More generally if $\mathcal{L}$ is the generator of a not necessarily self adjoint operator that generates a Markov semigroup, we can ask the following question. Can we estmate

$$P_x\left[\frac{1}{t}\int_0^t V(x(s))ds \geq \ell\right]$$

The starting point for such estimates is the Feynman-Kac formula that says

$$u(t,x) = E_x\left[\exp[\int_0^t V(x(s))ds]f(x(t))\right]$$

is the solution of

$$u_t = \mathcal{L}u + Vu; u(0,x) = f(x)$$

In particular, if $u(t,x) \equiv u(x)$ and $0 < c \leq u(x) \leq C < \infty$

$$0 = u_t = \mathcal{L}u - \frac{\mathcal{L}u}{u}u$$

and

$$E_x\left[\exp[\int_0^t \frac{-\mathcal{L}u}{u}(x(s))ds]\right] \leq \frac{u(x)}{c}$$

If we denote by $m(t,A)$, the empirical measure

$$m(t,A) = \frac{1}{t}\int_0^t \mathbf{1}_A(x(s))ds$$

and $Q_{t,x}$ the distribution of the empirical measure on the space $\mathcal{M}$ of all probability measures on our state space X, then the bound we have is

$$E^{Q_{t,x}}\left[\exp[-t\langle\frac{\mathcal{L}u}{u}, m\rangle]\right] \leq \frac{u(x)}{c}$$

By Tchebechev's inequality we can estimate

$$Q_{t,x}\big[E\big] \leq \frac{u(x)}{c}\exp\big[t\sup_{m\in E}\langle\frac{\mathcal{L}u}{u}, m\rangle\big]$$

Therefore

$$\limsup_{t\to\infty}\frac{1}{t}\log Q_{t,x}\big[E\big] \leq \sup_{m\in E}\langle\frac{\mathcal{L}u}{u}, m\rangle$$

Since $u \in \mathcal{D}^+$ is arbitrary

$$\limsup_{t\to\infty}\frac{1}{t}\log Q_{t,x}\big[E\big] \leq \inf_{u\in\mathcal{D}^+}\sup_{m\in E}\langle\frac{\mathcal{L}u}{u}, m\rangle$$

If N is small neighborhood of m, then

$$\lim_{N\downarrow m}\limsup_{t\to\infty}\frac{1}{t}\log Q_{t,x}\big[N\big]\le\inf_{u\in\mathcal{D}^+}\langle\frac{\mathcal{L}u}{u},m\rangle$$

The rate function for large deviation is the function

$$I(m)=-\inf_{u\in\mathcal{D}^+}\langle\frac{\mathcal{L}u}{u},m\rangle$$

With this rate function we have upper bound for small neighborhoods. Since the sum of a finite number of exponentials decays like the worst, this yields an upper bound immediately for compact sets K, which can be covered by a finite number of arbitrary small neighborhoods.

$$\limsup_{t\to\infty}\frac{1}{t}\log Q_{t,x}\big[K\big]\le-\inf_{m\in K}I(m)$$

If X is not compact some additional control is needed to prove exponential tightness, i.e.

$$\limsup_{K\uparrow X}\limsup_{t\to\infty}\frac{1}{t}\log Q_{t,x}\big[K^c\big]\le-\infty$$

Then we can estimate for any closed set C,

$$Q_{t,x}\big[C\big]\le Q_{t,x}\big[C\cap K\big]+Q_{t,x}\big[K^c\big]$$

Since the second term can be made to decay with a large exponential decay rate by the choice of K, our estimate for compact sets can now be extended to closed sets.

To prove exponential tightness, when X is not compact, for instance R^d, it is enough to get an estimate of the form

$$E^{P_x}\Big[\exp[\int_0^t V(x(s))ds]\Big]\le c(x)e^{at}$$

with $c(x)<\infty$ and $a<\infty$ for some $V(x)\ge 0$, $V(x)\to+\infty$ as $|x|\to\infty$. This would give us with

$$K_\ell=\{m:\int V(x)m(dx)\le\ell\}\subset\mathcal{M}$$

$$Q_{t,x}\big[K_\ell^c\big]\le Q_{t,x}\Big[m:\int V(x)m(dx)\ge\ell\Big]\le e^{-t\ell}E^{P_x}\Big[\exp[\int_0^t V(x(s))ds]\Big]\le c(x)e^{(a-\ell)t}$$

we can pick ℓ to be large and we will have our exponential tightness. For instance if

$$\mathcal{L}=\frac{1}{2}\Delta-<x,\nabla>$$

is the OU process, with $u(x)=e^{\sqrt{(1+x^2)}}$, it is not hard to see that $V(x)=-\frac{\mathcal{L}u}{u}\to+\infty$ as $|x|\to\infty$. We can add a constant to make it non negative.

Proving lower bounds involves changing the generator from $\mathcal{L}$ to $\widehat{\mathcal{L}}$, such that, μ is an (ergodic?) invariant measure for $\widehat{\mathcal{L}}$ and and the relative entropy of P_x to $\widehat{P}_x$ in time t is bounded by Ht. $\widehat{\mathcal{L}}$ may not be unique, but the optimal choice, i.e the smallest possible H will equal $I(\mu)$, providing the lower bound.

We will illustrate this in the context of diffusions on a d-torus.

$$\mathcal{L} = \frac{1}{2}\Delta + < b(x), \nabla >$$

and

$$\widehat{\mathcal{L}} = \frac{1}{2}\Delta + < c(x), \nabla >$$
$$d\mu = \phi(x)dx$$

$c(x)$ should be such that

$$\frac{1}{2}\Delta\phi = \nabla \cdot c\phi$$

and

$$H(c) = \frac{1}{2}\int \|c - b\|^2 \phi dx$$

What needs to be proven is the identity

$$-\inf_{u\in\mathcal{D}^+} \int \frac{\mathcal{L}u}{u}\,\phi\, dx = I(\phi) = \inf_{c:\frac{1}{2}\Delta\phi = \nabla\cdot c\phi} H(c)$$

Replacing u by e^{-v}, the left hand side can be written as

$$\sup_v \int [\mathcal{L}v - \frac{1}{2}|\nabla v|^2]\phi\, dx$$

The right hand side is rewritten as

$$\inf_c \sup_u \int [\frac{1}{2}\|c - b\|^2 + \widehat{\mathcal{L}}u]\phi\, dx$$

Note that $\sup_u$ is $+\infty$ unless $\frac{1}{2}\Delta\phi = \nabla \cdot c\phi$, in which case it is 0. We now calculate

$$\begin{aligned} RHS &= \inf_c \sup_u \int [\frac{1}{2}\|c - b\|^2 + \frac{1}{2}\Delta u + c(x)\cdot\ \nabla u]\,\phi\, dx \\ &= \sup_u \inf_c \int [\frac{1}{2}\|c - b\|^2 + \frac{1}{2}\Delta u + c(x)\cdot\ \nabla u]\,\phi\, dx \\ &= \sup_u \int [\mathcal{L}u - \frac{1}{2}|\nabla u|^2]\phi\, dx \\ &= LHS \end{aligned}$$

because the $\inf_c$ can be done pointwise and

$$\inf_c \frac{1}{2}[\|b - c\|^2 + c\cdot p] = b\cdot c - \frac{1}{2}\|p\|^2$$

Interesting counter example:

$$\mathcal{L} = \frac{1}{2}\frac{d^2}{dx^2} + a\,\frac{d}{dx}$$

If $\phi(x)$ satisfies

$$\frac{1}{2}\phi_{xx} = (c\,\phi)_x$$

then

$$\frac{1}{2}\phi_x = c(x)\,\phi(x) + k$$

$k = 0$, because $(c(x)-a)^2\phi, \phi \in L_1(R)$. This forces $\int c(x)\phi(x)dx = 0$ and

$$\int (c(x)-a)^2\,\phi(x)\,dx \geq a^2$$

In particular $I(\mu) \geq \frac{a^2}{2}$. There is a locally uniform exponential rate. But the total probability is 1.