Chapter 15

Limit Theorems

One of the important aspects of diffusion processes is its appearance as the limit of suitably scaled Markov Chains. Suppose we have on R^d a family of Markov Chains with transition probability $\{\pi_h(x, dy)\}$. We think of h as the time unit and $h \to 0$. We start with the Markov chain with transition probability $\pi_h(x, dy)$ and imbed it in a stochastic process as $\{x(nh)\}$. We can make it piecewise constant and in the space $D[[0, T]; R^d]$ we will have a family of probability measures $\{P_x^h\}$ indexed by the parameter h and the starting point $x(0) = x$. We want to determine conditions under which P_x^h will converge to a diffusion process P_x that corresponds to the generator

$$\mathcal{L} = \frac{1}{2} \sum_{i,j} a_{i,j}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_i b_i(x) \frac{\partial}{\partial x_i}$$

Theorem 15.1. *We begin by assuming that $\{a_{i,j}(x)\}, \{b_j(x)\}$ are continuous and $\mathcal{C}(a, b, 0, x)$ contains exactly one process P_x. Assume further that for any $\rho > 0$ and uniformly for x in any compact set of R^d,*

$$\lim_{h \to 0} \frac{1}{h} \int_{B(x,\rho)^c} \pi_h(x, dy) = 0 \tag{15.1}$$

Assume also that for some (and in view of (15.1) for any) $\rho > 0$,

$$\lim_{h \to 0} \frac{1}{h} \int_{B(x,\rho)} (y_i - x_i) \pi_h(x, dy) = b_i(x) \tag{15.2}$$

$$\lim_{h \to 0} \frac{1}{h} \int_{B(x,\rho)} (y_i - x_i)(y_j - x_j) \pi_h(x, dy) = a_{i,j}(x) \tag{15.3}$$

It follows, by Taylor expansion, that for any smooth bounded f

$$\lim_{h \to 0} \frac{1}{h} \int [f(y) - f(x)] \pi_h(x, dy) = (\mathcal{L}f)(x) \tag{15.4}$$

and the limit is uniform when x varies over compact subsets of R^d as well as when the function f varies over a class of uniformly smooth functions. The processes $P_x^h \to P_x$ on $D[[0,T];R^d]$ as $h \to 0$ uniformly as x varies over a compact set.

Proof. The proof consists of several steps.

Step 1. Let us first check that by (15.1) and Taylor's formula with reminder

$$
\begin{aligned}
\frac{1}{h}\int [f(y)-f(x)]\pi_h(x,dy) &= \frac{1}{h}\int_{B(x,\rho)} [f(y)-f(x)]\pi_h(x,dy) + \epsilon_\rho(h) \\
&= \frac{1}{h}\int_{B(x,\rho)} [\sum_i (D_i f)(x)(y_i - x_i) \\
&\qquad + \frac{1}{2}\sum_{i,j}(D_i D_j f)(x)(y_i - x_i)(y_j - x_j)]\pi_h(x,dy) \\
&\qquad + \epsilon(\rho)\frac{1}{h}\int_{B(x,\rho)} |y-x|^2 \pi^h(x,dy) + \epsilon_\rho(h) \\
&= (\mathcal{L}f)(x) + \epsilon_\rho(h) + C\epsilon(\rho)
\end{aligned}
$$

Letting $h \to 0$ and then $\rho \to 0$ establishes (15.4).

Step 2. First we use smooth truncation to stop the process when it exits from a large ball of radius 2ℓ around the origin. This is done by defining

$$
\pi_h^\ell(x,dy) = \phi(\frac{x}{\ell})\pi_h(x,dy) + (1-\phi(\frac{x}{\ell}))\delta_x(dy)
$$

where $0 \le \phi(\cdot) \le 1$ is a smooth function which is 1 on $B(0,1)$ and 0 in $B(0,2)^c$. Then

$$
\frac{1}{h}\int [f(y)-f(x)]\pi_h^\ell(x,dy) = \phi(\frac{x}{\ell})(\mathcal{L}f)(x)
$$

We will now show that the stopped processes $\{P_x^{h,\ell}\}$ is a tight family of measures on $D[[0,T];R^d]$. It is enough to prove that for each $\ell < \infty$ and $\delta > 0$,

$$
\lim_{t\to 0}\sup_{|x|\le 2\ell} P_x^{\ell,h}[\sup_{0\le s\le t}|x(s)-x(0)| \ge \delta] = 0
$$

Let $f_x(y)$ be a smooth function which is 0 at x and 1 outside $B(x,\delta)$. The smoothness of the function depends only on δ and consequently

$$
|\frac{1}{h}\int [f_x(y)-f_x(x)]\pi_h^\ell(x,dy)| \le C_\delta
$$

It is easy to verify that with respect to $(D[0,\infty),\mathcal{F}_{kh}^0,P_x^{h,\ell})$, with $f(y) = f_x(y)$

$$
Z_h^\ell(kh) = f(x(kh)) - f(x) - \sum_{j=0}^{k-1}\int [f(y) - f(x(jh))]\pi_h^\ell(x,dy)
$$

is a martingale. In particular if $\tau_\delta = \{\inf t : |x(t) - x(0)| \geq \delta\}$,

$$P_x^{h,\ell}[\tau_\delta \leq t] \leq E[f(x(\tau \wedge t))] \leq C_\delta t$$

and this holds uniformly in the ball $B(0, 2\ell)$. The process stops outside this ball. This proves tightness and by (15.1) the limiting measure is supported on $C[0, \infty)$.

Step 3. Since $Z_h^\ell(kh)$ is a martingale with respect to $\mathcal{F}_{kh}^0$, it follows that with respect to any limit Q^ℓ of $P_x^{h,\ell}$ will have the property that for any smooth f

$$f(x(t)) - f(x) - \int_0^t \phi(x(s))(\mathcal{L}f)(x(s))ds$$

is a martingale with respect to $(C[0,\infty), \mathcal{F}_t^0, Q^\ell)$. Since $\phi(x) = 1$ in the ball $B(0, \ell)$, Q^ℓ will coincide with P_x on $\mathcal{F}_{\tau_\ell}^0$. Since P_x exists it follows that

$$\lim_{\ell \to \infty} Q^\ell[\tau_\ell \leq t] = \lim_{\ell \to \infty} P_x[\tau_\ell \leq t] = 0$$

and this proves that $P_x^h \to P_x$. $\square$