Hormander's Theorem

Let $\mathcal{L}$ denote the operator

$$\mathcal{L} = \frac{1}{2}\sum_{i=1}^{k} X_i^2 + Y$$

where $X_1\,,\cdot,\,,X_k\,,Y$ are C^∞ vector fields in R^n. Assume that the Lie Algebra generated by $X_1\,,\cdot,\,,X_k\,,Y$ span R^n at every x. Then, if u is a distribution such that

$$Lu = f$$

and f is C^∞ in an open set G, it is always true that u is C^∞ in G.

Proof is in three steps.

Step 1. Let L have a fundamental solution $p(t\,,x\,,y)$ with the following properties:

$$\sup_{0<t\le 1}\sup_{|x-y|\ge\epsilon} |D_x^r D_y^s\, p(t\,,x\,,y)| \le C_{r,s}(\epsilon) < \infty$$

$$\int p(t\,,x\,,y) f(y) dy \to f(x)$$

in every C^r if $f \in C^r$. And the same is true of the adjoint.

$$\int p(t\,,x\,,y) f(x) dx \to f(y)$$

in every C^r if $f \in C^r$. In particular for $f \in C^r$,

$$\sup_{0\le t\le 1} \|P_t f\|_{(r)} \le C_r \|f\|_{(r)}$$

and

$$\sup_{0\le t\le 1} \|P_t^* f\|_{(r)} \le C_r \|f\|_{(r)}$$

Then $\mathcal{L}$ is hypoelliptic.

Proof: Let $x_0 \in G$. Let us find ϵ such that

$$\overline{B(x_0\,,3\epsilon)} \subset G$$

and a C^∞ function g which is 1 in $B(x_0\,,2\epsilon)$ and 0 outside $B(x_0\,,3\epsilon)$.

$$\mathcal{L}(gu) = g\mathcal{L}u + u\mathcal{L}g + < a\nabla g\,,\nabla u >= f + h$$

where h is a distribution supported in $B(x_0\,,2\epsilon)^c$

$$\begin{aligned} P_1(gu) - gu &= \int_0^1 P_t[\mathcal{L}(gu)]dt \\ &= \int_0^1 P_t\, f\, dt + \int_0^1 P_t\, h\, dt \end{aligned}$$

It is done.

Step 2. Hormander condition does not give smooth fundamental solution for the parabolic equation. We do the following trick. Introduce an extra variable x_{n+1} and a function

$$\rho(x_{n+1}) = 2 + \sin x_{n+1}$$

Define new vector fields

$$\widehat{X}_i = \sqrt{\rho(x_{n+1})} X_i$$

for $1 \leq i \leq k$,

$$\widehat{X}_{k+1} = \frac{\partial}{\partial x_{n+1}}$$

and

$$\widehat{Y} = \rho(x_{n+1}) Y$$

so that

$$\widehat{\mathcal{L}} = \frac{1}{2} \sum_{i=1}^{k} \widehat{X}_i^2 + \widehat{Y} = \rho(x_{n+1}) \mathcal{L} + \frac{1}{2} \frac{\partial^2}{\partial x_{n+1}^2}$$

One checks that $\widehat{\mathcal{L}}$ satisfies the assumptions so that the conditions are fulfilled. So $\widehat{\mathcal{L}}$ is Hypoelliptic in the Hormander sense.

Step 3. If $\mathcal{L} u = f$ in R^n then $\widehat{\mathcal{L}} u = \rho(x_{n+1}) f$ in R^{n+1} and if f is regular in an open set $G \subset R^n$, then $\rho(x_{n+1}) f$ is regular in $R \times G \subset R^{n+1}$. Therefore u which does not depend on x_{n+1} is regular in $R \times G \subset R^{n+1}$ and therefore in G.