Exit Problem. Consider

$$x_\epsilon(t) = x + \int_0^t b(x_\epsilon(s))ds + \sqrt{\epsilon}\beta(t)$$

and let $Q_{x,\epsilon}$ be the distribution of the solution x_ϵ. As $\epsilon \to 0$ the measure $Q_{x,\epsilon}$ concentrates on the trajectory which is the solution of

$$x(t) = x + \int_0^t b(x(s))ds$$

There is a large deviation principle for $\{Q_{x,\epsilon}\}$ on $C[[0,T];R^d]$.

$$Q_{x,\epsilon}(A) = \exp[-\inf_{\substack{f(\cdot)\in A\\ f(0)=x}} \frac{1}{2\epsilon}\int_0^T \|f'(t) - b(f(t))\|^2 dt + o(\frac{1}{\epsilon})]$$

More precisely for closed sets C

$$\limsup_{\substack{y\to x\\ \epsilon\to 0}} \epsilon \log Q_{y,\epsilon}(C) \le -\inf_{\substack{f(\cdot)\in C\\ f(0)=x}} \frac{1}{2}\int_0^T \|f'(t) - b(f(t))\|^2 dt$$

and for open sets G,

$$\liminf_{\substack{y\to x\\ \epsilon\to 0}} \epsilon \log Q_{y,\epsilon}(G) \le -\inf_{\substack{f(\cdot)\in G\\ f(0)=x}} \frac{1}{2}\int_0^T \|f'(t) - b(f(t))\|^2 dt$$

Let G be an open set containing a unique stable equlibrium point x_0 for the ODE

$$\dot{x}(t) = b(x(t))$$

i.e. any solution of the ODE starting from any point in the closure $\bar{G}$ tends to x_0 as $t \to \infty$, remaining in G for all $t > 0$. For instance assume that G is smooth and $b \neq 0$ on the boundary δG and points inward at every point. For any $x \in G$ and $z \in \delta G$ let

$$U(T, x, z) = \inf_{\substack{f: f(0)=x, f(T)=z\\ f(t)\in G \text{ for } t<T}} \frac{1}{2}\int_0^T \|f'(t) - b(f(t))\|^2 dt$$

and

$$U(x, z) = \inf_{T>0} U(T, z)$$

Let $z_0 \in \delta G$ be such that $U(x_0, z_0) < U(x_0, z)$ for all $z \in \delta G, z \neq z_0$. If τ is the exit time and $x(\tau)$ is the exit place from G, then for any $x \in G$ and any neighborhood N of z_0,

Theorem:

$$\lim_{\epsilon\to 0} Q_{x,\epsilon}[x(\tau) \notin N] \to 0$$

Remark. No matter where the process starts inside G intially it will follow the ODE, be driven towards x_0, slow down as it reaches x_0 and hang around there for a very long time.

Let us take two neighborhoods S_1, S_2 around x_0, with $x_0 \in S_1 \subset \bar{S}_1 \subset S_2$. It is not hard to see that $U(x,z)$ is a continuous function of x and z, and given N, we can pick S_1, S_2 such that

$$\inf_{x\in\delta S_2}\inf_{z\in N^c} U(x,z) \geq \sup_{x\in\delta S_2} U(x,z_0)+\eta$$

We will estimate the following probabilities: if τ' be the exit time from $G\cap\bar{S}_1^c$

$$\limsup_{\epsilon\to 0}\epsilon\sup_{x\in\delta S_2}\log Q_{x,\epsilon}[x(\tau')\in N^c] \leq -\inf_{x\in\delta S_2}\inf_{z\in N^c} U(x,z)$$

and

$$\liminf_{\epsilon\to 0}\epsilon\inf_{x\in\delta S_2}\log Q_{x,\epsilon}[x(\tau')\in N] \geq -\sup_{x\in\delta S_2} U(x,z_0)$$

This will do it. The picture is the process will sooner or later exit from $\bar{S}_1^c$. But most of the time it will be pulled back to x_0. There is a very small chance that it will exit in N and even smaller chance of exiting from N^c. So it is most likely to exit from N.

First we estimate the probability that exit from $\bar{S}_1^c$ takes too long.

$$\limsup_{T\to\infty}\limsup_{\epsilon\to 0}\epsilon\log\sup_{x\in\bar{S}_1^c} Q_{x,\epsilon}[\tau'\geq T] = -\infty$$

Otherwise there will be paths with $\int_0^T \|f'(t)-b(f(t))\|^2dt$ bounded and T large. This means there will be paths with $\int_0^T \|f'(t)-b(f(t))\|^2dt$ small and T large. This in turn means solutions of ODE remaining in $\bar{S}_1^c$ for too long. If the paths do not hang around for too long, the large deviation estimate applies and it is much more likely to exit from N, than from N^c.

A special case is the gradient flow, where $b(x) = -(\nabla V)(x)$. x_0 is a local minimum of V. Then it is not hard to see that $U(x_0,z) = 2[V(z)-V(x_0)]$.

Invariant distributions.

$$\mathcal{L} = \frac{1}{2}\sum_{i,j} a_{i,j}(x)D_{i,j} + \sum_j b_j(x)D_j$$

μ is probability measure on R^d such that

$$\int (\mathcal{L}u)(x)d\mu(x) = 0$$

for all smooth u with compact support. Suppose there is a unique process corresponding to $\mathcal{L}$, is μ an invariant distribution fro the process? Proof dpends on duality and consequently finding enough classical solutions for the equation

$$u_t = \mathcal{L}u$$

or the resolvent equation

$$\lambda u - \mathcal{L}u = f$$

which require ellipticity and Hölder continuity. Assume only that the coefficients are continuous, but the process is unique. If we know that $d\mu = \phi dx$ with $\phi \in L_q$ we can use the L_p theory in the elliptic case. To prove it in general requires several steps.

Invariance Principle.

Theorem: Suppose $\pi_h(x, dy)$ is a Markov Chain such that, for every smooth u with compact support

$$\frac{1}{h}\int [u(y) - u(x)]\pi_h(x, dy) \to (\mathcal{L}u)(x)$$

uniformy on compact sets, and there exists a unique process with out explosion for $\mathcal{L}$, then the interpolated Markov Chain converges to the process. In particular

$$\lim_{\substack{h\to 0\\ nh\to t}} \int f(y)\pi_h^n(x, dy) \to (T_t f)(x) = \int f(y)p(t, x, dy)$$

where p is the transition probability of the process corresponding to $\mathcal{L}$.

Proof: Step 1. Let us interpolate the Markov chain and call the process P_h. Let us take smooth cut off function $\phi^R(x)$ and define

$$\pi_h^R(x, y) = \phi^R(x)\pi_h(x, dy) + (1 - \phi^R(x))\delta_x(dy)$$

It is easy to see that

$$\frac{1}{h}\int [u(y) - u(x)]\pi_h^R(x, dy) \to (\mathcal{L}^R u)(x) = \phi^R(x)(\mathcal{L}u)(x)$$

uniformly in x. We will prove that the processes $P_{h,x}^R$ are tight. Let τ_ϵ be the exit time from the ball of radius ϵ for the process starting from x. We want to estimate

$$\sup_x \sup_{h\le 1} P_{h,x}^R[\tau_\epsilon \le \delta] = F(\epsilon, \delta)$$

If u_ϵ is a smooth function that is 1 in a ball of radius $\frac{\epsilon}{2}$ and 0 outside a ball of radius ϵ, $\|\mathcal{L}^R u_\epsilon(x)\| \le C_\epsilon$ and

$$\int [u(y) - u(x)]\pi_h^R(x, dy) \le C_\epsilon h$$

In particular

$$u(X(nh)) - u(x) - nhC_\epsilon$$

is a super-martingale under $P_{x,h}^R$ and

$$P_{x,h}^R[\tau_\epsilon \le \delta] \le E[u(\tau_\epsilon \wedge \delta)] \le \delta C_\epsilon$$

Let $\tau_1, \tau_2, \ldots, \tau_N$ be the successive times at which $X(nh)$ gets away a distance ϵ from the previous $x(\tau_i)$. We proceed till $\tau_N > T$. We estimate the following.

$$\sup_{\omega,h} E[e^{-\tau_{i+1}}|\mathcal{F}_{\tau_i}] \leq \rho < 1$$

$$P[N \geq k] \leq P[\tau_1 + \cdots + \tau_k \leq T] \leq e^T E[e^{-(\tau_1+\cdots+\tau_k)}] \leq e^T \rho^k$$

and

$$P[\min(\tau_1, \ldots, \tau_k) \leq \delta)] \leq k\delta C_\epsilon$$

From the locality of $\mathcal{L}$, it follows that

$$\pi_h^R(x, B(x,\epsilon)^c) = o(h)$$

Therefore

$$\sup_x P^R_{x,h}\left[\sup_{0\leq j\leq n} |X((j+1)h - X(jh)| \geq \epsilon\right] \to 0$$

as $h \to 0$. This is enough to control the oscillations. We can use the control on the modulus of continuity to prove tightness. If P^R_x is any limit it is a solution to the martingale problem for $\mathcal{L}^R$. This agrees with $\mathcal{L}$ until th exit itme from B_R the ball of radius R. Since there is no explotion if R is large τ_R is large, is bigger than T, with probability nearly one and so $P^R_{x,h}$ and $P_{x,h}$ are close and the limit of $P_{x,h}$ as $h \to 0$ is P_x.

Finally to prove that μ is the invariant measure we will construct a Markov Chain $\{\pi_h(x, dy)\}$, for which μ is inavariant and which converges to $\{P_x\}$. Given $\mathcal{L}$, we construct the resolvent

$$\Pi_h = (I - h\mathcal{L})^{-1}$$

on the range of D_h of bounded functions with two bounded derivatives under $(I - h\mathcal{L})$. The maximum principle guarantees that Π_h is well defined and is positivity preserving. We define a linear functional Λ on functions of two variables of the form

$$g(x,y) = v_0(y) + \sum_i u_i(x)w_i(y)$$

with u, w being bounded continuous functions and $w_i = v_i - h\mathcal{L}v_i \in D_h$, by

$$\Lambda(g) = \int v_0(y)d\mu(y) + \sum_{i=1}^n \int u_i(x)v_i(x)d\mu(x)$$

Suppose Λ is nonnegative and we extend it as a non negative linear functional. Then both the marginals of Λ are μ. [Note that we can take $v_1 = 1$ and the remaining v as 0. Then $g(x,y) = u_1(x)$]. If we take the r.c.p.d $\pi_h(x, dy)$, $\mu\pi_h = \mu$ and

$$\pi_h(v - h\mathcal{L}v) = v$$

for smooth v we have

$$\frac{1}{h}(\pi_h v - v) = \pi_h\mathcal{L}v \to \mathcal{L}v$$

for v with compact support.

Suppose $g(x, y) \geq 0$. Then consider the function

$$\inf_x \sum_{i=1}^{n} u_i(x) v_i = \Phi(\mathbf{v})$$

defined for $\mathbf{v} = (v_1, \ldots, v_n) \in R^n$. Φ is concave and

$$\Phi(\mathbf{v}(x) - t(\mathcal{L}\mathbf{v})(x))$$

is a convex function of t for all x. So is the integral

$$\psi(t) = \int \Phi(\mathbf{v}(x) - t(\mathcal{L}\mathbf{v})(x)) d\mu(x)$$

$$\psi'(0) = -\int \sum_i \Phi_{u_i}(\mathbf{v}(x))(\mathcal{L}v_i)(x) d\mu(x) \leq \int [\mathcal{L}\Phi(\mathbf{v})](x) d\mu(x) = 0$$

Therefore for $h \geq 0$,

$$\psi(h) = \int \Phi(\mathbf{v}(x) - h(\mathcal{L}\mathbf{v})(x)) d\mu(x) \leq \int \Phi(\mathbf{v}(x)) d\mu(x)$$

We can approximate Φ by smooth convex functions. Denote $v_i - h\mathcal{L}v_i = w_i$. Then

$$\begin{aligned} \int [v_0(x) + \sum u_i(x) v_i(x)] d\mu(x) &\geq \int [v_0(x) + \Phi(\mathbf{v}(x))] d\mu(x) \\ &\geq \int [v_0(x) + \Phi(\mathbf{w}(x))] d\mu(x) \end{aligned}$$

But

$$[v_0(y) + \Phi(\mathbf{w}(y))] = \inf_x [v_0(y) + \sum_i u_i(x) w_i(y)] = \inf_x g(x, y) \geq 0$$