_\lambda(\xi)$ has a bound independent of λ. For the second step, we notice that if $a + b = 2$ and $ab - h^2 \geq c > 0$, then

$$(a-1)^2 + h^2 = \frac{1}{2}[(a-1)^2 + (b-1)^2 + 2h^2] \leq 1 - c$$

$$\begin{aligned}
|(a-1)u + (b-1)v + 2hw|^2 &= |(a-1)(u-v) + 2hw|^2 \\
&\leq [(a-1)^2 + h^2][(u-v)^2 + 4w^2] \\
&\leq (1-c)[(u-v)^2 + 4w^2]
\end{aligned}$$

Therefore

$$\begin{aligned}
\|\frac{1}{2}\sum_{i,j=1,2}(a_{i,j}(x)-\delta_{i,j}(x))u_{i,j}\|^2 &\leq \frac{1-c}{4}\int_{R^2}[(u_{1,1}-u_{2,2})^2+4u_{1,2}^2]^2dx \\
&= \frac{1-c}{4}\int_{R^2}[(\xi_1^2-\xi_2^2)^2+4\xi_1^2\xi_2^2][\hat{u}]^2d\xi \\
&= \frac{1-c}{4}\int_{R^2}(\xi_1^2+\xi_2^2)^2\left[\frac{1}{\lambda+\frac{1}{2}(\xi_1^2+\xi_2^2)}\right]^2[\hat{f}]^2d\xi \\
&= (1-c)\int_{R^2}^2\left[\frac{(\xi_1^2+\xi_2^2)}{2\lambda+(\xi_1^2+\xi_2^2)}\right]^2[\hat{f}]^2d\xi \\
&\leq (1-c)\int_{R^2}^2[\hat{f}]^2d\xi = (1-c)\|f\|^2
\end{aligned}$$

The perturbation argument will now work and the rest of the existence and uniqueness argument proceeds like the one dimensional case.