Chapter 2

Stochastic Integration.

2.1 Brownian Motion as a Martingale

P is the Wiener measure on $(\Omega, \mathcal{B})$ where $\Omega = C[0,T]$ and $\mathcal{B}$ is the Borel σ-field on Ω. In addition we denote by $\mathcal{B}_t$ the σ-field generated by $x(s)$ for $0 \le s \le t$. It is easy to see tha $x(t)$ is a martingale with respect to $(\Omega, \mathcal{B}_t, P)$, i.e for each $t > s$ in $[0,T]$

$$E^P\{x(t)|\mathcal{B}_s\} = x(s) \quad \text{a.e.} \quad P \tag{2.1}$$

and so is $x(t)^2 - t$. In other words

$$E^P\{x(t)^2 - t\,|\mathcal{F}_s\} = x(s)^2 - s \quad \text{a.e.} \quad P \tag{2.2}$$

The proof is rather straight forward. We write $x(t) = x(s) + Z$ where $Z = x(t) - x(s)$ is a random variable independent of the past history $\mathcal{B}_s$ and is distributed as a Gaussian random variable with mean 0 and variance $t - s$. Therefore $E^P\{Z|\mathcal{B}_s\} = 0$ and $E^P\{Z^2|\mathcal{B}_s\} = t - s$ a.e P. Conversely,

Theorem 2.1. Lévy's theorem. *If P is a measure on $(C[0,T], \mathcal{B})$ such that $P[x(0) = 0] = 1$ and the the functions $x(t)$ and $x^2(t) - t$ are martingales with respect to $(C[0,T], \mathcal{B}_t, P)$ then P is the Wiener measure.*

Proof. The proof is based on the observation that a Gaussian distribution is determined by two moments. But that the distribution is Gaussian is a consequence of the fact that the paths are almost surely continuous and not part of our assumptions. The actual proof is carried out by establishing that for each real number λ

$$X_\lambda(t) = \exp\big[\lambda x(t) - \frac{\lambda^2}{2}t\big] \tag{2.3}$$

is a martingale with respect to $(C[0,T], \mathcal{B}_t, P)$. Once this is established it is elementary to compute

$$E^P\big[\exp\big[\lambda(x(t) - x(s))\big]|\mathcal{B}_s\big] = \exp\big[\frac{\lambda^2}{2}(t-s)\big]$$

which shows that we have a Gaussian Process with independent increments with two matching moments. The proof of (2.3) is more or less the same as proving the central limit theorem. In order to prove (2.3) we can assume with out loss of generality that $s = 0$ and will show that

$$E^P\big[\exp\big[\lambda x(t) - \frac{\lambda^2}{2}t\big]\big] = 1 \tag{2.4}$$

To this end let us define successively $\tau_{0,\epsilon} = 0$,

$$\tau_{k+1,\epsilon} = \min\big[\inf\big\{s : s \ge \tau_{k,\epsilon}, |x(s) - x(\tau_{k,\epsilon})| \ge \epsilon\big\}, t\,, \tau_{k,\epsilon} + \epsilon\big]$$

Then each $\tau_{k,\epsilon}$ is a stopping time and eventually $\tau_{k,\epsilon} = t$ by continuity of paths. The continuity of paths also guarantees that $|x(\tau_{k+1,\epsilon}) - x(\tau_{k,\epsilon})| \le \epsilon$. We write

$$x(t) = \sum_{k\ge 0}[x(\tau_{k+1,\epsilon}) - x(\tau_{k,\epsilon})]$$

and

$$t = \sum_{k\ge 0}[\tau_{k+1,\epsilon} - \tau_{k,\epsilon}]$$

To establish (2.4) we calculate the quantity on the left hand side as

$$\lim_{n\to\infty} E^P\big[\exp\big[\sum_{0\le k\le n}\big[\lambda[x(\tau_{k+1,\epsilon}) - x(\tau_{k,\epsilon})] - \frac{\lambda^2}{2}[\tau_{k+1,\epsilon} - \tau_{k,\epsilon}]\big]\big]\big]$$

and show that it is equal to 1. Let us cosider the σ-field $\mathcal{F}_k = \mathcal{B}_{\tau_{k,\epsilon}}$ and the quantity

$$q_k(\omega) = E^P\big[\exp\big[\lambda[x(\tau_{k+1,\epsilon}) - x(\tau_{k,\epsilon})] - \frac{\lambda^2}{2}[\tau_{k+1,\epsilon} - \tau_{k,\epsilon}]\big]\Big|\mathcal{F}_k\big]$$

Clearly, if we use Taylor expansion and the fact that $x(t)$ as well as $x(t)^2 - t$ are martingales

$$\begin{aligned}|q_k(\omega) - 1| &\le CE^P\big[\big[|\lambda|^3|x(\tau_{k+1,\epsilon}) - x(\tau_{k,\epsilon})|^3 + \lambda^2|\tau_{k+1,\epsilon} - \tau_{k,\epsilon}|^2\big]\Big|\mathcal{F}_k\big]\\ &\le C_\lambda\,\epsilon\, E^P\big[\big[|x(\tau_{k+1,\epsilon}) - x(\tau_{k,\epsilon})|^2 + |\tau_{k+1,\epsilon} - \tau_{k,\epsilon}|\big]\big|\mathcal{F}_k\big]\\ &= 2C_\lambda\,\epsilon\, E^P\big[|\tau_{k+1,\epsilon} - \tau_{k,\epsilon}|\big|\mathcal{F}_k\big]\end{aligned}$$

In particular for some constant C depending on λ

$$q_k(\omega) \le E^P\big[\exp\big[C\,\epsilon\,[\tau_{k+1,\epsilon} - \tau_{k,\epsilon}]\big]\big|\mathcal{F}_k\big]$$

and by induction

$$\begin{aligned}\limsup_{n\to\infty} E^P\big[\exp\big[\sum_{0\le k\le n}\big[\lambda[x(\tau_{k+1,\epsilon}) - x(\tau_{k,\epsilon})] - \frac{\lambda^2}{2}[\tau_{k+1,\epsilon} - \tau_{k,\epsilon}]\big]\big]\big]\\ \le \exp[C\,\epsilon\, t\,]\end{aligned}$$

Since $\epsilon > 0$ is arbitrary we prove one half of (2.4). Notice that in any case $\sup_\omega |q_k(\omega) - 1| \leq \epsilon$. Hence we have the lower bound

$$q_k(\omega) \geq E^P\big[\exp\big[-C\,\epsilon\,[\tau_{k+1,\epsilon} - \tau_{k\ \epsilon}]\big]\Big|\mathcal{F}_k\big]$$

which can be used to prove the other half. This completes the proof of the theorem. □

Exercise 2.1. Why does Theorem 2.1 fail for the process $x(t) = N(t) - t$ where $N(t)$ is the standard Poisson Process with rate 1?

Remark 2.1. One can use the Martingale inequality in order to estimate the probability $P\{\sup_{0\leq s\leq t} |x(s)| \geq \ell\}$. For $\lambda > 0$, by Doob's inequality

$$P\big[\sup_{0\leq s\leq t} \exp\big[\lambda x(s) - \frac{\lambda^2}{2}s\big] \geq A\big] \leq \frac{1}{A}$$

and

$$\begin{aligned}
P\big[\sup_{0\leq s\leq t} x(s) \geq \ell\big] &\leq P\big[\sup_{0\leq s\leq t} [x(s) - \frac{\lambda s}{2}] \geq \ell - \frac{\lambda t}{2}\big]\\
&= P\big[\sup_{0\leq s\leq t} [\lambda x(s) - \frac{\lambda^2 s}{2}] \geq \lambda\ell - \lambda^2 t2\big]\\
&\leq \exp[-\lambda\ell + \frac{\lambda^2 t}{2}]
\end{aligned}$$

Optimizing over $\lambda > 0$, we obtain

$$P\big[\sup_{0\leq s\leq t} x(s) \geq \ell\big] \leq \exp[-\frac{\ell^2}{2t}]$$

and by symmetry

$$P\big[\sup_{0\leq s\leq t} |x(s)| \geq \ell\big] \leq 2\exp[-\frac{\ell^2}{2t}]$$

The estimate is not too bad because by reflection principle

$$P\big[\sup_{0\leq s\leq t} x(s) \geq \ell\big] = 2\,P\big[x(t) \geq \ell\big] = \sqrt{\frac{2}{\pi\,t}}\int_\ell^\infty \exp[-\frac{x^2}{2\,t}]\,dx$$

Exercise 2.2. One can use the estimate above to prove the result of Paul Lévy

$$P\big[\limsup_{\delta\to 0} \frac{\sup_{\substack{0\leq s,t\leq 1\\ |s-t|\leq\delta}} |x(s) - x(t)|}{\sqrt{\delta\log\frac{1}{\delta}}} = \sqrt{2}\big] = 1$$

We had an exercise in the previous section that established the lower bound. Let us concentrate on the upper bound. If we define

$$\Delta_\delta(\omega) = \sup_{\substack{0\leq s,t\leq 1\\ |s-t|\leq\delta}} |x(s) - x(t)|$$

first check that it is sufficient to prove that for any $\rho < 1$, and $a > \sqrt{2}$

$$\sum_n P\big[\Delta_{\rho^n}(\omega) \geq a\sqrt{n\rho^n \log \frac{1}{\rho}}\big] < \infty \tag{2.5}$$

To estimate $\Delta_{\rho^n}(\omega)$ it is sufficient to estimate $\sup_{t \in I_j} |x(t) - x(t_j)|$ for $k_\epsilon \rho^{-n}$ overlapping intervals $\{I_j\}$ of the form $[t_j, t_j + (1+\epsilon)\rho^n\,]$ with length $(1+\epsilon)\rho^n$. For each $\epsilon > 0$, $k_\epsilon = \epsilon^{-1}$ is a constant such that any interval $[s\,,t]$ of length no larger than ρ^n is completely contained in some I_j with $t_j \leq s \leq t_j + \epsilon\rho^n$. Then

$$\Delta_{\rho^n}(\omega) \leq \sup_j \big[\sup_{t \in I_j} |x(t) - x(t_j)| + \sup_{t_j \leq s \leq t_j + \epsilon\rho^n} |x(s) - x(t_j)|\big]$$

Therefore, for any $a = a_1 + a_2$,

$$\begin{aligned}
&P\Big[\Delta_{\rho^n}(\omega) \geq a\sqrt{n\rho^n \log \frac{1}{\rho}}\Big] \\
&\qquad \leq P\Big[\sup_j \sup_{t \in I_j} |x(t) - x(t_j)| \geq a_1\sqrt{n\rho^n \log \frac{1}{\rho}}\Big] \\
&\qquad\quad + P\Big[\sup_j \sup_{t_j \leq s \leq t_j + \epsilon\rho^n} |x(s) - x(t_j)| \geq a_2\sqrt{n\rho^n \log \frac{1}{\rho}}\Big] \\
&\qquad \leq 2k_\epsilon \rho^{-n}\Big[\exp[-\frac{a_1^2\, n\rho^n \log \frac{1}{\rho}}{2(1+\epsilon)\rho^n}] + \exp[-\frac{a_2^2\, n\rho^n \log \frac{1}{\rho}}{2\epsilon\rho^n}]\Big]
\end{aligned}$$

Since $a > \sqrt{2}$, we can pick $a_1 > \sqrt{2}$ and $a_2 > 0$. For $\epsilon > 0$ sufficiently small (2.5) is easily verified.

2.2 Brownian Motion as a Markov Process.

Brownian motion is a process with independent increments, the increment over any interval of length t has the Gaussian distribution with density

$$q(t,y) = \frac{1}{(2\pi t)^{\frac{d}{2}}} e^{-\frac{\|y\|^2}{2t}}$$

It is therefore a Markov process with transition probability

$$p(t,x,y) = q(t, y-x) = \frac{1}{(2\pi t)^{\frac{d}{2}}} e^{-\frac{\|y-x\|^2}{2t}}$$

The operators

$$(T_t f)(x) = \int f(y) p(t,x,y) dy$$

satisfy $T_t T_s = T_s T_t = T_{t+s}$, i.e the semigroup property. This is seen to be an easy consequence of the Chapman-Kolmogorov equations

$$\int p(t,x,y) p(s,y,z) dy = p(t+s,x,z)$$

The infinitesimal generator of the semigroup

$$(Af)(x) = \lim_{t \to 0} \frac{T_t f - f}{t}$$

is easily calculated as

$$\begin{aligned}(Af)(x) &= \lim_{t \to 0} \frac{1}{t} \int [f(x+y) - f(x)] q(t,y) dy \\ &= \lim_{t \to 0} \frac{1}{t} \int [f(x+\sqrt{t}y) - f(x)] q(t,y) dy \\ &= \frac{1}{2}(\Delta f)(x)\end{aligned}$$

by expanding f in a Taylor series in $\sqrt{t}$. The term that is linear in y integrates to 0 and the quadratic term leads to the Laplace operator. The differential equation

$$\frac{dT_t}{dt} = T_t A = A T_t$$

implies that $u(t,x) = (T_t f)(x)$ satisfies the heat equation

$$u_t = \frac{1}{2}\Delta u$$

and

$$\frac{d}{dt} \int f(y) p(t,x,y) dy = \int \frac{1}{2}(\Delta f)(y) p(t,x,y) dy$$

In particular if E_x is expectation with respect to Brownian motion starting from x,

$$E_x[f(x(t)] - f(x) = E_x\left[\int_0^t \frac{1}{2}(\Delta f)(x(s)) ds\right]$$

By the Markov property

$$E_x\left[f(x(t) - f(x(s)) - \int_s^t \frac{1}{2}(\Delta f)(x(\tau)) d\tau \Big| \mathcal{F}_s\right] = 0$$

or

$$f(x(t) - f(x(0)) - \int_0^t \frac{1}{2}(\Delta f)(x(\tau)) d\tau$$

is a Martingale with respect to Brownian Motion.

It is just one step from here to show that for functions $u(t,x)$ that are smooth

$$u(t,x(t)) - u(0,x(0)) - \int_0^t [\frac{\partial u}{\partial t} + \frac{1}{2}\Delta u](s,x(s)) ds \tag{2.6}$$

is a martingale. There are in addition some natural exponential Martingales associated with Brownian motion. For instance for any $\lambda \in R^d$,

$$\exp[< \lambda, x(t) - x(0) > -\frac{1}{2}\|\lambda\|^2 t]$$

is a martingale. More generally for any smooth function $u(t,x)$ that is bounded away from 0,

$$u(t,x(t))\exp\left[-\int_0^t\left[\frac{\frac{\partial u}{\partial t}+\frac{1}{2}\Delta u}{u}\right](s,x(s))ds\right] \tag{2.7}$$

is a martingale. In particular if

$$\frac{\partial u}{\partial t}+\frac{1}{2}\Delta u+v(t,x)u(t,x)=0$$

then

$$u(t,x(t))\exp[\int_0^t v(s,x(s))ds]$$

is a Martingale, which is the Feynman-Kac formula. To prove (2.7) from (2.6), we make use of the following elementary lemma.

Lemma 2.2. *Suppose $M(t)$ is almost surely continuous martingale with respect to $(\Omega,\mathcal{F}_t,P)$ and $A(t)$ is a progressively measurable function, which is almost surely continuous and of bounded variation in t. Then, under the assumption that $\sup_{0\le s\le t}|M(s)|Var_{0,t}A(\cdot,\omega)$ is integrable,*

$$M(t)A(t)-M(0)A(0)-\int_0^t M(s)dA(s)$$

is again a Martingle.

Proof. The main step is to see why

$$E[M(t)A(t)-M(0)A(0)-\int_0^t M(s)dA(s)]=0$$

Then the same argument, repeated conditionally will prove the martingale property.

$$\begin{aligned}E[M(t)A(t)-M(0)A(0)] &= \lim\sum_j E[M(t_j)A(t_j)-M(t_{j-1})A(t_{j-1})]\\ &= \lim\sum_j E[M(t_j)A(t_{j-1})-M(t_{j-1})A(t_{j-1})]\\ &\quad + \lim\sum_j E[M(t_j)[A(t_j)-A(t_{j-1})]]\\ &= \lim\sum_j E[M(t_j)[A(t_j)-A(t_{j-1})]]\\ &= E\Big[\int_0^t M(s)dA(s)\Big]\end{aligned}$$

The limit is over the partition $\{t_j\}$ becoming dense in $[0,t]$ and ones uses the integrability of $\sup_{0\le s\le t}|M(s)|Var_{0,t}A(\cdot)$ and the dominated convergence theorem to complete the proof. ☐

Now, to go from (2.6) to (2.7), we choose

$$M(t) = u(t, x(t)) - u(0, x(0)) - \int_0^t [\frac{\partial u}{\partial t} + \frac{1}{2}\Delta u](s, x(s))ds$$

and

$$A(t) = \exp\left[- \int_0^t \left[\frac{\frac{\partial u}{\partial t} + \frac{1}{2}\Delta u}{u} \right](s, x(s))ds \right]$$

2.3 Stochastic Integrals

If $y_1, \ldots, y_n$ is a martingale relative to the σ-fields $\mathcal{F}_j$, and if $e_j(\omega)$ are random functions that are $\mathcal{F}_j$ measurable, the sequence

$$z_j = \sum_{k=0}^{j-1} e_k(\omega)[y_{k+1} - y_k]$$

is again a martingale with respect to the σ-fields $\mathcal{F}_j$, provided the expectations are finite. A computation shows that if

$$a_j(\omega) = E^P[(y_{j+1} - y_j)^2 | \mathcal{F}_j]$$

then

$$E^P[z_j^2] = \sum_{k=0}^{j-1} E^P\left[a_k(\omega)|e_k(\omega)|^2\right]$$

or more precisely

$$E^P\left[(z_{j+1} - z_j)^2 | \mathcal{F}_j\right] = a_j(\omega)|e_j(\omega)|^2 \qquad \text{a.e. P}$$

Formally one can write

$$\delta z_j = z_{j+1} - z_j = e_j(\omega)\delta y_j = e_j(\omega)(y_{j+1} - y_j)$$

z_j is called a martingale transform of y_j and the size of z_n measured by its mean square is exactly equal to $E^P\left[\sum_{j=0}^{n-1} |e_j(\omega)|^2 \, a_j(\omega)\right]$. The stochastic integral is just the continuous analog of this.

Theorem 2.3. *Let $y(t)$ be an almost surely continuous martingale relative to $(\Omega, \mathcal{F}_t, P)$ such that $y(0) = 0$ a.e. P, and*

$$y^2(t) - \int_0^t a(s\,, \omega)ds$$

is again a martingale relative to $(\Omega, \mathcal{F}_t, P)$, where $a(s\,, \omega)ds$ is a bounded progressively measurable function. Then for progressively measurable functions $e(\cdot\,, \cdot)$ satisfying, for every $t > 0$,

$$E^P\left[\int_0^t e^2(s)a(s)ds \right] < \infty$$

the stochastic integral

$$z(t) = \int_0^t e(s)dy(s)$$

makes sense as an almost surely continuous martingale with respect to $(\Omega, \mathcal{F}_t, P)$ *and*

$$z^2(t) - \int_0^t e^2(s)a(s)ds$$

is again a martingale with respect to $(\Omega, \mathcal{F}_t, P)$. *In particular*

$$E^P\big[z^2(t)\big] = E^P\big[\int_0^t e^2(s)a(s)ds\big] \tag{2.8}$$

Proof.
Step 1. The statements are obvious if $e(s)$ is a constant.

Step 2. Assume that $e(s)$ is a simple function given by

$$e(s\,,\omega) = e_j(\omega) \qquad \text{for } t_j \le s < t_{j+1}$$

where $e_j(\omega)$ is $\mathcal{F}_{t_j}$ measurable and bounded for $0 \le j \le N$ and $t_{N+1} = \infty$. Then we can define inductively

$$z(t) = z(t_j) + e(t_j\,,\omega)[y(t) - y(t_j)]$$

for $t_j \le t \le t_{j+1}$. Clearly $z(t)$ and

$$z^2(t) - \int_0^t e^2(s\,,\omega)a(s\,,\omega)ds$$

are martingales in the interval $[t_j, t_{j+1}]$. Since the definitions match at the end points the martingale property holds for $t \ge 0$.

Step 3. If $e_k(s\,,\omega)$ is a sequence of uniformly bounded progressively measurable functions converging to $e(s\,,\omega)$ as $k \to \infty$ in such a way that

$$\lim_{k\to\infty} \int_0^t |e_k(s)|^2 a(s)ds = 0$$

for every $t > 0$, because of the relation (2.8)

$$\lim_{k,k'\to\infty} E^P\Big[|z_k(t) - z_{k'}(t)|^2\Big] = \lim_{k,k'\to\infty} E^P\Big[\int_0^t |e_k(s) - e_{k'}(s)|^2 a(s)ds\Big] = 0.$$

Combined with Doob's inequality, we conclude the existence of a an almost surely continuous martingale $z(t)$ such that

$$\lim_{k\to\infty} E^P\Big[\sup_{0\le s\le t} |z_k(s) - z(s)|^2\Big] = 0$$

and clearly

$$z^2(t) - \int_0^t e^2(s)a(s)ds$$

is an $(\Omega, \mathcal{F}_t, P)$ martingale.

Step 4. All we need to worry now is about approximating $e(\cdot\,,\cdot)$. Any bounded progressively measurable almost surely continuous $e(s\,,\omega)$ can be approximated by $e_k(s\,,\omega) = e(\frac{[ks]\wedge k^2}{k}\,,\omega)$ which is piecewise constant and levels off at time k. It is trivial to see that for every $t > 0$,

$$\lim_{k\to\infty} \int_0^t |e_k(s) - e(s)|^2 a(s)\,ds = 0$$

Step 5. Any bounded progressively measurable $e(s\,,\omega)$ can be approximated by continuous ones by defining

$$e_k(s\,,\omega) = k \int_{(s-\frac{1}{k})\vee 0}^{s} e\,(u\,,\omega)du$$

and again it is trivial to see that it works.

Step 6. Finally if $e(s\,,\omega)$ is un bounded we can approximate it by truncation,

$$e_k(s\,,\omega) = f_k(e(s\,,\omega))$$

where $f_k(x) = x$ for $|x| \le k$ and 0 otherwise.

This completes the proof of the theorem. □

Suppose we have an almost surely continuous process $x(t\,,\omega)$ defined on some $(\Omega, \mathcal{F}_t, P)$, and progressively measurable functions $b(s,\omega), a(s,\omega)$ with $a \ge 0$, such that

$$x(t\,,\omega) = x(0\,,\omega) + \int_0^t b(s\,,\omega)ds + y(t\,,\omega)$$

where $y(t,\omega)$ and

$$y^2(t,\omega) - \int_0^t a(s,\omega)ds$$

are martingales with respect to $(\Omega, \mathcal{F}_t, P)$. The stochastic integral $z(t) = \int_0^t e(s)dx(s)$ is defined by

$$z(t) = \int_0^t e(s)dx(s) = \int_0^t e(s)b(s)ds + \int_0^t e(s)dy(s)$$

For this to make sense we need for every t,

$$E^P\big[\int_0^t |e(s)b(s)|ds\big] < \infty \quad \text{and} \quad E^P\big[\int_0^t |e(s)|^2 a(s)ds\big] < \infty$$

If we assume for simplicity that eb and e^2a are uniformly bounded functions in t and ω. It then follows, that for any $\mathcal{F}_0$ measurable $z(0)$, that

$$z(t) = z(0) + \int_0^t e(s)dx(s)$$

is again an almost surely continuous process such that

$$z(t) = z(0) + \int_0^t b'(s,\omega)ds + y'(t,\omega)$$

where $y'(t)$ and

$$y'(t)^2 - \int_0^t a'(s,\omega)ds$$

are martingales with $b' = eb$ and $a' = e^2a$.

Exercise 2.3. If e is such that eb and e^2a are bounded, then prove directly that the exponentials

$$\exp\big[\lambda(z(t) - z(0)) - \lambda\int_0^t e(s)b(s)ds - \frac{\lambda^2}{2}\int_0^t a(s)e^2(s)ds\big]$$

are $(\Omega, \mathcal{F}_t, P)$ martingales.

We can easily do the mutidimensional generalization. Let $y(t)$ be a vector valued martingale with n components $y_1(t), \cdots, y_n(t)$ such that

$$y_i(t)y_j(t) - \int_o^t a_{i,j}(s\,,\omega)ds$$

are again martingales with respect to $(\Omega, \mathcal{F}_t, P)$. Assume that the progressively measurable functions$\{a_{i,j}(t\,,\omega)\}$ are symmetric and positive semidefinite for every t and ω and are uniformly bounded in t and ω. Then the stochastic integral

$$z(t) = z(0) + \int_0^t < e(s), dy(s) >= z(0) + \sum_i \int_0^t e_i(s)dy_i(s)$$

is well defined for vector velued progressively measurable functions $e(s\,,\omega)$ such that

$$E^P\big[\int_0^t < e(s)\,, a(s)e(s) > ds\big] < \infty$$

In a similar fashion to the scalar case, for any diffusion process $x(t)$ corresponding to $b(s\,,\omega) = \{b_i(s\,,\omega)\}$ and $a(s\,,\omega) = \{a_{i,j}(s\,,\omega)\}$ and any $e(s\,,\omega)) = \{e_i(s\,,\omega)\}$ which is progressively measurable and uniformly bounded

$$z(t) = z(0) + \int_0^t < e(s)\,, dx(s) >$$

is well defined and is a diffusion corresponding to the coefficients

$$\tilde{b}(s\,,\omega) =< e(s\,,\omega)\,, b(s\,,\omega) > \quad \text{and} \quad \tilde{a}(s\,,\omega) =< e(s\,,\omega)\,, a(s\,,\omega)e(s\,,\omega) >$$

It is now a simple exercise to define stocahstic integrals of the form

$$z(t) = z(0) + \int_0^t \sigma(s\,,\omega)dx(s)$$

where $\sigma(s\,,\omega)$ is a matrix of dimension $m \times n$ that has the suitable properties of boundedness and progressive measurability. $z(t)$ is seen easily to correspond to the coefficients

$$\tilde{b}(s) = \sigma(s)b(s) \quad \text{and} \quad \tilde{a}(s) = \sigma(s)a(s)\sigma^*(s)$$

The analogy here is to linear transformations of Gaussian variables. If ξ is a Gaussian vector in R^n with mean μ and covariance A, and if $\eta = T\xi$ is a linear transformation from R^n to R^m, then η is again Gaussian in R^m and has mean $T\mu$ and covariance matrix TAT^*.

Exercise 2.4. If $x(t)$ is Brownian motion in R^n and $\sigma(s\,,\omega)$ is a progreessively measurable bounded function then

$$z(t) = \int_0^t \sigma(s\,,\omega)dx(s)$$

is again a Brownian motion in R^n if and only if σ is an orthogonal matrix for almost all s (with repect to Lebesgue Measure) and ω (with respect to P)

Exercise 2.5. We can mix stochastic and ordinary integrals. If we define

$$z(t) = z(0) + \int_0^t \sigma(s)dx(s) + \int_0^t f(s)ds$$

where $x(s)$ is a process corresponding to $b(s), a(s)$, then $z(t)$ corresponds to

$$\tilde{b}(s) = \sigma(s)b(s) + f(s) \quad \text{and} \quad \tilde{a}(s) = \sigma(s)a(s)\sigma^*(s)$$

The analogy is again to affine linear transformations of Gaussians $\eta = T\xi + \gamma$.

Exercise 2.6. Chain Rule. If we transform from x to z and again from z to w, it is the same as makin a single transformation from z to w.

$$dz(s) = \sigma(s)dx(s) + f(s)ds \quad \text{and} \quad dw(s) = \tau(s)dz(s) + g(s)ds$$

can be rewritten as

$$dw(s) = [\tau(s)\sigma(s)]dx(s) + [\tau(s)f(s) + g(s)]ds$$

2.4 Ito's Formula.

The chain rule in ordinary calculus allows us to compute

$$df(t\,,x(t)) = f_t(t\,,x(t))dt + \nabla f(t\,,x(t)).dx(t)$$

We replace $x(t)$ by a Brownian path, say in one dimension to keep things simple and for f take the simplest nonlinear function $f(x) = x^2$ that is independent of t. We are looking for a formula of the type

$$\beta^2(t) - \beta^2(0) = 2\int_0^t \beta(s)\,d\beta(s) \tag{2.9}$$

We have already defined integrals of the form

$$\int_0^t \beta(s)\,d\beta(s) \tag{2.10}$$

as Ito's stochastic integrals. But still a formula of the type (2.9) cannot possibly hold. The left hand side has expectation t while the right hand side as a stochastic integral with respect to $\beta(\cdot)$ is mean zero. For Ito's theory it was important to evaluate $\beta(s)$ at the back end of the interval $[t_{j-1}\,,t_j]$ before multiplying by the increment $(\beta(t_j) - \beta(t_{j-1})$ to keeep things progressively measurable. That meant the stochastic integral (2.10) was approximated by the sums

$$\sum_j \beta(t_{j-1})(\beta(t_j) - \beta(t_{j-1})$$

over successive partitions of $[0\,,t]$. We could have approximated by sums of the form

$$\sum_j \beta(t_j)(\beta(t_j) - \beta(t_{j-1}).$$

In ordinary calculus, because $\beta(\cdot)$ would be a continuous function of bounded variation in t, the difference would be negligible as the partitions became finer leading to the same answer. But in Ito calculus the differnce does not go to 0. The difference D_π is given by

$$\begin{aligned}
D_\pi &= \sum_j \beta(t_j)(\beta(t_j) - \beta(t_{j-1}) - \sum_j \beta(t_{j-1}(\beta(t_j) - \beta(t_{j-1})\\
&= \sum_j (\beta(t_j) - \beta(t_{j-1})(\beta(t_j) - \beta(t_{j-1})\\
&= \sum_j (\beta(t_j) - \beta(t_{j-1})^2
\end{aligned}$$

An easy computation gives $E[D_\pi] = t$ and $E[(D_\pi - t)^2] = 3\sum_j (t_j - t_{j-1})^2$ tends to 0 as the partition is refined. On the other hand if we are neutral and approximate the integral (2.10) by

$$\sum_j \frac{1}{2}(\beta(t_{j-1}) + \beta(t_j))(\beta(t_j) - \beta(t_{j-1})$$

then we can simplify and calculate the limit as

$$\lim \sum_j \frac{\beta(t_j)^2 - \beta(t_{j-1})^2}{2} = \frac{1}{2}(\beta^2(t) - \beta^2(0))$$

This means as we defined it (2.10) can be calculated as

$$\int_0^t \beta(s)\, d\beta(s) = \frac{1}{2}(\beta^2(t) - \beta^2(0)) - \frac{t}{2}$$

or the correct version of (2.9) is

$$\beta^2(t) - \beta^2(0) = \int_0^t \beta(s)\, d\beta(s) + t$$

Now we can attempt to calculate $f(\beta(t)) - f(\beta(0))$ for a smooth function of one variable. Roughly speaking, by a two term Taylor expansion

$$\begin{aligned} f(\beta(t)) - f(\beta(0)) &= \sum_j [f(\beta(t_j)) - f(\beta(t_{j-1}))] \\ &= \sum_j f'(\beta(t_{j-1})(\beta(t_j)) - \beta(t_{j-1})) \\ &\qquad + \frac{1}{2}\sum_j f''(\beta(t_{j-1})(\beta(t_j)) - \beta(t_{j-1}))^2 \\ &\qquad + \sum_j O|\beta(t_j)) - \beta(t_{j-1})|^3 \end{aligned}$$

The expected value of the error term is approximately

$$E\big[\sum_j O|\beta(t_j)) - \beta(t_{j-1})|^3\big] = \sum_j O|t_j - t_{j-1}|^{\frac{3}{2}} = o(1)$$

leading to Ito's formula

$$f(\beta(t)) - f(\beta(0)) = \int_0^t f'(\beta(s))d\beta(s) + \frac{1}{2}\int_0^t f''(\beta(s))ds \tag{2.11}$$

It takes some effort to see that

$$\sum_j f''(\beta(t_{j-1})(\beta(t_j)) - \beta(t_{j-1}))^2 \to \int_0^t f''(\beta(s))ds$$

But the idea is, that because $f''(\beta(s))$ is continuous in t, we can pretend that it is locally constant and use that calculation we did for x^2 where f'' is a constant.

While we can make a proof after a careful estimation of all the errors, in fact we do not have to do it. After all we have already defined the stochastic integral (2.10). We should be able to verify (2.11) by computing the mean square of the difference and showing that it is 0.

In fact we will do it very generally with out much effort. We have the tools already.

Theorem 2.4. *Let $x(t)$ be an almost surely continuous process with values on R^d such that*

$$y_i(t) = x_i(t) - x_i(0) - \int_0^t b_i(s, \omega) ds \tag{2.12}$$

and

$$y_i(t) y_j(t) - \int_0^t a_{i,j}(s, \omega) ds \tag{2.13}$$

are martingales for $1 \leq i, j \leq d$. For any smooth function $u(t\,, x)$ on $[0\,, \infty) \times R^d$

$$\begin{aligned} u(t\,, x(t)) - u(0\,, x(0)) = &\int_0^s u_s(s\,, x(s)) ds + \int_0^t < (\nabla u)(s\,, x(s))\,, dx(s) > \\ &+ \frac{1}{2} \int_0^t \sum_{i,j} a_{i,j}(s\,, \omega) \frac{\partial^2 u}{\partial x_i \partial x_j}(s\,, x(s)) ds \end{aligned}$$

Proof. Let us define the stochastic process

$$\begin{aligned} \xi(t) = &u(t\,, x(t)) - u(0\,, x(0)) - \int_0^s u_s(s\,, x(s)) ds \\ &- \int_0^t < (\nabla u)(s\,, x(s))\,, dx(s) > - \frac{1}{2} \int_0^t \sum_{i,j} a_{i,j}(s\,, \omega) \frac{\partial^2 u}{\partial x_i \partial x_j}(s\,, x(s)) ds \end{aligned} \tag{2.14}$$

We define a $d+1$ dimensional process $\tilde{x}(t) = \{u(t\,, x(t)), x(t)\}$ which is again a process with almost surely continuous paths satisfying relations analogous to (2.12) and (2.13) with $[\tilde{b}, \tilde{a}]$. If we number the extra coordinate by 0, then

$$\tilde{b}_i = \begin{cases} [\frac{\partial u}{\partial s} + \mathcal{L}_{s,\omega} u](s\,, x(s)) & \text{if } i = 0 \\ b_i(s\,, \omega) & \text{if } i \geq 1 \end{cases}$$

$$\tilde{a}_{i,j} = \begin{cases} < a(s\,, \omega) \nabla u\,, \nabla u > & \text{if } i = j = 0 \\ [a(s\,, \omega) \nabla u]_i & \text{if } j = 0, i \geq 1 \\ a_{i,j}(s\,, \omega) & \text{if } i, j \geq 1 \end{cases}$$

The actual computation is interesting and reveals the connection between ordinary calculus, second order operators and Ito calculus. If we want to know the parametrs of the process $y(t)$, then we need to know what to subtract from $v(t\,, y(t)) - v(0\,, y(0))$ to obtain a martingale. But $v(t,\, , y(t)) = w(t\,, x(t))$, where

$w(t,x) = v(t\,,u(t\,,x)\,,x)$ and if we compute

$$
\begin{aligned}
(\frac{\partial w}{\partial t} + \mathcal{L}_{s,\omega} w)(t\,,x) &= v_t + v_u[u_t + \sum_i b_i u_{x_i} + \sum_i b_i v_{x_i} + \frac{1}{2}\sum_{i,j} a_{i,j} u_{x_i,x_j}] \\
&\quad + v_{u,u}\frac{1}{2}\sum_{i,j} a_{i,j} u_{x_i} u_{x_j} + \sum_i v_{u,x_i} \sum_j a_{i,j} u_{x_j} \\
&\quad + \frac{1}{2}\sum_{i,j} a_{i,j} v_{x_i,x_j} \\
&= v_t + \tilde{\mathcal{L}}_{t,\omega} v
\end{aligned}
$$

with

$$
\tilde{\mathcal{L}}_{t,\omega} v = \sum_{i\geq 0} \tilde{b}_i(s\,,\omega) v_{y_i} + \frac{1}{2}\sum_{i,j\geq 0} \tilde{a}_{i,j}(s\,,\omega) v_{y_i,y_j}
$$

We can construct stochastic integrals with respect to the $d+1$ dimensional process $y(\cdot)$ and $\xi(t)$ defined by (2.14) is again an almost surely continuous process and its parameters can be calculated. After all

$$
\xi(t) = \int_0^t < f(s\,,\omega)\,, dy(s) > + \int_0^t g(s\,,\omega) ds
$$

with

$$
f_i(s\,,\omega) = \begin{cases} 1 & \text{if } i = 0 \\ -(\nabla u)_i(s\,,x(s)) & \text{if } i \geq 1 \end{cases}
$$

and

$$
g(s\,,\omega) = -[\frac{\partial u}{\partial s} + \frac{1}{2}\sum_{i,j} a_{i,j}(s\,,\omega)\frac{\partial^2 u}{\partial x_i \partial x_j}](s\,,x(s))
$$

Denoting the parameters of $\xi(\cdot)$ by $[B(s\,,\omega), A(s\,,\omega)]$, we find

$$
\begin{aligned}
A(s\,,\omega) &=< f(s\,,\omega)\,, \tilde{a}(s\,,\omega) f(s\,,\omega) > \\
&=< a\nabla u\,, \nabla u > -2 < a\nabla u\,, \nabla u > + < a\nabla u\,, \nabla u > \\
&= 0
\end{aligned}
$$

and

$$
\begin{aligned}
B(s\,,\omega) =< \tilde{b}\,, f > + g &= \tilde{b}_0(s\,,\omega) - < b(s\,,\omega)\,, \nabla u(s\,,x(s)) > \\
&\quad - [\frac{\partial u}{\partial s}(s\,,\omega) + \frac{1}{2}\sum_{i,j} a_{i,j}(s\,,\omega)\frac{\partial^2 u}{\partial x_i \partial x_j}(s\,,x(s))] \\
&= 0
\end{aligned}
$$

Now all we are left with is the following

Lemma 2.5. *If $\xi(t)$ is a scalar process corresponding to the coefficients* $[0,0]$ *then*

$$
\xi(t) - \xi(0) \equiv 0 \qquad \text{a.e.}
$$

Proof. Just compute

$$E[(\xi(t)-\xi(0))^2] = E[\int_0^t 0\,ds] = 0$$

□

This concludes the proof of the theorem. □

Exercise 2.7. Ito's formula is a local formula that is valid for almost all paths. If u is a smooth function i.e. with one continuous t derivative and two continuous x derivatives (2.11) must still be valid a.e. We cannot do it with moments, because for moments to exist we need control on growth at infinity. But it should not matter. Should it?

Application: Local time in one dimension. Tanaka Formula.

If $\beta(t)$ is the one dimensional Brownian Motion, for any path $\beta(\cdot)$ and any t, the occupation meausre $L_t(A\,,\omega)$ is defined by

$$L_t(A,\omega) = m\{s : 0 \le s \le t \;\;\&\;\; \beta(s) \in A\}$$

Theorem 2.6. *There exists a function $\ell(t\,,y,\,\omega)$ such that, for almost all ω,*

$$L_t(A,\omega) = \int_A \ell(t\,,y\,,\omega)\,dy$$

identically in t.

Proof. Formally

$$\ell(t\,,y\,,\omega) = \int_0^t \delta(\beta(s)-y)ds$$

but, we have to make sense out of it. From Ito's formula

$$f(\beta(t)) - f(\beta(0)) = \int_0^t f'(\beta(s))\,d\beta(s) + \frac{1}{2}\int_0^t f''(\beta(s))ds$$

If we take $f(x) = |x-y|$ then $f'(x) =$ sign x and $\frac{1}{2}f''(x) = \delta(x-y)$. We get the 'identity'

$$|\beta(t)-y| - |\beta(0)-y| - \int_0^t \text{ sign } \beta(s)d\beta(s) = \int_0^t \delta(\beta(s)-y)ds = \ell(t\,,y\,,\omega)$$

While we have not proved the identity, we can use it to define $\ell(\cdot\,,\cdot\,,\cdot)$. It is now well defined as a continuous function of t for almost all ω for each y, and by Fubini's theorem for almost all y and ω.

Now all we need to do is to check that it works. It is enough to check that for any smooth test function ϕ with compact support

$$\int_R \phi(y)\ell(t\,,y\,,\omega)\,dy = \int_0^t \phi(\beta(s))ds \tag{2.15}$$

The function

$$\psi(x) = \int_R |x-y|\phi(y)\,dy$$

is smooth and a straigt forward calculation shows

$$\psi'(x) = \int_R \text{ sign } (x-y)\phi(y)\,dy$$

and

$$\psi''(x) = -2\phi(x)$$

It is easy to see that (2.15) is nothing but Ito's formuls for ψ. □

Remark 2.2. One can estimate

$$E\Big[\int_0^t [\text{ sign } (\beta(s)-y) - \text{ sign } (\beta(s)-z)]d\beta(s)\Big]^4 \le C|y-z|2$$

and by Garsia- Rodemich- Rumsey or Kolmogorov one can conclude that for each t, $\ell(t\,,y\,,\omega)$ is almost surely a continuous function of y.

Remark 2.3. With a little more work one can get it to be jointly continuous in t and y for almost all ω.