Let $\sigma(s)$ and $b(s)$ be "smooth" progressively measurable functions of ω. Then so are

$$x(t) = \int_0^t \sigma(s) d\beta_k(s)$$

and

$$y(t) = \int_0^t b(s) ds$$

In fact

$$\mathcal{L}x(t) = \int_0^t [\mathcal{L}\sigma(s) - \sigma(s)] d\beta(s)$$

and

$$\mathcal{L}y(t) = \int_0^t \mathcal{L}b(s) ds$$

It is easily proved by approximating the integrals by a sum. One notes that

$$\mathcal{L}\sigma(s)[\beta(t) - \beta(s)] = [\mathcal{L}\sigma(s)][\beta(t) - \beta(s)] + \sigma(s)\mathcal{L}[\beta(t) - \beta(s)]$$

due to the independence of $\sigma(s)$ and $[\beta(t) - \beta(s)]$ and $\mathcal{L}\beta(t) = -\beta(t)$. If we do the Picard iteration

$$x_i^n(t) = x_i + \int_0^t \sum_k \sigma_{ik}(s, x^{n-1}(s)) d\beta_k(s) + \int_0^t b_i(s, x^{n-1}(s)) ds$$

then denoting by $\mathcal{L}x_i^n$ by X_i^n

$$\begin{aligned}
X_i^n(t) = &\int_0^t \sum_k < (\nabla_x \sigma_{ik})(s, x_i^{n-1}(s)), X^{n-1}(s) > d\beta_k(s) \\
&+ \int_0^t < (\nabla_x b_i)(s, x_i^{n-1}(s)), X^{n-1}(s) > ds \\
&+ \int_0^t \sum_k < \nabla_x^2 \sigma_{ik}(s, x_i^{n-1}(s)), A^{n-1}(s) > d\beta_k(s) \\
&+ \int_0^t < \nabla_x^2 b_i(s, x_i^{n-1}(s)), A^{n-1}(s) > ds
\end{aligned}$$

where

$$A_{ij}^n(s) = < Dx_i^n(s), Dx_j^n(s) >$$

If we calculate the derivative of $x_i^n(t)$ in some direction $h = \{h_k\}$ and denote the derivative by $D_h x_i^n(t)$ by $y_i^n(t)$, then

$$\begin{aligned}
y_i^n(t) = &\int_0^t \sum_k < (\nabla_x \sigma_{ik})(s, x_i^{n-1}(s)), y^{n-1}(s) > d\beta_k(s) \\
&+ \int_0^t < (\nabla_x b_i)(s, x_i^{n-1}(s)), y^{n-1}(s) > ds \\
&+ \int_0^t \sum_k \sigma_{ik}(s, x_i^{n-1}(s)) h_k(s) ds
\end{aligned}$$

We came across the equation

$$y_i^n(t) = \int_0^t \sum_k < (\nabla_x \sigma_{ik})(s, x_i^{n-1}(s)), y^{n-1}(s) > d\beta_k(s)$$
$$+ \int_0^t < (\nabla_x b_i)(s, x_i^{n-1}(s)), y^{n-1}(s) > ds$$

while considering the solution of the SDE as a flow and the solution with $y(0) = y$ represented the Jacobian of the flow. $x(0) \to x^n(t)$. The Jacobian is of course just a matrix and is given by $M_{ij}^n(t, 0)$. It is clear that the limit as $n \to \infty$ exists and is the gradient at time t of the solution of the SDE viewed as a flow. More generally we can start at time $s < t$ and $M(t, s) = \{M_{ij}(t, s)\}$ is the Jacobian and satisfies.

$$M(t_3, t_1) = M(t_3, t_2)M(t_2, t_1)$$

for $t_1 \leq t_2 \leq t_3$. By variation of parameters we can calculate in our case the limit of $y^n(t)$ exists and equals

$$D_h x(t) = y(t) = \int_0^t M(t, s)\sigma(s, x(s))h(s)ds$$

In other words $[Dx(t)](s) = M(t, s)\sigma(s, x(s))\mathbf{1}_{s \leq t}$. The Malliavin covariance $A(t) = \{A_{ij}(t)\}$ is given by

$$A(t) = \int_0^t M(t, s)a(s, x(s))M^*(t, s)ds$$

with $a = \sigma\sigma^*$. We see now that the solution $x(t)$ is "smooth".

In the elliptic case, $a \geq cI$.

$$A(t) \geq ct \inf_{\substack{0 \leq s \leq t \\ \|x\|=1}} \|M(t, s)x\|^2$$

We had estimates on the inverse of $M(s, t)$ while proving that the flow was a diffeomorphism. Hence elliptic equations have a nice fundamental solution.