For any noise if the viscosity is large enough there is a unique invariant measure. Let the equation be

$$du(t,x) = \gamma \Delta u - P(u \cdot D)u + dW(t,x)$$
$$u(0,x) = u_0(x)$$

If we take two solution $u(t), v(t)$ with different initial data, then the difference $w(t) = u(t) - v(t)$ satisfies

$$\begin{aligned}\frac{dw(t)}{dt} &= \gamma \Delta w(t) - B(u(t), u(t)) + B(v(t), v(t)) \\ &= \gamma \Delta w(t) - B(v(t) + w(t), v(t) + w(t)) + B(v(t), v(t)) \\ &= \gamma \Delta w(t) - B(v(t), w(t)) - B(w(t), v(t)) - B(w(t), w(t))\end{aligned}$$

and

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\|w(t)\|^2 &=< \dot{w}(t), w(t) > \\ &=< \gamma \Delta w(t) - B(v(t), w(t)) - B(w(t), v(t)) - B(w(t), w(t)), w(t) > \\ &=< \gamma \Delta w(t) - B(w(t), v(t)), w(t) > \\ &=< \gamma \Delta w(t) + B(w(t), w(t)), v(t) >\end{aligned}$$

Therefore

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\|w(t)\|^2 + \gamma \|w(t)\|_1^2 &\leq C\|w(t)\|_1 \|w(t)\| \|v(t)\|_1 \\ &\leq \frac{\gamma}{2}\|w(t)\|_1^2 + \frac{2C}{\gamma}\|w(t)\|^2 \|v(t)\|_1^2\end{aligned}$$

Using Poincaré inequality

$$\|w\|_1^2 \geq c\|w\|^2$$

we conclude that

$$\|w(t)\|^2 \leq \|w(0)\|^2 \exp\Big[\int_0^t [\frac{2C}{\gamma}\|v(s)\|_1^2 - \frac{c\gamma}{2}]ds\Big]$$

In order to prove that $\|w(t)\|$ goes to 0 as $t \to \infty$ it is enough to show that for any starting point u_0, a.e w.r.t P_{u_0},

$$\liminf_{t\to\infty} \int_0^t [\frac{2C}{\gamma}\|v(s)\|_1^2 - \frac{c\gamma}{2}]ds > +\infty$$

If $f(u) = \frac{1}{2}\int_{\mathcal{D}} |u(x)|^2 dx$ then

$$df(u) = -\gamma \|u\|_1^2 dt + kdt + < u, dW >$$

with k given by

$$kt = E[\int_{\mathcal{D}} |W(t,x)|^2 dx]$$

and

$$d[f(u)]^p = p[f(u)]^{p-1} df(u) + \frac{p(p-1)}{2}[f(u)]^{p-2}(df(u))^2$$

In particular

$$\frac{d}{dt} E\big[[f(u(t))]^p\big] \le E\Big[-\gamma\|u(t)\|_1^2 c_p [f(u(t))]^{p-1} + C_p[f(u(t))]^{p-1}\Big]$$

Using Poincaré inequality

$$\frac{d}{dt} E\big[[f(u(t))]^p\big] \le E\Big[-c\gamma[f(u(t))]^p + C_p[f(u(t))]^{p-1}\Big]$$

It is easy to show by induction on p, that

$$\sup_t E[[f(u(t))]^p] \le C_p(\|u_0\|)$$

We know that

$$\xi(t) = \|u(t)\|^2 - \|u(0)\|^2 + 2\gamma \int_0^t \|u(s)\|_1^2 ds - kt$$

is a martingale with quadratic variation

$$(d\xi(t))^2 = \Big[\int_{\mathcal{D}}\int_{\mathcal{D}} < \rho(x,y)u(t,x), u(t,y) > dxdy\Big] dt$$

Here $\rho(x,y)$ is the covariance of $dW(t,x)dW(t,y) = \rho(x,y)dt$. It is not hard to argue from here that for any initial u_0 in $L_2(\mathcal{D})$ almost surely with respect to the process P_{u_0} starting from u_0 we have

$$\lim_{t\to\infty} \frac{\xi(t)}{t} = 0$$

By Poincaré inequality this is like the diffusion

$$\frac{1}{2} x D^2 - \gamma x D$$

and $\frac{x(t)}{t} \to 0$ a.e. as $t \to \infty$. Then

$$\lim_{t\to\infty} [\frac{1}{t}\int_0^t \|u(s)\|_1^2 ds] = \frac{k}{2\gamma}$$

Enough if

$$\frac{Ck}{\gamma^2} < \frac{c\gamma}{2}$$

For the next step it is convenient to write the equation in terms of vorticity defined by a scalar

$$w(x) = D_y u - D_x v$$

In Fourier modes

$$w_k = ik_2 u_k - ik_1 v_k$$

Divergence free condition is

$$ik_1 u_k + ik_2 v_k = 0$$

providing a solution for (u, v), from w

$$u_k = -\frac{ik_2}{|k|^2} w_k, v_k = \frac{ik_1}{|k|^2} w_k$$

We will write the N-S equations for w_k.

$$\dot{w}_k = -\gamma |k|^2 w_k - \frac{1}{4\pi} \sum_{j+\ell=k} (j_2 \ell_1 - j_1 \ell_2) [\frac{1}{|\ell|^2} - \frac{1}{|j|^2}] w_j w_\ell$$

The randomly forced version will be

$$dw_k(t) = \Big[-\gamma |k|^2 w_k - \frac{1}{4\pi} \sum_{j+\ell=k} (j_2 \ell_1 - j_1 \ell_2) [\frac{1}{|\ell|^2} - \frac{1}{|j|^2}] w_j w_\ell \Big] dt + dz_k(t)$$

where $z_k = \bar{z}_k$.

$$\sum e^{i<k,x>} z_k(t) = \sum f_k(x) \beta_k(t)$$

with f_k being $\cos < k, x >$ or $\sin < k, x >$ depending on whether $k \in Z^+$ or Z^-. In other words

$$z_k(t) = \beta_k(t) + i\beta_{-k}(t) \text{ for } k \in Z^+$$

and

$$z_k(t) = \beta_{-k}(t) - i\beta_k(t) \text{ for } k \in Z^-$$

The goal is to still be able to prove the uniqueness of the invariant measure even if the viscosity is small and the noise is "on" only at a finite number of frequencies.

The new idea is to replace the strong Feller property by an "approximate" strong feller property. A some what strengthened version of the strong Feller property asserts that for any $t > 0$,

$$x \to p(t, x, \cdot)$$

is a continuous map in variation. The almost strong Feller property is a weakening of this. We need the notion of the following metric on probability measures on a metric space.

$$D(\alpha, \beta) = \sup_{\phi(\cdot) \in \mathcal{L}_1} | \int \phi(x) [d\alpha - d\beta] |$$

where $\mathcal{L}_1 = \{\phi : |\phi(x) - \phi(y)| \leq d(x,y)\}$. When $d(x,y)$ is the discrete metric with $d(x,y) = 1$ if $x \neq y$, D becomes the variational distance. It is not hard to see that if $d_n \uparrow d$, then $D_n \uparrow D$ as well.

We shall say that the transition probability P_t on a metric space is (X, ρ) is approximately strongly Feller at x if, there exists $d_n \uparrow d$ where d is the discrete metric and times $t_n \uparrow \infty$, such that

$$\lim_{\epsilon \to 0} \lim_{n \to \infty} \sup_{y : \rho(x,y) \leq \epsilon} D_n(P(t_n, x, \cdot), P(t_n, y, \cdot)) = 0$$

Note that ρ has nothing to do with d_n, d. Suppose $x \in X$ is such that for any neighborhood U of x and any $z \in X$, $P(s, z, U) > 0$ for some s and $P(\cdot, \cdot, \cdot)$ is approximately Feller at x, then the invariant measure ν is unique. Let if possible ν_1, ν_2 be two different invariant measures. We can assume they are orthogonal. Let $\phi(x)$ be a function which is 0 and 1 on the disjoint supports, so that $\int \phi(x) d\nu_1 = 1 - \int \phi(x) d\nu_2 = 0$. Clearly for any neighborhood U of x, $\nu_i(U) > 0$ for $i = 1, 2$ and

$$\inf_{x \in U} \int \phi(y) P(t, x, dy) = 1 - \sup_{x \in U} \int \phi(y) P(t, x, dy) = 0$$

which in turn implies that

$$\sup_{x, y \in U} D_n(P(t_n, x, \cdot), P(t_n, y, \cdot)) = 1$$

contradicting the approximate strong Feller property. We saw that strong Feller property at time t is a consequence of a gradient estimate of the form

$$\|(\nabla P_t \phi)(x)\| \leq C(\|x\|) \|\phi\|_\infty$$

The approximate strong Feller property will follow from

$$\|(\nabla P_t \phi)(x)\| \leq C(\|x\|)[\|\phi\|_\infty + \delta(t) \|\nabla \phi\|_\infty]$$

with $\delta(t) \to 0$ as $t \to \infty$.

The idea is to let the noise take care of small frequencies and the contraction the high frequencies. We have the equation

$$dw(t) = \gamma \Delta w(t) + \tilde{B}(w(t), w(t)) + dW(t)$$

The linearized equation for $\xi(t) = D_\xi w(t)$ is

$$d\xi(t) = \gamma \Delta \xi(t) + \widehat{B}(w(t), \xi(t));\ \xi(0) = \xi$$

Then

$$(D_\xi P_t \phi)(w) = E_w[< (\nabla \phi)(w(t)), \xi(t) >]$$

We have the integration by parts formula

$$E_w[< (\nabla\phi)(w(t)), \xi(t) >] = E_w[\phi(w(t)) \int_0^t v(s)dW(s)]$$

Let us choose a perturbation $v(t)$ adapted to $W(t)$ and perturb infinitesimally

$$dw(t) = [\gamma\Delta w(t) + \tilde{B}(w(t), w(t))]dt + dW(t) + hv(t)dt$$

Denoting the derivative by D^v, we have the integration by parts formula

$$D^v\phi(w(t)) =< \nabla\phi, \eta(t) >$$

and by Girsanov formula

$$E_w[D^v\phi(w(t))] = E_w[\phi(w(t)) \int_0^t S^{-1}v(s)dW(s)]$$

and $\eta(t)$ is the solution of

$$d\eta(t) = [\gamma\Delta\eta(t) + \widehat{B}(w(t), \eta(t)) + v(t)]dt; \eta(0) = 0$$

Getting a gradient estimate requires matching $\eta(t) = \xi(t)$ at some time t. That would lead to

$$(D_\xi P_t\phi)(w) = E_w[\phi(w(t)) \int_0^t S^{-1}v(s)dW(s)]$$

which is amenable to an estimate involving $\|\phi\|_\infty$. This requires invertiblility of the covariance operator S. If we have only noise of strength θ in low frequencies upto $|k| \le \ell$, then we need to do some thing. If we do not match $\eta(t)$ and $\xi(t)$ perhaps there is an error $\zeta(t) = \xi(t) - \eta(t)$. Then

$$(D_\xi P_t\phi)(w) = E_w[\phi(w(t)) \int_0^t S^{-1}v(s)ds] + E_w[(\nabla\phi)(w(t))\zeta(t)]$$

If we can keep $E_w|\int_0^t S^{-1}v(s)ds|$ bounded and $E_w[|\zeta(t)|] \to 0$ as $t \to \infty$ we are in business. We rewrite the equation

$$d\xi(t) = [\gamma\Delta\xi(t) + \widehat{B}(w(t), \xi(t))]dt;\ \xi(0) = \xi$$

in two pieces. The projections $\xi^L(t)$ and $\xi^H(t)$ into frequencies $|k| \le \ell$ and $|k| > \ell$. We denote by π^L and π^H the corresponding projections.

$$\begin{aligned}
d\xi^L(t) &= [\gamma\Delta\xi^L(t) + \pi^L\widehat{B}(w(t), \xi(t))]dt \\
d\xi^H(t) &= [\gamma\Delta\xi^H(t) + \pi^H\widehat{B}(w(t), \xi(t))]dt \\
\xi^L(0) &= \xi^L \\
\xi^H(0) &= \xi^H
\end{aligned}$$

Let us pick

$$v(s) = \frac{\zeta^L(s)}{\|\zeta^L(s)\|} + [\gamma\Delta\zeta^L(t) + \pi^L \widehat{B}(w(t), \zeta(t))]$$

where $\zeta(s) = \xi(s) - \eta(s)$ and

$$d\eta(t) = [\gamma\Delta\eta(t) + \widehat{B}(w(t), \eta(t)) + v(t)]dt; \eta(0) = 0$$

Note that $v(s)$ is in the range of S. We check the following:

$$\begin{aligned}
d\zeta^L(t) &= d\xi^L(t) - d\eta^L(t) \\
&= [\gamma\Delta\xi^L(t) + \pi^L \widehat{B}(w(t), \xi(t))]dt - [\gamma\Delta\eta^L(t) + \pi^L \widehat{B}(w(t), \eta(t))]dt - v(t)dt \\
&= [\gamma\Delta\zeta^L(t) + \pi^L \widehat{B}(w(t), \zeta(t))]dt - v(t)dt \\
&= -\frac{\zeta^L(s)}{\|\zeta^L(s)\|}dt
\end{aligned}$$

Since $\zeta^L(0) \leq \|\xi\| \leq 1$ we have $\zeta^L(t) = 0$ if $t \geq 1$. In addition we have for $t \geq 1$,

$$d\zeta^H(t) = [\gamma\Delta\zeta^H(t) + \pi^H \widehat{B}(w(t), \zeta^H(t))]dt$$

Now Δ has a decay rate of γN^2.