So far we have defined stochastic integrals with respect to processes $x(t)$ that have continuous paths and have certain martingales associated with them. $(\Omega, \mathcal{F}_t, P)$, $x(t,\omega) : \Omega \times [0,T] \to R^d$, $b(t,\omega) : \Omega \times [0,T] \to R^d$, $a(t,\omega) : \Omega \times [0,T] \to S_d^+$, progressively measurable and $x(t,\omega)$ is continuous a.e. If a and b are uniformly bounded, then we saw that the stochastic integrals

$$\xi(t) = \int_0^t < \sigma(s,\omega), dx(s) >$$

can be defined and is an almost surely continuous process $\Omega \times [0,T] \to R^n$ provided $\sigma : \Omega \times [0,T] \to R^n \otimes R^d$ is progressivley measurable and bounded. The parameters of ξ can be calculated according to the rules for computing means and variances under linear transformations. If $x(\cdot) \in [b,a]$ and $d\xi = \sigma dx$, then $\xi \in [\sigma b, \sigma a \sigma^*]$. Actaully, the class of processes can cover $[b,a]$ with the property

$$\int_0^T |b(s,\omega)| ds < \infty \; a.e.$$

and

$$\int \text{Tr } a(s,\omega) ds < \infty \; a.e.$$

Instead of Martingales, the expressions will be local martingales. $x(t)$ is a local martingale if there are stopping times $\tau_n \uparrow \infty$ such that $x(\tau_n \wedge t)$ is a martingale for every n. Example two dimesional Brownian motion. $\xi(t) = \log r(t)$

$$\log r(t) = \log r(0) + \int_0^t < \frac{x(s)}{r^2(s)}, \; dx(s) >$$

The trouble comes from 0. If $\tau_n = \{\inf t : r(t) \leq \frac{1}{n}\}$, then $\xi(\tau_n \wedge t)$ is seen to be a martingle. ξ is not. It is easy to see that $E[\xi(t)] \to \infty$ as $t \to \infty$. A bounded local martingale is a martingale. A nonnegative local martingale is a supermartingale. Itô's formula holds very generally, because it is an almost sure statement.

Stochastic Differential Equations.

Given $b(t,x)$ and $\sigma(t,x)$ and a Brownian motion $\beta(t)$ and $\xi(\omega) \in \mathcal{F}_s$, solve for $t \geq s$,

$$dx(t) = b(t,x(t))dt + \sigma(t,x(t))d\beta(t); x(s) = \xi(\omega)$$

Can assume that $s = 0$ and $\xi(\omega) = x_0$. Define iteratively

$$x_{n+1}(t) = x_0 + \int_0^t b(x_n(s))ds + \int_0^t < \sigma(x_n(s)), \; d\beta(s) >$$

Assume that σ and b are bounded and unifomly Lipshitz in x with a Lipshitz constant A. Then, fixing a time interval $[0,T]$,

$$x_{n+1}(t) - x_n(t) = \int_0^t [b(x_n(s)) - b(x_{n-1}(s))]ds + \int_0^t < \sigma(x_n(s)) - \sigma(x_{n-1}(s)), \; d\beta(s) >$$

Denoting by $\Delta_n(t) = E[\sup_{0\le s\le t} |x_n(s) - x_{n-1}(s)|^2]$, we have, from Doob's inequality

$$\Delta_{n+1}(t) \le 2TA^2 \int_0^t \Delta_n(s)ds + 8\int_0^t \Delta_n(s)ds \le C(T)\int_0^t \Delta_n(s)ds$$

By induction

$$\Delta_n(t) \le \frac{C(T)^n t^n}{n!}$$

Therefore for almost all ω, $x(t) = \lim_{n\to\infty} x_n(t)$, exists uniformly in t, and provides a solution of

$$x(t) = x_0 + \int_0^t b(s, x(s))ds + \int_0^t < \sigma(s, x(s)),\ d\beta(s) >$$

It is unique. If $x(t), y(t)$ are two solutions, then $\Delta(t) = E[|x(t) - y(t)]^2]$ satisfies

$$\Delta(t) \le C(T)\int_0^t \Delta(s)ds$$

and is 0. Clearly $x(\cdot) \in [b(s, x(s,\omega)), a(s, x(s,\omega))]$ with $a = \sigma\sigma^*$. One can easily verify that $x(t)$ is a Markov process, in fact a strong Markov process. The reason is that we have a "black box", we input x_s and Brownian increments and the output is $x(t)$ for $t \ge s$. Since the Brownian increments $\beta(t) - \beta(s)$ are independent of $\mathcal{F}_s$, we only need the value of $x(s,\omega)$ and the actual ω is unimportant. That is really the Markov property. $\sigma(s, x)$ is not unique. One can change $\sigma'(s, x) = \sigma(s, x)U(s, x)$ where U is an orthogonal matrix. The $\sigma\sigma^* = \sigma'\sigma'^*$. $d\beta'(s) = U^*(s, x(s))d\beta(s)$ defines another Brownian Motion. Therefore the two solutions have the same distribution.

Of course we can start with a solution on some $(\Omega, \mathcal{F}_t, P)$ where both x and β are given and are related by

$$x(t) = x(0) + \int_0^t b(s, x(s))ds + \int_0^t < \sigma(s, x(s)),\ dx(s) >$$

If b,σ are Lipshitz then x is measurable with respect to Brownian σ-field and is the same as the solution constructed earlier. Otherwise it is not clear. Such solutions are the same as solutions to the Martingale problem. Given $(\Omega, \mathcal{F}_t, P)$, $[b, a]$ and x, and any choice of σ with $\sigma\sigma^* = a$, there is a Brownian Motion β such that

$$dx = bdt + \sigma d\beta$$

If we assume that a is uniformly positive definite then we can define β as

$$\beta(t) = \int_0^t \sigma^{-1}(s, x(s))[dx(s) - b(s, x(s))ds]$$

It is easy to check that $\beta \in [0, I]$, because $\sigma^{-1}a\sigma^{-1\,*} = I$ and

$$dx = \sigma d\beta + bdt$$

The problem is when a can be degenerate. Then we have to go outside to find our Brownian motion. For instance $x(t) \equiv 0$ coreesponds to $a = b = 0$ and there is no Brownian motion on the space where there is only the zero path with probability 1. But we can take any Brownian motion and say

$$dx = 0 = 0\, d\beta$$

But we should use the new Brownian only we need it. This is done in two steps. First build a new Brownian motion by taking a product with Wiener space. Now we have a space with$x(t), \beta(t)$ corresponding to $[(b(s,\omega), 0), (a(s,\omega), I)]$. Let $Q(s,\omega)$ be the orthogonal projection on to the range of $a(s,\omega)$. If $\sigma\sigma^* = a$, then the range of σ is the same as the range of a and $\sigma^{-1}Q$ is well defined. We can define a new Brownian motion $\beta'(t)$ by

$$\beta'(t) = \int_0^t \sigma^{-1}(s,\omega)Q(s,\omega)[dx(s) - b(s,\omega)ds] + \int_0^t [I - Q(s,\omega)]d\beta(s)$$

then

$$\sigma^{-1}QaQ^*\sigma^{-1\,*} + I - Q = I$$

and

$$dx = \sigma d\beta' + b\, dt$$

Finally there us uniqueness theorem. If for some σ uniqueness holds in the sense that when ever $x(t), y(t)$ are two solutions on any $(\Omega, \mathcal{F}_t, P, \beta(\cdot))$ of

$$x(t) = x_0 + \int_0^t b(s, x(s))ds + \int_0^t < \sigma(s, x(s)),\ d\beta(s) >$$

$$y(t) = x_0 + \int_0^t b(s, y(s))ds + \int_0^t < \sigma(s, y(s)),\ d\beta(s) >$$

it follows that $x(t) \equiv y(t)$, then there is only one solution to the martingale problem for $[b, a]$ starting from x. The proof depends on a construction. Given $P_1, P_2, [a, b], x, \sigma$, i.e two solutions to the martingale problem for $[b, a]$ from the same starting point x_0 and a σ satisfying $\sigma\sigma^* = a$, we will construct $(\Omega, \mathcal{F}_t, x(\cdot), y(\cdot), \beta(\cdot))$ such that x and y are solutions with the same b, σ and the distribution of $x(t)$ is P_1 and that of $y(t)$ is P_2. Sinec $x(t) \equiv y(t)$, $P_1 = P_2$. the construction is staright forward. First construct $x(t), \beta(t)$ so that

$$dx(t) = \sigma(t, x(t))d\beta(t) + b(t, x(t))dt$$

This will produce a joint distribution of $\beta(\cdot)$ and $x(\cdot)$ we write this as $P(dw)Q^1_w(d\omega_1)$, the marginal of Brownian Motion and the conditional of $x(\cdot)$ given the Brownian motion. Similarly for y, $P(dw)Q^2_w(d\omega_2)$. Now we can put all three x, y, β on the same space aligning the Brownian trajectories, i.e. take the measure $P(dw)Q^1_w(d\omega_1) \otimes Q^2_w(d\omega_2)$. Make the processes x, y conditionally independent given β. One verifies that now we have two solutions on the same space.

Girsanov's formula.

If $b(t,x)$ is bounded and $a(t,x)$ be bounded and uniformly positive definite. P a solution to the martingale problem for $[0,a]$ starting from x.

$$\exp\left[\int_0^t < e(s,x(s)), dx(s) > -\frac{1}{2}\int < e(s,x(s)),\, a(s,x(s))e(s,x(s)) > ds\right]$$

is a martingale. Choose $e(s,x(s)) = \theta + a^{-1}(s,x(s))b(s,x(s))$.

$$\begin{aligned}\exp\Bigg[&\int_0^t < \theta + a^{-1}(s,x(s))b(s,x(s)), dx(s) > \\ &-\frac{1}{2}\int < \theta + a^{-1}(s,x(s))b(s,x(s)),\, a(s,x(s))[\theta + a^{-1}(s,x(s))b(s,x(s))] > ds\Bigg]\end{aligned}$$

is a martingale for every $\theta \in R^d$. This simplifies to

$$\begin{aligned}&\exp\Bigg[< \theta, x(t) - x > + \int_0^t < a^{-1}(s,x(s))b(s,x(s)), dx(s) > \\ &\quad - \int_0^t < \theta, b(s,x(s)) > ds - \frac{1}{2}\int_0^t < \theta, a(s,x(s))\theta > ds \\ &\quad - \frac{1}{2}\int_0^t < b(s,x(s)),\, a(s,x(s))[a^{-1}(s,x(s))b(s,x(s))] > ds\Bigg] \\ &= \exp\Bigg[\int_0^t < a^{-1}(s,x(s))b(s,x(s)), dx(s) > \\ &\quad - \frac{1}{2}\int_0^t < b(s,x(s)),\, a(s,x(s))[a^{-1}(s,x(s))b(s,x(s))] > ds \\ &\quad + < \theta, x(t) - x > - \int_0^t < \theta, b(s,x(s)) > ds \\ &\quad - \frac{1}{2}\int_0^t < \theta, a(s,x(s))\theta > ds\Bigg] \\ &= R(t,\omega)Y(\theta,t,\omega)\end{aligned}$$

If we set $\theta = 0$ then $Y = 1$ and $R(t,\omega)$ is a martingale. This defines a measure Q by $dQ = RdP$ and with respect to Q, $Y(\theta,t,\omega)$ are martingales. In other words Q is a solution for $[b,a]$. The steps are reversible so that there is a one to one correspondence between solutions of $[b,a]$ and $[0,a]$. Existence or uniqueness for one implies the same for the other.

Warning. If b is unbounded R may not be a martingale but only a supermartingale. This means that the paths explode and the total mass of Q is less than 1. In fact then

$$Q[\tau_\infty > t] = \int R(t,\omega)dP$$

Random Time Changes. On the space $[C[0,\infty]; X$ we define a family of transformations. Given a function $V(x) : X \to R$ which is meausrable and satisfies $0 < c_1 \leq V(x) \leq c_2 < \infty$, we define (stopping) times τ_t by

$$\int_0^{\tau_t} V(x(s))ds = t$$

and the transformation $\Phi_V : x(\cdot) \to y(\cdot)$ by

$$y(t) = x(\tau_t)$$

It is not hard to check that

$$\Phi_V \circ \Phi_U = \Phi_U \circ \Phi_V = \Phi_{UV}$$

If P is a solution to the martingale problem for $\mathcal{L}$ which is time homogeneous, then $Q = \Phi_U^{-1}$ is seen to be solution for $\frac{1}{V}\mathcal{L}$.

$$\int_0^{\tau_t} g(x(s))ds = \int_0^t \frac{g(y(s))}{V(y(s))}ds$$

Since τ_t are stopping times Doob's stopping theorem applies. Since we can go back and forth existence or uniqueness for $\mathcal{L}$ is equivalent to the same for $\frac{1}{V}\mathcal{L}$. In patitcular in $d = 1$ we can go from $[0, 1]$ to any $[b, a]$ with a bounded b and a bounded above and below by random time change and Girsanov.

PDE Methods.

If a is bounded and uniformly elliptic, b is bounded and they all satisfy Hölder conditions in t and x, then the PDE

$$u_t + \frac{1}{2}\sum a_{i,j}(t,x)u_{i,j} + \sum b_j(t,x)u_j = 0;\ u(T,x) = f(x)$$

has a classical solution, implying that the solutions to the martingale problem are unique. If we drop the assumption of Hölder continuity and assume only that a is continuous, then there are solutions in Sobolev spaces $W_p^{1,2}$. Then one has to show that for any solution to the martingale problem the functional

$$\Lambda(f) = E^P[\int_0^T f(s, x(s))ds]$$

is bounded in L_p. This can be done and implies uniquenss. Note that by Girsanov we can assume $b = 0$.

Localization.

We say that a solution to the martingale problem starting from x is unique untill the exit time τ_G from $G \ni x$, if any two solutions starting from x agree on $\mathcal{F}_{\tau_G}$. The localization principle says that if $[a, b]$ is such that for every x there is a neighborhood G, such that any two solutions starting from x agree until the exit time from G, then there is atmost one solution. This means that for given coefficients we can prove uniqueness by different methods at different points.