3 Multidimensional Versions

The problem of convergence of Fourier Series in several dimensions is more complicated because there is no natural truncation. If $n = \{n_1, \dots, n_d\}$ is a multi-index, then the sum

$$\sum_n a_n e^{in.x}$$

is natuarally computed by summing over finite stes D_N which are allowed to increase to Z^d. One tries to recover the function f by

$$f = \lim_{N\to\infty} \sum_{n\in D_N} a_n e^{in.x} \tag{1}$$

For smooth functions there is no problem because a_n decays fast. The degree of smoothness needed gets worse as dimension goes up. In d dimnsions we need $|a_n|$ to decay like $|n|^{-d+\delta}$ for some $\delta > 0$ to be sure of uniform convergence of the Fourier Series. On the other hand the orthogonality relations imply that in $f \in L_2$, the series converges in L_2 and again D_N can be arbitrary. However for $1 < p < \infty$ but different from 2 the situation is far from clear.

If we take $D_N = \{n : |n_j| \le N, j = 1, \dots, d\}$ the partial sum operator we need to look at is convolution by

$$\left[\frac{1}{2\pi}\right]^d \sum_{\substack{|n_j|\le N\\ j=1,\dots,d}} e^{i<n,x>} = \Pi_{j=1}^d \frac{\sin(N+\frac{1}{2})x_j}{2\pi \sin\frac{x_j}{2}}$$
$$= \Pi_{j=1}^d t_N(x_j)$$

The partial sum operator S_N is therefore the product

$$T^N = \Pi_{j=1}^d T_j^N$$

where T_j^N is the convolution in the variable x_j by the kernel $t_N(x_j)$. It is easy to see that as operators T_j^N have a bound that is uniform in N. The bound in the context of a single variable extends to d variables because t_j^N acts only on the single variable x_j. Therefore T_N have a uniform bound as well. Therefore we have with the choice of the cube $D_N = \{n : |n_j| \le N, j+1, \dots d\}$, we have convergence in L_p of the partial sums to f, for every $f \in L_p$ provided $1 < p < \infty$.

It is known that the result is false for any $p \neq 2$ if we choose $D_N = \{n : n_1^2 + \cdots + n_d^2 \leq N^2$.

We now look at Fourier Transforms on R^d. If $f(x)$ is a function in $L_1(R^d)$ its Fourier transform $\hat{f}(y)$ is defined by

$$\hat{f}(y) = \left(\frac{1}{\sqrt{2\pi}}\right)^d \int_{R^d} e^{i <x,y>} f(x) dx \tag{2}$$

We denote by $\mathcal{S}$ the class of all functions f on R^d that are infinitely differentiable such that the function and its derivitives of all orders decay faster than any power, i.e. for every $n_1, n_2, \ldots, n_d \geq 0$ and $k \geq 0$ there are constants $C_{n_1,n_2,\ldots,n_d,k}$ such that

$$|[(\frac{d}{dx_1})^{n_1}(\frac{d}{dx_1})^{n_2} \ldots (\frac{d}{dx_d})^{n_d} f](x)| \leq C_{n_1,n_2,\cdots,n_d,k}(1 + \|x\|)^{-k}$$

It is easy to show by repeated integration by parts that if $f \in \mathcal{S}$ so does $\hat{f}$.

Theorem 1. *The Fourier transform has the inverse*

$$f(x) = \left(\frac{1}{\sqrt{2\pi}}\right)^d \int_{R^d} e^{-i <x,y>} \hat{f}(y) dy \tag{3}$$

proving that the Fourier transform is a one to one mapping of $\mathcal{S}$ *onto itself.*

In addition the Fourier transform extends as a unitary map from $L_2(R^d)$ *onto* $L_2(R^d)$.

Proof. Clearly

$$g(x) = \left(\frac{1}{\sqrt{2\pi}}\right)^d \int_{R^d} e^{-i <x,y>} \hat{f}(y) dy$$

is well defined as a function in $\mathcal{S}$. We only have to identify it. We compute

g as

$$
\begin{aligned}
g(x) &= \left(\frac{1}{\sqrt{2\pi}}\right)^d \int_{R^d} e^{-i<x,y>} \hat{f}(y) dy \\
&= \lim_{\epsilon \to 0} \left(\frac{1}{\sqrt{2\pi}}\right)^d \int_{R^d} e^{-i<x,y>} \hat{f}(y) e^{-\epsilon \frac{\|y\|^2}{2}} dy \\
&= \lim_{\epsilon \to 0} \left(\frac{1}{\sqrt{2\pi}}\right)^d \int_{R^d} \left[\left(\frac{1}{\sqrt{2\pi}}\right)^d \int_{R^d} e^{i<z,y>} f(z) dz\right] e^{-i<x,y>} e^{-\epsilon \frac{\|y\|^2}{2}} dy \\
&= \lim_{\epsilon \to 0} \left(\frac{1}{2\pi}\right)^d \int_{R^d} \int_{R^d} e^{i<z-x,y>} f(z) e^{-\epsilon \frac{\|y\|^2}{2}} dy dz \\
&= \lim_{\epsilon \to 0} \left(\frac{1}{2\pi}\right)^d \int_{R^d} f(z) \left[\int_{R^d} e^{i<z-x,y>} e^{-\epsilon \frac{\|y\|^2}{2}} dy\right] dz \\
&= \lim_{\epsilon \to 0} \left(\frac{1}{\sqrt{2\pi\epsilon}}\right)^d \int_{R^d} f(z) e^{-\frac{\|z-x\|^2}{2\epsilon}} dz \\
&= f(x)
\end{aligned}
$$

Here we have used the identity

$$\frac{1}{\sqrt{2\pi}} \int_R e^{i\,xy} e^{-\frac{x^2}{2}} dx = e^{-\frac{y^2}{2}}$$

We now turn to the computation of L_2 norm of $\hat{f}$. We calculate it as

$$
\begin{aligned}
\|\hat{f}\|_2^2 &= \lim_{\epsilon \to 0} \int_{R_d} |\hat{f}(y)|^2 e^{-\frac{\epsilon \|y\|^2}{2}} dy \\
&= \lim_{\epsilon \to 0} \int_{R_d} \int_{R_d} \int_{R_d} f(x) \bar{f}(z) e^{i<x-z,y>} e^{-\frac{\epsilon \|y\|^2}{2}} dy dx dz \\
&= \lim_{\epsilon \to 0} \left(\frac{1}{\sqrt{2\pi\epsilon}}\right)^d \int_{R_d} \int_{R_d} f(x) \bar{f}(z) e^{-\frac{\|x-z\|^2}{2\epsilon}} dx dz \\
&= \lim_{\epsilon \to 0} \int_{R^d} f(x) [K_\epsilon \bar{f}\,](x) dx \\
&= \int_{R^d} |f(x)|^2 dx
\end{aligned}
$$

□

We see that the Fourier transform is a bounded linear map from L_1 to L_∞ as well as L_2 to L_2 with corresponding bounds $C = (\frac{1}{\sqrt{2\pi}})^d$ and 1. By the Riesz-Thorin interpolation theorem the Fourier transform is bounded from L_p into $L_{\frac{p}{p-1}}$ for $1 \le p \le 2$. If $\frac{1}{p} = 1.t + \frac{1}{2}(1-t)$ then $\frac{1}{2}(1-t) = 1 - \frac{1}{p} = \frac{p-1}{p}$. See exercise to show that for $f \in L_p$ with $p > 2$ the Fourier Transform need not exist.

For convolution operators of the form

$$(Tf)(x) = (k * f)(x) = \int_{R^d} k(x-y)f(y)dy \tag{4}$$

we want to estimate $\|T\|_p$, the operator norm from L_p to L_p for $1 \le p \le \infty$. As before for $p = 1, \infty$,

$$\|T\|_p = \int_{R^d} |k(y)|dy.$$

Let us suppose that for some constant C,

1. The Fourier transform $\hat{k}(y)$ of $k(\cdot)$ satisfies

$$\sup_{y \in R^d} |\hat{k}(y)| \le C < \infty \tag{5}$$

2. In addition,

$$\sup_{x \in R^d} \int_{\{y : \|x-y\| \ge C\|x\|\}} |k(y-x) - k(y)|dy \le C < \infty \tag{6}$$

We will estimate $\|T\|_p$ in terms of C. The main step is to establish a weak type $(1,1)$ inequality. Then we will use the interpolation theorems to get boundedness in the range $1 < p \le 2$ and duality to reach the interval $2 \le p < \infty$.

Theorem 2. *The function $g(x) = (Tf)(x) = (k * f)(x)$ satisfies a weak type $(1,1)$ inequality*

$$\mu\{x : |g(x)| \ge \ell\} \le C_0 \frac{\|f\|_1}{\ell} \tag{7}$$

with a constant C_0 that depends only on C.

We first prove a decomposition lemma that we will need for the proof of the theorem.

Lemma 1. *Given any open set $G \in R^d$ of finite Lebesgue measure we can find a countable set of balls $\{S(x_j, r_j)\}$ with the following properties. The balls are all disjoint. $G = \cup_j S(x_j, 2r_j)$ is the countable union of balls with the same centers but twice the radius. More over each point of G is covered at most 9^d times by the covering $G = \cup_j S(x_j, 2r_j)$. Finally each of the balls $S(x_j, 8r_j)$ has a nonempty intersection with G^c.*

Basically, the lemma says that it is possible to write G as a nearly disjoint countable union of balls each having a radius that is comparable to the distance of the center from the boundary.

Proof. Suppose G is an open set in the plane of finite volume. Let $d(x) = d(x, G^c)$ be the distance from x to G^c or the boundary of G. Let $d_0 = \sup_{x\in G} d(x)$. Since the volume of G is finite, G cannot contain any large balls and consequently d_0 cannot be infinite. We consider balls $S(x, r(x))$ around x of radius $r(x) = \frac{d(x)}{4}$. They are contained in G and provide a covering of G as x varies over G. All these balls have the property that $S(x, 5r(x))$intersects G^c. We select a countable subcover from this covering $\cup_{x\in G} S(x, r(x))$. We choose x_1 such that $d(x_1) > \frac{d_0}{2}$. Having chosen $x_1, \ldots, x_k$ the choice of x_{k+1} is made as follows. We consider the balls $S(x_i, r(x_i))$ for $i = 1, 2, \ldots, k$. Look at the set $G_k = \{x : S(x, r(x)) \cap S(x_i, r(x_i)) = \emptyset \text{ for } 1 \le i \le k\}$ and define $d_k = \sup_{x\in G_k} d(x)$. We pick $x_{k+1} \in G_k$ such that $d(x_{k+1}) > \frac{d_k}{2}$. We proceed in this fashion to get a countable collection of balls $\{S(x_j, r(x_j))\}$. By construction, they are disjoint balls contained in the set G of finite volume and therefore $r(x_j) \to 0$ as $j \to \infty$. Since, $d_j \le 2d(x_{j+1}) \le 8r(x_{j+1})$ it must also necessarily go to 0 as $j \to \infty$. Every $S(x_j, 5r(x_j))$ intersects G^c. We now worry about how much of G they cover. First we note that $G_0 \supset G_1 \supset \cdots \supset G_k \supset G_{k+1} \supset \cdots$. We claim that $\cap_k G_k = \emptyset$. If not let $x \in G_k$ for every k. Then $d_k \ge d(x) > 0$ for every k contradicting the convergence of d_k to 0. Since $x \in G_0 = G$, we can find $k \ge 1$ be such that $x \notin G_k$ but $x \in G_{k-1}$. Then $S(x, r(x))$ must intersect $S(x_k, r(x_k))$ giving us the inequality $|x - x_k| \le r(x) + r(x_k) \le \frac{d(x)}{4} + r(x_k) \le \frac{d_{k-1}}{4} + r(x_k) \le \frac{d(x_k)}{2} + r(x_k) = \frac{3}{2}r(x_k)$. Clearly $S(x_k, 2r(x_k)$ will contain x. Since $\frac{3}{2}r(x) < d(x)$ the enlarged ball is still within G. This means $G = \cup_k S(x_k, 2r(x_k))$. Now we worry about how often a point x can be covered by $\{S(x_k, 2r(x_k)\}$. Let for some k, $|x - x_k| \le 2r(x_k)$. Then by the triangle inequality $|d(x) - d(x_k)| \le 2r(x_k) = \frac{1}{2}d(x_k)$. This implies that for the ratio $\frac{r(x)}{r(x_k)} = \frac{d(x)}{d(x_k)}$ we have $\frac{1}{2} \le \frac{r(x)}{r(x_k)} \le \frac{3}{2}$ In particular any ball $S(x_j, 2r(x_j)$ that covers x, must have its center with in a distance of $4r(x)$ and the corresponding $r(x_j)$ must be in the range $\frac{2}{3}r(x) \le r(x_j) \le 2r(x)$. The balls $S(x_j, r(x_j)$ are then contained in $S(x, 6r(x))$ are disjoint and have a radius of atleast $\frac{2}{3}r(x)$. There can be atmost 9^d of them by considering the total volume. We can choose our norm in R^d to be $\max_i |x_i|$ and force the spheres to be cubes.

□

Proof of theorem. The proof is similar to the one-dimensional case with some modifications.

1. We let G_ℓ be the open set where the maximal function $M_f(x)$ satisfies $|M_f(x)| > \ell$. From the maximal inequality

$$\mu[G_\ell] \le C\frac{\|f\|_1}{\ell} \tag{8}$$

2. We write $G_\ell = \cup_j B_j = \cup_j S(x_j, 2r_j)$, a countable union of cubes according to the lemma.

3. If we let

$$\phi(x) = \sum_j \mathbf{1}_{B_j}(x)$$

then $1 \le \phi(x) \le 9^d$ on G_ℓ.

4. Let us define a weighted average m_j of $f(y)$ on B_j by

$$\int_{B_j} [f(y) - m_j]\frac{dy}{\phi(y)} = 0 \tag{9}$$

and write

$$\begin{aligned} f(x) &= f(x)\mathbf{1}_{G_\ell^c}(x) + \frac{1}{\phi(x)}\sum_j f(x)\mathbf{1}_{B_j}(x) \\ &= f(x)\mathbf{1}_{G_\ell^c}(x) + \frac{1}{\phi(x)}\sum_j m_j\mathbf{1}_{B_j}(x) + \frac{1}{\phi(x)}\sum_j [f(x) - m_j]\mathbf{1}_{B_j}(x) \\ &= h_0(x) + \sum_j h_j(x) \end{aligned} \tag{10}$$

5. For any cube B_j with center x_j there is a cube with 4 times its size and with the same center that contains a point $x'_j \in G^c_\ell$ with $|M_f(x'_j)| \leq \ell$. The cube $S(x'_j, 10r_j)$ contains B_j. Therefore with some constant depending only on the dimension

$$|m_j| \leq C_d \ell \tag{11}$$

Moreover on G^c_ℓ, $|f(x)| \leq M_f(x) \leq \ell$. Hence

$$\|h_0\|_\infty \leq \ell + C_d \ell = (C_d + 1)\ell \tag{12}$$

On the other hand

$$\begin{aligned} \|h_0\|_1 &\leq \|f\|_1 + C_d \ell \sum_j \mu[B_j] \\ &\leq \|f\|_1 + C_d^2 \ell \mu[G_\ell] \\ &\leq (1 + CC_d^2)\|f\|_1 \end{aligned} \tag{13}$$

and therefore

$$\|h_0\|_2^2 \leq (C_d + 1)\ell \|h_0\|_1 \leq C_1 \ell \|f\|_1 \tag{14}$$

From the boundedness of T from L_2 to L_2 this gives

$$\mu\{x : |(Th_0)(x)| \geq \ell\} \leq C_2 \frac{\|f\|_1}{\ell} \tag{15}$$

6. We now turn our attention to the functions $\{h_j\}$

$$\begin{aligned} w = T[\sum_j h_j] &= \sum_j \int_{B_j} [f(y) - m_j] k(x-y) \frac{dy}{\phi(y)} \\ &= \sum_j \int_{B_j} [f(y) - m_j][k(x-y) - k(x-x_j)] \frac{dy}{\phi(y)} \\ &\leq \sum_j \int_{B_j} |f(y) - m_j||k(x-y) - k(x-x_j)| dy \end{aligned} \tag{16}$$

We estimate $|w(x)|$ for $x \notin \cup_j U_j$ where U_j is the cube with the same center x_j as B_j but enlarged by a factor $C + 1$. In particular if $y \in B_j$

and $x \in U_j^c$, then $|y - x| \geq |x - x_j| - |y - x_j| \geq C|y - x_j|$.

$$\int_{\cap_j U_j^c} |w(x)|dx \leq \sum_j \int_{\cap_j U_j^c} [\int_{B_j} |f(y) - m_j||k(x-y) - k(x-x_j)|dy]dx$$
$$\leq \sum_j \int_{B_j} |f(y) - m_j|[\int_{E_j} |k(x-y) - k(x-x_j)|dx]dy \tag{17}$$

where $E_j \subset \{x : |x - y| \geq C|y - x_j|\}$. Therefore,

$$\int_{E_j} |k(x-y) - k(x-x_j)|dx$$
$$\leq \sup_{y,j} \int_{\{x:|x-y|\geq C|y-x_j|\}} |k(x-y) - k(x-x_j)|dx$$
$$\leq \sup_{y} \int_{\{x:|x-y|\geq C|y|\}} |k(x-y) - k(x)|dx$$
$$\leq C \tag{18}$$

giving us the estimate

$$\int_{\cap_j U_j^c} |w(x)|dx \leq C \sum_j \int_{B_j} |f(y) - m_j|dy$$
$$\leq C(\|f\|_1 + [\sup_j m_j] \sum_j \mu[B_j])$$
$$\leq C_1 \|f\|_1 \tag{19}$$

7. We put the pieces together and we are done.

□