Chapter 3

Diffusion Processes

3.1 What is a Diffusion Process?

When we want to model a stochastic process in continuous time it is almost impossible to specify in some reasonable manner a consistent set of finite dimensional distributions. The one exception is the family of Gaussian processes with specified means and covariances. It is much more natural and profitable to take an evolutionary approach. For simplicity let us take the one dimensional case where we are trying to define a real valued stochastic process with continuous trajectories. The space $\Omega = C[0,T]$ is the space on which we wish to construct the measure P. We have the σ-fields $\mathcal{B}_t = \sigma\{x(s) : 0 \le s \le t\}$ defined for $t \le T$. The total σ-field $\mathcal{B} = \mathcal{B}_T$. We try to specify the measure P by specifying approximately the conditional distributions $P[x(t+h)-x(t) \in A|\mathcal{B}_t]$. These distributions are nearly degenerate and and their mean and variance are specified as

$$E^P\big[x(t+h) - x(t)|\mathcal{B}_t\big] = h\, b(t,\omega)) + o(h) \tag{3.1}$$

and

$$E^P\big[(x(t+h) - x(t))^2|\mathcal{B}_t\big] = h\, a(t,\omega)) + o(h) \tag{3.2}$$

as $h \to 0$, where for each $t \ge 0$ $b(t,\omega)$ and $a(t,\omega)$ are $\mathcal{B}_t$ measurable functions. Since we insist on continuity of paths, this will force the distributions to be nearly Gaussian and no additional specification should be necessary. We will devote the next few lectures to investigate this.

Equations (3.1)and (3.2) are infinitesimal differential relations and it is best to state them in integrated forms that are precise mathematical statements.

We need some definitions.

Definition 3.1. *We say that a function $f : [0,T] \times \Omega \to R$ is progressively measurable if, for every $t \in [0,T]$ the restiction of f to $[0,t] \times \Omega$ is a measurable function of t and ω on $([0,t] \times \Omega, \mathcal{B}[0,t] \times \mathcal{B}_t)$ where $\mathcal{B}[0,t]$ is the Borel σ-field on $[0,t]$.*

The condition is somewhat stronger than just demanding that for each t, $f(t,\omega)$ is $\mathcal{B}_t$ is measurable. The following facts are elementary and left as exercises.

Exercise 3.1. If $f(t,x)$ is measurable function of t and x, then $f(t,x(t,\omega))$ is progressively meausrable.

Exercise 3.2. If $f(t,\omega)$ is either left continuous (or right continuous) as function of t for every ω and if in addition $f(t\,\omega)$ is $\mathcal{B}_t$ measurable for every t, then f is progressively measurable.

Exercise 3.3. There is a sub σ-field $\Sigma = \Sigma_{pm} \subset \mathcal{B}[0,T] \times \mathcal{B}_T)$ such that progressive measurability is just measurability with respect to Σ_{pm}. In particular standard operations performed on progressively measurable functions yield progressively measurable functions.

We shall always insist that the functions $b(\cdot\,,\cdot)$ and $a(\cdot\,,\cdot)$ be progressively measurable. Let us suppose in addition that they are bounded functions. The boundedness will be relaxed at a later stage.

We reformulate conditions 3.1 and 3.2 as

$$M_1(t) = x(t) - x(0) - \int_0^t b(s,\omega)ds \tag{3.3}$$

and

$$M_2(t) = [M_1(t)]^2 - \int_0^t a(s\,,\omega))ds \tag{3.4}$$

are martingales with respect to $(\Omega, \mathcal{B}_t, P)$.

We can then define a Diffusion Process corresponding to a, b as a measure P on $(\Omega, \mathcal{B})$ such that relative to $(\Omega, \mathcal{B}_t, P)$ $M_1(t)$ and $M_2(t)$ are martingales. If in addition we are given a probability measure μ as the initial distribution, i.e.

$$\mu(A) = P[x(0) \in A]$$

then we can expect P to be determined by a, b and μ.

We saw already that if $a \equiv 1$ and $b \equiv 0$, with $\mu = \delta_0$, we get the standard Brownian Motion. $a = a(t,x(t))$ and $b = b(t,x(t))$, we expect P to be a Markov Process, because the infinitesimal parameters depend only on the current position and not on the past history. If there is no explicit dependence on time, then the Markov Process can be expected to have stationary transition probabilities. Finally if $a(t,x) = a(t)$ is purely a function of t and $b(t,\omega)) = b_1(t) + \int_0^t c(t\,,s)x(s)ds$ is linear in ω, then one expects P to be Gaussian, provided μ is so.

Because the paths are continuous the same argument that we provided earlier can be used to establish that

$$\begin{aligned} Z_\lambda(t) &= \exp[\lambda M_1(t) - \frac{\lambda^2}{2}\int_0^t a(s\,,\omega)ds] \\ &= \exp[\lambda[x(t) - x(0) - \int_0^t b(s\,,\omega)ds] - \frac{\lambda^2}{2}\int_0^t a(s\,,\omega)ds] \end{aligned} \tag{3.5}$$

is a martingale with respect to $(\Omega, \mathcal{B}_t, P)$ for every real λ. We can also take for our definition of a Diffusion Process corresponding to a, b the condition that $Z_\lambda(t)$ be a martingale with respect to $(\Omega, \mathcal{B}_t, P)$ for every λ. If we do that we did not have to assume that the paths were almost surely continuous. $(\Omega, \mathcal{B}_t, P)$ could be any space suppporting a stochastic process $x(t\,, \omega)$ such that the martingale property holds for $Z_\lambda(t)$. If C is an upper bound for a, it is easy to check with $M_1(t)$ defined by equation (3.5)

$$E^P\left[\exp[\lambda[M_1(t) - M_1(s)]\right] \leq \exp[\frac{\lambda^2 C}{2}]$$

The lemma of Garsia, Rodemich and Rumsey will guarantee that the paths can be chosen to be continuous.

Let $(\Omega, \mathcal{F}, P)$ be a Probability space. Let $\mathbf{T}$ be the interval $[0, T]$ for some finite T or the infinite interval $[0, \infty)$. Let $\mathcal{F}_T \subset \mathcal{F}$ be sub σ-fields such that $\mathcal{F}_s \subset \mathcal{F}_t$ for $s, t \in \mathbf{T}$ with $s < t$. We can assume with out loss of generality that $\mathcal{F} = \vee_{t\in\mathbf{T}}\mathcal{F}_t$. Let a stochastic process $x(t\,, \omega)$ with values in R^n be given. Assume that it is progressively measurable with respect to $(\Omega\,, \mathcal{F}_t)$. We can easily gneralize the ideas described in the previous section to diffusion processe with values in R^n. Given a positive semidefinite $n \times n$ matrix $a = a_{i,j}$ and an n-vector $b = b_j$, we define the operator

$$(\mathcal{L}_{a,b}f)(x) = \frac{1}{2}\sum_{i,j=1}^{n} a_{i,j}\frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \sum_{j=1}^{n}\frac{\partial f}{\partial x_j}(x)$$

If $a(t\,, \omega) = a_{i,j}(t\,, \omega)$ and $b(t\,, \omega) = b_j(t\,, \omega)$ are progresssively measurable functions we define

$$(L_{t\,,\omega}f)(x) = (L_{a(t\,,\omega),b(t\,,\omega)}f)(x)$$

Theorem 3.1. *The following defintions are equivalent. $x(t\,, \omega)$ is a diffusion process correponding to bounded progressively measurable functions $a(\cdot\,, \cdot), b(\cdot\,, \cdot)$ with values in the space of symmetric positive semidefinite $n \times n$ matrices, and n-vectors if*

1. *$x(t\,, \omega)$ is surely continuous and*

$$y_i(t\,, \omega) = x_i(t\,, \omega) - x_i(0\,, \omega) - \int_0^t b(s\,, \omega)ds$$

and

$$z_{i,j}(t\,, \omega) = y_i(t\,, \omega)\, y_j(t\,, \omega) - \int_0^t a_{i,j}(s\,, \omega)ds$$

are $(\Omega, \mathcal{F}_t, P)$ martingales.

2. *For every $\lambda \in R^n$*

$$Z_\lambda(t\,, \omega) = \exp\left[< \lambda\,, y(t\,, \omega) > -\frac{1}{2}\int_0^t < \lambda, a(s\,, \omega)\lambda > ds\right]$$

is an $(\Omega, \mathcal{F}_t, P)$ martingale.

3. *For every* $\lambda \in R^n$

$$X_\lambda(t\,,\omega) = \exp\left[i < \lambda\,, y(t\,,\omega) + \frac{1}{2}\int_0^t < \lambda,\, a(s\,,\omega)\lambda > ds\right]$$

is an $(\Omega, \mathcal{F}_t, P)$ *martingale.*

4. *For every smooth bounded function* f *on* R^n *with atleast two bounded continuous derivatives*

$$f(x(t\,,\omega)) - f((x(0\,,\omega)) - \int_0^t (\mathcal{L}_{s,\omega} f)(x(s\,,\omega))ds$$

is an $(\Omega, \mathcal{F}_t, P)$ *martingale.*

5. *For every smooth bounded function* f *on* $\mathbf{T} \times R^n$ *with atleast two bounded continuous* x *derivatives and one bounded continuous* t *derivative*

$$f(t\,, x(t\,,\omega)) - f(0\,, (x(0\,,\omega)) - \int_0^t (\frac{\partial f}{\partial s} + \mathcal{L}_{s,\omega} f)(s\,, x(s\,,\omega))ds$$

is an $(\Omega, \mathcal{F}_t, P)$ *martingale.*

6. *For every smooth bounded function* f *on* $\mathbf{T} \times R^n$ *with atleast two bounded continuous* x *derivatives and one bounded continuous* t *derivative*

$$\exp\left[f(t\,, x(t\,,\omega)) - f(0\,, (x(0\,,\omega)) - \int_0^t (\frac{\partial f}{\partial s} + \mathcal{L}_{s,\omega} f)(s\,, x(s\,,\omega))ds \right.$$
$$\left. - \frac{1}{2}\int_0^t < (\nabla f)(s\,, x(s\,,\omega)), a(s\,,\omega)\,(\nabla f)(s\,, x(s\,,\omega)) > ds\right]$$

is an $(\Omega, \mathcal{F}_t, P)$ *martingale.*

7. *Same as (6) except that* f *is replaced by* g *of the form*

$$g(t\,, x) = < \lambda,\, x > + f(t\,, x)$$

where f *is as in (6) and* $\lambda \in R^n$ *is arbitrary.*

Under any one of the above definitions, $x(t\,,\omega)$ *has an almost surely continuous version satifying*

$$P\left[\sup_{0 \le s \le t} |y(s\,,\omega) - y(0\,,\omega)| \ge \ell\right] \le 2n \exp[\frac{-\ell^2}{Ct}]$$

for some constant C *depending only on the dimension* n *and the upper bound for* a*. Here*

$$y_i(t\,,\omega) = x_i(t\,,\omega) - x_i(0\,,\omega) - \int_0^t b_i(s\,,\omega)ds$$

Proof. (1) *implies* (2). This was essentially the content of Theorem and the comments of the previous section. Also we saw that the exponential inequality is a consequence of Doob's inequality.
(2) *implies* (3). The condition that $Z_\lambda(t)$ is a martingale can be rewritten as a whole collecction of identities

$$\int_A Z_\lambda(t\,,\omega)dP = \int_A Z_\lambda(s\,,\omega)dP \tag{3.6}$$

that is valid for every $t > s$, $A \in \mathcal{F}_s$ and $\lambda \in R^n$. Both sides of eqation (3.6) are well defined when $\lambda \in R^n$ is replaced by $\lambda \in \mathbf{C}^n$, with complex components and define entire functions of the n complex variables λ. Since they agree when the values are real, by analytic continuation, they must agree for all purely imaginary values of λ as well. This is just (3).
(3) *implies* (4). This part of the proof requires a simple lemma.

Lemma 3.2. *Let $M(t\,,\omega)$ be a martingale relative to $(\Omega\mathcal{F}_t, P)$ which has almost surely continuous trajectories and $A(t\,,\omega)$ be a progressively measurable process that is for almost all ω a continuous function of bounded variation in t. Assume that for every t the random variable $\xi(t\,,\omega) = \sup_{0\le s\le t} |M(t)|\, Var_{[0,t]}\, A(t\,,\omega)$ has a finite expectation. Then*

$$\eta(t) = M(t)A(t) - M(0)A(0) - \int_0^T M(s)dA(s)$$

is again a martingale relative to $(\Omega, \mathcal{F}_t, P)$.

Proof. (of lemma.) We need to prove that for every $s < t$,

$$E^P\Big[M(t)A(t) - M(s)A(s) - \int_s^t M(u)dA(u)\Big|\mathcal{F}_s\Big] = 0 \qquad \text{a.e.}$$

We can subdivide the interval $[s,t]$ into subintervals with end points $s = t_0 < t_1 < \cdots < t_N = t$, and approximate $\int_s^t M(u)dA(u)$ by $\sum_{j=1}^N M(t_j)[A(t_j) - A(t_{j-1})]$. The fact that A is continuous and $\xi(t)$ is integrable makes the approximation work in $L_1(P)$ so that

$$\begin{aligned}
E^P\left[\int_s^t M(u)dA(u)\Big|\mathcal{F}_s\right] &= \lim_{N\to\infty} E^P\left[\sum_{j=1}^N M(t_j)[A(t_j) - A(t_{j-1})]\Big|\mathcal{F}_s\right] \\
&= \lim_{N\to\infty} E^P\left[\sum_{j=1}^N [M(t_j)A(t_j) - M(t_j)A(t_{j-1})]\Big|\mathcal{F}_s\right] \\
&= \lim_{N\to\infty} E^P\left[\sum_{j=1}^N [M(t_j)A(t_j) - M(t_{j-1})A(t_{j-1})]\Big|\mathcal{F}_s\right] \\
&= E^P\left[M(t)A(t) - M(s)A(s)\right]
\end{aligned}$$

and we are done. We used the martingale property in going from the second line to the third when we replaced $M(t_j)A(t_{j-1})$ by $M(t_{j-1})A(t_{j-1})$ □

Now we return to the proof of the theorem. Let us apply the above lemma with $M_\lambda(t) = X_\lambda(t)$ and

$$A_\lambda(t) = \exp[i\int_0^t <\lambda\,, b(s)> ds - \frac{1}{2}\int_0^t <\lambda\,, a(s)\lambda> ds].$$

Then a simple computation yields

$$\begin{aligned} M_\lambda(t)A_\lambda(t) - M_\lambda(0)A_\lambda(0) - \int_0^t M_\lambda(s)dA_\lambda(s) \\ = e_\lambda(x(t) - x(0)) - 1 - \int_0^t (\mathcal{L}_{s,\omega}e_\lambda)((x(s) - x(0))ds \end{aligned}$$

where $e_\lambda(x) = \exp[i <\lambda\,, x>]$. Multiplying by $\exp[i <\lambda\,, x(0)>]$, which is essentially a constant, we conclude that

$$e_\lambda(x(t)) - e_\lambda(x(0)) - \int_0^t (\mathcal{L}_{s,\omega}e_\lambda)((x(s))ds$$

is a martingale. The above expression is just what we had to prove, except that our f is special namely, the exponentials $e_\lambda(x)$. But by linear combinations and limits we can easily pass from exponentials to arbitray smooth bounded functions with two bounded derivatives. We first take care of infinitely diffrentiable functions with compact support by Fourier integrals and then approximate twice differentiable functions with those.

(4) *implies* (3). The steps can be retraced. We start with the martingales defined by (4) in the special case of f being e_λ and choose

$$A_\lambda(t) = \exp[-i\int_0^t <\lambda\,, b(s)> ds + \frac{1}{2}\int_0^t <\lambda\,, a(s)\lambda> ds]$$

and do the computations to get back to the martingales of type (3).

(4) *implies* (5). This is basically a computation. Since $f(t\,, x)$ can be approximated by smooth functions and so we may assume with out loss of generality

more derivatives.

$$
\begin{aligned}
&E^P[f(t\,,x(t)) - f(s\,,x(s))|\mathcal{F}_s] \\
&\quad = E^P[f(t\,,x(t)) - f(t\,,x(s))|\mathcal{F}_s] + E^P[f(t\,,x(s)) - f(s\,,x(s))|\mathcal{F}_s] \\
&\quad = E^P[\int_s^t (\mathcal{L}_{u,\omega} f(t\,,\cdot))(x(u))du|\mathcal{F}_s] + E^P[\int_s^t \frac{\partial f}{\partial u}(u\,,x(s))du|\mathcal{F}_s] \\
&\quad = E^P[\int_s^t (\mathcal{L}_{u,\omega} f(u\,,\cdot))(x(u))du|\mathcal{F}_s] \\
&\qquad + E^P[\int_s^t (\mathcal{L}_{u,\omega}[f(t\,,\cdot) - f(u\,,\cdot)])(x(u))]du|\mathcal{F}_s] \\
&\qquad\quad + E^P[\int_s^t \frac{\partial f}{\partial u}(u\,,x(u))du|\mathcal{F}_s] \\
&\qquad\qquad + E^P[\int_s^t [\frac{\partial f}{\partial u}(u\,,x(s)) - \frac{\partial f}{\partial u}(u\,,x(u))]du|\mathcal{F}_s] \\
&\quad = E^P[\int_s^t [\frac{\partial f}{\partial u} + (\mathcal{L}_{u,\omega} f)](u\,,x(u))du|\mathcal{F}_s] + J
\end{aligned}
$$

where

$$
\begin{aligned}
J &= E^P[\int_s^t (\mathcal{L}_{u,\omega}[f(t\,,\cdot) - f(u\,,\cdot)])(x(u))du|\mathcal{F}_s] \\
&\qquad + E^P[\int_s^t [\frac{\partial f}{\partial u}(u\,,x(s)) - \frac{\partial f}{\partial u}(u\,,x(u))]du|\mathcal{F}_s] \\
&= E^P[\int_s^t \int_u^t (\frac{\partial f}{\partial v}\mathcal{L}_{u,\omega} f)(v\,,x(u))du\,dv|\mathcal{F}_s] \\
&\qquad - E^P[\int_s^t \int_s^u (\mathcal{L}_{v,\omega}\frac{\partial f}{\partial u})(u\,,(x(v))du\,dv|\mathcal{F}_s] \\
&= E^P\Big[\int\int_{s\le u\le v\le t} (\mathcal{L}_{u,\omega}\frac{\partial f}{\partial v})(v\,,(x(u))du\,dv \\
&\qquad - \int\int_{s\le v\le u\le t} (\mathcal{L}_{v,\omega}\frac{\partial f}{\partial u})(u\,,(x(v))du\,dv\Big] \\
&= 0.
\end{aligned}
$$

The two integrals are identical, just the roles of u and v have been interchanged.
(5) *implies* (4). This is trivial because after all in (5) we are allowed to take f to be purely a function of x.
(5) *implies* (6). This is again the lemma on multiplying a martingale by a function of bounded variation. We start with a function of the form $\exp[f(t\,,x)]$ and the martingale

$$
\exp[f(t\,,x(t))] - \exp[f(0\,,x(0))] - \int_0^t (\frac{\partial e^f}{\partial s} + \mathcal{L}_{s,\omega} e^f)(s\,,x(s))ds
$$

and use

$$A(t) = \exp\big[- \int_0^t (\frac{\partial f}{\partial s} + \mathcal{L}_{s,\omega} f)(s\,, x(s))ds$$
$$- \frac{1}{2}\int_0^t < (\nabla f)(s\,, x(s))\,, a(s)(\nabla f)(s\,, x(s)) > ds\big]$$

(6) *implies* (5). This just again reversing the steps.
(6) *implies* (7). The problem here is that the function $< \lambda\,, x >$ are unbounded. If we pick a function $h(x)$ of one variable to equal x in the interval $[-1.1]$ and then levels off smoothly we get easily a smooth bounded function with bounded derivatives that agrees with x in $[-1, 1]$. Then the sequence $h_(x) = kh(\frac{x}{k})$ clearly converges to x, $|h_k(x)| \leq |x|$ and more over $|h_k'(x)|$ is uniformly bounded in x and k and $|h_k''(x)|$ goes to 0 uniformly in k. We approximate $< \lambda\,, x >$ by $\sum_j \lambda_j h_k(x_j)$ and consider the martingales

$$\exp\Big[\sum_j \lambda_j h_k(x_j(t)) - \sum_j \lambda_j h_k(x_j(0)) - \int_0^t \psi_k^\lambda(s)ds\Big]$$

where

$$\psi_k^\lambda(s) = \int_0^t \sum_j \lambda_j b_j(s\,, \omega) h_k'(x_j(s))ds + \frac{1}{2}\int_0^t \sum_j a_{j,j}(s\,, \omega) h_k''(x_j(s))ds$$
$$+ \frac{1}{2}\int_0^t \sum_{i,j} a_{i,j}(s\,, \omega)\lambda_i\lambda_j h_i'(x_i(s) h_j'(x_j(s)ds$$

and converges to

$$\psi^\lambda(s) = \int_0^t \sum_j \lambda_j b_j(s\,, \omega)ds + \frac{1}{2}\int_0^t \sum_{i,j} a_{i,j}(s\,, \omega)\lambda_i\lambda_j ds$$

as $k \to \infty$. By Fatous's lemma the limit of nonnegative martingales is always a supermartingale and therefore in the limit

$$\exp\Big[< \lambda\,, x(t) - x(0) > - \int_0^t \psi^\lambda(s)ds\Big]$$

is a supermartingale. In particular

$$E^P\Big[\exp[< \lambda\,, x(t) - x(0) > - \int_0^t \psi^\lambda(s)ds]\Big] \leq 1$$

If we now use the bound on ψ it is easy to obtain the estimate

$$E^P[\exp[< \lambda\,, x(t) - x(0) >] \leq C_\lambda$$

This provides the necessary uniform integrability to conclude that in the limt we have a martingale. Once we have the estimate, it is easy to see that we can

approximate $f(t\,,x)+<\lambda\,,x>$ by $f(t\,,x)+\sum_j \lambda_j h_k(x_j)$ and pass to the limit, thus obtaining (7) from (6). Of course (7) implies both (2) and (6). Also all the exponential estimates follow at this point. Once we have the estimates there is no difficulty in obtainig (1) from (3). We need only take $f(x) = x_i$ and $x_i x_j$ that can be justified by the estimates. Some minor manipulation is needed to obtain the results in the form presented. □